An Alternative Analysis Procedure for Wave Spectra

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ABSTRACT

I review the present procedures used to calculate the spectra of wave data. I suggest a coarser rate of sampling of the data, the use of interpolation to search for the frequency of the maximum power and the calculation of smoother spectra over broader bandwidths, and recommend the use of the method of maximum entropy because it does these tasks effectively.

RÉSUMÉ

J'examine les méthodes actuelles qui servent à déterminer les spectres des données de mesure des vagues. Je propose une façon plus simple de prélever les données, l'utilisation du procédé d'interpolation pour trouver la fréquence de la puissance maximale et le calcul de spectres plus réguliers couvrant une largeur de bande plus grande; de plus, je recommande l'utilisation de la méthode d'entropie maximale qui permet d'effectuer efficacement ces travaux.

Recommendations

1. Decimate the wave records to a lower sampling rate.
2. Lower the sampling rate of the instrument in the field to the new rate of sampling established.
3. One should calculate smoother spectra over broader bandwidths for the wave data.
4. The peak period should be interpolated from the calculated values of the spectrum over at least three bands.
5. Secondary peaks should be picked out, identified and localized whenever they are significant.
6. Round off the value of the significant wave height to the centimeter not the decimeter in the displays.
7. Use the maximum entropy method program to calculate the spectrum of wave data.

CONTENTS

INTRODUCTION

The 30 records collected at Station 140 (Zapata Ugland) are used to test various forms of processing in order to evaluate programs devised to calculate power spectra.

1. SAMPLING RATE

For any band limited signal ($Z(s) = 0 \; |s| \geq s_f$), the maximum spacing between samples is:

$$\Delta t = 1/2 \; s_f$$

s_f is the folding frequency beyond which the spectrum $Z(s)$ is zero. All observed time series are band limited and s_f is determined by the design of the instrument and not by the signal to be measured. A properly designed instrument should respond to all the frequencies of interest in the phenomenon to be investigated. The folding frequency of the instrument can be found a posteriori from the record collected with it. For the 30 wave records collected at Station 140 (Zapata Ugland) Table 1 gives the threshold frequency beyond which the power is less than 1%, .1%, .01% and .001% of the peak power of the individual spectrum. These values can be used to locate approximately the folding frequency. The shortest period found for the 30 samples is 2.2, 1.4, 1.3 and 1.1 sec for the threshold values indicated. Clearly the instrument does not respond to any oscillation with a period less than 1 second. If this is physically acceptable and no change in design of the instrument is contemplated, the present sampling rate should be adjusted to the folding frequency s_f of the instrument as it is. Take:

$$s_f \sim 1/1.3 \text{ c/s to } .01\% \text{ of the peak power}$$

$$s_f \sim 1 \text{ c/s to } .001\% \text{ of the peak power}$$

which, according to the sampling theorem, implies spacings of:

$$\Delta t = .65 \text{ sec at } .01\% \text{ and } .50 \text{ sec at } .001\%.$$

Table 1. Search for the folding frequency by identifying the threshold frequency beyond which the power is less than 1%, .1%, .01% and .001% of the peak power in the individual spectrum. (The maximum entropy method program was used to obtain these results from Station 140).

Frequency (period) beyond which the power is less than the percentage indicated.

Record	< 1%		< .1%		< .01%		< .001%	
	s (c/sec)	T (sec)	s (c/sec)	T (sec)	s (c/sec)	T (sec)	s (c/sec)	T (sec)
1	.35	2.84	.64	1.56	.72	1.39	.92	1.09
2	.31	3.23	.59	1.71	.72	1.39	.78	1.28
3	.34	2.96	.63	1.58	.73	1.37	.79	1.27
4	.33	3.00	.61	1.63	.70	1.42	.78	1.28
5	.38	2.63	.65	1.55	.74	1.36	.85	1.18
6	.42	2.40	.67	1.49	.74	1.35	> .94	< 1.07
7	.40	2.51	.67	1.50	.79	1.26	.87	1.15
8	.41	2.45	.61	1.64	.72	1.39	.78	1.29
9	.38	2.67	.65	1.53	.72	1.39	.77	1.30
10	.37	2.74	.58	1.72	.71	1.40	.76	1.32
11	.30	3.39	.53	1.89	.70	1.43	.75	1.35
12	.34	2.96	.57	1.75	.72	1.39	.77	1.30
13	.35	2.84	.60	1.68	.72	1.39	.76	1.32
14	.31	3.18	.56	1.78	.72	1.39	.76	1.32
15	.34	2.92	.61	1.64	.73	1.38	.77	1.30
16	.31	3.23	.55	1.81	.72	1.39	.76	1.32
17	.28	3.56	.49	2.05	.70	1.43	.76	1.32
18	.29	3.50	.54	1.86	.71	1.40	.76	1.31
19	.32	3.09	.54	1.86	.70	1.42	.76	1.31
20	.38	2.63	.62	1.60	.73	1.38	.78	1.29
21	.46	2.18	.68	1.46	.75	1.33	> .94	< 1.07
22	.42	2.40	.68	1.48	.75	1.34	.82	1.23
23	.42	2.40	.69	1.44	.75	1.34	> .94	< 1.07
24	.45	2.25	.69	1.44	.75	1.33	> .94	< 1.07
25	.43	2.34	.67	1.49	.75	1.34	> .94	< 1.07
26	.38	2.60	.63	1.58	.73	1.38	.80	1.25
27	.29	3.44	.62	1.62	.69	1.44	.76	1.32
28	.31	3.18	.50	2.01	.68	1.46	.75	1.33
29	.33	3.05	.57	1.75	.70	1.42	.75	1.33
30	.27	3.68	.47	2.11	.69	1.45	.75	1.34

Range of frequencies

.28 → .46 .47 → .69 .68 → .79 .75 → .94 (c/s)

Maximum frequency

.46 .69 .79 .94 (c/s)

Minimum period

2.2 1.4 1.3 1.1 (sec)

To	< 1%	< .1%	< .01%	< .001% (max power)
Δt =	1.09	.72	.633	.532 (sec)
samples/sec =	.92	1.38	1.58	1.88

∿ 3 samples/2 sec

∿ 2 samples/sec

Taking 2 samples/sec insures that more than .99999 of the power is truly sampled and that there is danger of aliasing at most .00001 of the power present in the record.

The actual sampling rate is:

$$7.5 \text{ samples/sec}$$

so that:

$$\Delta t = 2/15 = .133333 \text{ sec.}$$

It is <u>excessive</u> by a factor of 5 at .01% and of 3.75 at .001%.

Recommendation

1. Decimate the wave records to a lower sampling rate.
2. Lower the sampling rate of the instrument in the field to the new rate of sampling established.

Implementation

I note, using the original spacing $\Delta t_o = .133333$ sec, that:

$$4\ \Delta t_o = .533 \text{ sec}$$

which is very close to the sampling time step of $\Delta t = .532$ sec corresponding to the folding frequency of $s_f = .532$ c/s at the threshold of .001%. I noted also that $5\ \Delta t_o = .667$ sec is larger than the step of $\Delta t = .633$ sec corresponding to the threshold of .01%. If we wish to keep the original time step in the installed electronic which must be 1/7.5 sec and avoid interpolation of the data to obtain a new time step of .633 sec, an obvious decimation procedure is to pick 1 out of 4 from the original data.

```
Present data   1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
               . . . . . . . . . . . . . . . . .
               ↑       ↑       ↑           ↑
Decimated
  data         .       .       .           .

               1 ←Δt→ 2       3           4 rate of sampling 1 sample/.533 sec
               Δt = .533 sec                                 1.875 sample/sec
                                                             7.5   samples/4 sec
                                                             15    samples/8 sec.
```

1.1 Advantages of the new rate of sampling

With a new rate of sampling of 7.5 samples/4 sec:

a) the possible aliasing of the record spectrum is reduced to some .001% of the power present;

b) the data presently on record can be decimated simply by picking 1 out of 4 of the original data without any need of interpolation;

c) the bulk of the data in storage is reduced by a factor of 4, thus reducing the space needed and cost of maintaining the archives;

d) the sampling rate of the instrument in the field being reduced by the same factor, its span of observations may be prolonged; and

e) the decimation reduces the CPU time by the same factor each time the data is handled by the computer.

2. OPTIMUM BANDWIDTH

The bandwidth over which the spectrum is resolved is determined by the duration of the observations with respect to the set of frequencies present in the record. The <u>exact</u> spectrum is given by:

$$Z(s) = \Delta t \sum_{j=-\infty}^{\infty} z(j\Delta t)\, e^{-2\pi i j \Delta t\, s} \qquad (1)$$

where the samples $\{z(j\Delta t)\}$ are taken at time intervals satisfying $\Delta t \leq 1/2s_f$. One may sample at any rate one likes provided $\Delta t \leq 1/2s_f$: the sum (1) will always represent the true spectrum (sampling theorem). In practice one tries to sample at the coarsest rate compatible with the sampling theorem so that aliasing is all but eliminated while the number of samples is kept to a strict minimum. We have seen that for the wave data records, a rate of 15 samples/8

seconds (1.875 samples/sec) insures that less than .001% of the power will be aliased.

The concept of minimum bandwidth arises at the moment one tries to estimate Z(s) from a <u>finite</u> set of samples:

$$Z_{2N+1}(s) = \Delta t \sum_{j=-N}^{N} z(j\Delta t)e^{-2\pi i j\Delta t\, s} \tag{2}$$

where we have a set of 2N+1 observations covering an interval of time $2T = 2N\Delta t$. I call the calculated spectrum Z_{2N+1} and I presume that it approximates the exact but unknown spectrum Z(s). In fact:

$$Z_{2N+1}(s) = \int_{-s_f}^{s_f} \frac{Z(s')}{2s_f} \frac{\sin\,[\pi(N+\frac{1}{2})\,(s-s')/s_f]}{\sin\,[\pi\frac{1}{2}\,(s-s')/s_f]}\, ds' \tag{3}$$

in terms of the true spectrum. The spectrum derived from the data is therefore a smeared version of the exact one and the estimates of Z(s) at closely spaced frequencies are not independent of each other.

If the spectrum consists of only two spectral lines, the true spectrum is given by:

$$Z(s) = a_1\delta(s-s') + a_2\delta(s-s')$$

where $\delta(s)$ is the Delta function and s_1, s_2 are the frequencies at which the spectral lines exist. The spectrum estimated from the sample will be:

$$Z_{2N+1}(s) = (1/2s_f)\left[a_1 \frac{\sin\,[(N+\frac{1}{2})\pi(s-s_1)/s_f]}{\sin\,[\frac{1}{2}\pi(s-s_1)/s_f]} + a_2 \frac{\sin\,[(N+\frac{1}{2})\pi(s-s_2)/s_f]}{\sin\,[\frac{1}{2}\pi(s-s_2)/s_f]}\right]$$

The function sin nx/sin x with $|x| < \pi$ is shown in Figure 1.

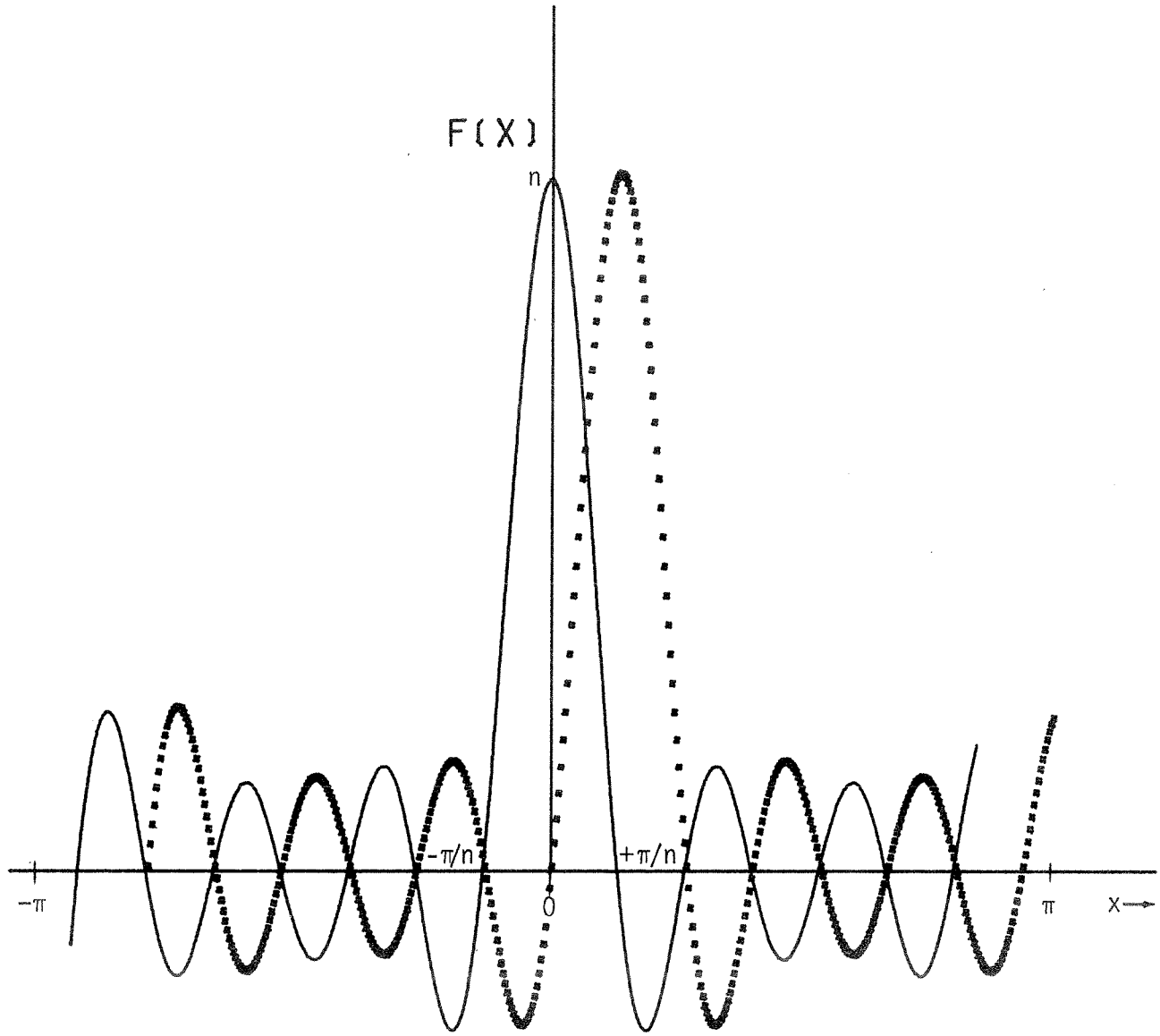

Fig. 1. Diagram explains the concept of the elementary frequency band. The dephasing (separation) between two spectral lines is proportional to the interval of observation. For the interval shown, the two lines can just be recognized from the diffraction pattern created by the truncation of the observations to n units of time. Reversely this determines the minimum spacing between spectral estimates for this interval of observations.

If the two lines lie within π/n cycles from each other, they will blur each other. They can only be resolved if their frequencies s_1 and s_2 satisfy:

$$(N+\tfrac{1}{2})\pi\left|(s_1-s_2)\right|/s_f \geq \pi$$

or

$$|s_1 - s_2| \geq 2\ s_f/(2N+1)$$

the minimum frequency spacing for which two lines can just be distinguished; it clearly depends on the interval of observation $2T = 2N\Delta t$. I call:

$$s_f/(2N+1) \equiv \Delta s$$

<u>the elementary frequency band</u>. Note:

$$\Delta s = s_f/(2N+1) \geq 1/(2\Delta t(2N+1)) \simeq 1/4T$$

since $s_f \leq 1/2\Delta t$ from the sampling theorem. It is a waste of time to try to resolve the spectrum at a spacing finer than:

$$\Delta s \simeq 1/4T \tag{4}$$

2T is the interval of observations. Equation (4) gives a guideline as to which size of bandwidth to select in the initial calculations of the spectrum; which says that it should be coarser than one over twice the interval of observations.

In practice we may calculate the spectrum over narrower bandwidths keeping in mind that the estimates are not independent of each other and that the computational task is increased unnecessarily.

The above considerations apply to a deterministic signal from which all noise is absent. If noise is present as is the case in all observations, an irreducible variability is introduced into the calculated estimates of the spectrum which does <u>not</u> decrease with more prolonged observations. It is necessary to reduce this variability by procedures which will be described later. In practice, a manipulated power spectrum will be presented over bandwidths of:

$$10 \rightarrow 20\ \Delta s$$

For wave records:

$$2T \sim 20 \text{ min}$$

$$\Delta s = 1/40 \text{ min} = 1/2400 \text{ sec} = .00042 \text{ c/s}$$

A practical bandwidth should be of the order of:

$$.00417 \rightarrow .00833 \text{ c/s}$$

if the whole series is retained. If the record is divided into 8 subsequences, the elementary bandwidth is .00336 c/s and the final spectrum should be presented over bandwidths of:

$$.0334 \rightarrow .0666 \text{ c/s}$$

The present bandwidth in the wave spectra program is:

$$.00833 \text{ c/s } (.5 \text{ c/s } /60)$$

which is <u>too fine</u> and gives a very jagged spectrum.

2.1 Smoothing of spectra

As previously mentioned, increasing the duration of the observations does not decrease the intrinsic variability of the spectral estimates although the estimates will become statistically independent at closer frequency intervals. The only way to reduce the variability of the estimates is to manipulate them by suitably devised operators called windows. The most elementary smoothing operator is the average operator. One may average:

a) successive estimates at the same frequency, or
b) estimates centered around a given frequency.

To get successive estimates of the spectrum at the same frequency for a given set of observations one must breakup the sequence into smaller sequences. Automatically this forces us into coarser resolution bands. To average estimates around a given frequency, these must be sufficiently independent statistically to make the average successful; if the true spectrum varies rapidly over these frequencies, there will be a loss of resolution.

This is unavoidable: we are caught between resolution and stability and the optimum situation consists in a compromise between the two extremes. I note first that a) called the "Bartlett" smoothing procedure, consists in the wave climate program to calculate:

$$\overline{Z}(s) = \frac{1}{8} \sum_{i=1}^{8} Z_i(s)$$

is <u>exactly</u> equivalent to calculating the spectrum of the <u>whole</u> set of data $\left\{z(j\Delta t)\right\}$ $j\varepsilon$ (-N,N) and applying the window:

$$N \left(\frac{\sin \pi s M}{\pi s M} \right)^2$$

where M = 2T/8, the length of the shorter series, to the spectral estimates so that:

$$\overline{Z}(s) = \int_{-\infty}^{\infty} Z(s-s')\ M \left(\frac{\sin \pi s' M}{\pi s' M} \right)^2 ds'$$

In practice the function $(\sin x/x)^2$ is significant only between the frequencies ± 1/M and one multiplies the discrete spectral estimates by its values at the discrete frequencies. In the case of the wave data 1/M = .003333 c/s and the spectrum could be initially calculated at a spacing of some .00126 c/s to have enough samples to weigh.

Procedure b) allows the calculation of the spectrum over narrower bands and the variance is reduced by a factor equal to the number of spectral estimates lumped into a single band. The program using procedure a) is the one currently applied to wave records. The program using b) is in general use throughout MEDS and I will call it FFT hereafter.

<u>Recommendation</u>

One should calculate smoother spectra over broader bandwidths for the wave data.

Implementation

This will be discussed after the comparison between conventional Fourier spectra and maximum entropy spectra.

3. COMPARISON BETWEEN CONVENTIONAL FOURIER POWER SPECTRA AND POWER SPECTRA DERIVED FROM MAXIMUM ENTROPY

A sequence of observations are given:

$$\{z(j\Delta t)\} \quad j = -N,\ldots,-1,0,1,\ldots,N$$

which has a spectrum:

$$Z(s)$$

which must be estimated from the samples. The conventional method of calculations is to evaluate the approximation to the spectrum:

$$Z(k\Delta s) = \Delta t \sum_{j=-N}^{N} z(j\Delta t)e^{-2\pi ij\Delta tks}$$

using the algorithms developed by Runge in 1903 which were explained in details in a set of lectures given in 1924; these were put into general use by Cooley and Tukey in 1965. The power spectrum is obtained by calculating:

$$\left|Z(k\Delta s)\right|^2 = a_k^2 + b_k^2$$

where a and b are the real and imaginary parts of $Z(k\Delta s)$ the spectral estimate at the frequency $k\Delta s$. As we have seen, the estimates are averaged over a few bands or over the same frequency using subsequences of observations in order to increase the stability.

There is a hidden hypothesis behind this procedure; it is that z is 0 beyond the span of observations, in the past and in the future. If the function $z(t)$ is defined only over the interval $(-T,T)$ its Fourier series represents a periodic extension of $z(t)$ over the whole t interval (Godin 1972).

In fact it is possible to avoid this assumption by going back to first principles. I treat the sequence $\{z(j\Delta t)\}$ as a bit of information; information is maximized by making it absolutely random. A filter is therefore devised that will transform the data into white noise. The resulting series being absolutely white, its power spectrum will be a constant, this constant being the variance of the noise. The original power spectrum (which is the one we actually want) is obtained by dividing this variance by the variance of the response of the filter used for the whitening. The procedure consists in searching for a filter whose convolution weights are:

$$1, d_2, d_3, \ldots$$

and which when applied to the data gives the sequence:

$$\sum_k d_k z\left[(j-k)\Delta t\right]$$

which should be white noise. This is done by recursion, starting with a one term predictor, two term predictor etc. Taking the cross correlation of the original observations with the filtered series, results in a set of equations determining the weights and the variance of the noise denoted by P_{k+1} where k is the number of weights in one sequence. These equations are:

$$\begin{pmatrix} c(0) & c(1) & \ldots & c(k) \\ c(1) & c(0) & \ldots & c(k-1) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ c(k) & c(k-1) & \ldots & (0) \end{pmatrix} \begin{pmatrix} 1 \\ d_2 \\ \\ \\ \\ d_{k+1} \end{pmatrix} = \begin{pmatrix} P_{k+1} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

where c(k) is the autocorrelation:

$$c(k) = \sum_{j=1}^{2N+1-k} \frac{1}{2N+1-k} z\left[(j-k)\Delta t\right] \cdot z(j\Delta t)$$

We get the power spectrum $|Z(s)|^2$ we are looking for, by noting that the spectrum of the filtered sequence is:

$$Z(s) + Z(s)d_2e^{-2\pi i\Delta ts} + Z(s)e^{-4}d_3{}^{\pi i\Delta ts} + \ldots$$

$$= Z(s)\left(1 + \sum_k d_k e^{-2\pi ik\Delta ts}\right)$$

Its power spectrum is:

$$|Z(s)|^2 \left| 1 + \sum_k d_k e^{-2\pi ik\Delta ts}\right|^2 = P_{k+1}$$

the variance or power spectrum of the noise which is determined from the autocorrelations by recursion.

The desired power spectrum is:

$$|Z(s)|^2 = \frac{P_{k+1}}{\left|1 + \sum_k d_k e^{-2\pi ik\Delta ts}\right|^2}$$

The number of lags k is found empirically: too many make the spectrum wiggly, not enough make it too smooth. The practical advantage of the maximum entropy method is that it gives an immensely better resolution of the peaks if they are present in a short sequence of data (see Figure 2). Both methods should give identical results for longer series. Using one in preference to the other is simply a matter of taste and of the quality of the programs available.

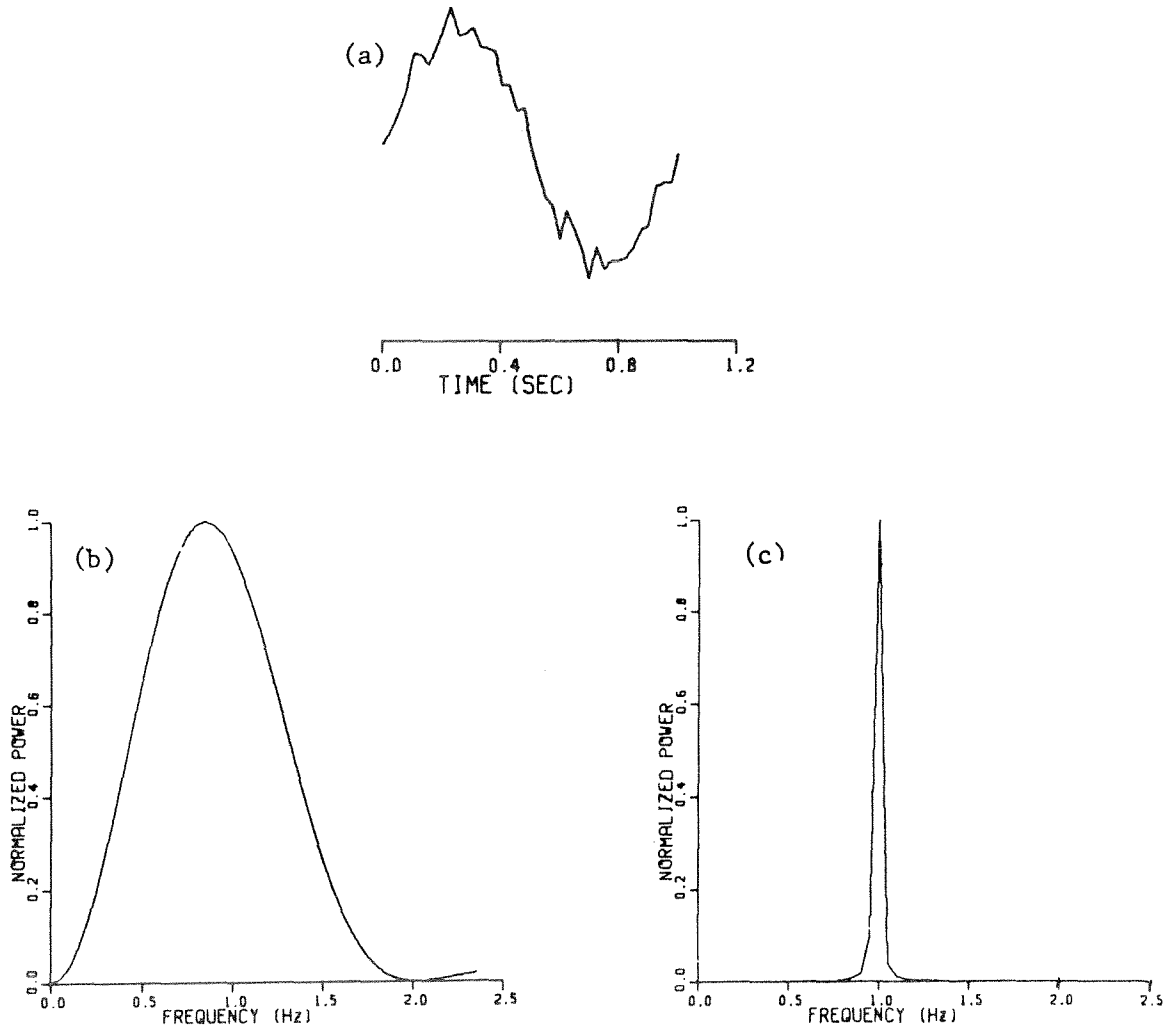

Fig. 2. Comparison between the power of resolution of the FFT and the MEM on a record consisting in a single period of a signal over which 10% of white noise has been added (from Ulrych 1972).

4. INSPECTION OF THE WAVE SPECTRA AT STATION 140 USING VARIOUS BANDWIDTHS FOR THE FAST FOURIER TRANSFORM AND THE MAXIMUM ENTROPY METHOD

I have calculated the spectra of the 30 records at Station 140 covering the interval 10:00 25/11/79 → 1:00 29/11/79 using:

a) the original wave climate program for bandwidths of
.00833 c/s the bandwidth actually used;
.01667 c/s (twice as coarse);
.0333 c/s (4 times); and
.0667 c/s (8 times);

b) the FFT program in current use at MEDS with a bandwidth of .0114 c/s (equivalent to a bandwidth of .0910 c/s in the wave climate program because the full length of the records is used in the calculation of the spectrum);

c) the Maximum Entropy Method program (MEM) as written by John Taylor. After some experiments some 43 terms were retained in the whitening filter and the spectrum was calculated over bandwidths of .004688 c/s.

The calculations did not stop at s = .5 c/s in b) and c) as in the wave program but continued until it was clear there no longer was any energy. The computing times for going through the 30 records of Station 140 were for the FFT and MEM:

	FFT (200 bands)	MEM (200 bands)	
CP	188.7	112.1	seconds
IO	139.2	80.4	seconds

Now review the spectra for appearance and for consistency in the calculation of the two parameters, the peak period T_p and the significant wave height SWH. The wave climate spectra, at the usual resolution, are peaky and probably contain spurious oscillations due to the overresolution. Those done with the coarser bandwidth of .01667 c/s give essentially the same information but with less peakiness. The coarser bandwidths do delete some information about the peaks. The FFT plots at .0114 c/s reproduce almost all the details of the standard plots of the wave climate and do exhibit some peakiness too. The MEM spectra are smoother (because of the limited number of weights), are normalized with the normalization clearly indicated, give the limits of confidence of the peaks in words and in graphic form in the log plot of the power. They have a very good appearance and had they contained the additional information of peak period and significant wave height, would give all the information needed for wave spectra. In appearance therefore, the MEM plots are much more appealing than those derived from conventional Fourier methods.

4.1 Location of peaks

It is quite important to locate the main peak of the wave spectrum accurately especially in the low frequencies because a slight change in location there affects the value of the peak period appreciably. Notice first that the practice of locating the peak period at the maximum value of the calculated spectrum is not correct: the peak of the spectrum may fall to the right or to the left of the band and in low frequencies, this may make a significant difference. Looking at record 2 of Station 140, the MEM located the maximum calculated value of the spectrum on frequency s = .098438 c/s. If we take this as the point where the spectrum is maximum as is the practice in wave climate, the peak period would be:

$$T_p = 10.16 \text{ sec}$$

In fact, the MEM program interpolates between the calculated values of the spectrum and locates the peak on s = .09575 c/s which gives a peak period of:

$$T_p = 10.44 \text{ sec.}$$

Recommendation

The peak period should be interpolated from the calculated values of the spectrum over at least three bands.

Implementation

One could use parabolic interpolation for instance. Write the curve of the spectrum as:

$$y = ax^2 + bx + c$$

where x is the frequency and a, b, c, are constants to be determined from the values of the spectrum around which the maximum is located.

At the maximum:

$$\partial y/\partial x = 0 \quad \text{at} \quad x = x_0 \quad \text{the location of the maximum}$$

$\partial y/\partial x = 0 \Rightarrow 2ax_0 + b = 0$ so that the frequency of the peak value is:

$$x_0 = -b/2a$$

Call the three frequencies x_1, x_2 and x_3. The central frequency is x_2 and we have:

$$x_1 = x_2 - d \qquad\qquad x_3 = x_2 + d$$

where d is the bandwidth. At the three points we have:

$$y_3 = ax_3^2 + bx_3 + c$$

$$y_2 = ax_2^2 + bx_2 + c$$

$$y_1 = ax_1^2 + bx_1 + c$$

The constant c is not needed. By elimination:

$$y_3 - y_2 = a(x_3^2 - x_2^2) + b(x_3 - x_2)$$

$$y_2 - y_1 = a(x_2^2 - x_1^2) + b(x_2 - x_1)$$

Using the bandwidth value d between the x's, we get:

$$x_2^2 - x_1^2 = 2dx - d^2 \qquad \text{where } x = x_2$$

$$x_3^2 - x_2^2 = 2dx + d^2$$

so that:

$$-2a = \frac{[(y_2-y_1) - (y_3-y_2)]}{d^2}$$

and

$$b = \frac{1}{2d}[(y_3-y_2) + (y_2-y_1)] + \frac{x}{d^2}[(y_2-y_1) - (y_3-y_2)]$$

so that:

$$x_o = x_2 + \frac{d}{2}\frac{[(y_3-y_2) + (y_2-y_1)]}{[(y_2-y_1) - (y_3-y_2)]} \equiv x_2 + e$$

The second term represents a correction term which locates the peak a bit more accurately to the right or the left of x_2 depending on whether it is positive or negative. Let us check this parabolic interpolation formula for record 2:

$$x_2 = .098438 \qquad y_1 = 900.6685 \qquad y_2 = 1000 \qquad y_3 = 694.8551 \qquad d = .004688$$

$$y_3-y_2 = -305.1449 \qquad\qquad (y_2-y_1) = 99.3315$$

$$e = \frac{.004688}{2}\frac{99.3315 - 305.1449}{99.3315 + 305.1449} = -.00119272$$

so that:

$$x_o = .098438 - .00119272 = .09724.$$

This value differs quite significantly from the x_o given in the listing, .0957483421, obtained by using a routine which was applied to samples over finer bandwidths; our interpolation used the coarser samples.

4.2 Secondary peaks

The diagrams indicate that there are significant secondary peaks at times in the spectrum. The MEM picks out all the peaks, the wave climate pick only one.

Recommendation

Secondary peaks should be picked out, identified and localized whenever they are significant.

Implementation

Decide on a threshold of significance of the secondary peak, 30%, 40%, 50% of the power of the main peak and whenever a peak reaches this level, identify it in the printout along with its corresponding period. There could be more than one secondary peak and the only criterion for picking them out is that they should exceed the threshold power.

4.3 Revision of the values obtained for the peak period and the significant wave height from various programs

We do not know the correct values of these variables and can only check the quality of the program calculating them by following the way they vary from record to record, either jaggedly or in a smooth natural way. In Table 2, I give the peak period obtained for the 30 records of Station 140 from the MEM without and with interpolation. The value in brackets is the period of the secondary peak whenever it is significant. Then the values derived by the FFT program and finally the values from the wave climate program from the coarsest band up to the fine one currently in use. Note a wide range in values of the peak period for the same record. Disregard the results from the wave climate program for the two coarsest bands. The double band and the single band tend to agree more closely and I assume that the match would have been even better if interpolation had been used.

Table 2. Peak period at Station 140. Estimates in seconds from various programs: a) MEM without and with interpolation, bandwidth d = .004688 c/s. b) FFT (no interpolation), bandwidth d = .0125 c/s. c) Wave climate program (no interpolation), bandwidths d = .067 c/s, d = .033 c/s, d = .0167 c/s, d = .00833 c/s (regular). Brackets () indicate secondary peak MEM.

Record	(a) MEM				(b) FFT	(c) Wave Climate			
	Without Interpolation		With			x8	x4	x2	x1
1	10.2		10.2		10.0	8.5	8.5	9.8	9.8
2	10.2		10.4		10.0	8.5	8.5	9.8	9.8
3	10.7		10.6		10.0	8.5	11.4	9.8	9.8
4	10.7		10.9		10.0	8.5	11.4	9.8	10.5
5	10.7	(6.9)	11.2	(6.8)	10.0	8.5	11.4	9.8	9.8
6	11.2	(8.9)	11.3	(8.9)	11.4	8.5	11.4	11.4	11.4
7	6.5	(10.7)	6.4	(10.6)	6.2	5.7	6.8	6.2	6.5
8	7.4	(6.5)	7.4	(6.4)	6.7	8.5	6.8	6.8	7.6
9	7.6	(10.2)	7.7	(10.0)	8.0	8.5	8.5	7.6	7.6
10	8.2	(10.2)	8.2	(10.4)	8.0	8.5	8.5	7.6	7.6
11	8.5		8.6		8.9	8.5	8.5	8.5	9.1
12	8.5	(10.2)	8.6	(10.4)	8.9	8.5	8.5	9.8	9.8
13	8.5	(11.2)	8.7	(11.4)	8.9	8.5	8.5	8.5	9.1
14	11.9	(7.6)	11.7	(7.7)	11.4	8.5	11.4	11.4	12.4
15	12.5	(7.9)	12.5	(8.0)	11.4	11.4	8.5	11.4	12.4
16	12.5	(10.2)	12.6	(10.0)	11.4	8.5	11.4	11.4	12.4
17	12.5	(9.7)	12.3	(9.9)	11.4	8.5	11.4	11.4	12.4
18	11.9	(9.7)	11.9	(9.6)	11.4	8.5	11.4	11.4	11.4
19	11.9	(7.4)	12.1	(7.4)	11.4	8.5	11.4	11.4	11.4
20	11.9	(7.4)	12.0	(7.4)	11.4	8.5	11.4	11.4	11.4
21	11.9	(9.3)	11.7	(9.3)	11.4	8.5	11.4	11.4	11.4
22	5.9	(10.7)	6.0	(10.9)	6.2	5.7	5.7	6.2	5.9
23	6.9	(10.7)	6.8	(10.4)	7.3	8.5	6.8	6.8	7.2
24	6.9	(10.7)	7.0	(10.6)	7.3	8.5	6.8	6.8	7.2
25	6.3	(10.7)	6.4	(10.5)	6.7	5.7	6.8	6.2	6.5
26	6.7	(7.9)	6.6	(7.8)	7.3	5.7	6.8	6.8	6.8
27	9.3		9.3		8.9	8.5	8.5	8.5	8.5
28	10.2		10.2		10.0	8.5	8.5	9.8	9.8
29	10.7	(8.5)	11.0	(8.7)	11.4	8.5	11.4	11.4	11.4
30	10.7	(7.4)	11.0	(7.2)	11.4	8.5	11.4	11.4	10.5

The FFT program gives results which agree at times with the wave climate results and at other times with those of the MEM program. I plotted the peak period obtained from the wave climate program and the MEM program in Figure 3. Part of the stilted behaviour of the graph of the wave climate results is probably due to the lack of interpolation of the peak. On the other hand, the results from the MEM vary smoothly and beautifully. I plotted also the peak period of the secondary peak (from MEM) when it was significant. The peak periods from the Zapata Ugland records are clustered around 11 and 7 seconds as if two types of waves tended to predominate there. The secondary peak corresponds to the alternative type when it stops predominating. In Figure 4 the plot of the peak period obtained from the double band and the fine band in the wave climate program. Table 3 gives the significant wave height SWH obtained from the same set of programs. In the case of the MEM we show two values, one from the integrated spectrum, the other from the variance of the data. Figure 5 shows a plot of the SWH from the wave climate and the MEM and in Figure 6, a plot of the SWH from the wave climate and from the variance. There is clearly no difficulty calculating this parameter. I note that in the standard plots of the wave climate spectra, the SWH is given in m and is rounded off to the dm which is 1/3 of a foot. In the files kept in feet, the SWH is rounded off to .1 foot: it means therefore a loss of accuracy, at least in the display, when converting from feet to meters.

Recommendation

Roundoff the SWH to the cm in the display spectra.

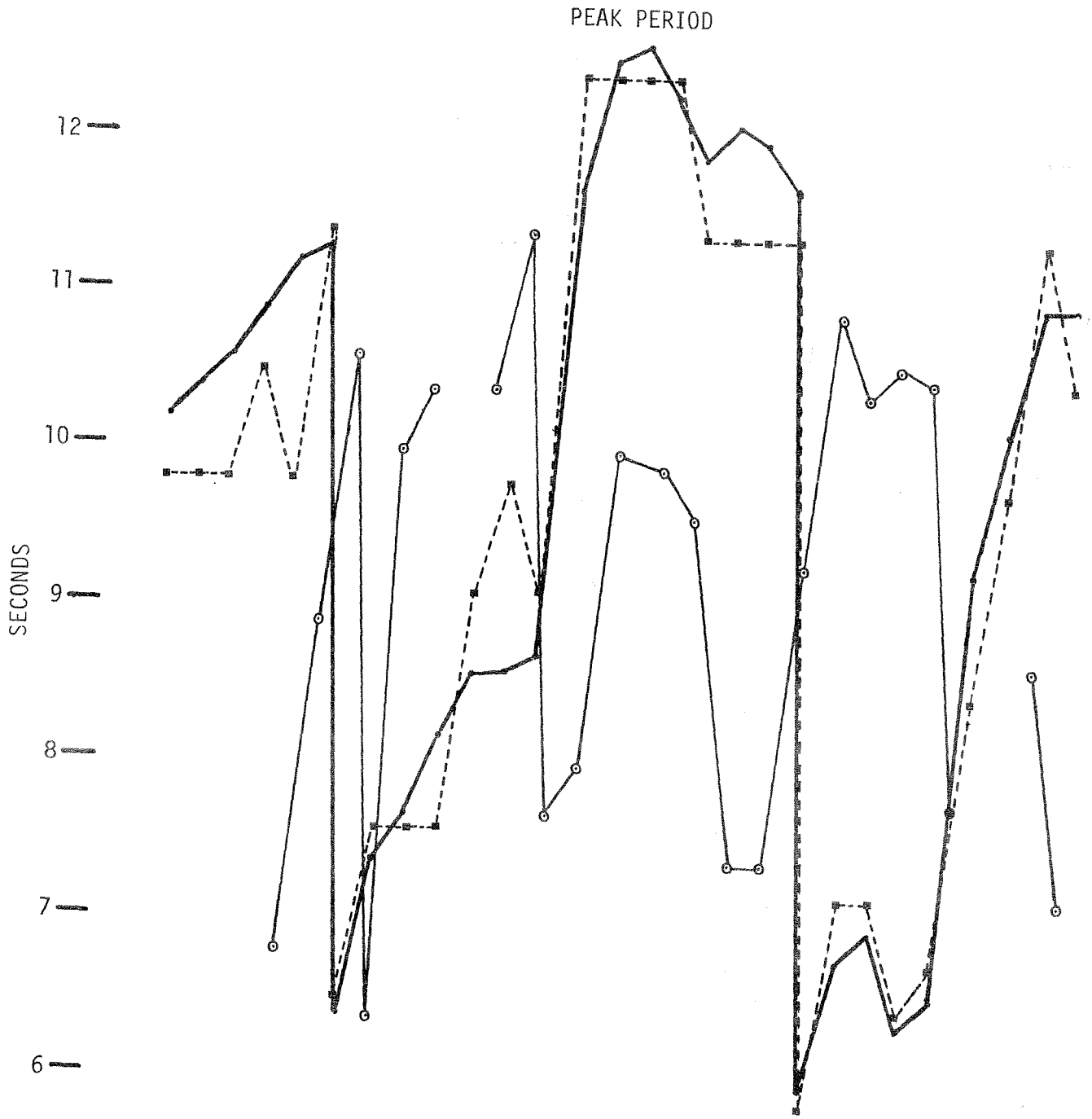

Fig. 3. Plot of the peak period for 30 records at Station 140 (Zapata Ugland) obtained from: (a) •——• the MEM (with interpolation) (b) ■-----■ the wave climate program (no interpolation) and (c) ⊙——⊙ the period of the secondary peak noted in the MEM output is also plotted when it is deemed to be significant.

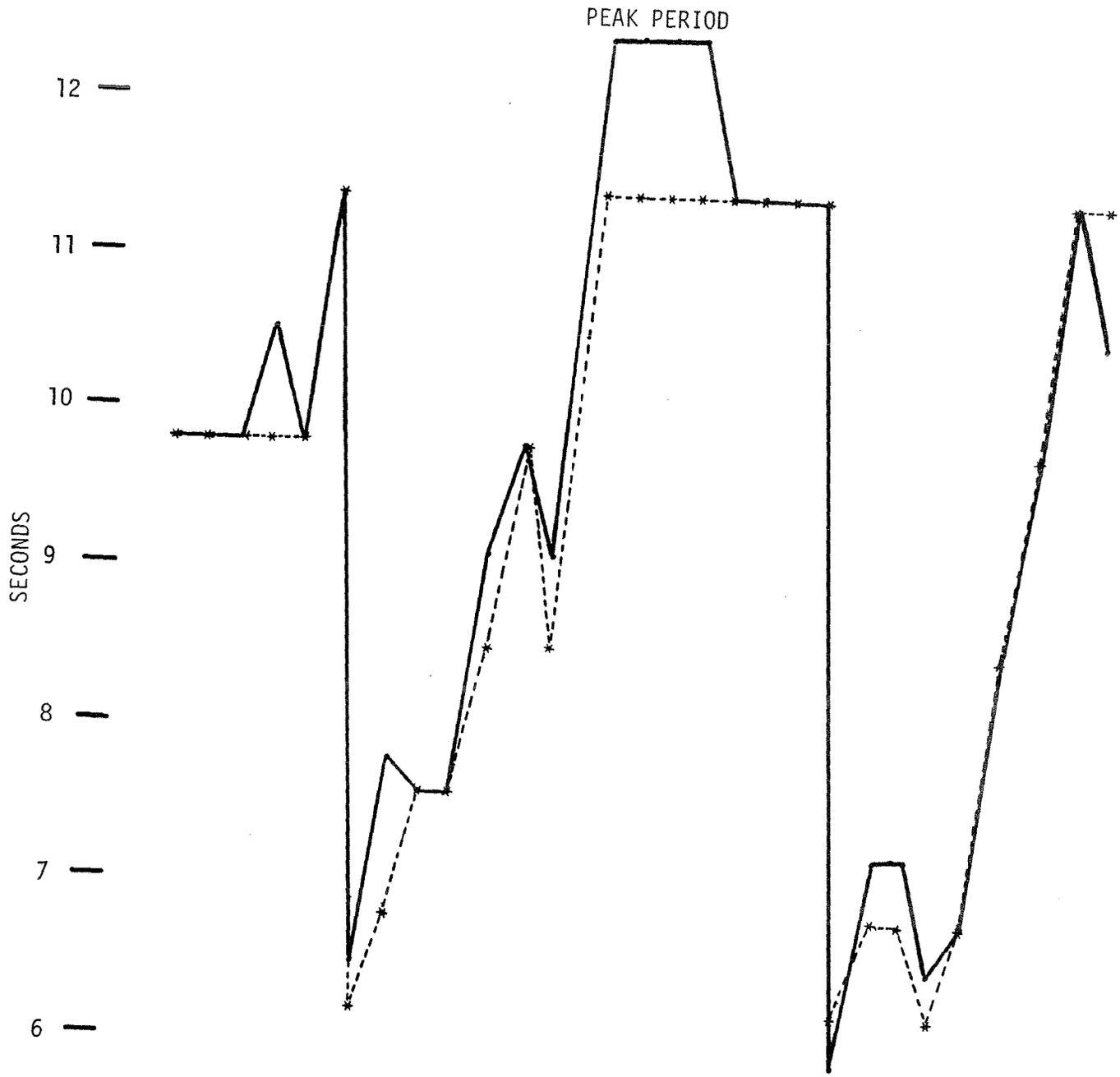

Fig. 4. Plot of the peak period for the same record evaluated from the wave climate program for single • and * bands without interpolation.

Table 3. Significant wave height at Station 140. Estimates in feet from various programs and from the variance of the data. a) The variance s^2 of the data so that SWH = 4s. b) The MEM. c) The wave climate program for the same bandwidths as for the peak period.

Record	(a) Variance	(b) MEM	(c) Wave Climate x8	x4	x2	x1
1	8.6	8.6				8.2
2	8.5	8.5	8.5	8.5	8.6	8.6
3	8.2	8.2	8.2	8.2	8.2	8.2
4	8.4	8.4	8.3	8.5	8.5	8.5
5	8.1	8.1	8.1	8.1	7.9	7.9
6	8.1	8.1	8.1	8.1	8.2	8.2
7	10.0	10.0	9.8	9.8	9.8	9.8
8	12.5	12.5	12.0	11.9	12.0	12.0
9	10.4	10.4	10.0	10.0	10.0	9.8
10	10.7	10.7	10.6	10.6	10.5	10.5
11	13.6	13.6	13.4	13.4	13.6	13.6
12	13.8	13.8	13.9	13.9	13.9	13.9
13	14.1	14.1	14.1	14.1	14.1	14.1
14	15.3	15.3	15.3	15.3	15.6	15.6
15	13.7	13.7	13.7	13.7	13.7	13.9
16	13.8	13.8	13.9	13.9	14.1	14.1
17	12.5	12.5	12.4	12.6	12.4	12.4
18	11.6	11.6	11.6	11.6	11.7	11.7
19	9.9	9.9	9.9	9.9	9.9	9.9
20	8.1	8.1	8.1	8.1	8.1	8.1
21	7.9	7.9	7.9	7.9	7.8	7.8
22	8.8	8.8	8.8	8.8	8.8	8.8
23	7.6	7.6	7.6	7.6	7.6	7.6
24	7.3	7.3	7.2	7.2	7.4	7.4
25	7.8	7.8	7.8	7.8	7.8	7.8
26	9.7	9.7	9.6	9.6	9.5	9.5
27	13.2	13.2	13.0	12.9	13.0	12.9
28	14.3	14.3	14.3	14.3	14.6	14.6
29	12.9	12.9	12.9	12.9	12.6	12.7
30	14.9	14.9	14.8	15.0	15.1	15.1

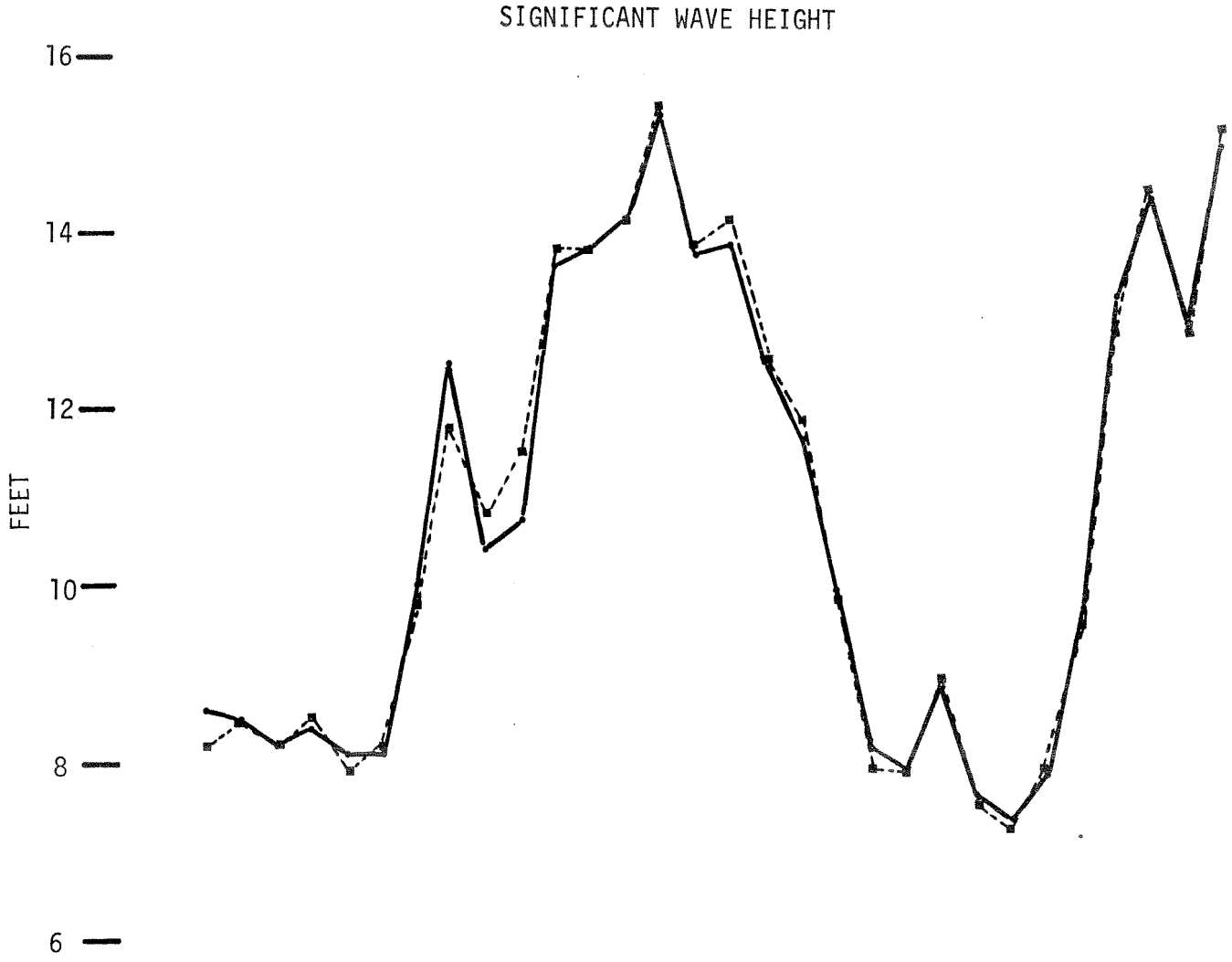

Fig. 5. Plot of the SWH obtained for Station 140 (Zapata Ugland) from: (a) •——• the MEM program and (b) ▪-----▪ the wave climate program.

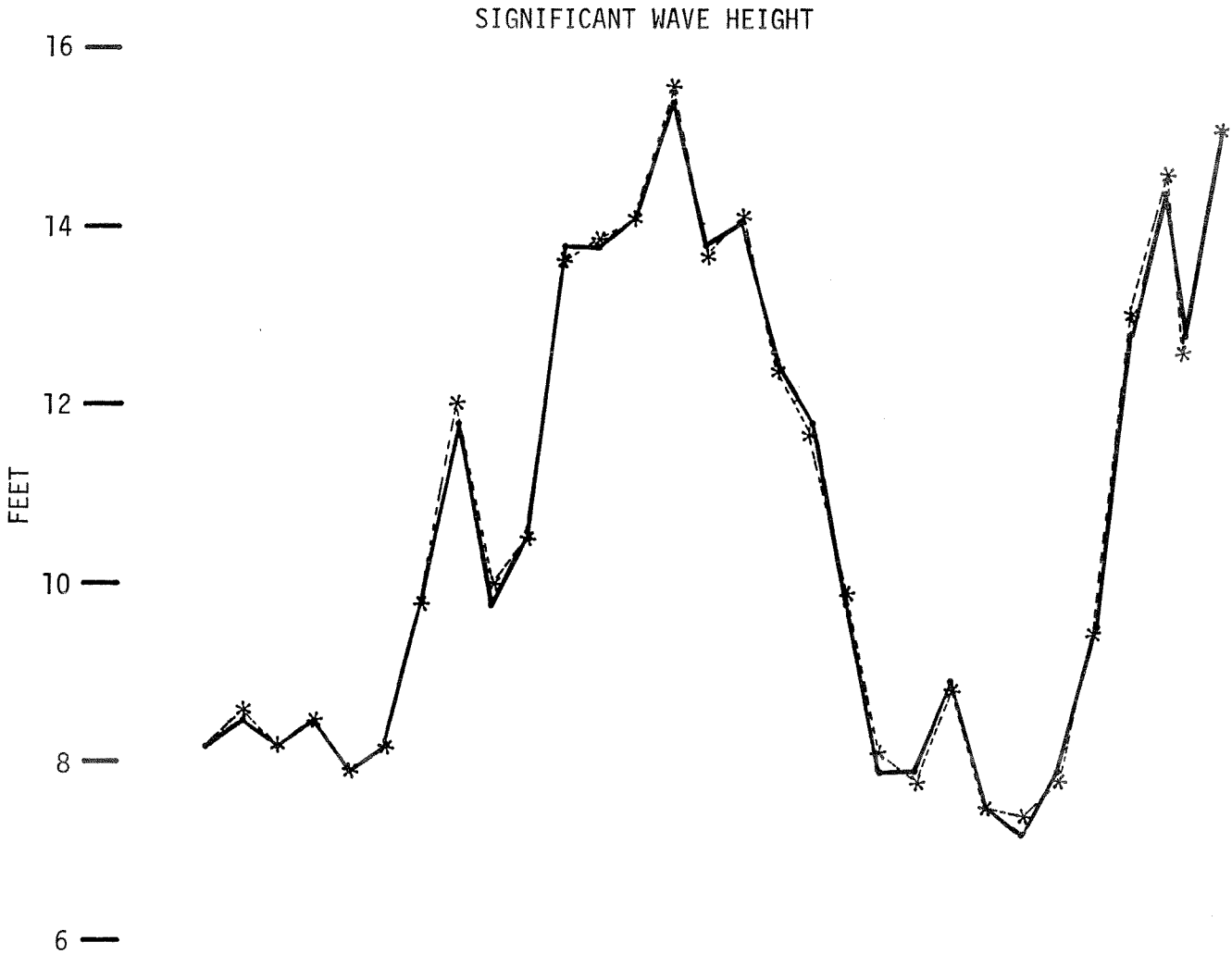

Fig. 6. Plot of the SWH obtained from • the wave climate program and * the value of the computed variance of the data. (60 bands, and d = .00833 c/s).

4.4 Discussion of the measured values of the peak period and of the significant wave height

There was no difficulty in obtaining comparable values of the SWH from the programs investigated, from the two bit programs to the most sophisticated. In fact, there is not even the need to compute the spectrum of the data to get the SWH as it can be derived directly from the variance of the observations.

But finding a good value of the peak period lies at the other extreme of difficulty. The spectrum varies very rapidly around the peak period and any crudeness in calculating the spectrum or in locating the peak will result in considerable uncertainty: we must seek the best spectral analysis program available and search for the peak with the help of the most accurate methods possible. In view of all this, I wish to make the following recommendation.

Recommendation

Use the MEM program to calculate the spectra of wave data. The reasons for this recommendation are mainly esthetic: the MEM spectra calculated are smooth and are not riddled with peaks which really add nothing to the information needed; the program is beautifully written with clear statements interspersed; the limits of confidence of the sample spectral values are given both in the printouts and in the log plots of the spectrum; normalization is used which makes it easy to compare a succession of spectra and the integrated spectrum is given explicitly. There is also a scientific reason to advocate its use: the method of maximum entropy is recognized as being especially good at locating peaks very precisely and a successful estimate of the peak period depends on this very ability of the spectrum program to do this adequately. Finally, the MEM program is already written, debugged and is ready to use; one simply has to push a button to put it in operation.

4.5 Limits of confidence of the peak

The problem of deciding whether a peak is significant or not, does not appear to be stringent in the case of wave records as peaks seem to be few and very sharp. Still, there exists a half baked theory which allows one to write limits of confidence for spectral estimates based on the duration of the observations and on the bandwidth chosen.

The variance of the power spectrum is:

$$\mathrm{Var}\left[|Z(s)|^2\right] \sim \frac{1}{2T} \int_{-s_f}^{s_f} |Z(s')|^4 \cdot W^2(s-s')\, ds'$$

where W(s) is the spectrum of the window used in the calculation of the spectrum (a window is always used when one calculates spectra, whether the analyst does it consciously or unconsciously). For a flat spectrum (noise) $|Z(s)|^2$ is a constant and can be taken out of the integral. The relative variance of the power spectrum is:

$$\frac{\mathrm{Var}\,[|Z(s)|^2]}{|Z(s)^2|^2} \sim \frac{1}{2T} \int_{-s_f}^{s_f} W^2(s')ds' = \frac{1}{2T} \int_{-T}^{T} W^2(t')dt'$$

For a Bartlett window (as unconsciously used in the wave climate program):

$$\frac{1}{2T} \int_{-T}^{T} W^2(t')dt' = \frac{2}{3} \cdot \frac{M}{2T}$$

where M is the maximum lag used (in the wave climate $M/2T = 1/8$) so that the relative variance of the calculated spectrum is $\sim$ 10%. Near a peak this may mean quite a lot. To get absolute estimates of the variance, the estimates of the spectrum of a white noise can be shown to have a chi-square distribution and these must probably be the limits shown in the MEM program.

5. SPECTRAL WIDTH

An immediate data product of the MEM program is an empirical definition of the "spectral width". The exact mathematical definition of the spectral width parameter is:

$$e_s^2 = \frac{m_o m_4 - m_2^2}{m_o m_4}$$

where the index on m indicates the order of the moment. The spectral width then depends on the inverse of the fourth moment. Vandal (1976) tried to correlate this spectral width with other wave data but he could not obtain any clear correlation. A simple empirical definition of the spectral width consists in stating that it is the frequency interval covered from 0 c/s to the frequency for which 90% of the integrated power is reached. For the 30 records at Station 140, the power falls very sharply in that region and the cutoff frequency is well defined. In Table 4 I give a listing of the periods for which the 90% power cutoff corresponds. By itself it does not mean much but when I plot it along the SWH and the peak period, a clearer pattern emerges. There is a good correlation between the SWH and $T_{90\%}$ with possible some time lag between the two (Figure 7). There is also some correlation between the peak period and $T_{90\%}$ as stronger wave activity is linked to longer periods and to a narrower range of frequencies (narrower spectral width). In Figure 8 I give a scatter diagram of the $T_{90\%}$, a measure of the spectral width and the SWH in order to follow the correlation more clearly. We get a correlation coefficient of .79 between the spectral width and the SWH while Vandal could not get any clear correlation with the theoretically defined spectral width. The regression line is:

$$T_{90\%} = 3.066 + \frac{.213}{\sqrt{2}} \text{SWH} \qquad \text{seconds}$$

for the 30 records at Zapata Ugland. There is a factor of $\sqrt{2}$ running loose in the latter two plots because it had been overlooked in the original wave climate program.

Table 4. Empirical spectral width at Station 140. Defined as the period at which the computed integrated power just reaches 90% of its total value.

Record	$T_{90\%}$ (sec)
1	4.96
2	4.64
3	4.44
4	4.54
5	4.54
6	4.10
7	4.03
8	4.85
9	4.44
10	4.54
11	4.85
12	4.85
13	5.08
14	5.20
15	5.20
16	5.47
17	5.33
18	5.47
19	5.08
20	4.64
21	3.95
22	4.18
23	3.68
24	3.81
25	3.95
26	4.35
27	5.20
28	5.08
29	4.85
30	5.33

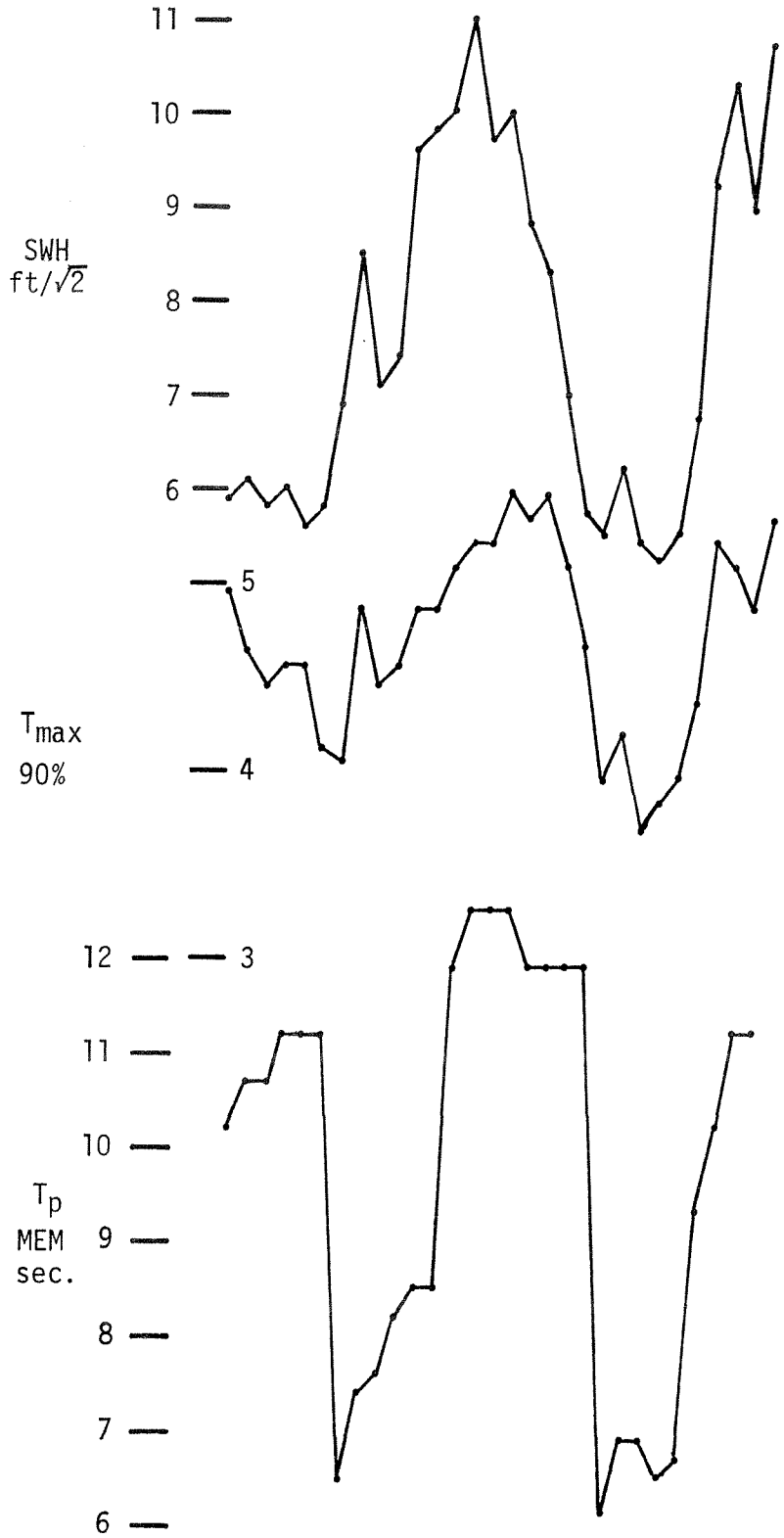

Fig. 7. Parallel plots of the SWH, $T_{90\%}$ and T_p for Station 140 (Zapata Ugland) to show the time change of the spectral width in comparison with that of the SWH and the peak period. The scale for the SWH is ft/√2.

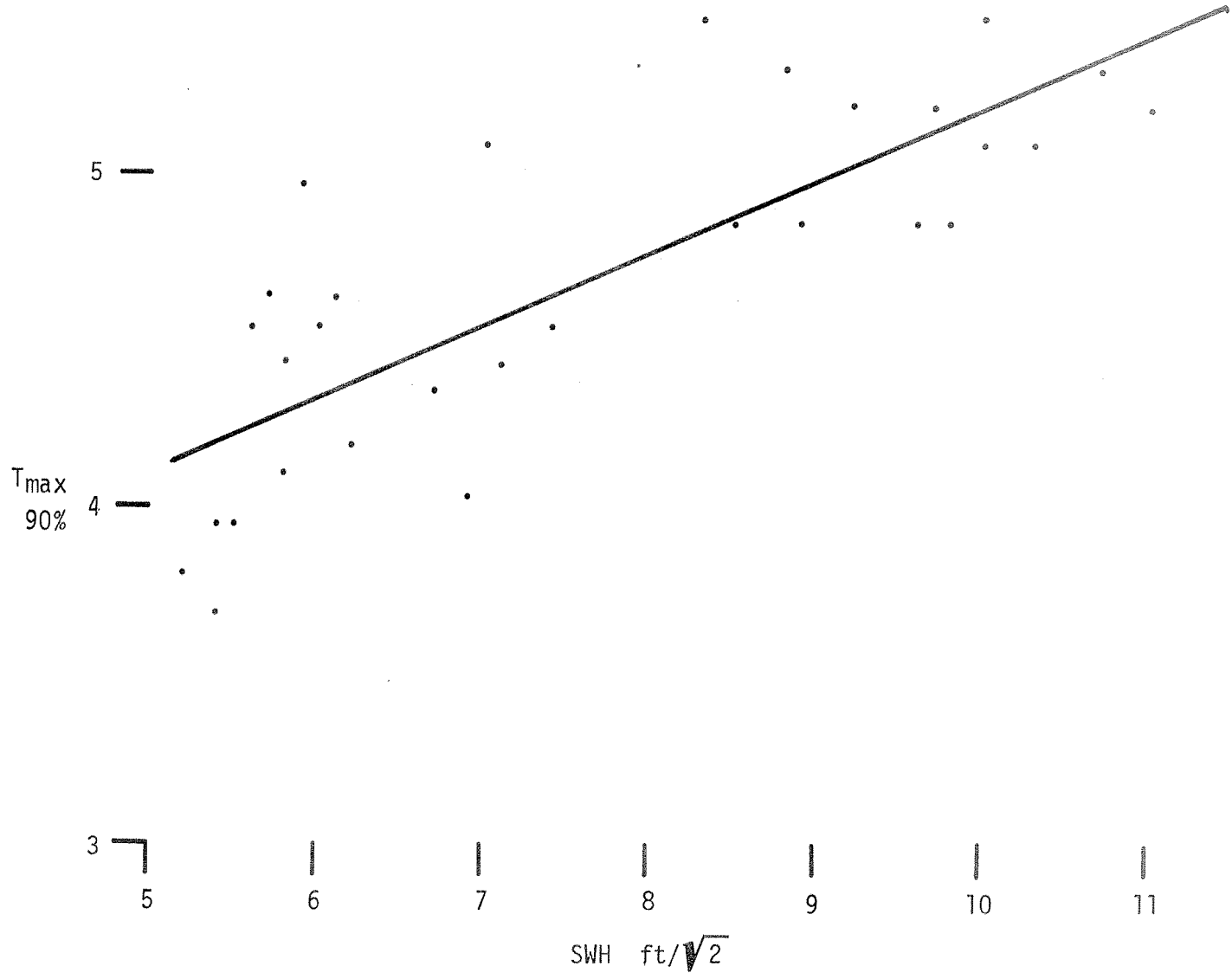

Fig. 8. Scatter diagram of $T_{90\%}$ vs SWH. The scale for the SWH is in feet/$\sqrt{2}$. The intercept is 3.066 sec and the slope is .213/$\sqrt{2}$ sec/ft.

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