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Theoretical Study of Wind-Generated Surface Ocean Wave-Current Interactions

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THEORETICAL STUDY OF WIND-GENERATED SURFACE OCEAN WAVE-CURRENT INTERACTIONS

by

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ABSTRACT

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A review is presented of wave-current interactions appropriate to wind-generated surface waves. We first survey ~~and~~ list available data sets. A discussion of mechanisms that can be important in wave-current interactions and which have significance for estimation of significant wave height is presented as well as the probable magnitude of wave-current interactions in Canadian waters. Finally, we make estimates of potential wave-current interaction effects on offshore structures and operations, and recommendations for further study and experimentation.

RÉSUMÉ

Resio, D.T. 1990. Theore~~c~~tical Study of Wind-Generated Surface Ocean Wave-Current Interactions. Canadian Contract Report of Hydrography and Ocean Sciences No. 37: iv + 29 pp.

Les interactions entre les vagues et les courants sont examinées en rapport avec les vagues de surface soulevées par le vent. Les ensembles de données disponibles sont d'abord examinés et énumérés. On présente ensuite une discussion des mécanismes qui peuvent être importants dans les interactions entre les vagues et les courants et qui sont importants pour l'estimation de la hauteur significative des vagues, ainsi que l'ordre de grandeur probable des interactions entre les vagues et les courants dans les eaux canadiennes. Enfin, des estimations des effets possibles des interactions entre les vagues et les courants sur les structures et travaux au large sont effectuées et des recommendations concernant des études et experiences plus approfondies sont présentées.

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1. INTRODUCTION

In many areas within Canadian waters strong currents and high wave conditions coexist. In spite of this, essentially all design and planning considerations for offshore structures and operations are now based primarily on wave considerations, along with wave estimates based on the assumption of no significant currents being present. The purpose of this report is to present the results of a preliminary analysis of the potential significance of wave-current interactions on prudent structure design and offshore operations. For the case of large, gravity structures, the important role of wave-current interactions may be to modify directly wave heights and periods. This in turn could influence design and operational decision-making. For compliant structures, pipelines, and vessels, combined wave and current effects on total forces, resonance, and operational limitations could directly control many aspects of design, planning, and operational considerations.

This report is divided into six parts as follows:

1. Introduction;
2. Review of wave-current interaction data sets;
3. Discussion of wave-current interaction mechanisms which are potentially significant to wave estimation;
4. Discussion of the probable magnitude of current-induced effects on waves in Canadian waters;
5. Estimates of potential wave-current interaction effects on offshore structure design and operations; and
6. Discussion of results and recommendations.

2. REVIEW OF WAVE-CURRENT INTERACTION DATA SETS

As part of this study, an extensive search was undertaken for field studies in Canadian waters which measured both waves and currents simultaneously. Unfortunately, even though some were planned, no such data have successfully been collected in Canadian offshore areas as yet. However, several published data sets were found for other areas; and references for these have been included in Appendix A.

A primary problem in using available data sets to help understand the effect of currents on wave conditions is the fact that it is extremely difficult to determine if the waves would have been different had no currents been present. In a controlled experiment, one should have measurements of wave conditions with and without currents or over a range of independently varying current conditions.

Because of this shortcoming, most studies of wave-current interactions to date have relied on theoretical assessments and modeling technologies to assess the role and magnitude of wave-current interaction effects. Two notable examples of this are the study by Stronach et al. (1986) which examined wave-current interaction effects in coastal waters of Northern British Columbia and the recent version of the WAM model by Tolman (1989).

3. DISCUSSION OF WAVE-CURRENT INTERACTION MECHANISMS WHICH ARE POTENTIALLY SIGNIFICANT TO WAVE ESTIMATION

3.1 MONOCHROMATIC WAVE CONSIDERATIONS

In the last twenty years or so, many good theoretical works have been written on the subject of wave-current interactions. A partial listing of such works is given in Appendix B. Also, an excellent annotated bibliography up through 1982 is given by Peregrine et al. (1983); and, a brief synopsis of some of the essential problems in this area is given in Peregrine and Jonsson (1983).

The purpose of this report is not to repeat the details of all of these reports available in the literature, but merely to document the primary aspects of wave-current-interaction theory as it pertains to potential problems in the development and utilization of resources in offshore and coastal areas. Appendix B and the material in Peregrine et al. provide references to most of the important publications in this area for individuals who are interested in specific detailed aspects of this problem.

Let us first examine a small amplitude monochromatic wave. The governing equation for such waves is

$$1.\quad Z(\underline{x},t) = a(\underline{x},t)\ \exp(-iP(\underline{x},t)),$$

where $a(\underline{x},t)$ is the space-time varying amplitude function, $P(\underline{x},t)$ is the phase function, and $Z(\underline{x},t)$ is the local surface elevation. The wavenumber vector, $\underline{k}$, can be defined as the gradient of the phase function, i.e.

$$2.\quad \underline{k} = \nabla P(\underline{x},t);$$

and the angular frequency, ω, is given by

3. $\omega = -\frac{\partial P}{\partial t}.$

Equations 2 and 3 can be combined to yield an equation for conservation of phase

4. $\frac{\partial \underline{k}}{\partial t} + \nabla\omega = 0.$

Equations 1-4 remain valid even when a wave field is superimposed on a non-zero current that is constant with depth. If one takes as a basis a reference coordinate system that is fixed to the bottom, one can define an intrinsic frequency, ω_0, for waves in that coordinate system. The apparent frequency, ω, for these waves when traveling on a superimposed current is given by

5. $\omega = \omega_0 - \underline{U}(\underline{x},t)\cdot\underline{k}$

where $\underline{U}(\underline{x},t)$ is the vertically-uniform, superimposed current.

In deep water the phase velocity can be written as

6. $c = \sqrt{\frac{g}{k}}$

where g is the acceleration due to gravity. Combining equations 5 and 6 provides us with an estimate of the ratio of the phase velocity in a region with a current to that in a region with zero current,

7. $c_u = c_0(1-\frac{U}{c_0})$

where the subscript "u" denotes the value of the apparent angular frequency in the region with non-zero currents and the subscript "0" denotes the value in the region with zero currents. From equations 6 and 7 we see that the wavenumber for a wave in an area with a superimposed current can be written as

8. $k_u = k_0(1-\frac{U}{c_0})^2$

where the subscript "u" again denotes the value of a parameter in the superimposed current system.

For simplicity let us first examine a current with its direction of flow parallel to the direction of wave propagation. In the general case of a flow which is not parallel to the wave propagation direction, one must consider the fact that the group velocity vector (the direction of energy propagation) is not along the same direction as the phase velocity vector (the direction of wave propagation). In this simple situation, as shown by Longuet-Higgins and Stewart (1961, 1964), the ratio of the energy in the region of currents can be written as

9. $E = E_0\, c_0^2/c(c+2U)$

for the case when the continuity of mass is preserved by upwelling from below. For the case when continuity is preserved by lateral spreading, Longuet-Higgins and Stewart show that the energy in the region of currents can be written as

10. $E = E_0\, c/(c+2U)$

Figure 1 shows a comparison of the amplification of wave energy in waves of 5 and 15 seconds encountering a range of opposing current velocities. As can be seen there, the two theories yield fairly similar results over a practical range of current speeds. Also, the importance of wave-current interactions is clearly shown, since energy amplification factors are seen to be potentially quite large.

Srokosz (1985) provides a good basic description of the effect that current variations with depth have on the above theoretical considerations. Analytical solutions for an arbitrary vertical current profile do not exist; however, if we allow for a simple linear shear in the current profile, i.e.

11. $\underline{U}(\underline{x},t,z) = \underline{U}(\underline{x},t,z) + bz$

where b is a real-valued slope, then the deep-water dispersion relationship can be written as

12. $(\omega-kU_0)^2 = [gk-b(\omega-kU_0)]$

In most cases, researchers have found that the shear in deep-ocean currents in the surface mixed layer is fairly small (Sanford _et al_., 1987; Gordon, 1982) and can be represented approximately as a slab-like profile. Consequently, this modification may not be too significant for a first approximation to the wave-current interaction problem. However, additional research into the

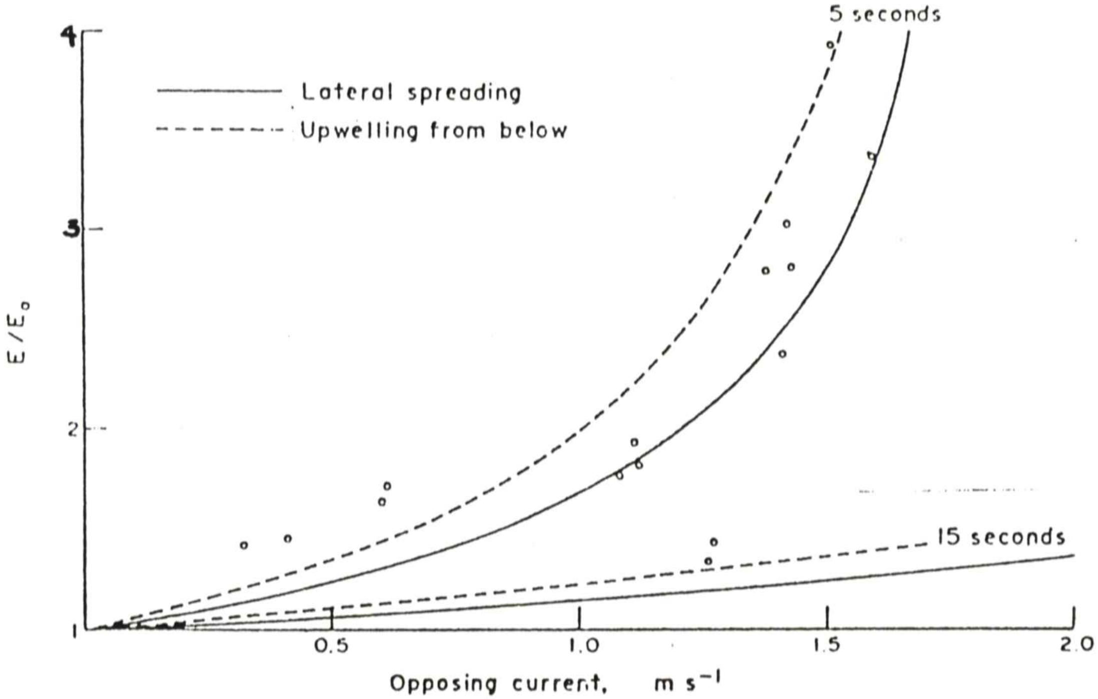

Fig. 1. Fractional energy increases for waves with 5- and 15-second periods using equations (5.1) and (5.2). Experimental points are also shown; these were determined using $\gamma = 0{\cdot}4$ s m^{-1} in equation (4.1).
(from: Schumann, 1976).

sensitivity of waves to naturally occurring current profiles appears to be needed.

If we now return to the simpler situation with a current that is constant with depth, we can examine the effects of varying angles between the current direction and the wave propagation direction. In this situation it is necessary to solve the complete system of Hamiltonian equations which describe the propagation of waves in the geometric-optics approximation. These can be written as

$$13.\quad \frac{\partial \underline{x}}{\partial t} = \frac{\partial \omega}{\partial \underline{k}}$$

$$14.\quad \frac{\partial \underline{k}}{\partial t} = -\frac{\partial \omega}{\partial \underline{x}}$$

$$15.\quad \frac{d\omega}{dt} = \frac{\partial \omega}{\partial t}$$

In such a physical system wave action, defined as

$$N = E/\omega$$

is conserved whereas wave energy is not (Whitham, 1965, 1967), and the relationship of energy propagating from a region of zero current into a region with non-zero currents can be written as

$$16.\quad E = \frac{\omega k}{\omega_0 k_0} E_0$$

Analogous to the situation with waves propagating into shallow water, waves will be refracted (wave rays will be bent) toward the region of slower propagation speed (Figure 1).

3.2 SPECTRAL WAVE CONSIDERATIONS

As shown in Appendix C, monochromatic and spectral concepts of wave propagation including the effects of refraction and shoaling, even if they are written in different forms, have a basic equivalency, as expected. However, in a monochromatic wave train, apparent wave frequency and wave propagation direction are single-valued parameters at any given point. In spectral waves, the wave train is assumed to consist of a continuum of energy distributed over a range of frequencies and propagation directions; consequently, the method of solution typically must consider energy propagation in frequency-direction space as well as in physical x-y space. Hence, a general form for action conservation in spectral waves passing through a region of variable currents can be written as (Tolman, 1989)

$$17. \quad \frac{\partial N(f,\theta)}{\partial t} + \nabla\cdot[(\underline{c}_g+U)N] + \frac{\partial}{\partial\theta}[c_\theta N] + \frac{\partial}{\partial f}[c_f N] = \frac{\Sigma S_k}{\omega}$$

where

N is the action density at frequency f and propagation direction θ;
∇ is a differential operator in x-y space;
$\underline{c}_g$ is the action propagation vector in x-y space (frame of reference moving with $\underline{U}$);
c_θ is the propagation velocity in spectral direction space;
c_f is the propagation velocity in spectral frequency space; and
S_k is the kth source-sink function for wave variance.

Equation 17 is formulated in terms of action density because, as previously pointed out, action is conserved in wave fields passing through variable current fields, whereas energy is not. Details of the derivation of the various propagation terms on the left hand side of equation 13 can be found in Mei (1983) and Tolman (1989).

3.2.1 Negligible Source Terms

Most research into the interactions of wave spectra and currents has examined the simple case in which it is assumed that all source terms on the right hand side of equation 17 are negligible relative to the propagation effects. Only in the last two to three years have spectral models which include non-zero source terms been used to simulate combined propagation-source effects. Let us begin here by examining the restrictive case in which all source terms are identically zero before moving to the more complicated situation.

It can be shown that if one had a "unidirectional" spectrum, i.e. a spectrum in which energy existed continuously through a range of frequencies but only at a single propagation direction, the behavior of such waves could be viewed in the context of equation 10. In such waves, the total energy given by

$$18. \quad E_0 = \iint E(f,\theta)\,dfd\theta = \iint \frac{N(f,\theta)}{\omega}dfd\theta$$

could be observed to behave in a fashion consistent with the independent treatment of each individual frequency element. For waves such as this (perhaps narrow-band swell), it is straightforward to visualize the variations in wave energy, as one follows the waves through a region of variable currents, as a simple analogue to equation 17.

For the case of a spectrum with energy distributed continuously through some range of both frequency and propagation direction (i.e. a directional spectrum), some insight on the evolution of wave energy is given by the work of Lavrenov and Ryvkin (1986). They derive analytical expressions for the variation of wave energy in a wave spectrum which propagates into an adverse current. Suffice it to say that the situation here becomes quite complex and the resulting wave heights in the region of adverse currents is very sensitive to the directional distribution of wave energy in the incident wave spectrum. Using a spectrum given by

19. $$E_0(k_0,\theta_0) = \begin{cases} \frac{8}{3\pi} \cos^4(\theta_0) E_0(k_0) & \text{for } |\theta| \leq \pi/2 \\ 0 & \text{for } |\theta| > \pi/2 \end{cases}$$

$$E_0(k_0 = E(\omega)\frac{\partial\omega}{k_0} = 1/2(n+1) m_0 \omega_p{}^n \omega^{-(nH)} (g/k_0)^{1/2} \text{ x}$$

$$\exp\{-\frac{n+1}{n}(\frac{\omega_p}{\omega})^n\}$$

where ω_p is the angular frequency of the spectral peak and n is a spectral width parameter, they showed that the general form of the spectrum inside the current region could be written as

20. $$E(z,\theta,U) = \frac{8}{3\pi}\alpha^n m_0(n+1)(1-\mu^2)^2 \exp\{-\frac{n+1}{n}\} \text{ x function of angle}$$

where z is given by

$$z = -U\sqrt{k/g}$$

and α represents a combined wave-current parameter. Based on equation 20, they derived an expression for the ratio of the wave height in the current region to the wave height in the region with zero currents as

21. $$\frac{H}{H_0} = f(\alpha,s) = [4\pi \int_0^\infty\!\!\int E(z,\theta,U)d\text{-}dz]^{1/2}$$

where s is the dimensionless current velocity given by

$$s = -U\omega_p/g.$$

Figure 2, from Lavrenov and Ryvkin, shows that wave height amplification factors ranging from 0.5 to 2.5 could be expected for typical wind sea spectra encountering currents in nature.

In general one must rely on numerical models to provide a more precise description of wave heights in a region of variable currents, even in the simplified case with no significant source terms. It should be noted at this point that all of the treatments of directionality given here have assumed that the variations in current speeds have been primarily one dimensional. If two-dimensional variations of currents are treated then the effects of divergence and convergence of wave rays must be considered. This adds considerable additional complexity to the analysis and can realistically be examined only via the application of numerical models when actual current fields are considered.

3.2.2 Kinematic Considerations

The addition of source terms to the analysis of wave behavior, in areas of space-time varying currents, adds another significant complication to our attempts to obtain parameterized solutions that can be easily understood in terms of their design and operational implications. One of the problems one encounters in attempting to do this is that, on one hand, propagation effects occur (for practical purposes) instantaneously, whereas source functions typically involve transient terms when a spectrum is perturbed from an equilibrium condition (as one might expect when crossing into a region of significantly different current speeds and/or directions). Very little research has investigated the importance of various source terms in waves propagating across a region of varying currents. Even in numerical models assumptions usually are made that effectively limit the primary response to propagation effects, since the form of the source functions used in available models are only weakly dependent on currents.

In Kitaigorodskii et al. (1975) it is suggested that the equilibrium range of a wind wave spectrum will always adjust to a consistent form in wavenumber space, i.e.

$$22a. \quad F(k) = \frac{\alpha k^{-3}}{2}$$

$$22b. \quad E(f) = \frac{\alpha g^2 f^{-5}}{(2\pi)^4} \phi(\omega_h)$$

where α is a universal constant and $\omega_h = \omega(h/g)^{1/2}$, with h being water depth. Gadzhiyev et al. (1978) present field evidence which supports the contention that both the "finiteness of depth and the presence of currents lead to an apparent decrease in the absolute value of the exponent in Phillips' law," when written in form 22b. They suggest a generalized form for the spectrum given by

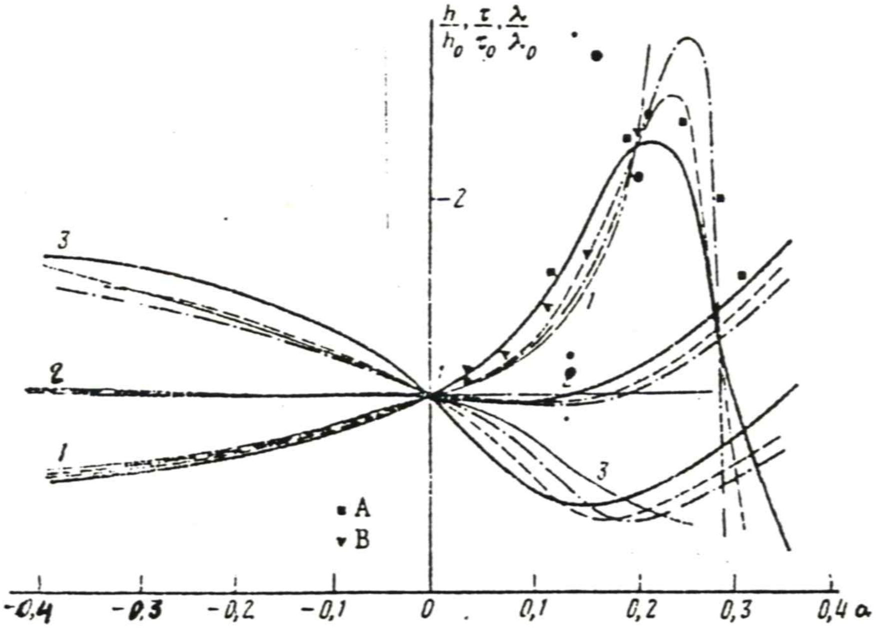

Fig. 2. Relative variation of the mean wave parameters in a current h/h_0 (curves 1), τ/τ_0 (curves 2), and λ/λ_0 (curves 3) for the parameter values $n = 5$ (heavy lines), $n = 8$ (dashed lines), $n = 10$ (dot-dashed lines), and $n = 15$ (thin continuous lines). A represents field data for h/h_0 by Zhevnovatyy; B are Scripps Institute data.
(from: Lavrenov and Ryvkin, 1986).

23. $$E(f) = \frac{\alpha g^2 f^{-5}}{(2\pi)^4} \phi(\omega_h, \omega_c)$$

where $\omega_c = \omega U/g$.

Kitaigorodskii *et al*.'s results were based on the theory by Phillips (1958) in which it was believed that wave breaking was the single mechanism responsible for maintaining this equilibrium range. Later works (Mitsuyasu, 1968; Hasselmann *et al*., 1973) have shown that the observed values for energy densities in the equilibrium range were dependent on wind speed and the frequency of the spectral peak. In this context, a balance between wind inputs and nonlinear energy fluxes through the spectrum appear to be responsible for maintaining a dynamic balance in the equilibrium range. This implication has since been found to be consistent with the observed behavior of waves propagating into shallow water in analyses of extensive data sets from around the world (Bouws *et al*., 1985, 1987). Subsequently, using a theoretical approach based on a dimensional analyis of wave-wave interactions and typical wind inputs, it has been found that the form of the wave spectrum might in the equilibrium range be better described as (Kitaigorodskii, 1983)

24. $$E(f) = \frac{\alpha' v g}{(2\pi)^3} f^{-4}$$

where v is the wind speed and α' is a dimensionless constant. This latter form is also consistent with the findings of Toba (1978). Resio (1987, 1988) has shown that the observations of changes in energy levels in, as well as the characteristic form of, shallow-water wave spectra are con-sistent with an extension of wave-wave interaction theory. Resio and Perrie (1989) extend the work of Resio to derive a slightly different form for the equilibrium range in a wind wave spectrum given by

25. $$E(f) = \alpha_* (v^2 c_p)^{1/3} g f^{-4}$$

In a recent study by Resio and Perrie (1990), it has been shown that in a wind wave spectrum the characteristic relaxation time toward an equilibrium level, at frequencies higher than 1.3 times the frequency of the spectral peak, is only about 20 wave periods or so. Hence, for a 10-second wave period, following even a large perturbation away from its equilibrium value, the dynamic mechanism responsible for maintaining an equilibrium will force the spectrum back into an equilibrium in only about 200 seconds.

The fundamental form for wave-wave interactions (in any depth of water or with superimposed non-zero currents) is

$$26. \quad \frac{\partial n_1(\underline{k}_1)}{\partial t} = \iiint [n_1 n_3 (n_2 - n_4) + n_2 n_4 (n_1 - n_3)] \; \delta(\underline{k}_1 + \underline{k}_2 + \underline{k}_3 + \underline{k}_4)$$

$$\times \; \delta(\omega_1 + \omega_2 + \omega_3 + \omega_4) \, d\underline{k}_2 d\underline{k}_3 d\underline{k}_4$$

where δ is the Dirac delta function. Equation 26 can be solved numerically or following the analysis of Resio (1988) can be solved for characteristic spectral shapes in the equilibrium range and the fluxes that are associated with these nonlinear interactions. A thorough discussion of the theoretical basis and the interpretation of these fluxes can be found in Resio (1988). Following along those lines it can be shown that, for a condition of nondivergence of these fluxes to exist, a spectrum must have the form

$$27. \quad n(\underline{k}) \sim \omega k^{-3}$$

which is consistent with the observations of Forristall (1981), Donelan et al. (1985), and Dobson et al. (1989). The magnitude of the fluxes found in the studies by Resio (1987, 1988) and Resio and Perrie (1989) along with equation 27 suggests that the equilibrium constraints on a spectrum may play an important role in modifying total wave energy when waves propagate from one current regime to another. In fact, from this generalized analysis of the dynamic balance in the equilibrium range (analogous to the arguments of Kitaigorodskii et al., 1975), one can hypothesize that the propagation of waves from an area of essentially zero currents into a region of opposing (following) currents should produce analogous results to waves propagating into shallower (deeper) water. Using equation 26 as a scaling index, similar to the "TMA" concept in shallow water (Bouws et al., 1985), one finds that, for storm waves (i.e. reasonably steep waves where the nonlinear fluxes are large), wave energy and wave heights might be expected to scale as

$$28a. \quad E_u = E_0 (k_{p_0}/k_{p_u})^2 \; \phi_1(\theta_u, \theta_w, \underline{g})$$

$$28b. \quad H_u = H_0 (k_{p_0}/k_{p_u}) \quad \phi_2(\theta_u, \theta_w, \underline{g})$$

respectively, where k_p is the wavenumber of the spectral peak, θ_u is the current direction, θ_w is the mean wave direction, $\underline{g}$ describes the spectral shape, and as in previous treatments the subscript "u" refers to the region with non-zero current and "0" refers to the region with zero current. Of course, this can be simply extended to consider the case for propagation between regions of differing currents. For the simple case of current vectors that are colinear with the direction of wave propagation, this reduces to

29a. $E_u \approx E_0(1-r)^2$

29b. $H_u \approx H_0(1-r)$

where r is the ratio U/c. For the case of a wind sea the directional spectrum can be rather broad; consequently, equations 29a and 29b are not strictly valid. However, at least as a first approximation, these equations might be used with the substitution of the wavenumber of the spectral peak, k_p, for k and the use of the mean wave direction in place of the monochromatic wave direction.

3.3 EXAMINATION OF THE EXPECTED MAGNITUDE OF RESPONSE IN WAVES DUE TO THE EFFECTS OF CURRENTS ON VARIOUS PROPAGATION AND SOURCE TERMS

Observations of spectra in nature indicate that storm waves typically have a much broader spectral width than is found in waves that have propagated some distance from their region of generation (i.e. swell). Also, wave steepness in swell is usually very much reduced from its values in an area of active wave generation, which implies that wave-wave interaction effects should be minimal. Given the additional fact that wind inputs for swell can be neglected, at least for practical purposes on the scale that we are treating here, we can hypothesize that propagation effects should dominate wave train evolution in swell passing through variable currents. Hence, we can expect amplifications of wave heights in the range of 0.5 to 3. Values in this range have been reported in studies by Mallory (1974), Schumann (1976), Battjes (1982), Lambrakos (1981), and Gonzalez (1984). Since all of these studies were performed in situations dominated by swell rather than locally-generated waves, it would appear that the neglect of source terms in analyses of swell-current interactions is justifiable as a first approximation.

The treatment of wind waves on variable currents is more complex since one must consider the effects of a larger directional spread in the wave energies, as well as the effects of several source terms such as

1. wind input,
2. nonlinear wave-wave interactions,
3. wave breaking, and
4. bottom dissipation (in shallow water).

It is likely that the direct effects of currents on wind inputs can be neglected in most areas, since the relevant scaling parameter for such interactions would be U/v which is usually in the range of 0.01 to 0.04. This is consistent with the analysis of Banner (1990) who also notes that direct current effects on wind inputs will be small. The effects of nonlinear wave-wave interactions and wave breaking are somewhat coupled via the analysis of the equilibrium range described previously. Hence, it is likely that this term will be significant in certain situations. The observations of Vincent (1979) in the southern North Sea are in fact consistent with the analysis presented here. In that study the ratio of wavenumbers in waves generated on following and opposing currents (tidal currents with speeds of slightly greater than 1 metre per second) can be shown to be about 0.85 (for an opposing current) to 1.17 (for a following current), assuming a wave period of 10 seconds. Using equation 29b, this suggests that the variations in wave height should be in the range of 0.85 to 1.17, which is consistent with Vincent's observations of tidal modulation of the waves in his study area. It should be noted that Vincent's observations indicate a decrease in wave heights on an opposing current and an increase of wave heights on a following current. This is opposite from the predictions based on propagation effects alone and is explained by Vincent in terms of a relatively one-dimensional propagation theory, but his theoretical results did not compare well with the observations. Since the waves in this area are primarily local wind waves, it seems more likely that an interpretation involving changing source-term balances in the equilibrium range is more appropriate for these observations.

A recent study by Tung and Cho (1989) used the Miche breaking criterion to investigate expected changes in the directional spectrum of waves due to combined current and depth effects. Their numerical findings suggest that the effect of currents on wave breaking (S_{diss}) can be significant.

4. THE POSSIBLE MAGNITUDE OF CURRENT EFFECTS ON WAVES IN SELECTED CANADIAN WATERS

A useful scaling parameter for analyzing the magnitude of current effects of waves is

$$30. \quad r = U/c_p$$

where c_p is the phase speed associated with the spectral peak, as was found in previous discussions of both propagation effects and source term modifications due to the presence of variable currents. For open-ocean areas along both the East Coast and West Coast of Canada, typical spectral peak frequencies range from about 0.06 Hz to 0.16 Hz, which translates into phase speeds associated with the spectral peak of 24.5 m/sec to 9.4 m/sec. Typical large-scale current velocities in these offshore areas are found to be 0.2 to 0.8 m/sec, with some localized areas at times exceeding 3 m/sec. Near coasts, in constricted straits, and in areas of high tidal fluctuations, velocities can exceed 6 m/sec.

Using the above analyses and the assumption that propagation effects dominate the evolution of wave heights, typical amplifications of waves encountering large-scale currents in open-ocean areas can be found to be in the range of 0.8 to 2.5. These numbers are probably typical for modifications of swell passing through regions of variable currents, with the 2.5 values occurring only in localized areas with strong gradients in the ambient current. In certain coastal areas and straits, modifications to the wave heights could be even larger.

If source-term effects are significant, it is more difficult to assess the overall pattern of changes in wave heights. Probably, near the edges of currents in areas of strong gradients, propagation effects might still dominate over source-term effects in wind waves, since the propagation effects are essentially instantaneous and the source terms can require many minutes to force a return to an equilibrium. However, as the waves propagate further into the current system, the relaxation to an equilibrium balance should begin to be very important. In this context, wave height amplifications due to combined source-term and propagation effects will probably exhibit a pattern similar to that discussed above for swell near areas of strong current gradients. In the interior of a given current regime and in regions of small current gradients, the variation in waves undergoing active generation will probably be consistent with equation 29b. In this latter case, variations of wave heights in the range of 0.8 to 1.3 might be expected in open-ocean areas and in the range of 0.6 to 1.8 in coast areas and in straits.

It is recognized that the analysis of expected variations in wave heights presented here only represents a "scoping" study of the potential problem and not a definitive presentation of the actual expected variations. The one area of significant departure from many of the classic earlier studies is that an analysis of current-induced source-term effects on the waves has been added. The prevailing theory for wave height modifications in waves propagating into shallow water for many years was that it depended essentially only on propagation effects. Consequently, shoaling and refraction were believed to be the primary mechanisms affecting the behavior of such waves. Computer programs based on ray-tracing (linear theory) and more involved finite-difference and finite element approaches, which neglected all source terms, were developed and tested against monochromatic wave theory or monochromatic laboratory data. These models have been found to provide reasonable results in many tests with swell, in which case the wave heights almost invariably become larger in shallow water due to the effects of shoaling. This is because refraction effects are almost always smaller than shoaling in areas with relatively smooth bottom contours. However, the results of Bouws et al. (1985), Resio (1988), and Vincent (1979) have clearly demonstrated that the tendency toward an equilibium in wave spectra is a dominant mechanism affecting wave height evolution in wind-wave spectra propagating into shallow water. Due to these findings it seems inadvisable to neglect the potential for such effects in wave-current interactions, particularly since these effects appear to be opposite in sign to the expected propagation effects, analogous to the case of waves coming into shallow water.

As a final point in this section, it is worthwhile to point out that some important problems relative to operational and design considerations might not be only related to variations in wave height parameters related to total energy, such as the significant (or zero-moment) wave height defined as

$$H_{m_0} = 4.01[\iint E(f,\theta)dfd\theta]^{1/2}$$

Other important variations in wave forces and wave characteristics are related to modifications of the combined current and wave velocity profiles (which can dramatically affect overturning moments, etc.) and modifications in the expected behavior of individual wave heights. This latter factor could be of tremendous importance in defining extreme wave crest height for structure design. Mallory (1974) has collected substantial data which suggest that so-called "freak" waves often occur in regions of large current gradients off the coast of South Africa in the Agulhas Current. Several theoretical accounts of the behavior of nonlinear wave groups have shown that spatial variations in wave heights could trigger the growth of large-amplitude fluctuations in wave envelopes (Alber, 1978). The behavior of these phase-linked nonlinear wave groups has been found to be governed by the nonlinear Shrodinger equation (Alber and Saffman, 1978; Yuen and Lake, 1975); and, based on analyses in laboratories, these groups have shown to exhibit potentially extremely destructive capabilities. Therefore, this area of wave-current interaction research should not be neglected.

5. ESTIMATES OF POTENTIAL WAVE-CURRENT INTERACTION EFFECTS ON OFFSHORE STRUCTURE DESIGN AND OPERATIONS

Based on the analysis shown here, it appears that the expected magnitude of current effects on design and operational wave heights could lie in the range of up to 200 percent. Since design forces and operational thresholds can be significantly altered by such variations, it would appear that such effects cannot be safely neglected.

Also, it is important to note that, in many areas around the globe, compliant structures are beginning to be used as safe, economical alternatives to gravity-based structures. If this trend continues and the concept begins to be explored for applications in Canadian waters, an improved understanding of waves, currents, and wave-current interactions could become critical to a wide range of design considerations. If such information were not available at that time, costly delays and incorrect estimation of design parameters might result.

In coastal and shallow-water regions, local currents can become even larger than those in deep-water, unbounded areas. Consequently, it is expected that certain operations (laying pipelines, coastal shipping, fishing, etc.) in such areas can be dramatically affected by wave-current interactions. Furthermore, as environmental concerns for understanding and protecting coastal areas continue to increase, it will become even more important to understand the anture and effect of these interactions (transports of oil spills, contaminated bottom materials, medical wastes, etc.).

6. DISCUSSION OF RESULTS AND RECOMMENDATIONS

Based on the theoretical considerations presented in this scoping report, it seems very likely that wave-current interactions need to be better understood to estimate design and operational criteria. An important question that critics of this conclusion might raise is "if the effects of currents are so important, how can past wave hindcasts (which neglect currents) have been so successful in reproducing measured wave conditions?" The answer to this is twofold. First, wave-current interactions are probably only important in certain sub-areas of Canadian waters and may be unimportant in many areas where comparisons have been made. Second, calibration efforts in wave hindcasts usually have a number of "tuning knobs." Hence, it is not surprising that hindcasts can be locally tuned (via altering model coefficients or slight modifications to uncertain wind fields) to match measured wave conditions rather well.

Based on this scoping study, the following recommendations are made:

1. A detailed theoretical analysis of the effects of currents on nonlinear wave-wave interactions and wave breaking should be undertaken;

2. A series of field efforts should be designed for sites where independently varying currents (tidal as opposed to wind-driven) of large magnitude relative to wave phase speeds occur and a large range of spectral shapes are expected;

3. A series of laboratory experiments should be designed to investigate specific wave-current interaction effects;

4. Due to the importance of understanding wave-current interactions, all measurements should be made with techniques which have been proven to be extremely accurate in their measurements of both currents and wave spectra, particularly in the presence of each other.

5. Due to the increasing importance of compliant structures in the development of offshore resources, a better understanding of the vertical structure of forces in combined wave-current systems is essential; consequently, laboratory and field investigations of this phenomena are required;

6. The potentially critical role of destructive nonlinear wave groups is such that an investigation of the increased likelihood of their occurrence (freak waves) in areas with variable currents is definitely advisable; and

7. Due to the relatively large expense of field experiments and laboratory tests (compared to numerical experiments), a set of numerical sensitivity tests should be conducted. These tests should be used to investigate a series of more realistic scenarios than the simple magnitude arguments presented here. They should also be used to test the relative importance of various source and propagation terms when their effects are acting in concert.

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APPENDIX A

APPENDIX A

LIST OF REFERENCES FOR WAVE-CURRENT INTERACTION DATA SETS

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APPENDIX B

APPENDIX B

PARTIAL LISTING OF REFERENCES ON THE SUBJECT OF WAVE-CURRENT INTERACTIONS

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APPENDIX C

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COMPARISON OF SPECTRAL AND MONOCHROMATIC REPRESENTATION FOR REFRACTION AND SHOALING

1. Since most engineers are more familiar with the monochromatic representations for refraction and shoaling, a comparison with the spectral representations is included here, along with a comparison of the differences predicted by the two methods. The traditional conservation of energy flux in monochromatic wave is written as

$$(c_g Eb) = \left(c_g Eb\right)_o \tag{C1}$$

where c_g is the group velocity, E is wave energy, b is the width between two adjacent wave rays, and the subscript o refers to initial conditions (deep water). Figure C1 shows two wave rays coming into a coast. The spacing between the rays can be shown to be equal to a projection of an orthogonal onto a plane parallel to the depth contours; hence the ratio of b to b_c can be defined in terms of the cosines of the angles θ and θ_o.

$$\frac{b}{b_o} = \frac{\cos \theta}{\cos \theta_o} \tag{C2}$$

where θ_o and θ are the initial angle and subsequent angles along the wave ray, respectively. This ratio b_o/b is commonly referred to as the refraction coefficient and the ratio c_{g_o}/c_g, the shoaling coefficient. From Equation C1 and C2, the energy along the ray can be written as

$$E = \left(\frac{\cos \theta_o}{\cos \theta}\right)\left(\frac{c_{g_o}}{c_g}\right) E_o \tag{C3}$$

2. If we now examine a narrow spectrum approaching in the limit a delta function in angle and frequency, Equation 12 yields

$$E = \int_0^{\infty} \int_0^{2\pi} E_2(f,\theta) \frac{(cc_g)_o}{(cc_g)} \left| \frac{\partial\theta}{\partial\theta_o} \right| d\theta_o df_o \tag{C4}$$

where $E_2(f,\theta)$ is the initial energy distribution given by

$$E_2(f,\theta) = E_o\delta(\theta-\theta')\delta(f-f')$$

where θ' is the central angle of the angular distribution and f' is the frequency of the spectral peak. Integrating over both delta functions gives

$$E = E_o \frac{(cc_{g_o})}{(cc_g)} \left| \frac{\partial\theta}{\partial\theta_o} \right| \tag{C5}$$

From Snell's law we have:

$$\sin\theta = \frac{c}{c_o} \sin\theta_o \tag{C6}$$

Differentiating both sides with respect to θ gives

$$\cos\theta = \frac{c}{c_o} \cos\theta_o \left| \frac{\partial\theta}{\partial\theta_o} \right|^{-1} \tag{C7}$$

which when rearranged provides an evaluation for the partial derivative in Equation C5.

$$\left| \frac{\partial\theta}{\partial\theta_o} \right| = \frac{c}{c_o} \frac{\cos\theta_o}{\cos\theta} \tag{C8}$$

Substituting this into Equation C5 gives

$$E = E_o \frac{\cos\theta_o}{\cos\theta} \frac{c_{g_o}}{c_g} \tag{C9}$$

which is identical with Equation C3, as it should be for this case.