Department of Finance

Ministère des Finances

Working Paper

Document de travail

Elasticities of Substitution in Computable General Equilibrium Models

by

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Working Paper 2012-02

* The author would like to thank, without implication, Claude Lavoie, Aled ab Iorwerth, Meenakshi Basant Roi, Madanmohan Ghosh, Bruno Roy, Michael Willox and Michael Garrard for their helpful comments and suggestions. Any and all errors are the responsibility of the author.

Abstract

This paper examines the different types of elasticities of substitution and estimation techniques used for calculating elasticities of substitution with a focus on their use in computable general equilibrium models. Two measures of elasticities of substitution are analyzed, the Allen elasticity and Morishima elasticity. The Morishima elasticity is found to better characterize the ability to substitute between inputs. Three techniques for estimating elasticities are analyzed, the trans-log, the generalized Leontief and the linear logit methods. The linear logit estimation technique is found to be the preferred method for estimating elasticities of substitution. Elasticity estimates from the literature are compiled and can be used as a guide for calibrating computable general equilibrium models.

Résumé

Cette étude examine différents types d'élasticité de substitution ainsi que les méthodes utilisées pour les estimer dans le but de les utiliser à l'intérieur de modèle d'équilibre général calculable. Deux élasticités de substitution sont analysées : l'élasticité de type Allen et l'élasticité de type Morishima. Il est démontré que l'élasticité de type Morishima est meilleure pour déterminer la substitution entre les intrants. Trois techniques pour estimer les élasticités sont analysées: la méthode translogarithmique, le Léontief généralisé et la méthode des logit-linéaires. La méthode des logit-linéaires est démontrée comme étant la meilleure façon d'estimer les élasticités de substitution. Différentes estimations d'élasticité provenant de la littérature sont compilées et peuvent être utilisées comme guide afin de calibrer les modèles d'équilibre général calculable.

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1. Introduction

Computable general equilibrium (CGE) models are tools used to analyze the economic impacts of various policy instruments or economic events. Traditionally, the models rely heavily on elasticity parameters entered into the models. Elasticities of substitution, for example, measure the ease or difficulty of substituting between inputs in production. Elasticities therefore play an important role in determining the economic cost impact of any policy.

Although these elasticities drive model results, the source of these elasticity parameters is rarely from the modeller themselves. Instead, most CGE models rely on elasticity parameters taken from either other CGE models or from econometric estimation from the literature. This paper assesses the different types of elasticities of substitution, the methods of calculating elasticities of substitution and the general properties of elasticities of substitution. This analysis is illustrated by examples taken from CGE models used to look at the use of energy in the economy.

The elasticities of substitution used in most CGE models are Allen elasticities (AES). The use of AES as an approximation of the direct elasticity of substitution has been called into question since the late 1970's. Economists now favour a measure known as the Morishima elasticity of substitution (MES). The MES directly measures the shape of the isoquant providing substantially more information on the substitutability between inputs than the AES. Both measures are analysed in detail in this paper. Although the Morishima elasticity is superior conceptually, it does not dramatically change the values of the elasticities.

There are several methods of econometrically estimating elasticities of substitution. The transcendental logarithmic (trans-log) cost function is widely used to estimate energy-capital elasticities of substitution. However, because of the importance of fuel switching in energy research, a number of studies estimate inter-fuel elasticities of substitution using the more advanced linear logit cost function. This paper describes these estimation techniques in detail and provides a guide for estimation possibilities should re-estimation be deemed necessary.

A significant number of studies at the aggregate level have been compiled to show properties of different estimation techniques and measurement types. These results can be used as a guide in assessing the validity of the elasticities currently used in CGE models. Unfortunately there are only a limited number of studies which estimate elasticities at the sectoral level. The literature summarized here is however useful in putting bounds on reasonable elasticities in any sensitivity analysis undertaken with CGE models.

The paper is broken into seven sections. Section two briefly describes the role of elasticities in CGE models. Section three provides detailed information on the different measures of the elasticities of substitution. Section four examines econometric methods used to estimate elasticities of substitution. Section five summarizes known econometric issues which arise when estimating elasticities of substitution. Section six provides estimation results from a number of studies from the literature. Section seven concludes.

2. The Role of Elasticities in CGE Models

Technically, the elasticity of substitution is defined as the change in the input ratio caused by a change in the marginal rate of technical substitution. Put more generally, elasticities of substitution measure the ease of substitution between inputs. For CGE models each industry requires specific elasticities of substitution at each stage of the production technology. The constant-elasticity-of-substitution (CES) is the underlying production technology in many CGE modes. Cobb-Douglas and Leontief functions are special cases of the CES function where elasticities of substitution are defined as 1 and 0 respectively. Estimates for these elasticity values are often taken from the literature and are open to criticism for being outdated, poorly estimated or as often is the case, arbitrary.

Figure 1 presents a generic production structure based on CES, Leontief, and Cobb-Douglas production functions and input nestings. At each stage of production an elasticity of substitution is required. As an example, suppose that the use of a particular fuel in some industry were penalized. Working up the nested structure, firms would substitute away from that fuel into others. The ease with which the firm can do this depends on technology, and this is captured in the estimate of the elasticity. Working further up the nesting structure, there will also be some substitution out of energy and into capital, through the use of more energy-efficient machinery and equipment. Taking the overall economy into account, there will also be some shift away by other manufacturing firms from using the good of that industry. The more difficult these substitutions are—or, in other words, the lower the elasticity is —the higher the economic cost of adjustment will be to adjust to the optimal production structure in the presence of the penalty.

The desirable characteristics of elasticity estimates for a computable general equilibrium model are:

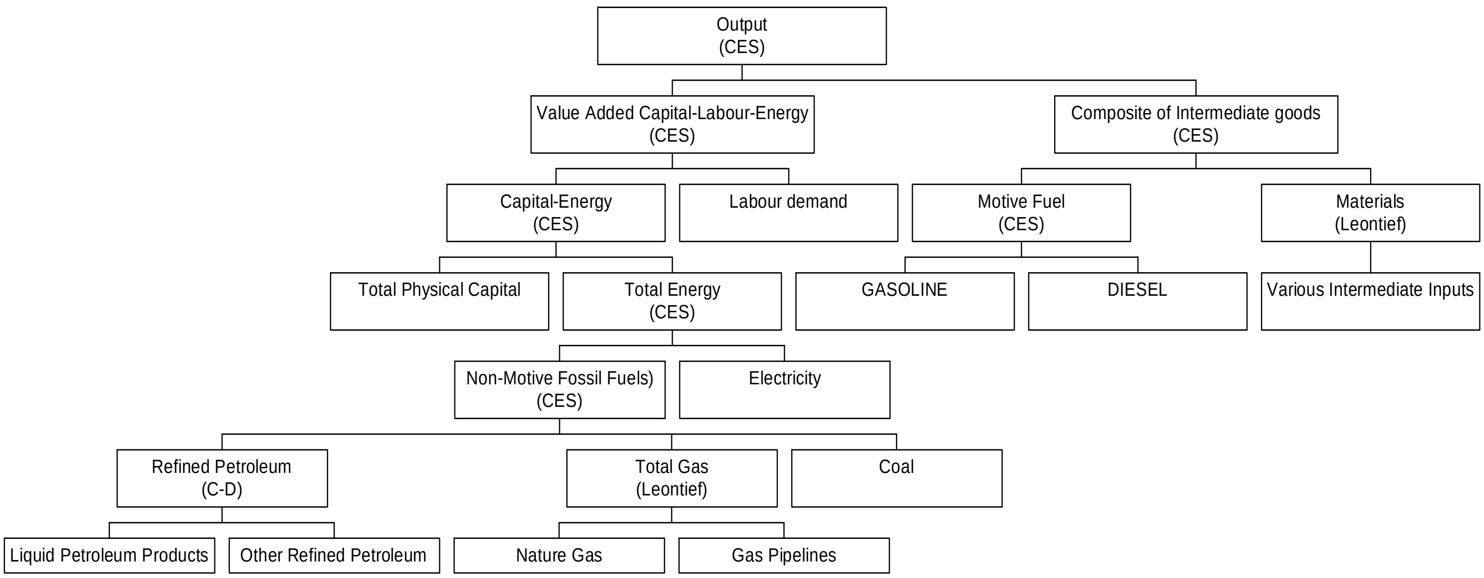
1. **Symmetry** The structure of most CGE models requires that all elasticities be symmetric: that is that the degree of substitutability from input A to input B must be the same as from input B to input A.;

2. **Substitutability** All elasticities must be non-negative so that all inputs are either substitutes or perfect complements.

3. **Consistency** Elasticities of substitution should consistently represent the long-run ease of substitution between inputs.

The symmetry and substitutability characteristics are requited for use in CES based models and are often imposed on the system of regression equations used to estimate elasticities. Consistency is a harder, but critically important characteristic to satisfy. CGE models that are calibrated to a steady state require long run elasticities of substitution to produce credible steady state results. The task of acquiring long run estimates can be difficult as their estimation relies heavily on the characteristics of the underlying data. These characteristics play a central role in choosing the optimal method of estimating and calculating elasticities of substitution and will be the focus of the remaining sections.

Figure 1: Generic Firm Production Structure[1]



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[1] This structure is similar to those found in Weyant and Hill 1999, "The Costs of the Kyoto Protocol: A Multi-Model Evaluation", The Energy Journal, Special Issue.

3. Explanation of the Technical Aspects

The measurement and estimation of elasticities of substitution has been well documented since Hicks introduced the theory in 1932. Since that time two separate measurements of substitution have dominated the literature, the Allen elasticity of substitution (Allen and Hicks (1934) and Allen (1938)) and the Morishima elasticity of substitution (Morishima (1967) and Blackorby and Russell (1975)).

Allen elasticities of substitution have become the subject of debate in the literature. Morishima elasticities of substitution (MES) have been identified as a more accurate measure of the ease of substitution between inputs than Allen elasticities (Blackorby and Russell (1989)). Support for Morishima elasticities has been gaining momentum with a growing number of researchers reporting Morishima elasticity estimates rather than or in conjunction with Allen elasticities (Falk and Koebel (1999), Nguyen and Streitwieser (1997) and Thompson and Taylor (1995)).

In brief, the primary difference between the AES and MES is that the MES measures the response in the ratio of inputs—rather than in only one of the inputs—from a price change in one of the inputs (Thompson and Taylor (1995)). The ability to measure the change in the input ratio provides more information about the curvature of the iso-quant which measures the ease of substitutability between inputs (Blackorby and Russell (1989)).

To give a broader explanation we turn to the direct calculation of the elasticity of substitution, derived from the production function at $\tilde{x}$, which is given by:

$$\sigma_{ij} \equiv \frac{d\ln(x_j / x_i)}{d\ln(f_i(\tilde{x}) / f_j(\tilde{x}))}$$

The elasticity of substitution is defined as the change in the input ratio from a change in the marginal rate of technical substitution. Unfortunately, in order to calculate the direct elasticity of substitution the functional from of the production function must be known. Estimating the exact structure of the production function is often not feasible and typically results in the use of very restrictive functional forms such as CES, Cobb-Douglas or Leontief production functions. Given the difficulty in estimating the production function it is generally easier to estimate the cost function instead, and then utilize duality to recover properties of the underlying production function (Jehle and Reny (2001)).

What results from estimating the cost function is an indirect measure of the elasticity of substitution. Both AES and MES are functions of the cross price elasticities of demand. Once the cross price elasticity of demand is estimated, the AES or MES calculation is applied to construct an estimate of the elasticity of substitution which is designed to emulate the curvature of the isoquant. A number of other measures of substitution have been proposed instead of the standard Morishima and Allen measures including the Pigou elasticity of complementarily, the Antonelle elasticity of complimentarily the Hicks-McFadden elasticity of substitution (Stern (2004)) and the bilateral elasticity of substitution (Thompson (1996)). However these elasticities are not commonly estimated and are will not be examined further[2].

3.1 Allen Elasticities of Substitution

The Allen elasticity of substitution is characterised by the following expression:

$$AES_{ij}(y,p) = \frac{C(y,p)C_{ij}(y,p)}{C_i(y,p)C_j(y,p)}$$

where C is the total cost function with input prices p and output y as independent variables and the subscripts i and j indicating the derivative of the cost function with respect to price of inputs i or j. The AES can also be written as a function of the cross price elasticity between inputs i and j and the total cost share of the price-changing input,

$$AES_{ij}(y,p) = \frac{\eta_{ij}(y,p)}{S_j(y,p)}$$

where the cross-price elasticity of demand (η_{ij}) is defined as:

$$\eta_{ij}(y,p) = \frac{\partial x_i(y,p)}{\partial p_j}\frac{p_j}{x_i(y,p)}$$

and $x_i\,(y, p)$ is the input factor i as a function of output and prices (Blackorby and Russell (1981), Jehle and Reny (2001)).

The AES can be interpreted as the percentage change in the quantity of input i for a one percent change in the price of input j. (Thompson and Taylor (1995)) The AES is symmetric for all goods i and j by definition since $C_{ij}(y,p) = C_{ji}(y,p)$ by Clairaut's theorem. The symmetric property of AES is useful as symmetry is a required for the constant elasticity of substitution (CES) production function used in many CGE models.

3.2 Morishima Elasticities of Substitution

The Morishima elasticity of substitution is explicitly a function of input prices.

$$MES_{ij}(y,p) = \frac{p_i C_{ij}(y,p)}{C_j(y,p)} - \frac{p_i C_{ii}(y,p)}{C_i(y,p)}$$

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[2] The McFadden shadow elasticity of substitution is briefly mentioned below as a symmetric transformation of the MES.

where the MES can also be written as a function of cross price and own price elasticities:

$$MES_{ij} = \eta_{ji}(y, p) - \eta_{ii}(y, p)$$

The cross price elasticity is as defined above and the own price elasticity is defined as

$$\eta_{ii} = \frac{\partial x_i(y, p)}{\partial p_i} \frac{p_i}{x_i(y, p)}$$

The MES_{ij} can be interpreted as the percentage change in the ratio of input *j* to input *i* for a change in the price of good *i*. (Thompson and Taylor (1995))

Morishima elasticities of substitution are typically not symmetric unless it is the case that $C_j(y, p) = C_i(y, p)$ and $C_{ij}(y, p) = C_{ii}(y, p)$. Blackorby and Russell (1981) define the necessary and sufficient conditions for the MES to be symmetric as:

1. If the technology has an implicit CES structure or an explicit Cobb Douglas structure, and

2. If there are only two inputs to production.

Under condition (1), a test can be conducted to test for the underlying production functions being either of the Cobb Douglas or the CES structure. This test is of importance considering the underlying production function used in many CGE models is defined to be a CES production function. The relationship between the AES and MES can easily be shown as:

$$MES_{ij} = S_i(AES_{ji} - AES_{ii}) \qquad \{\forall\ (i, j) \mid i \neq j\}$$

Given that the own price elasticity must always be negative and the cost share for all factors must be positive the AES_{ii} must be negative for all factors. Thus it is possible to have a Morishima substitute be an Allen complement.

3.3 Problems with using AES and MES

There are a number of problems with using AES and MES approximations to the actual underlying production function's elasticity of substitution. Both AES and MES are derived from cost functions rather than directly from the production functions themselves. This can cause some fundamental discrepancies between aspects of both AES and MES and the actual elasticity of substitution. The elasticity of substitution derived directly from the production function is defined at the point $\tilde{x}$ as:

$$\sigma_{ij} \equiv \frac{d\ln(x_j / x_i)}{d\ln(f_i(\tilde{x}) / f_j(\tilde{x}))} = \frac{d(x_j / x_i)}{x_j / x_i} \frac{f_i(\tilde{x}) / f_i(\tilde{x})}{d(f_i(\tilde{x}) / f_j(\tilde{x}))}$$

When the production function is quasiconcave the elasticity of substitution (σ) is always greater than or equal to zero (Jehle and Reny (2001)). This requires that under a quasiconcave production function all inputs are substitutes of one another. The CES family of production functions are known to be strictly quasiconcave when ρ<1 (Uzawa 1962). The CES production function is defined as:

$$f(x) = (a_1 x_1^{\rho} + a_2 x_2^{\rho} + \dots + a_n x_n^{\rho})^{1/\rho}$$

where x_i (i = 1, 2, 3,…..n) are the input variable and a_i (i = 1, 2, 3,….., n) are positive constants and $0 \neq \rho \leq 1$. The elasticity of substitutions for the CES production function is given as:

$$\sigma_{ij} = \frac{1}{1-\rho}$$

It can be shown that the elasticity of substitution from the CES production function is constant, symmetric, non-negative and equal for all pairs of inputs i and j (Uzawa 1962). Viewed in terms of elasticities the CES production function is particularly easy to use, but can also be very restrictive. Allen and Morishima elasticities of substitution are often reported to take on negative values (Stern (2004), Nguyen and Streitwieser (1997), Thompson and Taylor (1995) Andrikopoulos, Brox and Paraskevopoulos (1989), Prywes (1986)). A negative elasticity of substitution between two inputs is interpreted as a complementary relationship between the inputs. This presents a problem when using AES or MES elasticities in a CES production function which has a range of possible elasticities between zero and infinity

Anderson and Moroney (1992 and 1994) describe a case where Allen elasticities of substitution can be negative under a supposed CES production function by nesting inputs into separate processes. Under a nested production function it is possible to have complementary inputs. Such a case is not difficult to imagine especially in intermediate stages of production. Anderson and Moroney (1992) specifically describe how via the nesting of inputs, negative Allen elasticities of substitution can be calculated using a trans-log cost function[3]. However such nesting requires that certain separability conditions must be met. These conditions will be explored in detail below.

3.4 Choosing between the Morishima and Allen Elasticities of Substitution

The Allen elasticity of substitution began to come under scrutiny as a true measure of substitution after Blackorby and Russell ((1975), (1981) and (1989)) explored the properties of the Morishima measure of substitution more thoroughly. Their findings were extremely critical of the AES in favour of MES when the production function is generalised to more than two inputs.

[3] Estimation of the trans-log cost function in order to calculate Allen elasticities is the most common econometric technique employed in the literature. This technique is described below in detail.

> *"While the AES reduces to the original Hicksian concept in the two-dimensional case, in general it preserves none of the salient properties of the Hicksian notion. In particular, the Allen elasticity of substitution (i) is not a measure of the "ease" of substitution, or the isoquant, (ii) provides no information about relative factor shares (the purpose for which the elasticity of substitution was originally defined), and (iii) cannot be interpreted as a (logarithmic) derivative of a quantity ratio with respect to a price ratio (or the marginal rate of substitution). As a quantitative measure, it has no meaning; as a qualitative measure, it adds no information to that contained in the (constant output) cross-price elasticity. In short, the AES is (incrementally) completely uninformative."*
>
> Blackorby and Russell (1989, page 882 par.5)

Blackorby and Russell's strongly worded criticisms have however not prevented most researchers from continuing to report Allen elasticities. Blackorby and Russell (1989) show by example using a Leontief functional form with a nested two input Cobb-Douglas aggregator how the AES does not measure the curvature of the isoquant. The Cobb-Douglas production function has a direct elasticity of substitution equal to one. In their simple example they are able to show that the elasticity of substitution between the Cobb-Douglas aggregated inputs is not equal to one as is predicted by theory, and that the AES measure of elasticity of substitution is biased. They go on to show, again by example, how the MES does measure the shape of the isoquant for the same three input Leontief production function with a two input Cobb-Douglas aggregator by demonstrating that the MES between the Cobb-Douglas aggregated inputs is equal to unity.

Thompson and Taylor (1995) further call into question the use of AES as an appropriate measure of elasticity of substitution with regards to the substitutability between energy and capital. Firstly AES is sensitive to small changes in the cost share. This is easy to see when AES is represented as a ratio of cross price elasticity to the input cost share. Energy costs make up a relatively small portion of the total cost in some industries making AES subject to a high degree of variability when there are small changes in cost shares.[4] The MES is not subject to this variability because it is a direct calculation between the cross-price elasticities and the own price elasticity. Both own and cross-price elasticities are calculated using cost shares as shown in section 3.6. The MES does not weight price elasticities by their cost shares as the AES does.

A second reason for questioning the validity of AES with respect to energy-capital substitutability is that since output is held constant in the calculation of elasticities, changes in output are not examined across different sectors. The primary reason for studying the energy-capital substitutability is that the output of energy at some point may become restricted. Given this concern Thompson and Taylor (1995) suggest that an examination of the capital-to-energy ratio might be of greater use. The MES is a measure

[4] Thompson and Taylor (1995) suggest that energy cost shares are, "Considerably lower than 3% of total cost and less than 10% of valued added in most industries" p.556 par. 2

of the percentage change in the input ratio caused by a change in the price of one of the inputs making it of greater use when analysing the degree of energy-capital substitutability. Stern (2004) disagrees with Thompson and Taylor (1995) pointing out that Morishima elasticities of demand are not helpful in determining whether or not energy and capital are compliments or substitutes. The MES is not a direct measure of the effects caused by a change in the price of one input on the demand for another input. The issue of complementary and substitute inputs is not so much a matter of the measurement of the curvature of the isoquant as it is a matter of the effect of an input price change on input demand. This suggests that AES or cross-price elasticity is a better indicator of complementarity or substitutability. However the use of AES for this purpose is not required as AES always takes on the same sign as the cross price elasticity. In addition using MES for this purpose is subject to bias towards substitutability since the own price elasticity is always negative and typically larger in absolute value than any cross price elasticity. (Stern (2004) and Thompson (1997))

In this respect the choice of elasticity of substitution measure should be a function of the intended purpose of the resulting estimate. While the MES appears to be a truer measure of the direct elasticity of substitution, the AES is a better tool for characterizing inputs as complementary or substitutes. For the purpose of CGE modeling it would appear that the MES is the desired elasticity measure since the issue of complementary or substitution is not as important as the correct measure of the curvature of the isoquant.

Morishima elasticities of substitution do have one very notable flaw for the use in CES - based models, direct measures of CES elasticities of substitution are symmetric while MES estimates are not guaranteed to be symmetric and are often found to be asymmetric[5]. This amounts to the rejection the hypothesis that the underlying production function is of the CES form as explained above. One proposed way of getting around the asymmetry problem is to use what is known as the McFadden shadow elasticity of substitution which is defined as half the weighted average of the respective Morishima elasticities (Thompson (1997)).

$$MFSES_{ij} = \frac{1}{2}\frac{S_i MES_{ij} + S_j MES_{ji}}{S_i + S_j}$$

This measure however has not been adapted as a standard measure of substitution. Blackorby and Russell (1989) reiterate that asymmetry of the elasticity of substitution is consistent with the original theory proposed by Hicks. Regardless of this observation the use of the CES functional form in most CGE models requires that in order for MES to be used some kind of symmetric transformation must be applied to the MES estimates before they are inserted into the model.

[5] Blackorby and Russell 1981 provide conditions for symmetric MES given above. However the assumption that the underlying production function is of the CES form is routinely rejected as MES estimates are most commonly reported as asymmetric.

4. Estimating Different forms of the Cost Function

In addition to the different types of elasticities outlined in Section 3, there is also a range of possible ways of estimating them. Here we describe three approaches and their respective advantages and disadvantages.

4.1 The Transcendental Logarithmic Model:

A transcendental logarithmic (trans-log) cost function is the most often used functional form for estimating elasticities of substitution econometrically. The production function is not estimated directly given the difficulty of gathering technical engineering, productivity and structural information on the production function. Rather the duality between cost and production functions is utilized to estimate the required production function properties to compute elasticity estimates while taking advantage of the relative ease of estimating the cost function.

The trans-log production function is preferred to the CES or Cobb-Douglas production functions when calculating elasticity estimates because it places fewer restrictions on the functional form allowing for a more general measure of elasticity of substitution. The trans-log functional form allows for inputs to be compliments where as direct estimates of the CES or Cobb-Douglas production functions do not. Moreover elasticities derived from the CES production function require that the elasticity of substitution between any two inputs is the same as the elasticity of substitution between every other pair of inputs (Uzawa (1962)). The Cobb-Douglas functional form is more restrictive than the CES form given that the elasticity of substitution is equal to one for all pairs of inputs. This restriction makes estimating the true elasticities difficult considering the actual relationship between inputs is more likely to differ between all possible combinations of inputs than it is to be the same.

4.1.1 Estimation of Elasticities of Substitution:

The structure of the trans-log model is not econometrically difficult to estimate depending on data availability. For the most basic production function with *n* aggregate inputs ($x_1, \ldots x_n$) the estimation of the trans-log model can be broken down into the following steps. This model can easily be augmented to include further disaggregation of the aggregate variables as described below (Berndt and Wood (1975)).

1. Assume there exists a twice differentiable production function for output (Y) as a function of a set of inputs ($x_1, \ldots, x_n$), and that the production function $Y=f(x_1, \ldots, x_n)$ is exhibiting constant returns to scale and follows Hicks neutral technology change.
2. From duality between cost and production functions there must also exist a cost function $C=C(y, p_1, \ldots, p_n)$ which is twice differentiable. Where p_i are input prices and Y is output.

3. The cost function C=C(y, $p_{1,\ldots,}p_n$) can be expressed in the trans-log functional form.

$$\ln C = \ln \alpha_0 + \ln y + \sum_{i=1}^{n} \alpha_i \ln p_i + \frac{1}{2} \sum_{i=1}^{n} \gamma_{ii} (\ln p_i)^2 + \sum_{\substack{i=1 \\ j=1}}^{n} \gamma_{ij} \ln p_i \ln p_j$$

4. In order to analyse elasticities of substitution input demand functions must be constructed and interacted with the parameters of the trans-cost function. This is a two step process. First the trans-log cost function must be logarithmically differentiated

$$\frac{\partial \ln C}{\partial \ln p_i} = \frac{\partial C}{\partial p_i} \frac{p_i}{C} = \alpha_i + \sum_{i-1}^{n} \gamma_{ij} \ln p_j$$

   Utilizing Shephard's lemma the next step is to solve for the input demand functions.

$$x_i = \frac{\partial C}{\partial p_i} \qquad \forall\, i \in (1, \ldots\ldots n)$$

   These input demand functions can be substituted into the above equation to construct input cost shares as a function of prices.

$$\frac{\partial \ln C}{\partial \ln p_i} = \frac{x_i p_i}{C} = \alpha_i + \sum_{i=1}^{n} \gamma_{ij} \ln p_j$$

   This function is typically expressed as:

$$S_i = \frac{x_i p_i}{\widetilde{C}} = \alpha_i + \gamma_{ii} \ln p_i + \sum_{j \neq i}^{n} \gamma_{ij} \ln p_j$$

   Where $\widetilde{C}$ is the total cost of producing Y given as:

$$\widetilde{C} = \sum_{i=1}^{n} p_i x_i$$

5. Cost shares can be estimated stochastically using a multivariate regression technique.

$$S_{it} = \alpha_i + \gamma_{ii} \ln p_{it} + \sum_{j \neq i}^{n} \gamma_{ij} \ln p_{jt} + e_{it}$$

Input cost shares are simultaneously determined dependent variables. The sum of the cost shares for the *n* inputs must be one implying that the error terms are zero at every observation which implies that the error variance-covariance matrix is singular and non-diagonal. To avoid this problem *n*-1 of the cost share equations are estimated rather than the full system. Ordinary least squares is not the best linear unbiased estimator of the cost share system unless assumptions regarding the independence of the error terms across equations are invoked. Most researchers estimate the cost share system using either a seemingly unrelated least squares estimation procedure or an iterated three stage least squares procedure.

As stated above AES is defined as:

$$AES_{ij} = \frac{C(y,p)C_{ij}(y,p)}{C_i(y,p)C_j(y,p)}$$

Under the trans-log cost function Allen elasticities of substitution are defined as:

$$AES_{ii} = \frac{\gamma_{ii} + S_i^2 - S_i}{S_i^2} \qquad \forall i \in (1,\ldots..n)$$

$$AES_{ij} = \frac{\gamma_{ij} + S_i S_j}{S_i S_j} \qquad \forall\ (i,j) \in (1,\ldots..n) \mid i \neq j$$

AES as a function of cross price elasticity of demand is given as:

$$AES_{ij}(y,p) = \frac{\eta_{ij}(y,p)}{S_j(y,p)}$$

Substituting AES_{ij} and solving for η_{ij} as a function of cost shares gives:

$$\eta_{ii} = \frac{\gamma_{ii} + S_i^2 - S_i}{S_i} \qquad \forall\ i \in (1,\ldots..n)$$

$$\eta_{ij} = \frac{\gamma_{ij}}{S_i} + S_j \qquad \forall\ (i,j) \in (1,\ldots..n) \mid i \neq j$$

Once the coefficients in the cost share equations are estimated the Morishima elasticities can be easily derived.

$$MES_{ij} = \eta_{ji}(y,p) - \eta_{ii}(y,p) \qquad \forall\ (i,j) \in (1,\ldots..n) \mid i \neq j$$

$$MES_{ij} = \frac{\gamma_{ji}}{S_j} + S_i - \frac{\gamma_{ii} + S_i^2 - S_i}{S_i}$$

$$MES_{ij} = \frac{S_i \gamma_{ji} - S_j \gamma_{ii}}{S_j S_i} - 1$$

The MES unlike the AES does not have an 'own price' elasticity of substitution which is consistent with elasticity of substitution theory.

4.1.2 Problems with the Transcendental Logarithmic model:

The cost functions must satisfy three criteria in order to be considered well behaved (Hunt 1984):

1. The cost function is linearly homogeneous in factor prices,
2. The cost function is monotonic,
3. The cost function is concave.

The first criterion is satisfied if:

$$\sum_{i=1}^{n} \alpha_i = 1 \qquad \forall\, i \in (1,.....,n)$$

$$\sum_{j=1}^{n} \gamma_{ij} = 0 \qquad \forall\, (i,\, j) \in (1,.....,n)$$

The second criterion is satisfied if there are non-negative input levels. This will be the case if the fitted values for the input cost shares are non-negative. This criterion can not be guaranteed to be met with the trans-log cost function. This problem is well documented in the literature and is analogous to the same problem with linear probability models. Attempts to compensate for this problem are described below in the section on linear logit models.

The concavity criterion is satisfied if the corresponding Hessian matrix is negative semi-definite for every data point. However this condition is not typically tested due to difficulty in computing the Hessian matrix at every point. A simple test to ensure that all own-price elasticities are negative is sufficient given that positive own price elasticities are only present if there is non-concavity (Hunt (1984)). Positive own-price elasticities are inconsistent with theory because it is impossible for an input to be 'Giffen factor'[6] if firms are profit maximizing. If firms are profit maximizing under perfect competition producers will always face a downward sloping factor demand curve provided that they are not budget constrained (Else (1971)). Nonetheless positive own price elasticities are frequently reported from estimations using the trans-log model (Nguyen and Streitwieser (1997), Andrikopoulos et al. (1989), Magnus and Woodland (1987), Pindyck (1979)), and Hall (1986)).

Further complicating maters, symmetry of elasticities of substitution is imposed as a condition for a 'well behaved cost function'. This condition guarantees that the calculated AES will be symmetric. The symmetry condition is imposed by restricting the coefficients from the trans-log estimation such that (Berndt and Wood (1979)):

[6] A 'Giffen factor' would be analogous to a 'Giffen good' from consumer theory. However such a factor is not possible because unlike consumers, producers do not seek to maximize output and are not bound by a budget constraint. See Else (1971) pages: 30-31

$$\gamma_{ij} = \gamma_{ji} \qquad \forall (i, j) \in (1, \ldots, n)$$

These problems while potentially large under the trans-log model are easily testable. If the cost function satisfies the criteria for a 'well behaved cost function' it is generally accepted that the estimation technique is appropriate for estimating elasticities of substitution.

4.1.3 Variance under the Transcendental Logarithmic model

The variance of the elasticities derived from the trans-log method can not be calculated using the typical properties of the variance. In order to construct variances for the elasticities assumptions must be made regarding the non-stochastic nature of the cost shares. The variance of AES and MES are calculated at the mean cost share as they are for the AES and the MES themselves. In order for the variance to be constructed using the typical properties of the variance operator these cost shares must be assumed to be non-stochastic. Such assumptions are however completely invalid considering cost shares are estimated stochastically under the trans-log model (Kopp and Smith (1981), Moroney and Tapani (1981), and McKnown, Pourgerami and Von Hirshhausen (1991)).

Many researchers fail to report variance estimates in an attempt to side step this problem. This practice appears to have become generally accepted for researchers trying to estimate elasticities of substitution between capital and energy. However once the issues of separability of energy into different fuel categories becomes the focus of research variance estimates become of greater importance as described below. Variance equations for AES are not difficult to compute requiring only information on the variance of the respective coefficient from the cost share estimation and a measure of the cost share itself. The variance of the MES is slightly more complicated to calculate than the variance of the AES requiring the covariance between the γ_{ji} coefficient and the γ_{ii} coefficient from the cost share estimation (Frondel 2004).

$$Var(AES_{ii}) = \frac{Var(\gamma_{ii})}{S_i^4}$$

$$Var(AES_{ij}) = \frac{Var(\gamma_{ij})}{(S_i S_j)^2}$$

$$Var(MES_{ij}) = \frac{S_i^2 Var(\gamma_{ji}) + S_j^2 Var(\gamma_{ii})}{(S_j S_i)^2} - \frac{2Cov(\gamma_{ji}, \gamma_{ii})}{S_j S_i}$$

The statistical significance of the elasticity estimates plays an important role in interpreting the estimate. An elasticity of substitution which is statistically insignificant from zero implies that there is no substitutable relationship between the two inputs. The variance of the elasticity of substitution is essential for constructing tests for this significance.

4.2 The Linear Logit Model

A second model commonly used for estimating elasticities of substitution is the linear logit model. The linear logit model was first adapted for this purpose by Considine and Mount (1984) and more recently has become the standard alternative to the trans-log approach (Jones (1995), Bjørn and Jensen (2002), Urga and Walters (2003)). This model, while not as simple to construct as the trans-log model has two desirable properties which the trans-log does not possess. Firstly the linear logit model restricts the dependent variable between zero and one. This is essential for cost share estimation which is defined to be between zero and one. This is one of the primary arguments against the trans-log technique which only provides estimated cost shares on the interval [0,1] over a limited range of prices. When the estimated cost shares are negative the monotonicity criterion for a well behaved cost function is violated. The use of the linear logit model ensures that the monotonicity criterion is met.

The second attribute of the linear logit model which is preferable to the trans-log model is that the linear logit model typically yields more reasonable own price elasticities than the trans-log model (Considine (1989a)). As described above own price elasticities must be negative in order to abide with the theoretical properties governing cost minimizing firm behaviour. The concavity of the cost function is generally regarded to be preserved if own price elasticities are non-positive (Hunt (1984)). Considine (1989a) finds that the mathematical properties of the trans-log model of cost shares coincide with violations of concavity and that the linear logit model provides a more stable method for modeling cost shares.

4.2.1 Estimation of Elasticities of Substitution

The linear logit cost function again utilizes input shares and Shephard's lemma to estimate elasticities. Beginning from Shephards lemma the construction of linear logit model of cost shares is not difficult to construct.

$$S_i = \frac{x_i p_i}{\widetilde{C}} = \frac{p_i \dfrac{\partial C}{\partial p_i}}{\sum_{i=1}^{n} p_j \dfrac{\partial C}{\partial p_j}}$$

The method at this point is the same as it was with trans-log function. However, under the linear logit model a functional form is specified for the price weighted partial derivatives. Rather than using a linear form as in the trans-log case the linear logit method uses an exponential form defined as:

$$S_i = \frac{e^{f_i}}{\sum_{j=1}^{n} e^{f_j}}$$

Where *f* is defined as

$$f_i = \alpha_i + \gamma_{ii} p_i + \sum_{k \neq i}^{n} \gamma_{ik} p_k$$

This is what Considine (1989a) defines as the linear-logit model of cost shares. The calculation of own and price elasticities of substitution under the liner logit is a two step process.

1. An unconstrained share elasticity is calculated as:

$$H_{ik} = \frac{d \ln S_i}{d \ln p_j} = \gamma_{ij} - \sum_{k=1}^{n} S_k \gamma_{kj}$$

2. Elasticity estimates are defined to be:

$$\eta_{ij} = H_{ij} + S_j$$

$$\eta_{ii} = H_{ii} + S_i - 1$$

Given the definition of the Allen elasticity of substitution provided above

$$AES_{ij}(y, p) = \frac{\eta_{ij}(y, p)}{S_j(y, p)} \qquad \forall\, (i, j) \in (1, \ldots.., n)$$

Allen elasticities of demand are defined as:

$$AES_{ij} = \frac{H_{ij} + S_j}{S_j} = \frac{H_{ij}}{S_j} + 1$$

$$AES_{ii} = \frac{H_{ii} + S_i - 1}{S_i} = \frac{H_{ii} - 1}{S_j} + 1$$

Similarly given the definition of the Morishima elasticity of substitution provided above

$$MES_{ij} = \eta_{ji}(y, p) - \eta_{ii}(y, p) \qquad \forall\, (i, j) \in (1, \ldots.., n) \mid i \neq j$$

Morishima elasticities of substitution are defined as:

$$MES_{ij} = H_{ji} - H_{ii} + 1$$

The linear logit method is typically constructed with restrictions for a ‘well behaved cost function’ modeled explicitly in the estimation. Embedding these restrictions in the linear logit model further complicates the construction of elasticities of substitution compared to the relatively simple construction of elasticities of substitution under the trans-log model. The homogeneity condition is satisfied if:

$$\sum_{j=1}^{n} \gamma_{ij} = a \qquad \forall (i, j) \in (1,....,n)$$

Where a is an unknown scalier which can be set to zero as was the case under the trans-log conditions. (Considine and Mount (1984))

Considine and Mount (1984) impose a symmetry condition to ensure that the elasticities of substitution are symmetric. The symmetry condition is imposed at the mean cost share level which is where elasticities are typically calculated. The symmetry condition is imposed by the following constraint:

$$\tilde{\gamma}_{ij} = \tilde{\gamma}_{ji} \qquad \forall (i, j) \in (1,....,n) \mid i \neq j$$

where

$$\tilde{\gamma}_{ij} = \frac{\gamma_{ij}}{\bar{S}_i}$$

The AES at the point of symmetry which is point of the mean cost share is then defined as:

$$AES_{ii} = \frac{\left[-\sum_{j \neq i}^{n} \bar{S}_j \tilde{\gamma}_{jk} + \bar{S}_i - 1 \right]}{\bar{S}_i}$$

$$AES_{ij} = \tilde{\gamma}_{ij} + 1 \qquad \forall (i, j) \in (1,....,n) \mid i \neq j$$

The own and cross price elasticities at the point of symmetry are defined as:

$$\eta_{ii} = -\sum_{j \neq i}^{n} \bar{S}_k \tilde{\gamma}_{jk} + \bar{S}_j - 1$$

$$\eta_{ij} = \bar{S}_j \left[\tilde{\gamma}_{ij} + 1 \right] \qquad \forall (i, j) \in (1,....,n) \mid i \neq j$$

Once own and cross price elasticities are calculated the MES can be calculated at the point of symmetry as:

$$MES_{ij} = \bar{S}_i \left[\tilde{\gamma}_{ji} + 1 \right] + \sum_{j \neq i}^{n} \bar{S}_k \tilde{\gamma}_{jk} - \bar{S}_j + 1 \qquad \forall (i, j) \in (1,....,n) \mid i \neq j$$

As was the case with the trans-log method, in the linear logit model the cost shares are simultaneously determined requiring a multivariate regression approach for the estimation of the cost share equations. The linear logit method also suffers from a non singular error variance-covariance matrix as was the case with the trans-log method. As such n-1 equations are used in the multivariate logit cost share model. The input corresponding to the dropped share equation will be used as the numéraire good. When written in its stochastic form the logit system of cost share equations is expressed as follows allowing good m to be the numéraire good (Considine 1989a):

$$\ln\left(\frac{S_{it}}{S_{mt}}\right)=(\alpha_1-\alpha_m)-\left[\sum_{\substack{J=2\\ j\neq m}}\bar{S}_j\tilde{\gamma}_{1j}+(\bar{S}_1+\bar{S}_m)\tilde{\gamma}_{im}\right]\ln\left(\frac{P_{1t}}{P_{mt}}\right)+\sum_{\substack{j=2\\ j\neq m}}^{n}\left[(\tilde{\gamma}_{1j}-\tilde{\gamma}_{jm})\bar{S}_j\ln\left(\frac{P_{jt}}{P_{mt}}\right)\right]+(e_{nt}-e_{mt})$$

.

.

$$\ln\left(\frac{S_{nt}}{S_{mt}}\right)=(\alpha_n-\alpha_m)-\left[\sum_{\substack{j=1\\ j\neq m}}^{n-1}\bar{S}_j\tilde{\gamma}_{nj}+(\bar{S}_n+\bar{S}_m)\tilde{\gamma}_{im}\right]\ln\left(\frac{P_{nt}}{P_{mt}}\right)+\sum_{\substack{j=1\\ j\neq m}}^{n-1}\left[(\tilde{\gamma}_{nj}-\tilde{\gamma}_{jm})\bar{S}_j\ln\left(\frac{P_{jt}}{P_{mt}}\right)\right]+(e_{nt}-e_{mt})$$

Considine and Mount (1984) find that under full information maximum likelihood estimation the coefficient estimates do not vary based on the choice of the numeraire good. They also find that their results do not differ widely under the linear logit model from the trans-log model. This observation provides two inferences regarding the two models. First since estimates do not differ widely using separate methods there is stronger evidence that estimates from the trans-log and linear logit models closely approximate the actual elasticities of substitution of the underlying production function. Secondly since both methods yield similar results, the results from the more rigorous and relatively more difficult to estimate linear logit model aid in the argument for estimating elasticities of substitution through the more simple trans-log model.

4.2.2 Problems with the Linear Logit Model

The linear logit method for calculating elasticities is clearly theoretically superior to the trans-log method. Monotonicity of the cost function is always satisfied under the linear logit method. The linear logit method is also thought to provided more realistic estimates for own price elasticities than the trans-log method (Considine (1989a)). However the linear logit model does have flaws which have prevented it from becoming the primary method of estimation.

The main criticism of the linear logit model is that symmetry does not hold globally for all cost shares. While this is a valid concern it should be pointed out that the point of symmetry can be defined at the mean cost share. For the purpose of generating elasticity of substitution estimates this is really the only point of concern considering elasticity estimates are calculated at the mean cost share under both the trans-log and linear logit models. Moreover Considine (1989b) shows that if concavity is met at the point of symmetry then concavity is met globally. The use of the linear logit model trades off global symmetry for global concavity as compared to the trans-log model (Considine (1989a)).

The liner logit model is also slightly more difficult to compute than the trans-log model. This is the most likely reason for the lack of use of the linear logit model given its clear theoretical superiority. This is especially true when calculating own price elasticities required for the construction of Morishima elasticities of substitution. However such hesitance should not be considered a valid reason for failure to estimate elasticities using this theoretically superior method. To the contrary, an estimation of elasticities using both the trans-log and linear logit methods would aid in supporting evidence to the validity of the estimates themselves as approximations of the true elasticities of substitution which can only be derived from the production function.

The variances of the elasticities of substitution from the liner-logit model are difficult to calculate and are not fully explored. This makes statistical analysis of the estimates potentially problematic. Variance estimates could be calculated in a similar fashion as they are under the trans-log model which considers mean cost shares to be non-stochastic.

4.3 The Generalized Leontief Model

The generalized Leontief functional form is another flexible functional form cost function offered as an alternative to the trans-log function. The model is relatively more obscure than the linear logit model, and certainly a distant third in terms of commonly estimated functions used for estimating inter-fuel and energy-capital elasticities of substitution. The generalized Leontief function has recently been examined by Ryan and Wales (2000) finding that local concavity is more often violated under the generalized Leontief function than under the trans-log function.

4.3.1. Estimation of Elasticities of Substitution

As was the case with the trans-log and the linear logit cost functions, the generalized Leontief cost function is defined and Shephard's lemma is utilized to derive factor demand equations. From Diewert (1971) and Berndt (1991) the process for calculating elasticities of substitution from the generalized Leontief cost function is defined as follows.

The structure of the generalized Leontief cost function is a simple cost function with total cost as a function of input prices and a function of output *h(y).*

$$C(p,y) = h(y)\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}\, p_i^{1/2}\, p_j^{1/2}$$

The function *h(y)* is defined to be monotonic and continuous in *y* and is typically set to *h(y) = y* (Diewert (1971), Diewert and Wales (1987) and Berndt (1991)). Utilizing Shephard's lemma factor demand equations are expressed as:

$$\frac{\partial C}{\partial p_i} = x_i = y\sum_{j=1}^{n}\gamma_{ij}\left(\frac{p_j}{p_i}\right)^{1/2} \qquad \forall\, i \in (1,\ldots.,n)$$

In order to estimate parameter values stochastically the factor demand equations are divided by output *y* to create input per unit of output equations. Once this is complete an additive error term is applied to allow for a stochastic specification.

$$\frac{x_1}{y} = \gamma_{11} + \sum_{j \neq 1}^{n} \gamma_{1j} \left(\frac{p_j}{p_1} \right)^{1/2} + e_t$$

.

.

$$\frac{x_n}{y} = \gamma_{nn} + \sum_{j \neq n}^{n-1} \gamma_{nj} \left(\frac{p_j}{p_n} \right)^{1/2} + e_t$$

This structure is unlike both the trans-log and linear logit models which use cost share equations to estimate parameters rather than direct input demand estimations. This structure has one clear advantage in that the input per unit of output is not constrained on the interval [0,1]. As such the resulting error covariance matrix is non-singular which allows for the estimation of the full system of *n* input demand functions rather than the (*n*-1) system required for both the trans-log and linear logit models. Cross equation restrictions are applied to the system to impose symmetry on parameters such that:

$$\gamma_{ij} = \gamma_{ji} \qquad \forall\, (i, j) \in (1,....,n)$$

The system is simultaneously determined and as such a multivariate regression technique should be applied. A seemingly unrelated regression is again the most reasonable candidate for estimation of the system.

The cross price elasticities are defined as:

$$\eta_{ii} = \frac{1}{2}\left(\frac{y}{x_i}\right)\left(\gamma_{ii} - \sum_{j}^{n} \gamma_{ij} \left(\frac{p_j}{p_i}\right)^{1/2}\right) \qquad \forall\, i \in (1,....,n)$$

$$\eta_{ij} = \frac{1}{2}\left(\frac{y}{x_i}\right)\left(\gamma_{ij} \left(\frac{p_j}{p_i}\right)^{1/2}\right) \qquad \forall\, (i, j) \in (1,....,n) \,|\, (i \neq j)$$

Allen elasticities of substitution are defined as:

$$AES_{ii} = -\frac{1}{2} \frac{yC(p,y)\sum_{j \neq i}^{n} \gamma_{ij}\, p_j^{1/2}\, p_i^{-3/2}}{x_i^2} \qquad \forall\, i \in (1,....,n)$$

$$AES_{ij} = \frac{1}{2} \frac{yC(p,y)\gamma_{ij}\,(p_i\, p_j)^{-1/2}}{x_i\, x_j} \qquad \forall\, i, j \in (1,.....,n) \,|\, i \neq j$$

Where the cost function *C(p,y)* is defined as the generalized Leontief function provided above with *h(y)=y*.

The Morishima elasticity of substitution is defined as:

$$MES_{ij} = \frac{1}{2}\left(\frac{y}{x_j}\right)\left(\gamma_{ji}\left(\frac{p_j}{p_i}\right)^{1/2}\right) + \frac{1}{2}\left(\frac{y}{x_i}\right)\sum_{j\neq i}^{n}\gamma_{ij}\left(\frac{p_j}{p_i}\right)^{1/2} \qquad \forall\,(i,j)\in(1,....,n)\,|\,i\neq j$$

The elasticities of substitution calculated from the generalized Leontief model are not constant elasticities of substitution, in that they are functions of the level of input as well as output. These elasticities are slightly more complicated to construct as compared to the trans-log model. The elasticities must be calculated using the fitted values of C as well as *(x_i/y)* and not the observed values. As such the elasticity estimates must be constructed from their simplified form.

$$\eta_{ii} = \frac{-\frac{1}{2}\sum_{j\neq i}^{n}\gamma_{ij}\left(\frac{p_j}{p_i}\right)^{1/2}}{a_i} \qquad \forall\, i\in(1,....,n)$$

$$\eta_{ij} = \frac{1}{2}\frac{\gamma_{ij}\left(\frac{p_j}{p_i}\right)^{1/2}}{a_i} \qquad \forall\,(i,j)\in(1,....,n)\,|\,i\neq j$$

$$AES_{ii} = -\frac{\frac{1}{2}C(y,p)\sum_{j\neq i}^{n}\gamma_{ij}\,p_j^{1/2}\,p_i^{-3/2}}{ya_i^2} \qquad \forall\, i\in(1,....,n)$$

$$AES_{ij} = \frac{1}{2}\frac{C(y,p)\gamma_{ij}\left(\frac{P_i}{P_j}\right)^{-1/2}}{ya_ia_j} \qquad \forall\,(i,j)\in(1,....,n)\,|\,i\neq j$$

$$MES_{ij} = \frac{1}{2}\frac{\gamma_{ji}\left(\frac{p_j}{p_i}\right)^{1/2}}{a_j} + \frac{\frac{1}{2}\sum_{j\neq i}^{n}\gamma_{ij}\left(\frac{p_j}{p_i}\right)^{1/2}}{a_i} \qquad \forall\,(i,j)\in(1,....,n)\,|\,i\neq j$$

4.3.2. Problems with the Generalized Leontief Model

The generalized Leontief cost function is one of the functional forms which Berndt and Wood (1975) describe as sufficiently flexible for estimating energy-capital elasticities of substitution. This specification has not however been widely adapted for use in estimating elasticity of substitution with most researchers favouring the trans-log or the econometrically superior linear logit model to the generalized Leontief model.

The most notable difference between this type of estimation and the cost share estimation techniques is that information on the actual number of units of the inputs and level of output is required. This additional data requirement further adds to the already difficult problem of obtaining historical data documented below. In addition different definitions of the unit of input can cause discrepancies between studies.

The elasticities of substitution constructed from the generalized Leontief model are clearly more difficult to construct compared to the trans-log model. The variable nature of the elasticity estimates also further complicates the case for using Allen or Morishima elasticities of substitution calculated from cost functions in models which are CES function based. Calculating the variance of the elasticity estimates is also difficult. Berndt (1991) states that the distributional properties of the elasticity estimates have not been sufficiently derived for use in employing statistical inferences.

To further complicate matters Ryan and Wales (2000) found that where no restrictions on the concavity of the cost function were imposed results using the generalized Leontief function violated concavity at 100% of the observations while concavity was only violated at 24% of the observations under the trans-log form. The issue of concavity is important because in order for duality to be applied the cost function must be considered "well behaved". A sufficient condition for concavity of the generalized Leontief cost function would be for:

$$\gamma_{ij} \geq 0 \qquad \forall (i, j) \in (1, \ldots, n) \mid i \neq j$$

Imposing such a constraint is possible, but it would constitute all inputs to be substitutes, thus breaking the flexibility of the generalized Leontief form (Ryan and Wales (2000)).

There is however a clear benefit to using the generalized Leontief over the trans-log and logit models regarding ease of estimation. Since the error covariance matrix is non-singular the full system of *n* input per unit output functions can be estimated at once. Under both the trans-log and linear logit modes *n*-1 equations must be estimated, the numéraire good must be varied and then the *n*-1 equations must be re-estimated. Moreover estimates must be checked for robustness to changes in the numéraire good once both systems have been estimated. In this light the generalized Leontief appears to be somewhat less computationally intensive in terms of the initial estimation of parameters.

5. Econometric Issues

In addition to the theoretical issues with each estimation approach outlined in the previous section, there are also problematic issues stemming from the types of data used for estimation.

5.1 Long Run - Short Run Dichotomy

5.1.1 Causes of the Dichotomy

There is a well documented dichotomy between elasticities of substitution depending on the type of data used in constructing estimates. Estimates calculated using cross-sectional data often yield different results for the capital-energy elasticity of substitution than when time series data is used. Time series results typically yield capital and energy as complements while cross section and panel results more often yield capital and energy as substitutes (Griffin and Gregory (1976), Apostolakis (1990), Hisnanick and Kyer (1995) and Thompson and Taylor (1995)).

Griffin and Gregory (1976) attempt to explain the dichotomy between time-series and cross-sectional approaches by creating a distinction between long-run and short-run elasticities of substitution. Long run elasticities of substitution take into account changes in technology allowing for greater substitutability between inputs due to the change in the price of one of the inputs. When changes in technology are taken into account it is expected that short-run elasticities of substitution would yield capital and energy as complements. However because of changes in technology spurred on by changes in the price of energy, the long-run relationship between capital and energy would be one of substitution rather than of complement. Long-run elasticities are estimated with the use of cross-sectional data while short run elasticities are more appropriately estimated using time series data. The justification of this distinction is that time-series data captures dynamic adjustments caused by changing relative prices as well as technological progress and external shocks, while cross sectional data typically excludes these dynamic changes (Griffin and Gregory (1976)). Essentially many of the short-run disequilibrium effects caused by market forces are washed out in cross sectional data, but are still picked up in time series data. For this reason time series data are better for estimating short-run effects. Cross-sectional data, however, will still contain some information on short term disturbances; estimates using cross-sectional data should only be considered an approximation of the true long-run estimate (Kuh (1959)).

This dichotomy, however, is not universally observed. Solow (1987) points out that the differing results from empirical studies might be caused by different assumptions about the structure of the production process. This stems from whether to include materials as a separable factor of production from the typical three input model with capital, labour and energy (KLE) as the only factors of production. The capital, labour, energy and materials (KLEM) model more often yields a complement elasticity of substitution between capital and energy (Berndt and Wood (1979)).

Cross-sectional and time series results differ across measures of elasticity of substitution as well. As described above it is generally the case that MES are more likely to be positive than AES since the own price elasticity is always negative and generally greater than the cross price elasticity. Thompson and Taylor (1995) find that not only do MES estimates yield substitutability more often, but also that there is no evidence of a cross-sectional time series dichotomy. In addition they also find that MES estimates show far less variability than the corresponding AES estimates. This indicates that MES is a more stable measure of elasticity of substitution than AES over time.

5.1.2. Attempts Separate Long-Run and Short-Run Elasticities

There have been several attempts to estimate long-run and short-run elasticities of substitution separately beyond simply using cross sectional and time series data. Typically this involves adding lagged cost shares, time trends, and lagged quantity terms to the trans-log and linear logit cost share functions. The most successful short-run–long-run estimation technique has been to use a dynamic linear logit cost share function (Considine and Mount (1984), Jones (1996 and 1996), Ugra and Walters (2003)).

The dynamic linear logit model offered by Considine and Mount (1984) includes a total output variable as well as a lagged quantity input variable. The inclusion of the lagged quantity variable allows for the distinction between short-run and long-run elasticities of substitution. The cost share function under the dynamic model is:

$$S_i = \frac{e^{f_i}}{\sum_{j=1}^{n} e^{f_j}}$$

where f is defined as

$$f_i = \alpha_i + \gamma_{ii} p_{i(t)} + \sum_{k \neq i}^{n} \gamma_{ik} p_{k(t)} + g_i Y_t + \lambda Q_{i(t-1)}$$

Once the coefficients are estimated and price elasticities are constructed the long-run elasticity of substitution can be defined as:

$$\eta_{ij}^{LR} = \frac{\eta_{ij}^{SR}}{1-\lambda}$$

The short run cross price elasticity is the same elasticity as described above in the section on the linear logit function.

Considine and Mount (1984) find that using this specification and comparing long run to short run elasticities that the Le Chatelier principle governing short-run – long-run relationships is satisfied. The Le Chatelier principle was proposed for use in economics by Samuelson (1947) and in essence it states that the elasticities of factor demand should be smaller in the short-run than they are in the long-run. This can be attributed to the fact that production factors may be fixed in the short-run but not in the long-run (Hatta (1987)). Samuelson (1947) shows this mathematically; however the reasoning behind why the principle should hold can be easily arrived at conceptually.

5.2 Finite Sample Size Problems

Using regression analysis to estimate elasticities of substitutions is typically plagued with problems from the finite sample size of industry data. For time-series regressions, data is only available on an annual basis. Berndt and Wood (1975) construct a time series spanning 25 years beginning in 1947 which would allow for 60 observations if the time series were expanded to include observations up to 2006. The KLEMS data base for Canadian data ranges from 1961-2003 which provides for 43 observations. However results from a 'long' time series should be analysed with caution. Elasticities of substitution are not necessarily constant over time if there is technological change in the production process. Debertin et al. (1990) find that machinery and energy go from being complements between 1960 and 1969 to substitutes between 1970 and 1979 in the American agricultural sector.

Cross-sectional regressions suffer from a similar problem. Aggregate data is commonly analysed at the state level for US studies and at the country level for international studies. Again there is some degree of difficulty in assuming identical technologies across regions when analysing elasticities at the international level. Pooled cross-sectional time series regressions are frequently estimated to combat small sample size. However due to incomplete data, balanced panel regression analysis is not commonly conducted.

Recently micro-level data has been available at the company level which not only allows for a greater number of observations, but has also allowed for more sophisticated panel econometric techniques to be employed (Bjørner and Jensen (2002) and Arnberg and Bjørner (2007)). It should be noted however that results for micro-level studies have shown smaller estimates for own-price elasticities than they do under macro level studies implying that estimates from macro data may not be directly comparable to those from micro data (Bjørner and Jensen (2002)).

5.3 Pseudo Data

Griffin (1977) presents an alternative method of estimating long-run elasticities of substitution using a method developed by Klein (1953) which involves the creation and use of pseudo data rather than actual historical data. This approach, used by Griffin throughout the late 1970's and early 1980's came under harsh criticism by Maddala and Roberts ((1980) and (1981)). The pseudo data approach to estimating elasticities has become a relatively obscure method and should not be considered a standard method of estimation. However Bataille et al. (2006) and Bataille (1998) utilized pseudo data to estimate elasticities of substitution for Canadian industries.

Pseudo data is created by utilizing engineering or 'process' models for specific industries. The estimation of elasticities of substitution using pseudo data is a two step process. First the engineering model is constructed using actual historical data on the engineering processes used in manufacturing. Griffin (1977) describes the linear programming problem as follows:

$$\min\ P_z' X_z \qquad z = (1,...., T)$$

Subject to:

$$AX_z \leq b$$

Where P_z is an n x 1 vector of input prices, and X_z is an n x 1 vector of process activity levels. Engineering data is used to construct the A matrix which is an m x n matrix of technical coefficients. The right hand side of the constraint is the m x 1 vector b which constrains output to fixed levels depending on the desired output mix. The model is solved using base prices from actual price data. Once the base case is completed by optimizing demand for input vector X_1, prices are then varied to generate new input mixes. These new input mixes can be interpreted as pseudo data.

Once the pseudo data has been created the second step is to estimate elasticities of substitution using the standard econometric methods described above. Both Griffin and Bataille use the trans-log method for this stage. Direct calculations of elasticities from the process model are not possible because of the non-differentiable nature of process models (Griffin (1979)).

There are several benefits to using pseudo data which make the method an attractive alternative to time series estimations. The process model enables long-run analysis by relaxing capacity constraints. This relieves concerns that are described above regarding the short run nature time series data. Prices are constructed externally to the process model. Exogenously setting prices ensures that prices are orthogonal to each other relieving concerns over multicollinearity. These prices can also be constructed with much larger variation than found in time series data providing for better estimates of coefficients from the trans-log estimation (Griffin (1979)).

The primary concern over the use of pseudo data which is pointed out by Griffin himself as well as Smith and Vaughan (1979) and Mandela and Roberts (1979 and 1981) is that there must be a great deal of confidence placed in the structure and accuracy of the process model. Engineering models are not only costly to construct, but also open to error based on required assumptions regarding industry specific production structures. In fact the difficulty of gathering data and constructing engineering models is the primary reason why duality of the cost function is used to obtain information about the production process.

Bataille et al. (2006) used the Canadian integrated modelling system (CIMS) as their process model. CIMS is a full equilibrium system which can be used to simulate pseudo data at the disaggregated industry level. Bataille (1998) used the Intra-Sectoral Technology Use Model (ISTUM) model, a partial equilibrium model, to generate pseudo data. ISTUM was the precursor to CIMS.

5.4 Cost Share Driven Changes in Elasticities of Substitution

Frondel and Schmidt (2002) analyse the structure of the trans-log functional form and find that changes in elasticities over time and across regions are primarily driven by changes in the input cost shares. Under the trans-log functional form cross price elasticities are a function of cost shares for both goods *i* and *j* as well as a coefficient determined by the cost share equation. Frondel and Schmidt (2002) maintain that the cross price elasticity will be approximately equivalent to the cost share of good *j* if the cost share is relatively larger than the coefficient γ_{ij} from the trans-log regression. They not only draw on empirical findings to support this observation, but also provide an economic explanation for this relationship.

$$\eta_{ij} = \frac{\gamma_{ij}}{S_i} + S_j \qquad \forall\ (i, j) \in (1,.....n) \mid i \neq j$$

The larger the cost share S_i is, the harder it will be to substitute to input *i* from input *j* when input *j*'s price increases. This argument considers the importance of inputs in the production process. Effectively if S_i is large this implies that input *i* is an important factor of production and so changes in the price of input j are not likely to have as strong an effect on the demand for input *i*, as they would if input *i* was relatively unimportant. The elasticity of substitution is affected by the importance of the inputs in question. This argument may hold some merit in the two input case however extending this reasoning into the multi-input case becomes more tedious.

In any respect Frondel's analysis regarding the importance of analysing cost shares is correct. Both AES and MES are functions of input cost shares and are sensitive to change in cost shares. Where cost shares are observed to show large variability over time the use of a time-series estimates for AES and MES should be applied with caution. This is true especially for energy cost shares which are known to be small allowing for large changes in AES, as a result of relatively small changes in the cost shares themselves (Thompson and Taylor (1995)). The same caution applies to cross sectional studies where cost shares can differ widely across regions. Since AES, MES and cross price elasticities are typically constructed at the mean cost share values the actual elasticity in a given region or time may not always be represented by the average elasticity estimate if there is a high variance in cost shares across regions or over time.

5.5 The Separability and Nesting of Inputs

The separability of inputs has been debated in relation to inter-fuel and energy-capital elasticities of substitution since Berndt and Wood (1979 and 1981) and Griffin and Gregory (1979a and 1981) began to examine the extent to which intermediate materials as factor inputs can be separated from capital, and labour, as well as energy. The cost functions used to calculate elasticities typically assume weak separability of inputs. This assumption is essential for analyzing elasticities of substitution. The primary weakly separable inputs are often further broken into sub sets of separable inputs of the aggregate input. This allows for analysis of substitutability at a lower level of input aggregation. For the purposes of energy policy the aggregate energy input can be analysed in terms of subsets of specific fuels. This level of desegregations allows for the analysis of inter-fuel elasticities of substitution.

5.5.1 Net and Gross Elasticities of Substitution:

Berndt and Wood (1979) propose partitioning aggregate inputs (K,L,E,M) from the master output function Y=F(K,L,E,M) into two mutually exclusive input subsets known as input nests.

$$K^* = f(K, E)$$
$$L^* = h(L, M)$$

The master output function has the duel cost function of:

$$C = G(Y, P_K, P_L, P_E, P_M)$$

Similarly the sub functions for K* and L* have dual cost structures of:

$$C_{k^*} = G(K^*, P_K, P_E)$$
$$C_{L^*} = G(L^*, P_L, P_M)$$

Where the partitioned cost functions C_X are linear functions of input prices and input quantities:

$$C_{K^*} = P_K K + P_M M$$
$$C_{L^*} = P_L L + P_M M$$

This allows for price elasticities to be constructed holding the quantities K* and L* fixed and allowing inputs in K* and L* to adjust to their cost minimizing levels holding all inputs outside the partition fixed. The cross price elasticities calculated from the sub-functions are said to be gross price elasticities. Cross price elasticities calculated from the master function hold total output fixed and allow all inputs to adjust to their cost minimizing level known as net elasticities of substitution. The relationship between the two elasticities can be defined as:

$$\eta_{ij} = \eta_{ij}^* + S_{jX}\eta_{XX}$$

Where η is the net cross price elasticity, η* is the gross cross price elasticity S_{jx} is the cost share of input j to the X* input (K* or L*) and η_{xx} is the own price elasticity of X* on the isoquant for total output. From this relationship it is clear that the net elasticity must be smaller than the gross elasticity since η_{xx} is always negative and S_{jx} is always positive. Berndt and Wood (1979) note that gross elasticities are more in line with engineering concepts of elasticities while net elasticities coincide with economic interpretations of elasticities. For this reason gross elasticities are sometimes called engineering elasticities while net elasticities are known as economic elasticities.

Griffin and Gregory (1976) partition the aggregate inputs (K,L,E,M) into a [(K,E,L), (M)] nesting structure. They find that the gross (E,K) elasticity of substitution from the (K,E,L) nest shows energy capital substitutability, while the net (E,K) elasticity of substitution shows a complementary relationship between energy and capital. Griffin (1981) further suggests that the relevant nesting and analysis of gross-net elasticities of substitution could provide more insight to the energy-capital substitutability-complementary problem than the long-run vs. short-run hypothesis.

5.5.2. Inter-fuel Elasticities of Substitution

Inter-fuel elasticities are of particular importance in environmental models. The techniques for estimating inter-fuel elasticities of substitution are the same as in the aggregate case. The aggregate energy input is partitioned to allow for analysis of individual energy inputs. The four most common energy inputs offered for analysis are oil, electricity, coal, and natural gas.

Interfuel elasticities of substitution do have one notable difference from their aggregate input counterparts. Complementary relationships between aggregate inputs are not all together uncommon, especially in the energy-capital case described above. However complementary relationships between fuels are more difficult to reconcile. Taheri and Stevenson (2002), Jones (1996 and 1995), Andrikopoulos et al (1989), and Fuss (1977) find cases where fuel inputs are compliments to one another. In cases where fuel inputs are known to show a complementary relationship, efforts should be made to ensure that the estimates represent long-run rather then short-run elasticities. Also in such cases the statistical significance of the estimate should be examined to ensure that the estimate truly does represent a complementary relationship.

5.5.3. Tests for Separability

The standard definition of weak separability states that:

Given a set of inputs in N={1,….,n} and supposing that these inputs can be partitioned into M>1 mutually exclusive and exhaustive subsets N_1, …N_m,, the production function is said to be weakly separable if the MRS between two inputs from the same subset is independent of inputs which are part of another subset:

$$\frac{\partial(MRS_{ij})}{\partial x_k} = 0 \qquad \forall(i,j)\in N_m \text{ and } k \notin N_m$$

This definition follows Berndt and Christensen (1973) who further explore the relationship between elasticities of substitution and separability of production and resulting duel cost functions. They show that given a duel cost function the underlying production function will be weakly separable if and only if:

$$AES_{ik} = AES_{jk} \qquad \forall (i, j) \in N_m \; and \; k \notin N_m$$

As stated above the statistical properties of the Allen elasticities have not been fully explored. As such a great deal of caution must be applied when testing this condition. Under a trans-log cost function the condition can be simplified to show a general case where this condition is always met involving only parameters from the trans-log estimation for which statistical properties are well defined.

$$AES_{ik} = AES_{jk}$$

$$\frac{\gamma_{ik}}{S_i S_k} + 1 = \frac{\gamma_{jk}}{S_j S_k} + 1$$

$$\frac{\gamma_{ik}}{S_i} = \frac{\gamma_{jk}}{S_j}$$

This will always hold for the trivial case where:

$$\gamma_{il} = \gamma_{jk} = 0 \qquad \forall (i, j) \in N_m \; and \; k \notin N_m$$

which implies that:

$$(AES_{ik} = AES_{jk} = 1) \qquad \forall (i, j) \in N_m \; and \; k \notin N_m$$

This condition is known as the linear separability condition which is applied by Berndt and Christensen (1973) and later by Garofalo and Malhotra (1988) to test for the appropriate nesting structure of inputs. If the inputs are not found to be weekly separable then they can not be disaggregated into separate inputs or input functions. If inputs are deemed weakly separable when they are in fact not, elasticities can not be constructed with confidence.

6. Summary of Empirical Results in the Literature

6.1 Inter-fuel Elasticities of Substitution

Inter-fuel elasticities of substitution are presented here from a number of prominent studies in the literature. The majority of researchers report either Allen elasticities of substitution or price elasticities. Where prices elasticities are reported Morishima elasticities are constructed by the author from the formulas provided above. The summary statistics of for these estimates are located in Tables 1 to 8. Several studies make effort to explicitly estimate long-run-short-run elasticities; summary statistics from these studies can be found in tables 1 to 4. Tables 5 to 8 display estimates by estimation method, data type and type of elasticity for the aggregate manufacturing sector.

Tables 1 and 3 confirm that Morishima elasticities are generally smaller in magnitude for short-run compared to long-run estimates. Tables 2 and 4 indicate the same is true for Allen Elasticities except in the case of coal and natural gas which are complements. Tables 5 and 6 show that there are significant differences between Allen elasticities estimated from linear-logit and trans-log models. Tables 7 and 8 confirm that elasticities of substitution estimated from panel data indicate a higher degree of substitutability than when time series data is used. Tables 9 and 10 also support this result for trans-log models however the sample is fragmented in the panel data case.

Allen elasticities are generally smaller than Morishima elasticities when comparing models of the same type. This result was expected as Morishima elasticities are known to be generally larger than Allen elasticities (Stern (2004) and Thompson (1997)).

Table 1. Short-Run Morishima Elasticities of Substitution

	N	Mean	Median	Min	Max	SD
Coal-Oil	5	0.30	0.29	0.12	0.53	0.18
Coal-Electricity	6	0.34	0.30	0.11	0.73	0.25
Coal-Natural Gas	6	0.26	0.26	0.06	0.51	0.19
Oil-Coal	5	0.42	0.41	0.23	0.64	0.18
Oil-Electricity	5	0.16	0.13	0.11	0.28	0.07
Oil-Gas	5	0.32	0.29	0.22	0.44	0.09
Electricity-Coal	6	0.77	0.30	0.11	3.33	1.26
Electricity-Oil	5	0.11	0.10	0.06	0.16	0.04
Electricity-Natural Gas	6	0.47	0.30	0.15	1.50	0.52
Natural Gas -Coal	6	0.08	0.09	-0.14	0.28	0.13
Natural Gas-Oil	5	0.32	0.31	0.23	0.45	0.09
Natural Gas-Electricity	6	0.33	0.27	0.16	0.54	0.16

Source: Jones (1995), Urga and Walters (2003), and Taheri (1994).

Table 2. Short-Run Allen Elasticities of Substitution

	N	Mean	Median	Min	Max	SD
Coal-Oil	3	0.69	0.60	0.35	1.12	0.39
Coal-Electricity	3	0.47	0.47	0.13	0.81	0.34
Coal-Natural Gas	3	-1.03	-0.79	-1.87	-0.44	0.75
Oil-Coal	3	0.55	0.64	0.35	0.67	0.18
Oil-Electricity	3	-0.03	-0.04	-0.10	0.04	0.07
Oil-Gas	3	0.46	0.50	0.34	0.56	0.12
Electricity-Coal	3	0.44	0.43	0.13	0.74	0.30
Electricity-Oil	3	-0.04	-0.06	-0.10	0.04	0.08
Electricity-Natural Gas	3	0.09	0.23	-0.34	0.38	0.38
Natural Gas -Coal	3	-0.61	-0.44	-1.02	-0.38	0.35
Natural Gas-Oil	3	0.39	0.34	0.31	0.52	0.12
Natural Gas-Electricity	3	0.44	0.25	0.23	0.83	0.34

Source: Urga and Walters (2003)

Table 3. Long-Run Morishima Elasticities of Substitution						
	N	Mean	Median	Min	Max	SD
Coal-Oil	5	0.44	0.45	0.29	0.58	0.11
Coal-Electricity	6	0.52	0.41	0.29	1.03	0.27
Coal-Natural Gas	6	0.36	0.29	0.17	0.70	0.19
Oil-Coal	5	0.67	0.64	0.38	0.92	0.21
Oil-Electricity	5	0.22	0.36	-0.32	0.46	0.32
Oil-Gas	5	0.57	0.49	0.25	0.90	0.27
Electricity-Coal	6	0.71	0.45	0.26	2.28	0.78
Electricity-Oil	5	0.22	0.23	0.09	0.35	0.12
Electricity-Natural Gas	6	0.64	0.51	0.19	1.71	0.54
Natural Gas -Coal	6	-0.03	0.17	-1.09	0.35	0.55
Natural Gas-Oil	5	0.57	0.42	0.35	0.89	0.26
Natural Gas-Electricity	6	0.55	0.57	0.25	0.82	0.23
Source Jones (1995), Urga and Walters (2003), and Taheri (1994).						

Table 4. Long-Run Allen Elasticities of Substitution						
	N	Mean	Median	Min	Max	SD
Coal-Oil	3	1.00	1.12	0.64	1.23	0.31
Coal-Electricity	3	0.60	0.49	0.48	0.83	0.20
Coal-Natural Gas	3	-1.48	-1.60	-1.86	-0.99	0.45
Oil-Coal	3	0.86	0.68	0.67	1.23	0.32
Oil-Electricity	3	0.05	0.01	-0.04	0.16	0.10
Oil-Gas	3	0.78	0.56	0.55	1.23	0.39
Electricity-Coal	3	0.57	0.49	0.49	0.74	0.15
Electricity-Oil	3	0.04	0.04	-0.10	0.16	0.13
Electricity-Natural Gas	3	0.34	0.38	-0.20	0.85	0.53
Natural Gas -Coal	3	-1.05	-1.01	-1.60	-0.54	0.53
Natural Gas-Oil	3	0.71	0.52	0.38	1.23	0.46
Natural Gas-Electricity	3	0.65	0.85	0.25	0.85	0.34
Source Urga and Walters (2003)						

Table 5. Allen Elasticities of Substitution from Linear Logit models using Time Series Data						
	N	Mean	Median	Min	Max	SD
Coal-Oil	3	0.68	0.45	0.35	1.23	0.48
Coal-Electricity	3	0.43	0.49	0.13	0.66	0.27
Coal-Natural Gas	3	-0.89	-0.63	-1.6	-0.44	0.63
Oil-Electricity	3	0.03	0.04	-0.12	0.16	0.14
Oil-Gas	3	0.72	0.61	0.34	1.23	0.46
Electricity-Natural Gas	3	0.53	0.5	0.23	0.85	0.31
Source Urga and Walters (2003).						

Table 6. Allen Elasticities of Substitution from Trans-log Models using Time Series data						
	N	Mean	Median	Min	Max	SD
Coal-Oil	6	0.33	0.66	-2.16	1.12	1.24
Coal-Electricity	6	0.86	0.78	0.47	1.84	0.5
Coal-Natural Gas	6	0.24	-0.99	-1.87	7.92	3.79
Oil-Electricity	6	0	-0.04	-0.1	0.26	0.13
Oil-Gas	6	0.67	0.55	0.5	1.32	0.32
Electricity-Natural Gas	6	0.12	0.26	-0.34	0.38	0.31
Source Urga and Walters (2003), Mangus and Woodland (1987)						

Table 7.Morishima Elasticities of Substitution from Linear Logit Models using Time Series data						
	N	Mean	Median	Min	Max	SD
Coal-Oil	9	0.41	0.31	-0.05	1.18	0.38
Coal-Electricity	9	0.41	0.32	0.03	1.15	0.36
Coal-Natural Gas	9	0.23	0.22	-0.40	0.81	0.34
Oil-Coal	9	0.47	0.30	0.19	0.96	0.32
Oil-Electricity	9	0.19	0.12	0.03	0.46	0.18
Oil-Gas	9	0.41	0.30	0.18	0.90	0.26
Electricity-Coal	9	0.66	0.42	0.11	1.87	0.62
Electricity-Oil	9	0.24	0.28	0.05	0.60	0.19
Electricity-Natural Gas	9	0.43	0.26	-0.23	1.12	0.44
Natural Gas -Coal	9	0.09	0.10	-0.54	0.35	0.27
Natural Gas-Oil	9	0.55	0.52	0.23	0.89	0.23
Natural Gas-Electricity	9	0.61	0.76	0.21	0.91	0.28
Source: Jones (1995), Urga and Walters (2003), and Considine (1989).						

Table 8.Morishima Elasticities of Substitution from Linear Logit Models using Panel Data						
	N	Mean	Median	Min	Max	SD
Coal-Oil	16	1.69	1.68	0.97	2.57	0.55
Coal-Electricity	16	1.28	1.26	0.73	1.84	0.45
Coal-Natural Gas	16	1.26	1.24	0.71	1.83	0.45
Oil-Coal	16	2.96	2.97	2.33	3.90	0.39
Oil-Electricity	16	2.57	2.67	1.94	3.62	0.42
Oil-Gas	16	3.37	3.32	2.52	4.70	0.54
Electricity-Coal	16	0.77	0.77	0.39	1.15	0.27
Electricity-Oil	16	1.15	1.15	0.90	1.38	0.13
Electricity-Natural Gas	16	0.19	0.18	-0.05	0.41	0.15
Natural Gas -Coal	16	1.06	1.06	0.73	1.50	0.22
Natural Gas-Oil	16	1.95	1.78	1.26	3.62	0.60
Natural Gas-Electricity	16	0.93	0.91	0.56	1.40	0.23
Source Jones (1996).						

Table 9.Morishima Elasticities of Substitution from Trans-Log Models using Time Series Data						
	N	Mean	Median	Min	Max	SD
Coal-Oil	23	1.22	1.40	0.05	3.14	0.79
Coal-Electricity	23	1.11	1.29	0.08	2.13	0.67
Coal-Natural Gas	21	1.67	0.95	-0.33	7.29	1.83
Oil-Coal	23	0.64	0.51	-0.45	2.09	0.55
Oil-Electricity	23	0.34	0.21	-0.32	1.99	0.49
Oil-Gas	23	0.31	0.29	-0.86	1.42	0.41
Electricity-Coal	23	0.55	0.52	-0.40	2.02	0.51
Electricity-Oil	23	0.27	0.16	-0.10	0.93	0.26
Electricity-Natural Gas	23	0.18	0.19	-1.70	1.20	0.59
Natural Gas -Coal	23	0.99	0.50	-0.82	2.91	1.12
Natural Gas-Oil	23	0.71	0.45	-0.20	2.13	0.61
Natural Gas-Electricity	23	0.76	0.52	-0.10	2.23	0.59
Sourced: Jones (1996), Mangus and Woodland (1987), Urga and Walters (2003), Pyndick (1979), Considine (1989), Fuss (1977).						

Table 10.Morishima Elasticities of Substitution from Trans-Log Models using Panel Data						
	N	Mean	Median	Min	Max	SD
Coal-Electricity	3	0.91	0.98	0.73	1.03	0.16
Coal-Natural Gas	3	0.31	0.51	-0.28	0.70	0.52
Oil-Electricity	2	5.37	5.37	3.16	7.58	3.13
Oil-Gas	2	6.02	6.02	3.53	8.51	3.52
Electricity-Coal	3	2.13	2.28	0.76	3.33	1.29
Electricity-Oil	2	3.39	3.39	2.08	4.70	1.85
Electricity-Natural Gas	5	1.77	1.71	0.90	2.80	0.69
Natural Gas -Coal	3	-0.24	0.06	-1.09	0.29	0.74
Natural Gas-Oil	2	4.99	4.99	3.36	6.62	2.31
Natural Gas-Electricity	5	1.59	0.77	0.54	3.73	1.39
Source: Taheri and Stevenson (2002), Taheri (1994), Bousquet and Ladoux (2006)						

Unfortunately there are a limited number of studies which have examined inter-fuel elasticities at the disaggregated industry level. Results from two studies are located in Tables 11 to 13. Both of these studies used the trans-log model for estimation. The Andrikopoulos et. al. (1989) estimation uses Canadian data from the province of Ontario from 1962 to 1982 to produce Allen elasticities of substitution for seven industry classifications. The Mangus and Woodland (1987) estimates use Dutch data from 1958 to 1976, producing both Allen and Morishima elasticities of substitution.

From these tables it is evident that once industries are examined at the disaggregated level large differences can be found between various sectors. In all three tables it can be observed that there are complementary relationships between certain fuel types in some industries while in others they are substitutable. There are also differences between the two studies. Tables 11 and 12 show the differences between Allen elasticity estimates for the food and beverage, paper and allied products, primary metals and chemical industries. There are several consistent findings between the two studies, notably in the case of complementary fuels. Results for the chemical industry show that coal and electricity as well as oil and electricity are complementary fuels. The same is also true for coal and oil in the primary metals and paper industries.

Table 11. Allen Elasticities by Industry Classification (Andrikopoulos et al, 1989)

	Food and Beverage	Paper and Allied products	Non-Metalic	Primary Metals	Chemical	Transport Equipment	Other Manufacturing
Coal-Oil	0.84	-0.51	-4.26	-1.64	9.60	4.22	1.96
Coal-Electricity	0.57	0.71	-1.88	1.75	-1.89	2.27	-2.79
Coal-Natural Gas	5.16	-1.49	5.53	2.91	6.09	7.24	19.20
Oil-Electricity	1.60	2.04	1.49	1.22	-5.89	0.70	1.28
Oil- Natural Gas	3.29	1.41	2.57	2.89	0.55	0.40	2.41
Electricity-Natural Gas	-1.10	-0.41	-0.58	-0.54	0.60	-0.40	-0.70

Table 12. Allen Elasticities by Industry Classification (Mangus and Woodland 1987)

	Food	Textiles	Paper	Chemical	Building Materials	Machinery and Equipment	Total	Market
Coal-Oil	-2.54	-4.28	-1.5	-0.45	-0.48	-0.36	-2.16	-0.56
Coal-Electricity	1.04	2.82	0.7	-1.89	2.33	1.11	1.84	-0.35
Coal-Natural Gas	5.27	8.03	5.56	2.63	2.97	1.07	7.92	2.71
Oil-Electricity	0.58	0.22	0.39	-1.33	-0.06	0.16	0.26	-0.76
Oil- Natural Gas	1.44	1.36	1.26	0.86	1.06	1.09	1.32	0.89
Electricity-Natural Gas	0.03	0.19	0.35	0.2	0.25	0.12	0.28	0.18

After comparing AES to MES estimates from the Mangus and Woodland (1987) study it is evident that the two units of measurement represent different elasticities of substitution across all industry classifications. Given the number of estimates from this study it is possible to examine the relationship between the elasticity measurements. One of the assertions outlined above was that Morishima elasticities of substitution are generally greater than Allen elasticities of substitution. Comparing estimates form tables 12 and 13 it is clear that this is not universally true in this case. Only 58 percent of the Morishima estimates are greater than their Allen counterparts from this study.

Table 13. Morishima Elasticities by Industry Classification (Mangus and Woodland 1987)								
	Food and Beverage	Textiles	Paper	Chemical	Building Materials	Machinery and Equipment	All Industries	Market
Coal-Oil	1.05	1.51	1.82	0.25	1.52	0.66	1.75	0.50
Coal-Electricity	1.26	1.76	2.00	0.10	1.86	0.75	1.92	0.51
Coal-Natural Gas	1.51	1.94	2.39	0.55	1.93	0.75	2.19	0.83
Oil-Coal	-0.33	-1.14	0.09	-0.22	0.21	0.19	-0.45	-0.18
Oil-Electricity	0.68	0.34	0.57	-0.53	0.32	0.32	0.42	-0.16
Oil-Gas	0.96	0.72	0.79	0.26	0.60	0.54	0.81	0.35
Electricity-Coal	0.56	1.33	0.49	-1.01	0.92	0.68	0.87	-0.30
Electricity-Oil	0.43	0.31	0.40	-0.89	0.35	0.21	0.33	-0.38
Electricity-Natural Gas	0.27	0.29	0.39	-0.57	0.42	0.19	0.33	-0.12
Natural Gas -Coal	2.52	2.76	2.99	1.46	1.83	0.61	2.91	1.55
Natural Gas-Oil	1.26	1.13	1.35	0.89	1.10	0.61	1.25	0.96
Natural Gas-Electricity	0.80	0.85	1.00	0.67	0.79	0.41	0.99	0.71

6.2 Aggregate Elasticities of Substitution for factors of production

Tables 14 to 19 show summary statistics for various elasticities, models and data types for the aggregate manufacturing industry. A number of stylized facts can be observed from comparing these results.

Table 14. AES from a Trans-Log Model using Pooled Data						
	N	Mean	Median	Min	Max	SD
Energy-Capital	21	0.85	1.02	-0.70	1.77	0.48
Energy-Labour	21	0.98	0.87	0.05	2.42	0.43
Capital-Labour	21	0.65	0.69	0.06	1.43	0.33
Labour-Materials	1	1	1	1	1	.
Energy-Materials	1	0.58	0.58	0.58	0.58	.
Capital-Materials	1	0.85	0.85	0.85	0.85	.
Source: Griffin (1976), Pindyck (1979), Garofalo and Malhotra (1988)						

Comparing tables 14 and 15 it is clear that there is a dichotomy between elasticities estimates from time series and pooled data sources. The energy-capital elasticity of substitution has a mean of 0.85 using pooled data while the same elasticity of substitution has a mean of -2.63 using time series data. This shows that using time series data capital and energy are compliments rather than substitutes. This finding is in line with previous studies described above which claim that the use of pooled data provides estimates of long-run elasticities of substitution which would yield substitutable relationship between capital and energy.

Table 15. AES from a Trans-Log Model using Times Series Data						
	N	Mean	Median	Min	Max	SD
Energy-Capital	13	-2.63	-3.22	-3.95	2.68	1.77
Energy-Labour	13	0.62	0.64	0.08	0.84	0.18
Labour-Energy	13	0.62	0.64	0.08	0.84	0.18
Capital-Labour	13	1.03	1.01	0.37	1.56	0.29
Labour-Materials	10	0.59	0.59	0.57	0.61	0.01
Energy-Materials	10	0.80	0.81	0.74	0.85	0.05
Capital-Materials	10	0.48	0.50	0.34	0.58	0.08
Source: Berndt and Wood (1975), Frondel (2004), Hunt (1984), and Hunt (1986).						

In order to examine differences between AES and MES measures tables 15 and 16, 14 and 17, and 18 and 19 are compared to one another. Comparing estimates in this manner shows the differences between AES and MES, while holding the model and data types constant. There are some rather notable differences between MES and AES when using the trans-log model with time series data. Most notably the energy-capital elasticities of substitution show a complementary relationship in the AES case but a substitutable relationship in the MES case. This is a particularly important observation for constructing elasticity of substitution estimates for use in CGE models using CES production technologies which require that all inputs are substitutes. Tables 14 and 17 show remarkably similar estimates for capital-energy elasticities of substitution with a mean AES of 0.85 and mean MES of 0.83 for energy-capital and 0.61 for capital-energy. Both tables 14 and 17 show a substitutable relationship between capital and energy. This indicates that in the long-run elasticities case, as estimated using pooled data, AES and MES estimates may become more similar to one another than for the short-run elasticities, as estimated using time series data.

Table 16. MES from a Trans-Log Model using Time Series Data						
	N	Mean	Median	Min	Max	SD
Energy-Capital	18	0.32	0.32	0.14	0.55	0.13
Capital-Energy	18	0.37	0.32	-0.07	1.10	0.32
Energy-Labour	18	0.53	0.56	0.29	0.66	0.13
Labour-Energy	18	0.69	0.64	0.10	1.13	0.24
Capital-Labour	18	0.58	0.55	0.06	1.35	0.35
Labour-Capital	18	0.66	0.70	0.33	1.51	0.35
Labour-Materials	15	0.63	0.60	0.54	0.80	0.09
Materials-Labour	15	0.60	0.61	0.41	0.71	0.09
Energy-Materials	15	0.56	0.53	0.25	0.76	0.18
Materials-Energy	15	0.51	0.56	0.06	0.71	0.21
Capital-Materials	15	0.38	0.46	0.00	0.54	0.18
Materials-Capital	15	0.40	0.39	0.13	0.58	0.16
Source: Krinsky and Robb (1991), Anderson (1981), Frondel (2004), Berndt and Woods (1975), Hunt (1984), and Hunt (1986).						

Table 17. MES from a Trans-Log Model using Pooled Data						
	N	Mean	Median	Min	Max	SD
Energy-Capital	11	0.87	0.92	0.44	0.94	0.15
Capital-Energy	11	0.61	0.70	0.01	0.76	0.23
Energy-Labour	11	0.93	0.91	0.84	1.11	0.08
Labour-Energy	11	0.78	0.71	0.53	1.71	0.31
Capital-Labour	11	0.53	0.51	0.19	0.96	0.18
Labour-Capital	11	0.51	0.48	0.17	0.86	0.17
Labour-Materials	1	0.93	0.93	0.93	0.93	.
Materials-Labour	1	0.47	0.47	0.47	0.47	.
Energy-Materials	1	0.47	0.47	0.47	0.47	.
Materials-Energy	1	0.36	0.36	0.36	0.36	.
Capital-Materials	1	1.33	1.33	1.33	1.33	.
Materials-Capital	1	0.61	0.61	0.61	0.61	.
Source: Garofalo and Malhotra (1988), Griffin (1976) and Fuss(1977)						

Tables 18 and 19 show the differences between MES and AES estimates under a linear logit model using time series data. Despite the fact that the estimates in these tables are sourced from the same author and result from identical regressions there are large differences between estimates. This could be due to the use of time series data rather than pooled data. Contrary to prediction, elasticities of substitution are shown to be larger using AES than MES. The lack of studies which employ the linear logit model in estimating energy-capital elasticities of substitution make it difficult to draw any conclusions from these tables, however energy and capital appear as substitutes.

Table 18. MES from a Linear Logit Model using Time Series Data						
	N	Mean	Median	Min	Max	SD
Energy-Capital	5	0.26	0.26	0.05	0.47	0.16
Capital-Energy	5	0.56	0.31	0.11	1.79	0.69
Energy-Labour	5	0.23	0.04	-0.12	0.67	0.35
Labour-Energy	5	0.51	0.38	-0.17	1.26	0.61
Capital-Labour	5	0.49	0.33	-0.09	1.51	0.60
Labour-Capital	5	0.40	0.26	-0.02	0.89	0.43
Labour-Materials	5	0.63	0.43	0.25	1.29	0.42
Materials-Labour	5	0.72	0.35	0.20	1.76	0.69
Energy-Materials	5	0.25	0.17	0.04	0.56	0.23
Materials-Energy	5	0.44	0.23	0.01	1.14	0.51
Capital-Materials	5	0.54	0.31	0.05	1.85	0.75
Materials-Capital	5	0.77	0.55	-0.10	2.78	1.18
Source: Considine (1989) and Considine (1990).						

The final comparison which can be made from these tables is to compare tables 16 and 18 and 15 and 19. This allows for a comparison of model type holding elasticity and data type constant. Tables 16 and 18 show similar results for energy-capital elasticities; both models suggest a substitutable relationship of similar magnitudes. Tables 15 and 19, however, show fairly significant differences between estimates. Capital and energy are relatively strong complements under the trans-log model while they are relatively strong substitutes under the linear logit model.

Table 19. AES from a Linear Logit Model using Time Series Data

	N	Mean	Median	Min	Max	SD
Energy-Capital	2	1.39	1.39	0.46	2.32	1.32
Energy-Labour	2	-0.99	-0.99	-1.65	-0.33	0.93
Capital-Labour	2	-1.25	-1.25	-2.09	-0.41	1.19
Labour-Materials	2	0.76	0.76	0.34	1.17	0.59
Energy-Materials	2	0.43	0.43	0.14	0.71	0.40
Capital-Materials	2	2.02	2.02	0.66	3.38	1.92

Source: Considine (1989).

Table 20. Energy-Capital Elasticities of Substitution by Industry

Industry	Bataille (2006)	Bataille (1998)	Walton (1981)	Denny (1981) (US)	Denny (1981) (Canada)	Field (1980) E,K	Field (1980) K,E	Prywes (1986)	Kemfert (1998)*	Van der Werf (2007)*
Agriculture	.	.	.	.	.	1.099	-0.013	.	.	.
Mining	0.18	0.03	.	.	.	.	.	.	.	.
Coal	0.39	.	.	.	.	.	.	.	.	.
Petroleum Refining	-0.09	0.12	1.95	.	.	2.709	2.702	-0.54	.	.
Power Generation	0.33	.	.	.	0.32	.	.	.	.	.
Pulp & Paper	0.32	0.34	.	-2.74	1.93	.	.	-4.53	.	0.9675
Cement	0.33	.	0.83	.	.	.	.	.	.	.
Iron & Steel	0.12	0.1	0.48	2.43	9.6	.	.	-2.99	0.34	.
Smelting	.	.	.		.	0.968	0.333	.	.	.
Chemical	0.04	0.11	0.65	0	13.82	0.749	-0.035	-2.09	0.93	.
Other Manufacturing	0.03	0.06	0.99	.	.	.	.	.	.	.
Transport	.	.	.	0.67	-9	0.794	-0.376	0.11	0.61	0.9966

Table 20 shows Allen elasticities of substitution by industry classification with the exception of Field (1980) where Morishima estimates are provided and Kempfert (1998) and Van der Wolf (2007) who provide direct estimates from a CES functional form. The industrial classifications are designed to comply with those used in many CGE models. Where exact classifications could not be matched elasticities were assumed to correspond to their closest classification. Battaille (2006 and 1998) use pseudo data to estimate elasticities of substitution, but for the reasons stated above, it is not expected that these estimates would match those calculated using actual data.

It is clear that there is a large variation of estimates across sectors and across studies. The differences across sectors within the same studies indicate that a single energy-capital elasticity of substitution should not be applied to all sectors. Given the apparent variance across studies it is difficult to reach a conclusion as to which elasticity best represents the true elasticity of substitution.

7. Conclusion

The findings in this paper show that elasticities of substitution can differ significantly and systematically depending on the type of elasticity, estimation technique employed and data type available. CGE modellers rely heavily elasticity estimates from the literature. Elasticities of substitution are one of the most critical and sensitive parameters in CGE models. Understanding what the elasticity means and how it is estimated can provide useful information to modellers in choosing the right source for elasticity estimates and aids in the interpretation of model results.

Certain methods of calculating and estimating elasticities of substitution are preferred to others. The preferred case would be one where Morishima elasticities are estimated using the linear logit model with panel data. Morishima elasticities of substitution provide a better measure of the curvature of the isoquant and therefore substitutability.

The symmetry problem inherent with the use of Morishima elasticities could be mitigated by constructing either McFadden shadow elasticities or by using the asymmetry property of Morishima elasticities for sensitivity analysis in model calibration. Cross-sectional or pooled data is generally accepted as the appropriate type of data for obtaining estimates of long-run elasticities. The linear logit model insures that the monotonicity condition is met in addition to providing more accurate estimates of the own price elasticity.

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