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Abstract

The network pattern of financial linkages is important in many areas of banking and finance. Yet bilateral linkages are often unobserved, and *maximum entropy* serves as the leading method for estimating counterparty exposures. This paper proposes an efficient alternative that combines information-theoretic arguments with economic incentives to produce more realistic interbank networks that preserve important characteristics of the original interbank market. The method loads the most probable links with the largest exposures consistent with the total lending and borrowing of each bank, yielding networks with *minimum density*. When used in a stress-testing context, the minimum-density solution overestimates contagion, whereas maximum entropy underestimates it. Using the two benchmarks side by side defines a useful range that bounds the cost of systemic stress present in the true interbank network when counterparty exposures are unknown.

JEL classification: G21, L14, D85, C63
Bank classification: Econometric and statistical methods; Financial institutions; Financial stability

Résumé

Le réseau de liens financiers est important dans de nombreux segments des secteurs bancaire et financier. Pourtant, dans bien des cas, les liens bilatéraux ne sont pas observés et la méthode de l'*entropie maximale* est la plus couramment utilisée pour estimer les expositions au risque de contrepartie. Dans leur étude, les auteurs proposent une solution de rechange efficace qui combine des arguments relevant de la théorie de l'information et des arguments économiques pour créer des réseaux interbancaires hypothétiques plus réalistes qui préservent les principales caractéristiques du marché interbancaire original. La méthode consiste à tenir compte des liens les plus probables en fonction de la taille des expositions entre les banques, selon le total des prêts et des emprunts de chacune, afin de générer des réseaux ayant une *densité minimale*. Dans le contexte des tests de résistance, la méthode axée sur la densité minimale surestime la contagion, alors que celle de l'entropie maximale la sous-estime. En ayant recours aux deux modèles en parallèle, on peut ainsi définir une fourchette utile qui délimite le coût attribuable aux tensions systémiques présentes dans le réseau interbancaire réel lorsque les expositions au risque de contrepartie sont inconnues.

Classification JEL : G21, L14, D85, C63
Classification de la Banque : Méthodes économétriques et statistiques; Institutions financières; Stabilité financière

1. Introduction

Interbank contagion is a fundamental channel in many of the stress tests gauging systemic risk. Yet in practice the interbank network at the core of these simulations often remains unobserved: because interbank loans are generally arranged over the counter, the bilateral exposures are often known only by the immediate counterparties of each trade. In some jurisdictions, bilateral positions can be obtained from regulatory filings or credit registers. More often, however, central banks and regulators do not observe the network because banks do not report their bilateral exposures. In those cases, the leading method is for researchers to fill in the blanks as evenly as possible, using the available information on each bank's total interbank lending (Upper 2011, Elsinger et al. 2013). This approach, known as *maximum entropy*, effectively assumes that banks diversify their exposures by spreading their lending and borrowing across *all* other active banks.

The maximum-entropy (ME) approach for filling in the blanks, however, can be misleading when the result is employed in network analysis or financial stress tests. The practical shortcoming is that applying ME tends to create *complete* networks which obscure the true structure of linkages in the original network. Key concepts in network analysis, such as degree (number of connections), become meaningless as a result of using ME. Furthermore, when such estimated networks are used for purposes of stress testing, ME introduces a bias (Mistrulli 2011, Markose et al. 2012). The related conceptual shortcoming is that applying ME is optimal from an information-theoretic perspective only if *nothing else* is known about the network. But we do know that interbank networks are typically sparse, because interbank activity is based on relationships (Cocco et al. 2009) and smaller banks use a limited set of money center banks as intermediaries (Craig and von Peter 2014). Indeed, most banks would find it prohibitively costly in terms of information processing and risk management to lend to *every* active bank in the system.

This paper proposes an alternative benchmark, one that minimizes the number of links necessary for distributing a given volume of loans on the interbank market. Our *minimum-density* method, in contrast to maximum entropy, is based on the economic rationale that interbank linkages are costly to maintain. Our procedure determines a pattern of linkages for allocating interbank positions that is efficient in the sense of minimizing these costs. Intuitively, our approach identifies the most probable links and loads them with the largest possible exposures consistent with the total lending and borrowing observed for each bank that could be obtained from the bank balance sheet. This link prediction method uses only this balance-sheet data, combined with elements of information theory and economic rationale, to determine a sparse network without using any information from unobserved bilateral positions of the original network.

Even as the stress-testing literature recognizes that the use of ME induces a bias, few viable alternatives have been proposed. In their study of the credit default swap market, Markose et al.

(2012) distribute exposures over a subset of possible linkages, where the degree of each bank is proportional to its market share. This method leads to a bilateral network that is more concentrated than the ME solution. Technically, our paper is more closely related to Mastromatteo et al. (2012), who provide an efficient algorithm to sample from a distribution of potential networks that satisfy structural constraints. Assuming a multinomial logit, or Gibbs-Boltzmann, probability distribution, they employ a belief-propagation algorithm to compute "marginal" probabilities for the likelihood that a link exists between two banks. Our paper, in contrast, constructs an optimal multinomial logit sampling distribution that draws on ideas from network science and information theory. We provide a heuristic algorithm to reconstruct interbank networks that incorporate priors over the set of links between banks that are more likely to arise.[1]

In section three, we confront these two estimated benchmarks with the German interbank network where we have detailed data on the linkages between 1,800 banks. We find that our minimum-density solution preserves some of the true network's structural features better than maximum entropy does, especially when using a solution less aggressive than the minimum-density benchmark. This makes our method a reasonable alternative benchmark for estimating missing counterparty exposures, one that does not wipe out the sparse structure of linkages that is so central to network analysis.

The final section shows that systemic risk clearly depends on the pattern of interlinkages. We contrast the results from a standard stress test on the German banking system with those obtained using the two alternative benchmarks instead. We find that the maximum-entropy approach underestimates contagion, whereas our minimum-density method overestimates it, often to a lesser extent. Using the two benchmarks in this context thus helps identify a range of possible stress-test outcomes and also facilitates robust systemic risk analysis through repeated application. This makes the case for using both benchmarks when gauging how financial linkages affect systemic risk.

While these findings are relevant for central banks and regulators, our approach may be of independent interest. In finance and various other disciplines, situations arise where a network of interest is not fully observed, or has yet to be designed. Networks in transportation, financial markets or international trade are much sparser – and for good reason – than maximum entropy would have us believe. Our minimum-density method may thus provide a meaningful alternative in various areas, guided by the simple economic rationale of minimizing the cost of additional links.

[1] Wang (2013) also proposes methods for estimating interbank networks using daily balance-sheet data for Singapore banks.

2. Minimum density as an alternative to maximum entropy

2.1. The case for an alternative to maximum entropy

Consider a system of N banks engaged in interbank lending and borrowing. The matrix $\mathbb{X} \in [0,\infty)^{N\times N}$ represents gross interbank positions, where the typical element X_{ij} represents the amount bank i lends to bank j. Such networks are directed and valued. For each bank i, the row sum of $\mathbb{X}$ shows total interbank assets, and the column sum tallies bank i's interbank liabilities vis-à-vis all other banks,

$$\begin{cases} \text{Interbank assets:} & A_i = \sum_{j=1}^{N} X_{ij} \\ \text{Interbank liabilities:} & L_i = \sum_{j=1}^{N} X_{ji} \end{cases}. \tag{1}$$

Matrix $\mathbb{X}$ is the network of bilateral exposures that is needed for additional analysis. In many situations, however, the bilateral linkages are unknown. National authorities in most jurisdictions do not observe the full interbank network, because banks do not report any bilateral positions or only disclose their largest exposures. However, the authorities do generally have balance-sheet information at their disposal, including the total interbank assets A_i and liabilities L_i of each bank i.

Suppose, for ease of exposition, that no bilateral positions are observed.[2] In this simple case, the authorities only know how much each bank lends and borrows on the interbank market overall (A_i and L_i), but not to whom. Since information on counterparty exposures (X_{ij}) is essential for further analysis, it is necessary to resort to a method for filling in the interbank matrix, given knowledge of the "marginals," $\{A_i, L_i\}$.

The standard approach in the literature is to estimate a matrix by maximum entropy (Upper 2011, Elsinger et al. 2013). Intuitively, this approach spreads exposures as evenly as possible, consistent with the marginals, and thereby fills all cells between active banks. Formally, the method solves for the bilateral exposures $\{E_{ij}\}$ that minimize the relative entropy function $\sum_{i,j} E_{ij} \ln\left(E_{ij}/Q_{ij}\right)$ subject to constraints (1), and relative to prior information, $\{Q_{ij}\}$ on bilateral exposures, if available. Since entropy is a measure of probabilistic uncertainty, this approach is optimal when selecting a solution in the sense of using least information (MacKay 2003). Entropy optimization is widely used across disciplines (Fang et al. 1997), and is straightforward to implement using a standard iterative algorithm (RAS), which can be generalized to handle additional constraints (Blien and Graef 1997, Elsinger et al. 2013).

In the interbank context, the maximum-entropy approach delivers an unrealistic network

[2] This is without loss of generality, since the method below can accommodate more information with exclusion restrictions on the prior distribution Q below.

structure, and one that tends to understate contagion in systemic stress tests. In the simple case where only the marginals (1) are known, the maximum-entropy solution takes the form of a gravity equation where the estimate E_{ij} is proportional to the product of marginals $A_i L_j$. To the extent that these marginals are positive (that is, all banks both borrow and lend to at least some bank in the network), ME produces a complete network where each bank lends to all other banks. Such a network structure is rather atypical for interbank markets, and when such a network is used in stress testing it tends to introduce a bias that underestimates the true extent of contagion (Mistrulli 2011, Markose et al. 2012). There are theoretical reasons why a complete market structure tends to be more robust to contagion (Allen and Gale 2000). Simulations on more realistic networks show that greater diversification through interlinkages indeed lowers the probability of a crisis, but may also raise its severity when a crisis occurs (Nier et al. 2007).

Our approach for deriving an opposite benchmark starts from the premise that establishing and maintaining network linkages is costly. Banks do not spread their borrowing and lending across the entire system, since the costs in terms of information processing, risk management and creditworthiness checks would be prohibitive for all but the largest banks. In reality, interbank activity occurs through relationships (Cocco et al. 2009), and interbank networks are *sparse* as a result, often with less than 1% of potential bilateral linkages in active use (see Bech and Atalay 2010 for the United States, and Craig and von Peter 2014 for Germany). These relationships are also *disassortative*: less-connected banks are more likely to trade with well-connected banks than with other less-connected banks (Bech and Atalay 2010, Iori et al. 2008). This reflects the economic rationale that smaller banks, rather than transacting with each other, typically use a small set of money center banks as intermediaries (Craig and von Peter 2014). Similar observations hold for dealer networks in financial markets (Li and Schürhoff 2013) or in international trade (Helpman et al. 2008, Ahn et al. 2011). This is why the paper proposes an alternative benchmark to estimate a network, the minimum density (MD).

2.2. An illustration

A simple example helps to illustrate the arguments made so far. Imagine a simple interbank market consisting of seven banks. In Figure 1, this market is represented by the top-left matrix; out of seven banks, two only lend (D,E), one only borrows (F), and the remaining four banks intermediate.[3] The authorities only observe individual bank balance sheets, but no bilateral transactions – they only see what each bank lends or borrows in total on the interbank market (the marginals, bottom left-hand matrix). If they estimate counterparty exposures by maximum

[3] Banks A-C are core banks that play a central role in intermediating between periphery banks (Craig and von Peter 2014).

entropy, they obtain the matrix at the top right. That solution differs substantially from the original $\mathbb{X}$; while it preserves the marginals of each bank, it fills all the linkages between active banks with positive exposures. The more banks are active, the less ME will preserve of the original network structure. Our MD solution, on the other hand, minimizes the number of linkages while also respecting all marginals and net positions, and leads to a sparse and concentrated interbank structure.

Actual Data

True Network

	A	B	C	D	E	F	G	A_i
A	0	3	1	0	0	1	2	**7**
B	2	0	2	0	0	0	1	**5**
C	1	1	0	0	0	1	0	**3**
D	1	0	0	0	0	0	0	**1**
E	0	0	2	0	0	0	1	**3**
F	0	0	0	0	0	0	0	**0**
G	0	1	0	0	0	0	0	**1**
L_i	**4**	**5**	**5**	**0**	**0**	**2**	**4**	**20**

Observable Interbank Market

	A	B	C	D	E	F	G	A_i
A								**7**
B								**5**
C								**3**
D								**1**
E								**3**
F								**0**
G								**1**
L_i	**4**	**5**	**5**	**0**	**0**	**2**	**4**	**20**

Estimated Networks

Maximum Entropy Solution

	A	B	C	D	E	F	G	A_i
A	0	2.53	2.18	0	0	0.74	1.55	**7**
B	1.72	0	1.6	0	0	0.54	1.14	**5**
C	0.98	1.06	0	0	0	0.31	0.65	**3**
D	0.25	0.27	0.23	0	0	0.08	0.17	**1**
E	0.75	0.81	0.7	0	0	0.24	0.5	**3**
F	0	0	0	0	0	0	0	**0**
G	0.3	0.32	0.28	0	0	0.09	0	**1**
L_i	**4**	**5**	**5**	**0**	**0**	**2**	**4**	**20**

Minimum Density Solution

	A	B	C	D	E	F	G	A_i
A	0	3	0	0	0	0	4	**7**
B	3	0	2	0	0	0	0	**5**
C	0	2	0	0	0	1	0	**3**
D	0	0	0	0	0	1	0	**1**
E	0	0	3	0	0	0	0	**3**
F	0	0	0	0	0	0	0	**0**
G	1	0	0	0	0	0	0	**1**
L_i	**4**	**5**	**5**	**0**	**0**	**2**	**4**	**20**

Figure 1: Illustrative example comparing the maximum-entropy and minimum-density solutions for a hypothetical interbank market.

The original network in this example has a density of 33%, with 14 out of the 42 potential bilateral links being used. The ME solution shows far higher density (62%), whereas our solution reaches the lowest density (21%) attainable with those marginals. While ME spreads interbank activity as evenly as possible, MD does the opposite by concentrating exposures on the smallest possible set of links. This illustrates that the two benchmarks depart from the true network in opposite directions in how they trade off the number versus the size of interbank linkages.

2.3. Minimum density: the formal problem

As an efficient alternative to maximum entropy, our approach minimizes the total number of linkages necessary for allocating interbank positions, consistent with total lending and borrowing observed for each bank.[4] Let c represent the fixed cost of establishing a link. Then the minimum-density approach can be formulated as a constrained optimization problem for the matrix $\mathbb{Z}$,

$$\begin{aligned} &\min_{\mathbb{Z}} \ c\sum_{i=1}^{N}\sum_{j=1}^{N} \mathbf{1}_{[Z_{ij}>0]} \qquad \text{s.t.} \\ &\textstyle\sum_{j=1}^{N} Z_{ij} = A_i \quad \forall i = 1,2,..N \\ &\textstyle\sum_{i=1}^{N} Z_{ij} = L_j \quad \forall j = 1,2,..N \\ &Z_{ij} \geq 0 \qquad\qquad \forall i,j\,, \end{aligned} \tag{2}$$

where the integer function $\mathbf{1}$ equals one only if bank i lends to bank j. The economic nature of this problem shares similarities with network design problems in transportation science and communication networks. Minimizing the cost of transporting a given volume of goods from origins to destinations appears analogous to moving money between banks. In our case, the capacities of transportation hubs (banks) are constrained by two marginals $\{A_i, L_i\}$, and the fixed cost of building new roads (credit relationships) must be considered as well. Such network design problems have been studied for decades and are known to be non-deterministic polynomial-time hard except in very special cases (Campbell and O'Kelly 2012 provide a survey).

We propose a link prediction method that combines elements of information theory and economic rationale. In problem (2), the *value* for any configuration of linkages $\mathbb{X}$ can be expressed as follows. In our objective function, which is used to assign links, we first soften the constraints by assigning penalties for *deviations* from the marginals,

$$\begin{aligned} AD_i &\equiv \left(A_i - \sum_j Z_{ij}\right), \\ LD_i &\equiv \left(L_i - \sum_j Z_{ji}\right), \end{aligned}$$

where LD_i measures bank i's current deficit; i.e., how much its bilateral borrowing falls short of the total amount it needs to raise, L_i, which is also the amount to be matched by the solution being constructed, $\mathbb{Z}$. Defining these deviations helps to make the optimization problem smooth, so that the only non-smooth part lies in the cost of links. When they are introduced into the

[4] Related to this, minimally connected networks arise as the *efficient* solution in economic models where agents trade off the costs and rewards of forming links (Goyal 2007, Jackson 2008).

objective function, the problem maximizes

$$V(\mathbb{Z}) = -c \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{1}_{[Z_{ij}>0]} - \sum_{i=1}^{N} \left[\alpha_i \, AD_i^2 + \delta_i \, LD_i^2\right]. \tag{3}$$

Sparse $\mathbb{Z}$ networks that minimize the deviations from the marginals have higher values and are desirable.

In addition to being sparse, interbank networks are disassortative: small banks seek to match their lending and borrowing needs through relationships with larger banks that are well placed to satisfy those needs. The relevant measure of size here is a bank's current surplus AD_i and deficit LD_i to be met in the interbank market. We codify this information through the set of probabilities $Q \equiv \{Q_{ij}\}$ for relationships between i and j. The probability that i lends to j increases if either i is a large lender to a small borrower j, or i is a small lender to a large borrower j. This is achieved by the following criterion:

$$Q_{ij} = \max\left\{\frac{AD_i}{LD_j}, \frac{LD_j}{AD_i}\right\}. \tag{4}$$

According to the objective function, equation (3), networks with a low density have higher values. At the same time, the beliefs Q suggest that small banks typically have links with larger banks. Our link allocation procedure incorporates these two features as a trade-off between finding sparse and disassortative network solutions. Weisbuch et al. (2000) propose a simple specification to capture this trade-off. Defining $P(\mathbb{Z})$ as the probability distribution over network configurations, this is derived by maximizing the sum of two terms. The first is the value of networks, $\sum_{\mathbb{Z}} P(\mathbb{Z})V(\mathbb{Z})$ – networks that have a higher value in equation (3) and fewer links should be more likely. To maximize the gains exploring disassortative solutions, these authors propose maximizing $\theta R(P \parallel Q)$, where θ is a scaling parameter and R is the relative entropy between the Q and the optimal distribution P. This is derived as the solution to

$$\max_{P} \quad \sum_{\mathbb{Z}} P(\mathbb{Z})V(\mathbb{Z}) + \theta R(P \parallel Q).$$

This objective function for $P(\mathbb{Z})$ may also be derived by drawing on the insights of Hansen and Sargent (2001) where there is uncertainty about the prior Q and one seeks a robust choice function P.[5] The solution to this problem can be obtained from the first-order conditions as

$$P(\mathbb{Z}) \propto Q(\mathbb{Z}) \, \mathrm{e}^{\theta V(\mathbb{Z})}, \tag{5}$$

which, when normalized over all possible network configurations, yields the multinomial logit

[5] See also Mattsson and Weibull (2002) and Strzalecki (2011).

choice function. What this expression states is analogous to other settings of probabilistic choice: a candidate $\mathbb{Z}$ has a higher likelihood of being chosen if the departure from the prior Q raises the value of the objective (3) which defines the minimum-density problem.

The solution would be straightforward if one could enumerate all possible network configurations $\mathbb{Z}$, and rank them according to $P(\mathbb{Z})$. But the computational complexity of such problems rises exponentially with N due to the number of possible subsets (2^N), even before allocating monetary values to each link. Since exhaustive search is impossible – certainly for our application with 1,800 banks – we develop a heuristic procedure for allocating links.

2.4. Implementation

The pseudo-code for our heuristic is provided in the appendix. The Matlab code is available from the authors upon request. The MD heuristic proceeds as follows:

- At each iteration, a link (i,j) is selected with probability Q_{ij}, where Q_{ij} is defined in (4).
- The exposure Z_{ij} is loaded with the maximum value that this pair of banks can transact given their current asset and liability positions, i.e., $Z_{ij} = \min\{AD_i, LD_j\}$.
- If adding this link increases the value function, $V(\mathbb{Z}+Z_{ij}) > V(\mathbb{Z})$, the allocation is retained.
- If, however, the addition of Z_{ij} diminishes the value function:
  - we retain the link as long as the network including Z_{ij} is more likely than without the link, i.e., with probability $P(\mathbb{Z}+Z_{ij})/P(\mathbb{Z}) \simeq \exp\{V(\mathbb{Z}+Z_{ij}) - V(\mathbb{Z})\}$.
  - The link is otherwise rejected.
- Finally, once positions have been updated, we proceed to the next iteration until the total interbank market volume has been allocated.[6]

The MD solution $\mathbb{Z}$ that this procedure yields and the ME solution are at opposite extremes of the spectrum, both potentially far from the true $\mathbb{X}$. One advantage of our method is that we can depart parametrically from the MD benchmark and obtain less-aggressive solutions that distribute interbank exposures over more links. This is achieved by scaling the loadings for selected links, $\lambda \times \min\{AD_i, LD_j\}$, using the parameter $\lambda < 1$. This forces the procedure to select more links until the overall interbank volume is reached. Such a "low-density solution," which we label $\mathbb{Y}$, features lower concentration and more density than the strict MD benchmark $\mathbb{Z}$, and as such may be closer to the original network $\mathbb{X}$.

[6]We also include a small probability of link deletion to allow the algorithm to cover the entire space of possible network configurations. This ensures that the state space is ergodic (see Proposition 1 in Anand et al. 2012).

3. Comparing estimated benchmarks with the true network

3.1. The true interbank network

This section assesses the maximum-entropy and minimum-density solutions against a large real-world network, the German interbank market. Our reference network $\mathbb{X}$ contains the "true" bilateral interbank positions observed for the German banking system. The Deutsche Bundesbank compiles a set of comprehensive banking statistics on large loans and concentrated exposures (*"Gross- und Millionenkreditstatistik"*) comprising all positions between financial institutions in the amount of at least 1.5 million euros or 10% of their liable capital. To obtain a consistent and self-contained network, we consolidate banks by ownership at the bank holding company level (*"Konzern"*) and exclude cross-border linkages. We use data from the second quarter of 2003, a choice that avoids the crisis period and issues with backward data revisions.

The resulting interbank network $\mathbb{X}$ is a square matrix with 3.16 million cells containing the observed bilateral interbank exposures among 1,779 active banks, amounting to 855 billion euros in total value. The network is sparse, with a density of 0.59% (18,624 active links). It is best described as a core-periphery structure in which most banks do not lend to each other directly but through core banks acting as intermediaries (Craig and von Peter 2014 elaborate).

3.2. The estimated benchmarks

The bilateral counterparty exposures can also be estimated by the two benchmark methods described above, using only information on every bank's total interbank lending and borrowing (the marginals). The ME solution ($\mathbb{E}$) yields an almost complete network with 92.8% of potential links being used. This is a density 158 times that of the original network. Since 92% of banks are active both as lenders and borrowers in the interbank market (and thus have positive marginals), ME fails to place any zeros between these pairs. The original structure of linkages is essentially lost as a result, in the way that Figure 1 had illustrated.

Our MD approach, at the other extreme, determines a solution ($\mathbb{Z}$) with density of 0.11% (3,450 links). This alternative benchmark thus allocates 100% of interbank volume using just a sixth of the links in the original network. The MD solution also respects all the marginals in equation (2), so that each bank lends and borrows as much in total as in the true interbank network that $\mathbb{Z}$ is trying to match (i.e., $\sum_j Z_{ij} = A_i$ and $\sum_i Z_{ij} = L_i$, so $AD_i = LD_i = 0 \forall i$).

The reason for this efficiency is that the MD algorithm identifies the most probable links and loads them with the largest possible exposures consistent with banks' total lending and borrowing in the interbank market (using $\lambda = 1$). As a result, the largest links consistently account for more value than was the case in the original network. Figure 2 shows how aggressively $\mathbb{Z}$ concentrates exposures on the largest links in the network, so that the total interbank volume is already fully allocated by the time we reach 0.185, or 18.5% of the number of links in the original network $\mathbb{X}$. It is apparent that the ME algorithm takes far longer to allocate the total interbank volume, since it places a positive value on virtually all pairs of banks. This illustrates how ME and MD differ in trading off the number versus the size of interbank linkages.

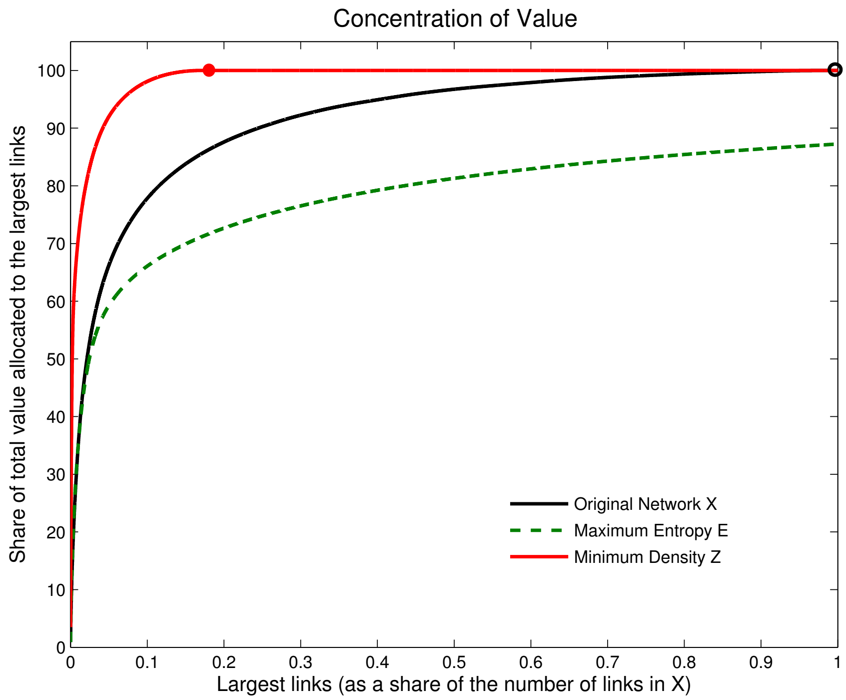

Figure 2: The figure shows the concentration of value on the largest links for the different networks. The x-axis ranks bilateral linkages (in descending order of size) and expresses the first n links as a share of the total number of links in the original network $\mathbb{X}$ (18,624). The y-axis shows the cumulative share of value allocated to the largest n links, relative to the total interbank volume. The dots indicate at which point 100% of volume has been reached. For $\mathbb{X}$ this is at unity, for $\mathbb{Z}$ this occurs at 0.185, whereas $\mathbb{E}$ needs 158 times the number of links in $\mathbb{X}$ before reaching 100% of interbank volume.

This obviously affects the most fundamental statistic in network analysis, the *degree* of a node (number of connections). Degree distributions are widely used as a diagnostic tool to characterize different types of networks, e.g. for distinguishing scale-free networks from Erdös-Rényi random graphs. The degree of individual nodes, and the degree distribution of the network as a whole, typically become meaningless after using ME. Whereas banks in the true network have only 11 counterparties on each side of their balance sheet on average, the ME solution $\mathbb{E}$ has nearly all banks lending to each other. This makes its degree distribution basically degenerate, a vertical line in Figure 3. The MD solution $\mathbb{Z}$ errs on the low side, but also assigns a high degree to the most important banks in the market. As a result, its degree distribution broadly retains the shape of the true interbank network $\mathbb{X}$, up to a factor reflecting its lower density. The mismatch is lessened when considering a less-aggressive solution to the MD problem, a "low-density" solution with density similar to that of $\mathbb{X}$.

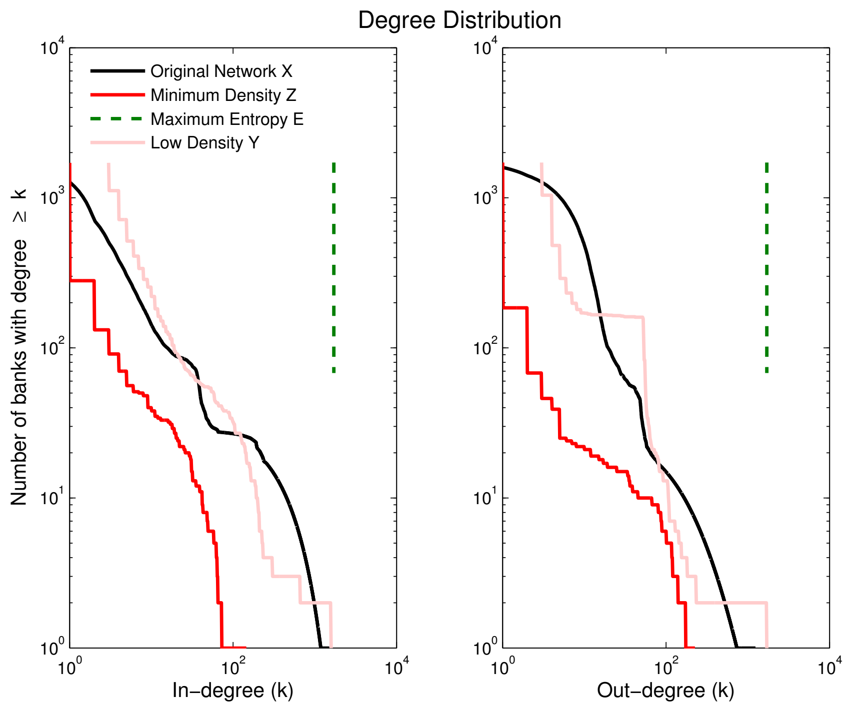

Figure 3: The figure displays the degree distribution in its cumulative form, showing the number of banks with a degree greater than the number shown on the x-axis, on a double log scale. A straight line would indicate a Pareto cumulative indicative of a power law distribution. The degree distribution of the original network $\mathbb{X}$ has been smoothed to preserve the confidentiality of individual bank data, and shows averages at the end points, instead.

3.3. Comparing network features

A more formal comparison is provided in Table 1, where the ME and MD benchmarks are evaluated against the original interbank market on several key network statistics. The end result is that MD preserves key features of the original network better than ME, especially when a low-density solution is selected. The first three lines of the table restate what was already apparent from Figures 2 and 3, namely that ME distributes exposures so widely that the estimate $\mathbb{E}$ has orders of magnitude more links than the original network $\mathbb{X}$. In so doing, ME also distorts a number of other characteristic features.

The fact that interbank relations often involve smaller less-connected banks trading with larger well-connected banks gives rise to negative *assortativity*.[7] In the original interbank network, assortativity is clearly negative (-0.53). Similar results were found for the federal funds market (Bech and Atalay 2010) and for the Italian interbank market (Iori et al. 2008). Disassortativity is nearly as strong for $\mathbb{Z}$, hence the MD solution preserves this important characteristic of interbank markets.[8]

The focus of banks on a few select counterparties comes out most clearly in the *dependence*

[7] The assortativity coefficient is the Pearson correlation coefficient of degree between pairs of linked nodes.
[8] This is not the case by construction. The assortativity built into the link prediction model related banks with large or small surplus or deficits, which need not coincide with high or low degree.

Network Characteristic	$\mathbb{E}$ Max Entropy	$\mathbb{X}$ True Network	$\mathbb{Z}$ Min Density	$\mathbb{Y}$ Low Density
Density, in %	92.8	0.59	0.11	0.61
Degree (average)	1649	10.5	1.94	10.9
Degree (median)	1710	6	1	4
Assortativity	-0.03	-0.53	-0.40	-0.32
Dependence when borrowing, %	12.2	84.7	97.3	93.4
Dependence when lending, %	7.2	45.1	97.4	87.2
Clustering local average, %	99.8	33.4	0.03	7.62
Core size, % banks	92.6	2.5	1.1	2.1
Error score, % links	14.6	9.2	41.2	35.7

Table 1: Comparing basic network features of benchmark estimates with those of the original German interbank network.

measures listed next in Table 1.[9] Under minimum density, the largest linkage on average accounts for close to 100% of a bank's total interbank borrowing. This reflects the efficiency with which the MD algorithm identifies counterparties that can single-handedly satisfy the funding needs of smaller banks. In the original network, high dependence on a single counterparty is also quite frequent when borrowing, though less so when lending. Since the ME solution spreads links indiscriminately across counterparties, it preserves very little of this feature – only to the extent that banks with large L_i borrow more from banks with large A_i.

One network feature that appears to be lost under MD is *local clustering*, the propensity of nodes to form cliques. The local clustering coefficient averages the probabilities that two neighbors of a node are themselves connected (Jackson 2008).[10] In matching big lenders with small borrowers and vice versa, MD tends to generate star-like networks where the extent of clustering is low. Indeed, it is part of the efficiency of the MD solution that transitive relationships are replaced by a single link where possible, obviating the need for smaller banks to form additional local relationships for their lending or funding needs. At the other extreme, clustering under ME trivially equals 100% among active banks. Therefore, both ME and MD fail to preserve local clustering.

The final rows of Table 1 consider the extent to which the different networks exhibit *tiering*.[11] The German interbank network is highly tiered, with only 9.2% of network links inconsistent with a perfectly tiered structure (error score). Indeed, 98% of interbank volume has one of the 44 core banks (2.5% of banks in the system) on either side of the transaction. The MD network

[9] Our dependence measure calculates what share of each bank's total borrowing (or lending, respectively) is transacted with its single largest counterparty, and averages this ratio across all active banks.

[10] The measure has been computed with MIT's Matlab Tools for Network Analysis available at: `http://strategic.mit.edu/downloads.php?page=matlabnetworks`.

[11] An interbank market is tiered when few banks intermediate between other banks that do not transact directly with each other; how close an interbank market is to such a core-periphery structure can be measured using block modelling methods (Craig and von Peter 2014).

retains this structure, even if its core is less than half of the original size because exposures are concentrated on fewer links. Since core banks continue to be linked to each other, however, the interbank network remains a single market in the sense of a giant connected component. In the ME network, on the other hand, the core-periphery structure largely disappears: the core trivially includes all active banks (more than 90% of the banks in the system), leaving only inactive banks in the periphery. This suggests that the hierarchical structure of interbank networks is erased when applying maximum entropy.

We briefly elaborate the point that our method also allows one to derive solutions that are less aggressive than the MD benchmark $\mathbb{Z}$. These low-density solutions, $\mathbb{Y}$ are obtained by lowering $\lambda < 1$, the extent to which selected links are loaded in $\lambda \times \min\{AD_i, LD_j\}$. Figure 4 illustrates one way of doing so, in order to show how selected network properties can be brought closer to those of the original network $\mathbb{X}$. In particular, it is possible to raise density monotonically (thick solid line) simply by varying the parameter λ, or the time at which it is set equal to one in the algorithm. The network $\mathbb{Y}$ with density similar to $\mathbb{X}$ also resembles $\mathbb{X}$ in terms of other network features (as in Table 1). This can be useful provided the researcher estimating counterparty exposures has information about the density of the interbank market $\mathbb{X}$, or the average number of links that banks maintain. However, there are many ways in which departures from the MD benchmark can be specified and programmed, leading to different results – so we do not pursue this refinement of MD further in what follows.

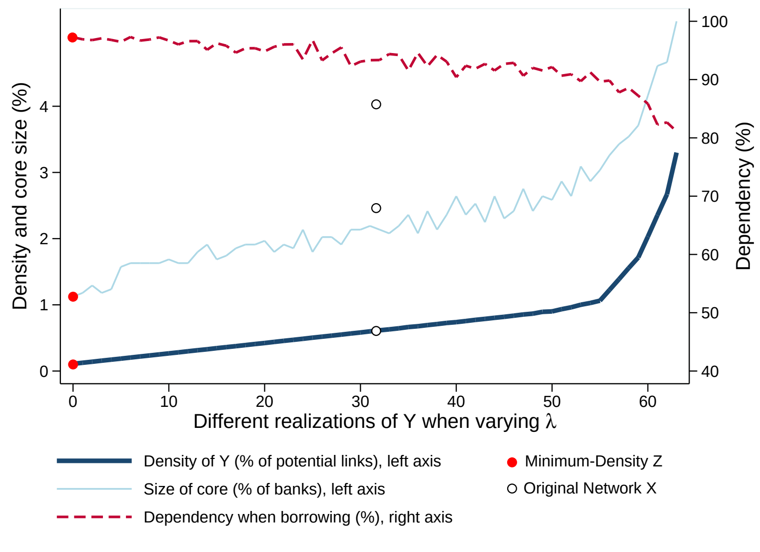

Figure 4: This figure shows three network features for 65 different low-density solutions $\mathbb{Y}$. The implementation here sets $\lambda = 0.5$ for the first k links being filled by the algorithm and $\lambda = 1$ thereafter, with k raised from 0 to 100,000 in 65 (unequally spaced) steps. The first realization (at $k = 0$) is the MD network $\mathbb{Z}$ with the network features shown as red dots (as in Table 1). The black circles indicate the values for the original network $\mathbb{X}$, plotted at the point where a comparable low-density network $\mathbb{Y}$ reaches a density similar to $\mathbb{X}$ (at k=16,000).

We conclude that the MD approach preserves some of the original network's structural features better than the ME benchmark does. This makes the MD approach a promising alternative to ME for filling in missing counterparty exposures, especially if the estimated matrix is used in network analysis where the pattern of linkages is of central interest. The next section explores an important application where both the pattern and size of linkages matter.

4. Performance in system stress tests

In this section, we contrast the results from a systemic stress test on the German banking system with those obtained using the alternative estimated networks. We opt for a standard simulation methodology, since the focus here is on variation in the simulation *inputs*, not on breaking new ground in stress methodology. Three building blocks in any stress-testing exercise are (i) the trigger event, (ii) the contagion mechanism, and (iii) data on bank balance sheets and counterparty exposures, where we consider the estimated networks using maximum entropy and minimum density, alongside the "true" observed interbank network.

4.1. Stress-test ingredients

Regarding the trigger event, the majority of studies focus on the unanticipated failure of individual banks (Upper 2011, Mistrulli 2011), and we follow this approach here. There are many ways to design a stress test in which banks hold common exposure to a variety of factors, such as stock market declines, credit risk, etc. Since all of these approaches cause losses to all banks simultaneously, we take a shortcut and directly hit their regulatory capital ratios. Our experiment combines a single failure with an across-the-board decline of regulatory capital of 2 percent. Banks go into default when their regulatory capital dips below a common regulatory requirement of 6 percent.

We run four sets of simulations, one for each interbank network separately, using maximum entropy ($\mathbb{E}$) and minimum density ($\mathbb{Z}$), as well as an example of low density ($\mathbb{Y}$), and compare the results to those from the true interbank market ($\mathbb{X}$). Each set of simulations proceeds as follows: in each run, we let a single bank i fail exogenously to trigger the contagion process, and solve for the clearing vector to obtain (a) the number and identity of banks that default as a result of contagion (excluding the initial failure), and (b) the total assets and interbank liabilities of the defaulting banks. From these 1,779 runs (one for each bank), we report the average number of contagious defaults and total bank assets affected by contagion (as in Mistrulli 2011).

We use two methods for simulating how losses cascade through the banking system. The first method is the sequential default algorithm traditionally used in the stress-testing literature (including Furfine 1999, and Mistrulli 2011). It assigns a deadweight loss proportional to loss-given-default (LGD) after each default to each of the surviving holders of an asset, where their capital is reduced by the exposure times the LGD. While easy to implement, this method ignores the fact that subsequent defaults induce additional losses to banks that had already defaulted in earlier rounds of the process. Hence we also employ the clearing vector methodology of Eisenberg and Noe (2001), which has gained traction in recent years (as discussed in Upper 2011, and Elsinger et al. 2013). The Eisenberg-Noe methodology is internally consistent: it determines the fixed point at which contagion comes to a halt, and in solving the simultaneity problem also delivers a unique solution. However, in determining the system's LGD endogenously, the Eisenberg-Noe methodology assumes that all assets can be liquidated at book value to meet the liabilities of defaulting banks – the more contagion, the less tenable this assumption becomes. To capture possible distress selling and bankruptcy costs, we allow for an additional deadweight loss of β percent assessed on the liabilities of defaulting banks, and step up its value from 0% to 35% in separate simulations (where $\beta = 0$ leads to the original Eisenberg-Noe fixed point). This generalizes the results and allows one to compute the overall liquidation and bankruptcy cost to the system as a whole. This is a true deadweight loss, net of all the redistributions that occur through the contagion process.

4.2. Results from the sequential default algorithm

Figure 5 summarizes the results of the standard stress test with single bank failures and an exogenous LGD. We first describe the reference results for the German interbank network (black bold lines), before turning to those for the estimated benchmarks. The simulations suggest that about one bank defaults on average as a result of the failure of an arbitrary single bank, or 2–3 banks once bankruptcy costs exceed a few percent (left panel). This low average is largely due to the fact that the results reported are averaged over 1,779 banks. In contrast with the average, the worst case produces more than 1,200 contagious defaults (70% of banks in the system). It is a well-known feature of complex networks with highly connected hubs that they are robust to failure but vulnerable to targeted attack (Albert et al. 2000). For the vast majority of individual bank failures, contagion does not result, in that the initial bank failure did not cause any other bank to default. If one were to look at the number of banks failing *conditional* on contagion occurring, the numbers on the left-hand axis increase by a factor of 100 (for low LGDs) down to a factor of 50 (for high LGDs): for high LGDs, more than 500 banks default on average.[12]

When the same stress test is run on the counterparty exposures estimated by maximum entropy, the results *underestimate* the extent of contagion (Figure 5, green lines). The entropy solution spreads exposures so widely that more small banks fail at low LGDs, if contagion occurs, but

[12] At the highest LGDs, 29 cases produce more than 3 contagious defaults, including 19 with more than 100 banks, and 14 of these cause more than 1,200 defaults.

they will incur smaller losses on the system. The entropy solution seems to optimize a network in a direction that makes contagion happen more frequently (and with small shocks to the system), but mostly involves smaller banks that rarely bring down the entire system.[13]

The MD solution $\mathbb{Z}$ (as well as most variants $\mathbb{Y}$) almost always *overestimates* the true extent of contagion (red and light red lines). The bias is small for the number of banks, but somewhat greater when measuring contagion by the total assets of defaulting banks (Figure 5, center panel). The same ranking holds for the system-wide deadweight loss (right panel). *Conditional* on contagion, more banks fail in network $\mathbb{Z}$ than in the true interbank network $\mathbb{X}$, for most of the LGD range.[14] This illustrates a fairly general property of the minimum-density solution: it produces more contagion than both the original network and the entropy estimate. However, for the highest LGDs representing extreme stress, the more diversified entropy estimate $\mathbb{E}$ lies far below the original $\mathbb{X}$ result, whereas our $\mathbb{Z}$ estimate tracks it quite closely while slightly overstating contagion. The fact that our $\mathbb{Z}$ has fewer links actually reduces some of the banks' exposure to contagion.[15] Moderate contagion is more likely to happen, but there is sometimes a buffer from complete system collapse, although this buffer is small.

As a general rule, our minimum-density solution $\mathbb{Z}$ provides an upper bound on system-wide stress, whereas the entropy estimate $\mathbb{E}$ yields a lower bound. These two bounds are illustrated in the figures by the grey shaded area denoted 'range.' Only for extreme scenarios with LGDs approaching 60% does the original network move outside the range defined by these two benchmarks. The $\mathbb{Z}$ (and all $\mathbb{Y}$ variants) that we tested always provided an effective upper bound for system-wide stress.

4.3. Results from the clearing vector methodology

Figure 6 shows that the Eisenberg-Noe clearing vector methodology produces similar stress-test results for all networks, with the same qualitative character of the results from the sequential default algorithm. In particular, ME underestimates systemic stress substantially, while MD overestimates the true extent of contagion: this again gives rise to a useful range that contains the estimate of systemic risk in the true interbank market. The exposures ME places between the major banks are not as large as the true exposures in the interbank market or in our MD estimate (recall Figure 2). For this reason, our $\mathbb{Z}$ generally overestimates contagion, providing an

[13] At the highest LGDs, the maximum number of banks failing is less than 1,160, which happens in only nine cases. Indeed, in the 35 cases with contagion, only the top nine cases involve contagion affecting more than three banks.

[14] The gentler stress tests with 10 percent LGD show contagion with 72 banks under $\mathbb{Z}$, compared to 6 for $\mathbb{X}$, and only 4 with the widely diversified entropy estimate $\mathbb{E}$.

[15] All cases cause less than 1,000 defaults in the most extreme scenarios, whereas 13 cause over 900 defaults, and 99 cause over 3 defaults.

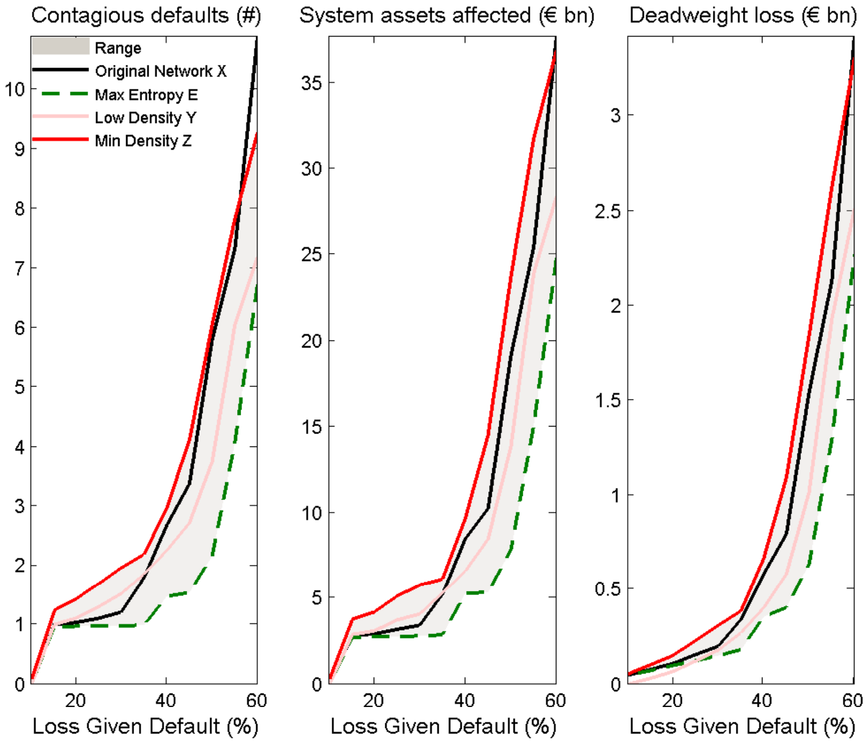

Figure 5: Results of the first stress test using the *sequential default algorithm*. The three lines compare the different average results obtained by using the "true" interbank network (black) and the maximum-entropy (green), minimum-density (red) and low-density (light red) solutions. The stress tests are run separately for the different levels of bankruptcy costs shown on the x-axis. The left panel plots the number of banks that default as a result of contagion (excluding the initial failure). The center panel expresses the extent of contagion by aggregating the total assets of the banks that end up in default. The right panel displays the deadweight loss arising from distress selling and bankruptcy costs assessed on the liabilities of defaulting banks according to a LGD. In all cases, individual bank failures are accompanied by a decline in the regulatory capital ratio by 2 percentage points at all banks, and banks default when their regulatory capital ratio falls below 6 percent. The losses are passed on to the other banks in the interbank market through a straight proportional LGD.

upper bound for virtually all stress-test results.[16] The intermediate results for our low-density solution $\mathbb{Y}$ sometimes mitigate this bias – the results are very close to the estimate of systemic risk in the true interbank network.[17]

4.4. *Interpretation of results*

Our findings for the German interbank network are in line with Mistrulli's (2011) results for Italy. In Mistrulli's direct comparison, the ME solution also underestimates systemic risk at low loss-given-default while overstating it at the higher end. We confirm his first result, and find that it is robust to a change in methodology from the sequential default algorithm to a (generalized) clearing vector method. There are several reasons in our context why there is more contagion under minimum density than under maximum entropy. By minimizing density, the MD network

[16]There is a small region where the number of banks affected under $\mathbb{Z}$ is slightly smaller than the number in the original network.

[17]However, as is true of several $\mathbb{Y}$ matrices we computed, the results were not uniform. While $\mathbb{Y}$ often comes close to the results for $\mathbb{X}$ for some range of bankruptcy costs, it can also diverge in the highest range, as shown in Figure 6.

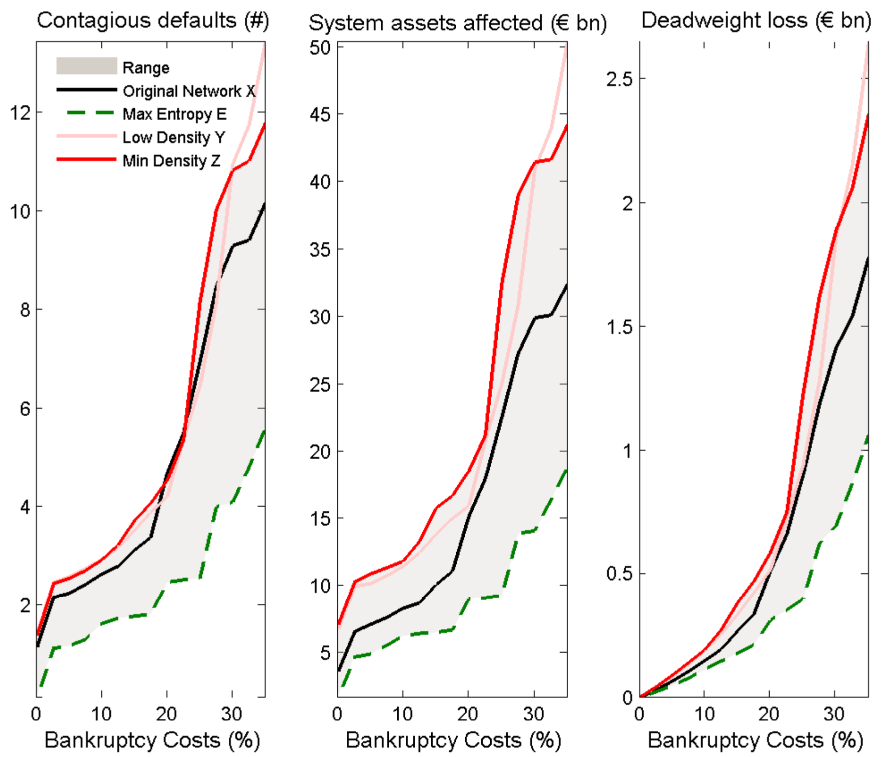

Figure 6: Results of the second stress-test using the *Eisenberg-Noe clearing vector methodology*, with an additional bankruptcy shown on the x-axis. The left and center panels are analogous to Figure 5. The right panel shows the deadweight loss associated with the bankruptcy costs passed on to all of the banks of the system (including defaulting banks).

focuses the concentration of exposures onto fewer links; this also means that the greater loss transmitted by a given link is more likely to exceed the capital of the lending bank and thereby cause its default.[18] To some extent, *concentration effect* is balanced by the fact that the scope of contagion is somewhat limited by the *sparsity effect*: a lower number of linkages in $\mathbb{Z}$ also reduces the conduits for the propagation of losses. This countervailing effect does not offset the effect of higher concentration owing to the *negative assortativity* in both the MD and the true network. The failure of small banks is largely inconsequential, because they do not cause systemic stress in any of the networks. The stress following a large bank failure, on the other hand, affects many small banks beyond their ability to withstand the losses, and this outcome dominates the averaged results in Figures 5 and 6. In our tests, greater diversification (as in the entropy estimate $\mathbb{E}$) does not raise the severity of a crisis if one occurs – unlike what is predicted by some simulations (Nier et al. 2007) or theoretical results (Gai et al. 2011).

Overall, our stress-test findings suggest that the minimum-density approach, while delivering an economically meaningful alternative to maximum entropy, also leads to a reasonable estimate

[18]The minimum-density solution has the single-minded goal of minimizing the system-wide cost of linkages, but ignores the value of diversifying exposures. Banks, in reality, would individually take this value of diversification into account in order to limit their exposure to systemic risk. In a more general optimization problem, one might find that banks that economize on links also choose to limit their exposure to systemic banks.

of *overall systemic risk* in stress tests. Using both benchmarks helps identify *a useful range* of possible stress-test results when the true counterparty exposures are unknown (shaded areas in Figures 5 and 6). The alternatives can also be compared in terms of the rankings they predict on the systemic importance of individual banks. Consider the 100 largest banks ranked by the damage their failure causes (as estimated from the true interbank network) in terms of contagious failures, total affected assets and other measures of contagion. The MD and ME results generally lead to similar rankings, with Kendall and Spearman rank correlations close to 80% relative to the "true" ranking. This suggests that both benchmarks deliver fairly reliable rankings of *systemically important banks*.

5. Conclusion

The pattern and size of linkages are of central importance in many areas of network analysis and finance. Yet bilateral counterparty exposures are often unknown, and maximum entropy serves as the leading method for filling in the blanks. This paper proposes an economically meaningful alternative that combines known network features with information-theoretic arguments to produce more realistic interbank networks. Our minimum-density solution preserves some characteristic features of the original interbank market. Moreover, in the context of systemic stress testing, the use of a minimum-density network as input yields an upper bound on the cost of systemic stress where the maximum-entropy benchmark produces a lower bound. The resulting range contains the true cost of systemic stress of the German banking system, and such a range can be useful in applications where the true counterparty exposures are unknown. Indeed, the minimum-density approach is more suitable for robustness analysis: whereas the maximum-entropy network is unique, our link prediction method is stochastic and thus capable of generating many low-density networks for repeated application in stress tests. This is a useful feature in a context where networks are not fully observed yet the structure and size of linkages is of great consequence for financial stability.

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Appendix A. MD Pseudo-code

The primitives of our procedure are two Markov processes – one for adding new links and value to the interbank network (with probability $1-\epsilon$), and the second to delete links, and correspondingly value, from the network (with probability ϵ). The intuition for this design is based on the notion of ergodicity, in that from any network structure, $\mathbb{X}$, through a finite series of link additions and deletions, we can obtain any other network $\mathbb{X}'$. As demonstrated in Proposition 1 of Anand et al. (2012), this ensures that the distribution $P(\mathbb{X})$ over network configurations is stationary, and hence our Metropolis algorithm is well defined. As a consequence of being stochastic, our algorithm can generate multiple realizations of minimum-density networks, which allows for robustness in conducting stress-test exercises. Algorithm 1 provides the pseudo-code for our procedure.

begin

(S) : $\mu^{(0)} = \{(i,j)\}_{i,j=1}^{N}$ and $\nu^{(0)} = \{\emptyset\}$.

(P) : $Q_{ij}^{(0)} = \max\left\{\frac{AD_i^{(0)}}{LD_j^{(0)}}, \frac{LD_j^{(0)}}{AD_i^{(0)}}\right\}$, $\forall\,(i,j) \in \mu^{(0)}$.

(C) : $\tau = 1$.

while

$$V(\mathbb{X}) < 0.999 \times \sum_{i=1}^{N} AD_i^{(0)}$$

do

(D) : $\rho \in [0,1]$ at random.

if $\rho < \epsilon$ **then**

Remove link.

(L) : $(i,j) \in \nu^{(\tau-1)}$ with probability $1/|\nu^{(\tau-1)}|$.

(M) : $AD_i^{(\tau)} = AD_i^{(\tau-1)} + X_{ij}$ and $LD_j^{(\tau)} = LD_j^{(\tau-1)} + X_{ij}$.

(A) : $X_{ij} = 0$.

(S) : $\mu^{(\tau)} = \mu^{(\tau-1)} \cup (i,j)$ and $\nu^{(\tau)} = \nu^{(\tau-1)} \setminus (i,j)$.

end

else

Add link.

(P) : $Q_{ij}^{(\tau-1)} = \max\left\{\frac{AD_i^{(\tau-1)}}{LD_j^{(\tau-1)}}, \frac{LD_j^{(\tau-1)}}{AD_i^{(\tau-1)}}\right\}$, $\forall\,(i,j) \in \mu^{(\tau-1)}$.

(L) : (i,j) with probability $Q_{ij}^{(\tau-1)}$.

(A) : $X_{ij} \leftarrow \lambda \times \min\left\{AD_i^{(\tau-1)}, LD_j^{(\tau-1)}\right\}$.

(A) : $\mathbb{X}' = \mathbb{X} + X_{ij}$.

(D) : $\psi \in [0,1]$ at random.

if $V(\mathbb{X}') > V(\mathbb{X}) \parallel \psi < \exp\left(\theta\left[V(\mathbb{X}') - V(\mathbb{X})\right]\right)$ **then**

(A) : $\mathbb{X} = \mathbb{X}'$.

(M) : $AD_i^{(\tau)} = AD_i^{(\tau-1)} - X_{ij}$ and $LD_j^{(\tau)} = LD_j^{(\tau-1)} - X_{ij}$. (S) : $\mu^{(\tau)} = \mu^{(\tau-1)} \setminus (i,j)$ and $\nu^{(\tau)} = \nu^{(\tau-1)} \cup (i,j)$.

end

end

(P) : $Q_{ij}^{(\tau)} = \max\left\{\frac{AD_i^{(\tau)}}{LD_j^{(\tau)}}, \frac{LD_j^{(\tau)}}{AD_i^{(\tau)}}\right\}$, $\forall\,(i,j) \in \mu^{(\tau)}$.

(C) : $\tau \leftarrow \tau + 1$.

end

end

Algorithm 1: Minimum-Density algorithm to allocate interbank networks. The labels for the different lines are, (S) : update sets, (P) : update priors, (C) : update counter, (D) : draw random number, (L) : pick link, (A) : assign value and (M) : update marginals.