DEVELOPMENT OF SYSTEM ORDER REDUCTION TECHNIQUES APPLICABLE TO THIRD GENERATION SPACECRAFT

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by (1)
G. West-Vukovich

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SUMMARY

This report is an extension of [1], which dealt with the application of closed-loop model order reduction methods to flexible spacecraft dynamics and control. Here the main method of [1], cost-decoupled coordinates, is extended to discrete-time systems and to continuous-time systems with state estimators in the feedback loop. In the latter case a numerical study is performed on "ZSAT," which is compared with the state feedback case performed in [1]. An extension to modal cost analysis for nonideal inputs is then presented. Lastly, environmental torques on ZSAT are discussed, and calculations of these influences presented.

PREFACE

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Mrs. J. Hughes typed this report.

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Units and Spelling

This report uses S.I. units and North American spelling.

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1. INTRODUCTION

This report is a continuation of [1], which discussed model reduction for spacecraft, and related topics. Several aspects of this subject are extended here, particularly those dictated by requirements of actual implementation of some of the proposed methods in [1]. The topics are somewhat diverse, so that each major section in this report is independent of the others.

The method of cost-decoupled coordinates is pointed out in [1] as an attractive model reduction method. Since it is most likely that any control scheme will contain discrete-time elements during actual implementation, it is of interest to determine how the method of cost-decoupled coordinates is affected by application to discrete-time systems, and what modifications are necessary. This subject is discussed in Section 2, where it is shown that the method is basically unaltered from the continuous-time procedure; in fact, once the required correspondences have been made, the method in the discrete-time case is identical to that in the continuous-time case.

In [1] the concept of cost-decoupled coordinates was developed under the assumption of availability of the state -- generally an unattainable ideal. Section 3 of this report considers what happens when it is necessary to estimate the state. It is shown that some changes are required to the method discussed in [1], particularly in the evaluation of controller competence. Some degradation in performance from the state feedback case for the "ZSAT" ('Lazy-Z' MSAT configuration) model is shown to occur.

Model reduction by modal cost analysis is intuitively appealing and has generally proven to be a worthwhile method. It bases the significance of modes on modal frequency, modal damping, the importance of modes in the output, and on excitation of modes. This last criterion is based on the rather ideal notions of impulsive or white-noise input excitation. Section 4 of this report discusses modifications to modal cost analysis for more realistic inputs -- those with a finite contour in the frequency domain. This also includes the case of nonideal actuators.

Section 5 contains an extension to the modeling of ZSAT. Disturbance torques to be included in the equations of motion of ZSAT due to gravity gradient and solar radiation are developed. The gravity gradient torque consists of a constant term and an attitude dependent term. The solar radiation torque term displays the variation of this disturbance on ZSAT with orbital anomaly.

2. COST-DECOUPLED COORDINATES FOR DISCRETE-TIME SYSTEMS

The method of model reduction via cost-decoupled coordinates for *continuous-time* systems was discussed in [1] along with criteria for evaluation of the quality of reduced models. It often happens that when dealing with a physical system it is desirable to observe and deal with the system only at a sequence of *discrete* instants rather than at *all* instants of time. If the system model is linear and time invariant, the sampling instants equidistant, and if the sampled quantity remains constant between sampling instants, then a particularly simple correspondence exists between the continuous-time model and its discrete-time counterpart. Representing the continuous-time system as the usual first order matrix differential equation

$$\dot{\underline{x}}_c = \underline{A}_c\underline{x}_c + \underline{B}_c\underline{u}_c \tag{2.1a}$$

$$\underline{y}_c = \underline{C}_c\underline{x}_c \tag{2.1b}$$

the corresponding discrete-time equation is a first order matrix difference equation

$$\underline{x}_d(k+1) = \underline{A}_d\underline{x}_d(k) + \underline{B}_d\underline{u}_d(k) \tag{2.2a}$$

$$\underline{y}_d(k) = \underline{C}_d\underline{x}_d(k) \tag{2.2b}$$

where

$$\underline{A}_d = e^{\underline{A}_c\Delta} \quad , \quad \underline{B}_d = \left[\int_0^{\Delta} e^{\underline{A}_c\tau}\, d\tau\right] \underline{B}_c \tag{2.3a}$$

$$\underline{C}_d = \underline{C}_c \tag{2.3b}$$

and $\Delta = t_{k+1} - t_k$, the sampling interval.

It is of interest to see how to apply the method of cost-decoupled coordinates to a discrete-time system (2.2) and what modifications are necessary from the continuous-time case; this is what is done in this section. Since the method of cost-decoupled coordinates is intimately related to the optimal linear regulator problem with quadratic cost criteria, it behooves us at this juncture to adumbrate the optimal control problem for discrete-time systems.

2.1 Optimal Control for Discrete-Time Systems

We begin with system (2.2), in which for the remainder of this section we shall drop the subscript 'd', it being understood that we are discussing discrete time systems. Our object, analogous to that in the continuous-time case, is to find a control input $\underline{u}(k)$ which minimizes the following performance index:

$$V = \sum_{k=k_0}^{k_1-1} [\underline{y}^T(k+1)\underline{Q}\underline{y}(k+1) + \underline{u}^T(k)\underline{R}\underline{u}(k)] \tag{2.4}$$

It is to be noted that unlike the continuous-time case, the arguments of $\underline{y}$ and $\underline{u}$ are different. This is because the initial value of the output, $\underline{y}(k_0)$, is unalterable over the first sampling interval, and can therefore not influence the performance index. Similarly the final value of the input, $\underline{u}(k_1)$, influences the output only beyond the terminal time and can also therefore be neglected. Another interesting contrast between (2.4) and its continuous-time counterpart is that the matrix $\underline{R}$ in (2.4) need not be positive definite. In the continuous-time case a positive definite $\underline{R}$ rules out the possibility of infinitely large control inputs taking the state to zero in an infinitely short time. In the discrete-time case it is not possible to drive the state to zero in an infinitely short time, thus allowing the requirement of positive

definite $\underline{R}$ to be relaxed.

We wish to consider the infinite time problem, so we accordingly modify (2.4) to

$$V = \sum_{b=o}^{\infty} [\underline{y}^T(k+1)\underline{Qy}(k+1) + \underline{u}^T(k)\underline{Ru}(k)] \tag{2.5}$$

which can be rewritten as

$$\underline{V} = \lim_{k\to\infty} \left\{ \underline{x}^T(k+1)\underline{C}^T\underline{QCx}(k+1) \right\} + \sum_{k=0}^{\infty} [\underline{x}^T(k)\underline{C}^T\underline{QCx}(k) + \underline{u}^T(k)\underline{Ru}(k)] \tag{2.6}$$

Since we are assuming that our controller has successfully stabilized the system, the first term in (2.6) vanishes.

Assuming now that feedback of the form

$$\underline{u}(k) = - \underline{Gx}(k) \tag{2.7}$$

has been applied, (2.6) becomes

$$V = \sum_{k=o}^{\infty} \{\underline{x}^T(k)[\underline{C}^T\underline{QC} + \underline{G}^T\underline{RG}]\underline{x}(k)\} \tag{2.8}$$

Recognizing that the solution to the state difference equation

$$\underline{x}(k+1) = \underline{\bar{A}x}(k) \tag{2.9}$$

where

$$\underline{\bar{A}} = \underline{A} - \underline{BG}$$

is given by

$$\underline{x}(k) = \underline{\bar{A}}^k\underline{x}(o)$$

allows us to write (2.8) as

$$V = \underline{x}^T(0)\underline{P}\underline{x}(0)$$

$$= \underline{x}^T(0) \left\{ \sum_{k=o}^{\infty} \bar{\underline{A}}^{k^T} [\underline{C}^T\underline{Q}\underline{C} + \underline{G}^T\underline{R}\underline{G}]\bar{\underline{A}}^k \right\} \underline{x}(0) \tag{2.10}$$

In a manner pleasingly analogous to the continuous-time situation in which an integral can be evaluated by solving a Liapunov equation, the sum in (2.10) can be evaluated by solving the following discrete-time matrix Liapunov equation:

$$\underline{P} = \bar{\underline{A}}^T\underline{P}\bar{\underline{A}} + \underline{C}^T\underline{Q}\underline{C} + \underline{G}^T\underline{R}\underline{G} \tag{2.11}$$

Thus far we have only discussed a performance measure, but we have not yet considered the minimization of this performance index. This minimization is accomplished by the controller in (2.7) with $\underline{G}$ given by

$$\underline{G} = \underline{R} + \underline{B}^T[\underline{C}^T\underline{Q}\underline{C} + \hat{\underline{P}}]\underline{B}\}^{-1}\underline{B}^T[\underline{C}^T\underline{Q}\underline{C} + \hat{\underline{P}}]\underline{A} \tag{2.12}$$

where $\hat{\underline{P}}$ is the solution of

$$\hat{\underline{P}} = \underline{A}^T[\underline{C}^T\underline{Q}\underline{C} + \hat{\underline{P}}][\underline{A} - \underline{B}\underline{G}] \tag{2.13}$$

Substituting (2.12) into (2.13) yields

$$\hat{\underline{P}} = \underline{A}^T\{(\underline{C}^T\underline{Q}\underline{C} + \hat{\underline{P}}) - (\underline{C}^T\underline{Q}\underline{C} + \hat{\underline{P}})\underline{B}[\underline{R} + \underline{B}^T(\underline{C}^T\underline{Q}\underline{C} + \hat{\underline{P}})\underline{B}]^{-1}\underline{B}^T(\underline{C}^T\underline{Q}\underline{C} + \hat{\underline{P}})\}\underline{A} \tag{2.14}$$

which is the discrete-time version of the Matrix Riccati Equation. The corresponding optimal cost is

$$V = \underline{x}^T(0)\hat{\underline{P}}\underline{x}(0) \tag{2.15}$$

Equation (2.14) can be more simply expressed by making the substitution

$$\underline{\bar{P}} \triangleq \underline{\hat{P}} + \underline{C}^T\underline{QC} \tag{2.16}$$

which yields

$$\underline{\bar{P}} = \underline{A}^T\underline{\bar{P}A} - \underline{A}^T\underline{\bar{P}B}[\underline{R} + \underline{B}^T\underline{\bar{P}B}]^{-1}\underline{B}^T\underline{\bar{P}A} + \underline{C}^T\underline{QC} \tag{2.17}$$

and the corresponding cost

$$V = \underline{x}^T(0)\underline{\bar{P}x}(0) \tag{2.18}$$

Let us refer to (2.11) for a moment. With $\underline{G}$ the optimal feedback, the correspondence between the Matrix Liapunov Equation (which is true for any stabilizing feedback), and the Matrix Riccati Equation (2.17) (which is true for the optimal feedback) becomes clear. The optimal feedback

$$\underline{G} = (\underline{R} + \underline{B}^T\underline{\bar{P}B})^{-1}\underline{B}^T\underline{\bar{P}A} \tag{2.19}$$

transforms (2.11) into (2.17), so that as in the continuous-time case the Matrix Liapunov Equation becomes the Matrix Riccati Equation.

The preceding derivation demonstrates that the closed-loop cost for any stabilizing feedback can be obtained by solving a discrete-time Liapunov equation, and that the minimum (optimal cost) is given by the full-order state feedback case, when the Liapunov equation becomes a Riccati equation. One would expect then, as in the continuous-time case, that as the model is successively reduced in order, the truncated feedback deviates more and more from the optimal.

2.2 Model Truncation

The philosophy followed in [1] of transforming a system into a set of coordinates for which the optimal cost matrix is diagonalized

can be repeated for discrete-time systems. For a complete discussion of the continuous-time case the reader is referred to [1]. Given the system (2.2), the performance index (2.6) and the solution to the optimal control problem (2.7), (2.17), (2.19), an orthonormal transformation

$$\underline{x} = \underline{T}\hat{\underline{x}} \tag{2.20}$$

can be defined such that the total cost (2.18) becomes

$$V = \hat{\underline{x}}^T(0)\underline{T}^T\underline{\bar{P}T}\hat{\underline{x}}(0) \tag{2.21}$$

We now define

$$\underline{\Lambda}_p \triangleq \underline{T}^T\underline{\bar{P}T} \tag{2.22}$$

which is a diagonal matrix whose diagonal elements represent the relative importance of corresponding state variables to the overall model. We note that in the usual case unique initial conditions can not be specified, so that rather than the *exact* performance measure represented by (2.21) we would more fruitfully consider an *average* performance measure, and the *expected* value of V. Following [1] we have

$$E\{V\} = \text{trace } \underline{\Lambda}_p \tag{2.23}$$

where $E\{\cdot\}$ denotes the expected value of $\{\cdot\}$.

Applying (2.20) to (2.2) and (2.7) results in

$$\begin{aligned} \hat{\underline{x}}(k+1) &= \hat{\underline{A}}\hat{\underline{x}}(k) + \hat{\underline{B}}\underline{u}(k) \\ \underline{y}(k) &= \hat{\underline{C}}\hat{\underline{x}}(k) \\ \underline{u}(k) &= -\hat{\underline{G}}\hat{\underline{x}}(k) \end{aligned} \tag{2.24}$$

where

$$\hat{\underline{A}} \triangleq \underline{T}^T \underline{A}\underline{T}, \qquad \hat{\underline{B}} \triangleq \underline{T}^T \underline{B}$$

$$\hat{\underline{C}} \triangleq \underline{C}\underline{T} \quad , \qquad \hat{\underline{G}} \triangleq \underline{G}\underline{T}$$

This transformed system can be partitioned into retained and truncated partions, determined by the relative sizes of corresponding elements of $\underline{\Lambda}_p$,

$$\begin{bmatrix} \underline{x}_R(k+1) \\ \underline{x}_T(k+1) \end{bmatrix} = \begin{bmatrix} \underline{A}_{RR} & \underline{A}_{RT} \\ \underline{A}_{TR} & \underline{A}_{TT} \end{bmatrix} \begin{bmatrix} \underline{x}_R(k) \\ \underline{x}_T(k) \end{bmatrix} + \begin{bmatrix} \underline{B}_R \\ \underline{B}_T \end{bmatrix} \underline{u}(k) \tag{2.25}$$

$$\underline{y}(k) = [\underline{C}_R \quad \underline{C}_T] \begin{bmatrix} \underline{x}_R(k) \\ \underline{x}_T(k) \end{bmatrix} \tag{2.26}$$

$$\underline{u}(k) = [-\underline{G}_R \quad -\underline{G}_T] \begin{bmatrix} \underline{x}_R(k) \\ \underline{x}_T(k) \end{bmatrix} \tag{2.27}$$

We also partition the diagonal cost matrix $\underline{\Lambda}_p$:

$$\mathrm{diag}(\underline{\Lambda}_{pR}, \underline{\Lambda}_{pT}) \triangleq \underline{\Lambda}_p \tag{2.28}$$

2.3 Truncated Model Evaluation

The sum (2.6) can be evaluated by solving an algebraic (Liapunov) equation. Since this equation is analogous to the continuous-time evaluation of an integral by solution of a Liapunov equation we can, without requiring a great leap of faith on the part of the reader, define a

model error index (MEI) and controller quantity index (CQI) in a manner entirely similar to that in the continuous-time case. The MEI is again defined as

$$\text{MEI} = \frac{\text{trace } \Lambda_p - \text{trace } \Lambda_{pR}}{\text{trace } \Lambda_p} \tag{2.29}$$

The MEI is a measure of the reduction of fidelity of the truncated system from the original model with respect to a *specific* performance measure. Again, as in the continuous-time case, it is desirable to assess in some measure the capabilities of controllers based on truncated models in handling the original model. A controller designed for a truncated system subsequently applied to the original system will have, instead of (2.27), the following form.

$$\underline{u}_R(k) = [-\underline{G}_R \quad \underline{0}] \begin{bmatrix} \underline{x}_R(k) \\ \underline{x}_T(k) \end{bmatrix} \tag{2.30}$$

which when applied to (2.25), (2.26) results in

$$\begin{bmatrix} \underline{x}_R(k+1) \\ \underline{x}_T(k+1) \end{bmatrix} = \begin{bmatrix} \underline{A}_{RR} - \underline{B}_R\underline{G}_R & \underline{A}_{RT} \\ \underline{A}_{TR} - \underline{B}_T\underline{G}_R & \underline{A}_{TT} \end{bmatrix} \begin{bmatrix} \underline{x}_R(k) \\ \underline{x}_T(k) \end{bmatrix} \tag{2.31}$$

The corresponding performance index is a modified version of (2.6):

$$V = \sum_{k=0}^{\infty} [\underline{x}^T(k)\underline{C}^T\underline{QCx}(k) + \underline{u}_R^T(k)\underline{Ru}_R(k)] \tag{2.32}$$

which has the associated discrete-time Liapunov equation:

$$\underline{\Pi} = \underline{\bar{A}}^T \underline{\Pi} \underline{\bar{A}} + \begin{bmatrix} \underline{C}_R^T \\ \underline{C}_R^T \end{bmatrix} \underline{Q} [\underline{C}_R \quad \underline{C}_T] + \begin{bmatrix} \underline{G}_R^T \\ \underline{0} \end{bmatrix} \underline{R} [\underline{G}_R \quad \underline{0}] \tag{2.33}$$

where $\underline{\bar{A}}$ is the system matrix in (2.31). Taking note of the comments preceding (2.23) we define the cost expectation

$$E\{V\} = \text{trace } \underline{\Pi} \tag{2.34}$$

and the corresponding measure of suboptimality, the controller quality index, is defined as

$$\text{CQI} = \frac{\text{trace } \underline{\Lambda} - \text{trace } \underline{\Lambda}_p}{\text{trace } \underline{\Lambda}_p} \tag{2.35}$$

Here again, we would expect an increase in CQI with a decrease in model order.

In conclusion then, we can say that the method of cost-decoupled coordinates is identical in the discrete-time case to the continuous-time version, with the same performance measures, after the required correspondences have been made.

3. COST-DECOUPLED COORDINATES FOR OBSERVER STABILIZED SYSTEMS

The method of cost-decoupled coordinates was discussed in [1] where it was assumed that the state was available for feedback, and a numerical case study of model reduction was performed using the 'Lazy - Z' MSAT configuration (ZSAT). This section extends the theory of [1] to the case in which an estimate of the state is used rather than the state itself, and the results of a numerical study for this case parallel to that in [1] are presented.

3.1 Observer Review

As is well known, a controllable system can be stabilized by

state feedback, and it is common policy (at least in theory) to use an *estimate* of the state rather than the state itself if the latter is unavailable and the system is observable.

We start with the usual linear time-invariant multivariable system in first order differential form,

$$\begin{aligned} \dot{\underline{x}} &= \underline{Ax} + \underline{Bu} \\ \underline{y} &= \underline{Cx} \end{aligned} \tag{3.1}$$

which is assumed controllable and observable.

If the state $\underline{x}$ is unavailable, an estimate of it can be generated by using an *observer*, which is a dynamic system of the following form:

$$\dot{\hat{\underline{x}}} = \underline{A\hat{x}} + \underline{Bu} + \underline{K}[\underline{y} - \underline{C\hat{x}}] \tag{3.2}$$

where $\hat{\underline{x}}$ is the estimate of the state, and for many operations is used as though it *is* in fact the state. One might, for example, wish to apply state feedback

$$\underline{u} = -\underline{Gx} \tag{3.3}$$

to a system, where $\underline{G}$ is selected so that the closed-loop system has certain desired properties. If $\underline{x}$ is unavailable, the feedback

$$\hat{\underline{u}} = -\underline{G\hat{x}} \tag{3.4}$$

is applied instead, where x converges asymptotically to $\underline{x}$, depending on the $\underline{K}$ selected.

This interconnection results in the following closed-loop augmented system:

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\hat{\underline{x}}} \end{bmatrix} = \begin{bmatrix} \underline{A} & -\underline{BG} \\ \underline{KC} & \underline{A} - \underline{KC} - \underline{BG} \end{bmatrix} \begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix} \tag{3.5}$$

By the well known *Separation Property*, the controller and observer gains, $\underline{G}$ and $\underline{K}$ respectively, can be designed independently of one another.

3.2 Optimal Control for Observer Stabilized Systems

The performance criterion associated with the infinite time optimal state feedback problem

$$V = \int_0^\infty (\underline{y}^T \underline{Qy} + \underline{u}^T \underline{Ru})dt \tag{3.6}$$

is no longer applicable when the state is unavailable, but must be modified to

$$\hat{V} = \int_0^\infty (\underline{y}^T \underline{Qy} + \hat{\underline{u}}^T \underline{R}\hat{\underline{u}})dt \tag{3.7}$$

where $\hat{u}$ is given by (3.4). This can be written in terms of the closed-loop augmented system (3.5) as

$$\hat{\underline{V}} = [\underline{x}_0^T \quad \hat{\underline{x}}_0^T] \int_0^\infty e^{\bar{\underline{A}}^T t} \begin{bmatrix} \underline{C}^T \underline{QC} & \underline{0} \\ \underline{0} & \underline{G}^T \underline{RG} \end{bmatrix} e^{\bar{\underline{A}}t} \, dt \begin{bmatrix} \underline{x}_0 \\ \hat{\underline{x}}_0 \end{bmatrix} \tag{3.8}$$

where $\begin{bmatrix} \underline{x}_0 \\ \hat{\underline{x}}_0 \end{bmatrix}$ is the initial condition on the controller and observer,

and $\bar{\underline{A}}$ is the system matrix in (3.5). This equation has the following Liapunov equation associated with it:

$$\underline{\bar{A}}^T\underline{P} + \underline{P}\underline{\bar{A}} = \begin{bmatrix} -\underline{C}^T\underline{QC} & \underline{O} \\ \underline{O} & -\underline{G}^T\underline{RG} \end{bmatrix} \tag{3.9}$$

where

$$\hat{\underline{V}} = [\underline{x}_o^T \quad \hat{\underline{x}}_o^T]\ \underline{P} \begin{bmatrix} \underline{x}_o \\ \hat{\underline{x}}_o \end{bmatrix} \tag{3.10}$$

is the cost associated with a particular initial condition.

An immediate question which comes to mind is, will a $\underline{G}$ which minimizes (3.6) also in some sense minimize (3.7)? In a stochastic setting with (3.7) appropriately modified, an affirmative answer is afforded by the deceptively simple *Separation Theorem* which tells us that the gain which minimizes (3.6) will also minimize the modified (3.7) when applied to the output of the appropriate *optimal observer*. In deterministic case, it can still be stated that for a *particular* observer the feedback which minimizes (3.6) also minimizes (3.7) with the cost given by (3.10). This feedback is of the form (3.4) in which $\underline{G}$ is provided by the standard result.

$$\underline{G} = \underline{R}^{-1}\underline{B}^T\underline{P} \tag{3.11}$$

where $\underline{P}$ is the solution to the Algebraic Riccati equation

$$\underline{A}^T\underline{P} + \underline{PA} + \underline{C}^T\underline{QC} - \underline{PBR}^{-1}\underline{B}^T\underline{P} = \underline{O} \tag{3.12}$$

3.3 Model Truncation

Thus far our discussion has been valid for *any* set of state variables. Henceforth, we shall assume that our system is in cost-decoupled coordinates. As considered in [1] in some detail, it is often desirable to control a system with a controller designed for a reduced or truncated version of the system. System (3.1) can be partitioned into retained and truncated portions as

$$\begin{bmatrix} \dot{\underline{x}}_R \\ \dot{\underline{x}}_T \end{bmatrix} = \begin{bmatrix} \underline{A}_{RR} & \underline{A}_{RT} \\ \underline{A}_{TR} & \underline{A}_{TT} \end{bmatrix} \begin{bmatrix} \underline{x}_R \\ \underline{x}_T \end{bmatrix} + \begin{bmatrix} \underline{B}_R \\ \underline{B}_T \end{bmatrix} \underline{u} \tag{3.13}$$

$$\underline{y} = [C_R \quad C_T] \begin{bmatrix} \underline{x}_R \\ \underline{x}_T \end{bmatrix} \tag{3.14}$$

where

$$\begin{bmatrix} \underline{x}_R \\ \underline{x}_T \end{bmatrix} \triangleq \underline{x} \qquad \begin{bmatrix} \underline{A}_{RR} & \underline{A}_{RT} \\ \underline{A}_{TR} & \underline{A}_{TT} \end{bmatrix} \triangleq \underline{A} \tag{3.15}$$

$$\begin{bmatrix} \underline{B}_R \\ \underline{B}_T \end{bmatrix} \triangleq \underline{B} \; ; \quad [\underline{C}_R \quad \underline{C}_T] \triangleq \underline{C} \tag{3.16}$$

The rationale of this partitioning according to the method of cost-decoupled coordinates is discussed in detail in [1] and will not be repeated here.

Controller design is based on the truncated system

$$\begin{aligned} \dot{\underline{x}}_R &= \underline{A}_{RR}\underline{x}_R + \underline{B}_R\underline{u} \\ \underline{y}_R &= \underline{C}_R\underline{x}_R \end{aligned} \tag{3.17}$$

so that when this controller is subsequently applied to (3.13), the closed-loop system will behave satisfactorily; in particular it must be stable. Since our controller design draws on optimal control methods, it is desirable to obtain a measure of suboptimality, or reduction in quality of the controlled system from the 'best' or optimal, as was done in [1].

3.4 Closed-Loop System and COQI

An observer based on (3.17) will have the form

$$\dot{\hat{\underline{x}}}_R = \underline{A}_{RR}\hat{\underline{x}}_R + \underline{B}_R\underline{u}_R + \underline{K}_R[\underline{y}_R - \underline{C}\hat{\underline{x}}_R] \tag{3.18}$$

and a controller for (3.17) based on the state estimate from (3.18) will be of the form

$$\underline{u}_R = -\underline{G}_R\hat{\underline{x}}_R \tag{3.19}$$

It is pointed out in [1] that $\underline{G}_R$ can be obtained either by solving the optimal control problem for (3.17), or, since the system is assumed to be in cost-decoupled coordinates, more simply by taking an appropriate truncation of $\underline{G}$ in (3.11).

In accordance with our intention to control (3.13), (3.14), with (3.19), we can express the augmented system with truncated feedback by combining (3.13), (3.14), (3.18) and (3.19) to give

$$\begin{bmatrix} \dot{\underline{x}}_R \\ \dot{\underline{x}}_T \\ \dot{\hat{\underline{x}}}_R \end{bmatrix} = \begin{bmatrix} \underline{A}_{RR} & \underline{A}_{RT} & -\underline{B}_R\underline{G}_R \\ \underline{A}_{TR} & \underline{A}_{TT} & -\underline{B}_T\underline{G}_R \\ \underline{K}_R\underline{C}_R & \underline{O} & \underline{A}_{RR}-\underline{K}_R\underline{C}_R-\underline{B}_R\underline{G}_R \end{bmatrix} \begin{bmatrix} \underline{x}_R \\ \underline{x}_T \\ \hat{\underline{x}}_R \end{bmatrix} \tag{3.20}$$

$$\underline{y} = [\underline{C}_R \quad \underline{C}_T \quad \underline{O}] \begin{bmatrix} \underline{x}_R \\ \underline{x}_T \\ \hat{\underline{x}}_R \end{bmatrix} \tag{3.21}$$

Another consequence of the input (3.19) being truncated is that the performance index with which we are dealing is no longer (3.7) but

$$\hat{V}_R = \int_0^\infty [\underline{y}^T\underline{Q}\underline{y} + \hat{\underline{u}}_R^T\underline{R}\hat{\underline{u}}_R]dt \tag{3.22}$$

This cost can be evaluated by solving

$$\hat{\underline{A}}^T \underline{\Pi} + \underline{\Pi}\hat{\underline{A}} = \begin{bmatrix} -\underline{C}^T\underline{Q}\underline{C} & \underline{0} \\ \underline{0} & -\underline{G}_R^T R \underline{G}_R \end{bmatrix} \tag{3.23}$$

for $\underline{\Pi}$, where $\hat{\underline{A}}$ is the system matrix in (3.20). Furthermore,

$$\hat{V}_R = [\underline{x}_R^T(0) \quad \underline{x}_T^T(0) \quad \hat{\underline{x}}_R^T(0)]\, \underline{\Pi} \begin{bmatrix} \underline{x}_R(0) \\ \underline{x}_T(0) \\ \hat{\underline{x}}_R(0) \end{bmatrix} \tag{3.24}$$

Expressions (3.20) - (3.23) enable us to produce one measure of comparison of controllers based on various truncations, analogous to the CQI of [1]. For a particular observer and set of initial conditions, the relative minimum of $\hat{V}_R$ will occur for the case of no truncation, when (3.19) is identical to (3.4) and (3.20) reduces to (3.5) (in appropriate coordinates of course), and $\underline{G}_R$ becomes the full-order optimal gain. We would expect, as in the state feedback case, that as the model is successively truncated the controller based on these reduced models would perform increasingly poorly, with $\hat{V}_R$ increasing from its optimal value ($\hat{V}$), finally becoming unequal to the task of stabilizing the full-order system.

Following the reasoning set forth in [1] we can obtain an estimate (expected value of $\hat{V}_R$ in (3.24). In this case we do not assume that all initial conditions are unknown, as we adopt the common practice of setting the observer initial condition to zero. We then find

$$E\left\{\hat{V}_R\right\} = E\left\{[\underline{x}^T \quad \underline{\phi}^T] \begin{bmatrix} \underline{\Pi}_{11} & \underline{\Pi}_{12} \\ \underline{\Pi}_{21} & \underline{\Pi}_{22} \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{0} \end{bmatrix}\right\} = \text{trace } \underline{\Pi}_{11} \tag{3.25}$$

where $E\{\cdot\}$ is the expected value of $\{\cdot\}$, and $\underline{\Pi}_{11}$ is the appropriate partition of the solution of (3.23). We also define a similar partition,

$\underline{P}_{11}$, of $\underline{P}$ in (3.10).

This immediately suggests a performance measure which closely parallels the CQI of [1]: the controller/observer quality index (COQI)

$$\text{COQI} = \frac{\text{trace } \underline{\Pi}_{11} - \text{trace } \underline{P}_{11}}{\text{trace } \underline{P}_{11}} \tag{3.26}$$

3.5 Numerical Results

We have decided the method of model reduction by cost-decoupled coordinates for observer stabilized systems in the previous pages, and while it looks attractive, whether or not it works has yet to be resolved. Although one can make theoretical arguments in support of the method, clearly the empirical approach is called for to settle the issue; after all, how better to judge the method than by trying it out! It would be most instructive to run a series of exhaustive tests on a variety of models, but we must settle for a relevant model of reasonable complexity. This is the modified ZSAT model of [1] which was used in [1] for a numerical study of model reduction using cost-decoupled coordinates for systems stabilized by state feedback. ZSAT is a third generation spacecraft -- large, flexible, and with significant flexible motion interaction with rigid body motion -- and as such is of sufficient significance and complexity to be of interest to us. The open-loop eigenvalues of the model are given in Table 3.1, and can be seen to correspond to 5 rigid body modes and 11 elastic modes which are all passively damped. A controller for the spacecraft was obtained by using standard optimal control methods, and the system transformed to cost-decoupled coordinates. These details, including a full description of the ZSAT model are contained in [1].

Use of an observer is simple in principle: merely append system (3.2) to system (3.1). In reality, however, the matter is not quite so simple, as selection of $\underline{K}$ is not straightforward and is an active area of research in its own right. Under certain conditions (modeled noises) there exists an elegant theory for producing an *optimal* $\underline{K}$

Table 3.1

Open-Loop Eigenvalues of ZSAT

$-0.11746 \times 10^{1} \pm 0.13804 \times 10^{2} j$

$-0.52390 \times \pm 0.10046 \times 10^{2} j$

$-0.75050 \times 10^{-1} \pm 0.15514 \times 10^{1} j$

$-0.60635 \times 10^{-2} \pm 0.10873 \times 10^{1} j$

$-0.80592 \times 10^{-2} \pm 0.10227 \times 10^{1} j$

$-0.21109 \times 10^{-1} \pm 0.77943\ j$

$-0.55312 \times 10^{-2} \pm 0.69018\ j$

$-0.85574 \times 10^{-2} \pm 0.55632\ j$

$-0.85210 \times 10^{-3} \pm 0.15118\ j$

$-0.17045 \times 10^{-2} \pm 0.23952\ j$

$-0.92301 \times 10^{-3} \pm 0.12435\ j$

$0.0 \pm 0.0\ j$ (repeated 5 times)

(optimal in a very specialized sense). In the usual case there are no real guidelines for constructing a $\underline{K}$, and although the phrase 'choose $\underline{K}$ to make the observer poles slightly faster than the controller poles' is often heard, finding a $\underline{K}$ which merely stabilizes the observer is no mean feat! In the present study $\underline{K}$'s were obtained by choosing arbitrary 'noise' matrices and treating the problem as though an optimal controller were needed. While an 'ideal' observer eigenvalue pattern might not result using this method, at least observer stability is guaranteed. The state 'noise' matrix was chosen as $\underline{1}_{32}$ while the observation noise matrix was $10^{-5} \times \underline{1}_{11}$. It was also found useful to multiply the resulting $\underline{K}$ by 10^5. Here, of course, we were making quite free with observer construction and gains; in a 'real world' problem we would naturally be forced to respect constraints of all sorts.

With observer feedback one would expect some degradation in system performance from the state feedback case, and this suspicion is borne out here. While in the state feedback case in [1] the full order system could be satisfactorily controlled (stabilized) by a controller designed for the 15th order truncated model, in the observer feedback case the smallest acceptable design model is of 21st order. Tables 3.2, 3.3 and 3.4 display eigenvalues of the composite system (3.20) for the 21st, 20th and 32th order systems respectively. It can be seen that the full-order system controlled by the 21st order controller seems no 'less stable' (slower) than when it is handled by the controller designed for the full order system: both are solidly stable. In contrast, when the controller designed for the 20th order system is applied to the full-order system, the resulting composite system is healthily unstable. Obviously some essential item of information was truncated out of model at this point.

If the observer were speeded up enough the state estimate would converge virtually instantaneously to the state, and the results in this study would approach those of [1]. It was found, however, that increasing observer speed requires more than simply magnifying observer gain because very large gains caused the controlled system to become unstable.

Table 3.2

Eigenvalues of ZSAT with 21st-Order Controller

Real	Imag
-0.9403D+07	0.0000D+00
-0.5098D+06	0.0000D+00
-0.4096D+06	0.0000D+00
-0.4426D+06	0.0000D+00
-0.2988D+06	0.0000D+00
-0.2119D+06	0.0000D+00
-0.1087D+06	0.0000D+00
-0.1570D+05	0.0000D+00
-0.1132D+05	0.0000D+00
-0.7999D+04	0.0000D+00
-0.6258D+02	0.0000D+00
-0.1175D+01	0.1380D+02
-0.1175D+01	-0.1380D+02
-0.5239D+00	0.1005D+02
-0.5239D+00	-0.1005D+02
-0.7494D-01	0.1551D+01
-0.7494D-01	-0.1551D+01
-0.6407D-02	0.1087D+01
-0.6407D-02	-0.1087D+01
-0.9501D-02	0.1023D+01
-0.9501D-02	-0.1023D+01
-0.2174D-01	0.7793D+00
-0.2174D-01	-0.7793D+00
-0.1536D-01	0.6812D+00
-0.1536D-01	-0.6812D+00
-0.4756D-01	0.6665D+00
-0.4756D-01	-0.6665D+00
-0.7196D+00	0.0000D+00
-0.7114D+00	0.0000D+00
-0.8422D-01	0.5667D+00
-0.8422D-01	-0.5667D+00
-0.5873D+00	0.1429D-01
-0.5873D+00	-0.1429D-01
-0.2573D+00	0.2980D+00
-0.2573D+00	-0.2980D+00
-0.3123D-01	0.2392D+00
-0.3123D-01	-0.2392D+00
-0.1786D+00	0.0000D+00
-0.1388D-02	0.1554D+00
-0.1388D-02	-0.1554D+00
-0.6547D-02	0.1453D+00
-0.6547D-02	-0.1453D+00
-0.1447D-01	0.1280D+00
-0.1447D-01	-0.1280D+00
-0.1013D+00	0.8942D-01
-0.1013D+00	-0.8942D-01
-0.5125D-02	0.0000D+00
-0.4749D-01	0.0000D+00
-0.5030D-01	0.0000D+00
-0.5266D-01	0.6153D-01
-0.5266D-01	-0.6153D-01
-0.5576D-01	0.8458D-01
-0.5576D-01	-0.8458D-01

Table 3.3

Eigenvalues of ZSAT with 20th-Order Controller

Real	Imag
-0.9405D+07	0.0000D+00
-0.5098D+06	0.0000D+00
-0.4060D+06	0.0000D+00
-0.4399D+06	0.0000D+00
-0.2988D+06	0.0000D+00
-0.2121D+06	0.0000D+00
-0.7796D+05	0.0000D+00
-0.1570D+05	0.0000D+00
-0.1151D+05	0.0000D+00
-0.8017D+04	0.0000D+00
-0.6968D+02	0.0000D+00
-0.1175D+01	0.1380D+02
-0.1175D+01	-0.1380D+02
-0.5239D+00	0.1005D+02
-0.5239D+00	-0.1005D+02
-0.7494D-01	0.1551D+01
-0.7494D-01	-0.1551D+01
-0.6178D-02	0.1087D+01
-0.6178D-02	-0.1087D+01
-0.2627D-01	0.1021D+01
-0.2627D-01	-0.1021D+01
-0.2182D-01	0.7798D+00
-0.2182D-01	-0.7798D+00
-0.1700D-01	0.6878D+00
-0.1700D-01	-0.6878D+00
0.2560D-02	0.6380D+00
0.2560D-02	-0.6380D+00
-0.2332D+00	0.5737D+00
-0.2332D+00	-0.5737D+00
-0.7196D+00	0.0000D+00
-0.6579D+00	0.0000D+00
-0.6217D+00	0.0000D+00
-0.4151D+00	0.1683D+00
-0.4151D+00	-0.1683D+00
-0.3125D-01	0.2392D+00
-0.3125D-01	-0.2392D+00
-0.1353D-02	0.1554D+00
-0.1353D-02	-0.1554D+00
0.4641D-01	0.1143D+00
0.4641D-01	-0.1143D+00
0.1697D-02	0.1297D+00
0.1697D-02	-0.1297D+00
-0.1013D+00	0.8944D-01
-0.1013D+00	-0.8944D-01
-0.8924D-01	0.8625D-01
-0.8924D-01	-0.8625D-01
-0.1227D+00	0.0000D+00
-0.5273D-01	0.6149D-01
-0.5273D-01	-0.6149D-01
0.1439D-01	0.0000D+00
-0.4819D-02	0.0000D+00
-0.6051D-01	0.0000D+00

Table 3.4

Eigenvalues of ZSAT with Full-Order Controller

Real	Imag
-0.9999D+07	0.0000D+00
-0.6224D+06	0.0000D+00
-0.4986D+06	0.0000D+00
-0.4098D+06	0.0000D+00
-0.3255D+06	0.0000D+00
-0.1405D+06	0.0000D+00
-0.1792D+06	0.0000D+00
-0.6867D+05	0.0000D+00
-0.1986D+06	0.0000D+00
-0.6964D+05	0.0000D+00
-0.5985D+02	0.0000D+00
-0.1175D+01	0.1380D+02
-0.1175D+01	-0.1380D+02
-0.5239D+00	0.1005D+02
-0.5239D+00	-0.1005D+02
-0.1832D+01	0.1056D+02
-0.1832D+01	-0.1056D+02
-0.1952D+01	0.0000D+00
-0.7509D-01	0.1551D+01
-0.7509D-01	-0.1551D+01
-0.6163D-02	0.1087D+01
-0.6163D-02	-0.1087D+01
-0.4660D-01	0.1079D+01
-0.4660D-01	-0.1079D+01
-0.9442D-02	0.1023D+01
-0.9442D-02	-0.1023D+01
-0.3118D+00	0.9020D+00
-0.3118D+00	-0.9020D+00
-0.2175D-01	0.7794D+00
-0.2175D-01	-0.7794D+00
-0.1511D-01	0.6813D+00
-0.1511D-01	-0.6813D+00
-0.5144D-01	0.6695D+00
-0.5144D-01	-0.6695D+00
-0.8862D-01	0.5706D+00
-0.8862D-01	-0.5706D+00
-0.7367D+00	0.0000D+00
-0.6934D+00	0.0000D+00
-0.6869D+00	0.0000D+00
-0.6475D+00	0.0000D+00
-0.5612D+00	0.4009D-01
-0.5612D+00	-0.4009D-01
-0.4827D+00	0.0000D+00
-0.1867D+00	0.2938D+00
-0.1867D+00	-0.2938D+00
-0.2893D-01	0.2381D+00
-0.2893D-01	-0.2381D+00
-0.3353D+00	0.0000D+00
-0.3386D+00	0.0000D+00
-0.4742D-02	0.1532D+00
-0.4742D-02	-0.1532D+00
-0.2590D-01	0.1324D+00
-0.2590D-01	-0.1324D+00
-0.1779D+00	0.0000D+00
-0.9632D-01	0.1008D+00
-0.9632D-01	-0.1008D+00
-0.9917D-01	0.9809D-01
-0.9917D-01	-0.9809D-01
-0.5791D-01	0.8762D-01
-0.5791D-01	-0.8762D-01
-0.5437D-01	0.5411D-01
-0.5437D-01	-0.5411D-01
-0.4970D-01	0.3997D-01
-0.4970D-01	-0.3997D-01

The model error index (MEI) is a measure of reduction in truncated model fidelity from the full-order model (see [1] for details). The MEI's for progressively reduced models are plotted against model order in Fig. 3.1. The MEI increases with the amount of information lost as the model is reduced. There is no corresponding figure for COQI vs. model order because numerical difficulties in the solution of (3.23) were encountered with the Stewart-Bartels algorithm even for the smallest (21st order) stable case. The most interesting item of information, however, is the point at which instability occurs, which is already determined.

4. HABLANI EXTENSION OF MODAL COST ANALYSIS

The derivation of Model Cost Analysis as presented in [1] can proceed from the assumption of an impulsive input or equivalently from a white noise input. Since these are both rather ideal concepts, one wonders how the idea of modal costs would be affected by more 'real' inputs. In [2] Hablani has developed modal cost analysis from the frequency domain point of view, and in so doing extends the method of [1] to include the case when the input is not white noise (equal power at all frequencies), but rather has a nonuniform frequency profile. This of course leaves scope for the inclusion of actuator dynamics (actuators in [1] were assumed to be perfect i.e., infinitely fast, or possessing an infinite bandwidth). We no longer necessarily assume actuators affect affect all modes equally, but rather that actuators can have nonuniform power spectral densities, and therefore mode excitation depends on frequency. The outcome of the analysis here is a result identical to that of [1] except that the input excitation term is modified, as one would anticipate.

4.1 Development of Modal Cost Analysis

We begin with a mechanical system expressed in modal coordinates. The system consists of N equations of the form

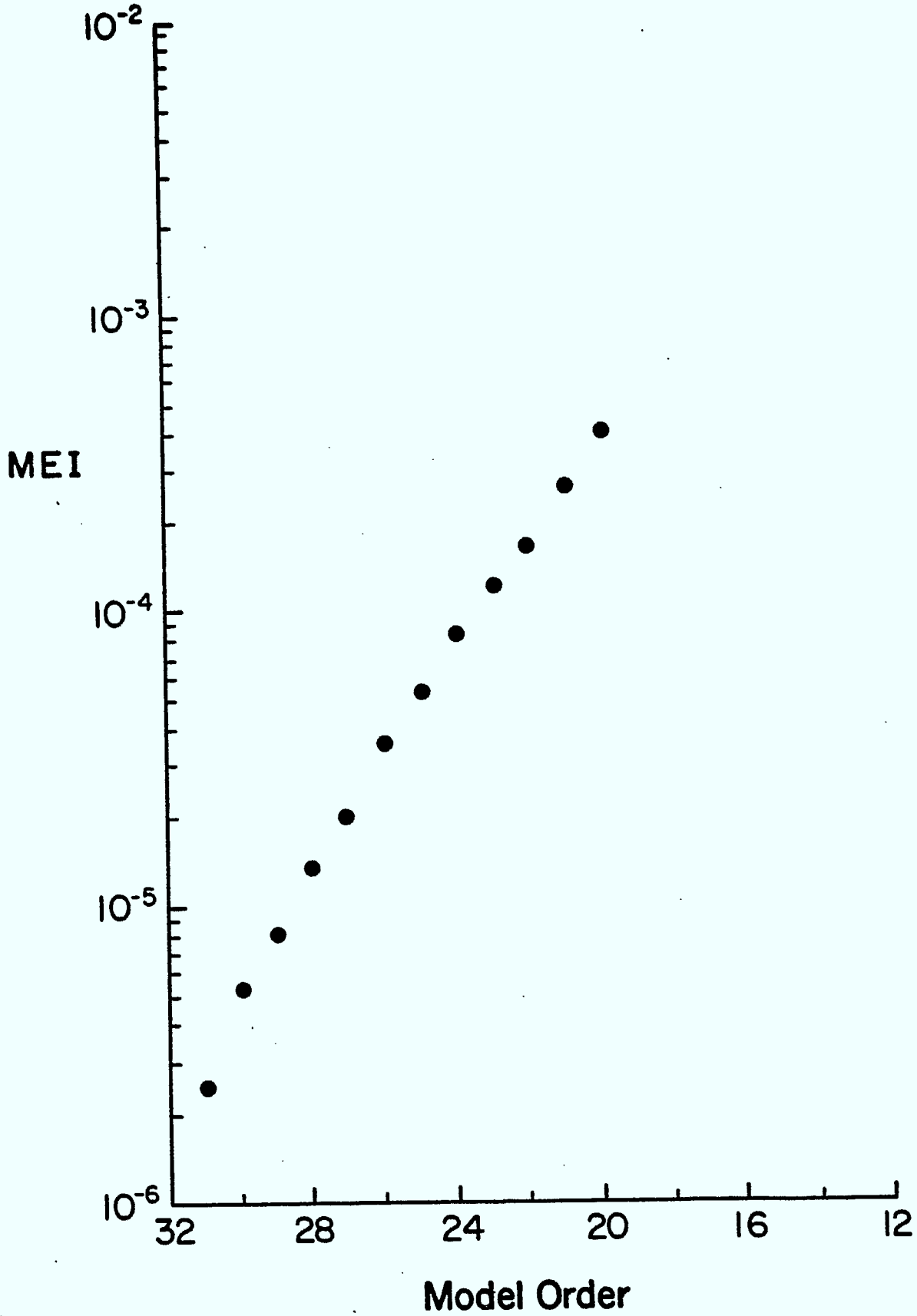

Fig. 3.1: Model Error Index vs Model Order

$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = \underline{\hat{b}}_i^T\underline{u} \quad , \qquad i = 1,\ldots N \tag{4.1}$$

where ζ_i and ω_i are the damping and natural frequency of mode i repectively, $\underline{u}$ is a vector of input (disturbances or control inputs) and $\underline{\hat{b}}_i$ is an input distribution vector. This set of modal equations can be collected and written in standard first order matrix differential form by defining

$$\underline{x} \triangleq \mathrm{col}\{\underline{\eta},\dot{\underline{\eta}}\} \tag{4.2}$$

$$\underline{A} \triangleq \begin{bmatrix} \underline{O} & \underline{1} \\ -\underline{\Omega}^2 & -2\underline{Z}\underline{\Omega} \end{bmatrix} \tag{4.3}$$

$$\underline{B} \triangleq \begin{bmatrix} \underline{O} \\ \underline{\hat{B}} \end{bmatrix} \tag{4.4}$$

where

$$\underline{\eta} \triangleq \mathrm{col}\{\eta_1,\ldots,\eta_N\} \tag{4.5}$$

$$\underline{\Omega} \triangleq \mathrm{diag}\,\{\omega_1,\ldots,\omega_N\} \tag{4.6}$$

$$\underline{Z} = \mathrm{diag}\{\zeta_1,\ldots,\zeta_N\} \tag{4.7}$$

$$\underline{\hat{B}} \triangleq \begin{bmatrix} \underline{\hat{b}}_1^T \\ \vdots \\ \hat{b}_N^T \end{bmatrix} \tag{4.8}$$

giving

$$\dot{\underline{x}} = \underline{Ax} + \underline{Bu} \tag{4.9}$$

In the usual case it is necessary to define a companion to (4.9) to relate state variables to 'important' output variables. This is of the form

$$\underline{y} = \underline{Cx} \tag{4.10}$$

where

$$\underline{C} = [\hat{\underline{C}} \quad \hat{\dot{\underline{C}}}] \triangleq [\hat{\underline{c}}_1, \ldots, \hat{\underline{c}}_N, \hat{\dot{\underline{c}}}_1, \ldots, \hat{\dot{\underline{c}}}_N] \tag{4.11}$$

Equation (4.10) relates *modal* coordinates to output variables, thus $\hat{\underline{C}}$ and $\hat{\dot{\underline{C}}}$ include the modal matrix used to transform from physical to model coordinates. Note that $\hat{\dot{\underline{C}}}$ is a constant matrix associated with the time derivative of $\underline{\eta}$. We shall, for the remainder of the development, assume that the input vector $\underline{u}$ is a zero-mean weakly stationary process having a power spectral density matrix $\underline{\Sigma}_u(\omega)$. We take as a measure of cost the following criterion, which is appropriate for stochastic systems:

$$V = E\{\underline{y}^T \underline{Qy}\} \tag{4.12}$$

where $E\{\cdot\}$ is the expected value of $\{\cdot\}$ and $\underline{Q}$ is an output weighting matrix, determined by the relative importance of the outputs.

We are assured by linear system theory that with $\underline{u}$ a zero-mean weakly stationary process, $\underline{x}$ and therefore $\underline{y}$ will also be weakly stationary zero-mean processes. We shall make use of this fact in a moment. We first note that under the condition of *light damping*, for which this analysis is intended to be valid, inter-mode coupling becomes negligible [3] so that any two modal coordinates can be considered independent processes. For zero-mean independent processes η_i, η_j, $i \neq j$

$$E\{\eta_i(t)\eta_j(t + \tau)\} = E\{\eta_i\}E\{\eta_j\} = 0 \tag{4.13}$$

for any t and τ. The same holds true for any pair $\dot{\eta}_i$, $\dot{\eta}_j$, $i \neq j$.

We further assume that η_i and $\dot{\eta}_i$ are independent. We have then that the variance matrix $\underline{R}_x(\tau)$ of the process $\underline{x}$ is diagonal,

$$\underline{R}_x(\tau) = \text{diag}[E\{\eta_1(t)\eta_1(t+\tau)\},\ldots,E\{\eta_N(t)\eta_N(t+\tau)\},E\{\dot{\eta}_1(t)\dot{\eta}_1(t+\tau)\},$$

$$\ldots,E\{\dot{\eta}_N(t)\dot{\eta}_N(t+\tau)\}] \tag{4.14}$$

The power spectral density matrix of the process $\underline{x}$ is the Fourier transform of the variance matrix of $\underline{x}$, or,

$$\underline{\Sigma}_x(\omega) \triangleq \int_{-\infty}^{\infty} \underline{R}_x(\tau)e^{j\omega\tau}d\tau$$

$$= \text{diag}[\Sigma_{\eta_1}(\omega),\ldots\Sigma_{\eta_N}(\omega),\Sigma_{\dot{\eta}_1}(\omega),\ldots,\Sigma_{\dot{\eta}_N}(\omega)] \tag{4.15}$$

where

$$\Sigma_{\eta_i}(\omega) \triangleq \int_{-\infty}^{\infty} E\{\eta_i(t)\eta_i(t+\tau)\}e^{j\omega\tau}d\tau \tag{4.15a}$$

and $\Sigma_{\dot{\eta}_i}(\omega)$ is corresponding power spectral density of $\dot{\eta}_i$.

Returning now to our performance index (4.12), the fact that $\underline{y}$ is a weakly stationary zero-mean process allows us to write

$$V = E\{\underline{y}^T\underline{Qy}\}$$

$$= E\{\underline{x}^T\underline{C}^T\underline{QCx}\}$$

$$= \text{trace}\ [\underline{C}^T\underline{QC}\int_{-\infty}^{\infty}\underline{\Sigma}_x(\omega)\frac{d\omega}{2\pi}$$

$$= \text{trace}\ [\underline{C}^T\underline{QC}\ \text{diag}[f_1,\ldots,f_N,\dot{f}_1,\ldots,\dot{f}_N]] \tag{4.16}$$

where

$$f_i \triangleq \int_{-\infty}^{\infty} \Sigma_{\eta_i}(\omega)\frac{d\omega}{2\pi} \tag{4.17a}$$

$$\dot{f}_i \triangleq \int_{-\infty}^{\infty} \Sigma_{\dot{\eta}_i}(\omega)\frac{d\omega}{2\pi} \tag{4.17b}$$

Recognizing (4.11) and collecting terms with the same subscripts, we can rewrite (4.16) as

$$V = [\underline{\hat{c}}_1^T \underline{Q}\,\underline{\hat{c}}_1 f_1 + \underline{\dot{\hat{c}}}_1^T \underline{Q}\,\underline{\dot{\hat{c}}}_1 \dot{f}_1] + \ldots + [\underline{\hat{c}}_N^T \underline{Q}\,\underline{\hat{c}}_N f_N + \underline{\dot{\hat{c}}}_N^T \underline{Q}\,\underline{\dot{\hat{c}}}_N \dot{f}_N] \tag{4.18}$$

It now remains for us to evaluate f_i and $\dot{f}_i$, which requires us to find the power spectral densities of η_i and $\dot{\eta}_i$. We shall concentrate first on η_i.

Equation (4.1) can be expressed in the frequency domain (with zero initial conditions) as

$$\eta_i(s) = \frac{\underline{\hat{b}}_i^T \underline{u}(s)}{s^2 + 2\zeta_i\omega_i s + \omega_i^2} \tag{4.19}$$

By a well known result in Linear System Theory, the power spectral density of η_i is

$$\Sigma_{\eta_i}(\omega) = \frac{\underline{\hat{b}}_i^T \underline{\Sigma}_u(\omega)\underline{\hat{b}}_i}{[(-\omega^2 + \omega_i^2) + j2\zeta_i\omega_i\omega][(\omega^2 + \omega_i^2) - j2\zeta_i\omega_i\omega]} \tag{4.20}$$

where we recall that $\underline{\Sigma}_u(\omega)$ is the power spectral density of the input process and $j \triangleq \sqrt{-1}$. To calculate f_i therefore, we must evaluate terms of the form

$$f_i = \int_{-\infty}^{\infty} \frac{\underline{\hat{b}}_i^T \underline{\Sigma}_u(\omega)\underline{\hat{b}}_i\, d\omega}{2\pi[(-\omega^2 + \omega_i^2) + j2\zeta_i\omega_i\omega][(\omega^2 + \omega_i^2) - j2\zeta_i\omega_i\omega]} \tag{4.21}$$

This integral can be evaluated via residue calculus by using the formula

$$\int_{-\infty}^{\infty} g(x)dx = 2\pi j \sum_k \mathrm{Res}(a_k) \tag{4.22}$$

where $\mathrm{Res}(a_k)$ are the residues of $g(z)$ at the points a_k, which are the poles of $g(z)$ in the upper half complex plane, with z the complex variable $x + jy$. The poles in the integral of (4.21) due to the structural modes are

$$z_{1,2} = \pm\omega_i + j\zeta_i\omega_i \tag{4.23}$$

$$z_{3,4} = \pm\omega_i - j\zeta_i\omega_i \tag{4.24}$$

The latter two poles are in the lower half complex plane and are therefore not taken into consideration in the evaluation of (4.21) via (4.22). It can be shown that for (4.21)

$$\mathrm{Res}(a_1) = \mathrm{Res}(z_1) = \frac{\hat{\underline{b}}_i^T \underline{\Sigma}_u(z_1)\hat{\underline{b}}_i}{16\pi j\zeta_i\omega_i^3(1 + j\zeta_i)} \tag{4.25}$$

$$\mathrm{Res}(a_2) = \mathrm{Res}(z_2) = \frac{\hat{\underline{b}}_i^T \underline{\Sigma}_u(z_2)\hat{\underline{b}}_i}{16\pi j\zeta_i\omega_i^3(1 - j\zeta_i)} \tag{4.26}$$

The assumption of light damping ($\zeta_i \to 0$) now suggests that these two residues greatly overpower the residues due to any singularities of $\underline{\Sigma}_u(z)$. We have therefore

$$f_i \doteq 2\pi j[\mathrm{Res}(a_1) + \mathrm{Res}(a_2)] \tag{4.27}$$

Substituting (4.25) and (4.26) into (4.27) and recognizing that $\zeta_i \to 0$ means that

$$(1 + j\zeta_i) \doteq 1 \tag{4.28}$$

$$a_1 = \omega_i + j\zeta_i\omega_i \doteq \omega_i \tag{4.29}$$

$$a_2 = -\omega_i + j\zeta_i\omega_i \doteq -\omega_i \tag{4.30}$$

and also that $\underline{\Sigma}_u(-\omega_i) \doteq \underline{\Sigma}_u^T(\omega_i)$

yields

$$f_i = \frac{1}{8\zeta_i\omega_i^3} \hat{\underline{b}}_i^T[\underline{\Sigma}_u(\omega_i) + \underline{\Sigma}_u^T(\omega_i)]\hat{\underline{b}}_i \tag{4.31}$$

The evaluation of $\dot{f}_i$ proceeds along parallel lines. Using the transfer function from the input to $\dot{\eta}_i(t)$ for (4.1) gives

$$s\eta_i(s) = \frac{s\hat{\underline{b}}_i^T\underline{u}(s)}{s^2 + 2\zeta_i\omega_i s + \omega_i^2} \tag{4.32}$$

so that

$$\Sigma_{\dot{\eta}_i}(\omega) = \frac{\omega^2\hat{\underline{b}}_i^T\underline{\Sigma}_u(\omega)\hat{\underline{b}}_i}{[(-\omega^2 + \omega_i^2) + j2\zeta_i\omega_i\omega][(\omega^2 + \omega_i^2) - j2\zeta_i\omega_i\omega]} \tag{4.33}$$

Repeating the arguments leading from (4.20) to (4.31) results in

$$\dot{f}_i = \frac{1}{8\zeta_i\omega_i} \hat{\underline{b}}_i^T[\underline{\Sigma}_u(\omega_i) + \underline{\Sigma}_u^T(\omega_i)]\hat{\underline{b}}_i \tag{4.34}$$

We now return to (4.18), which we can rewrite as a sum of modal costs,

$$V = \sum_{i=1}^{N} \nu_i \tag{4.35}$$

where

$$\nu_i \triangleq \hat{\underline{c}}_i^T\underline{Q}\hat{\underline{c}}_i f_i + \hat{\dot{\underline{c}}}_i^T\underline{Q}\hat{\dot{\underline{c}}}_i \dot{f}_i \tag{4.36}$$

Substituting (4.31) and (4.34) into (4.36) results in

$$\nu_i = \frac{\hat{\underline{c}}_i^T Q \hat{\underline{c}}_i + \omega_i^2 \dot{\hat{\underline{c}}}_i^T Q \dot{\hat{\underline{c}}}_i}{8\zeta_i \omega_i^3} \hat{\underline{b}}_i^T [\underline{\Sigma}_u(\omega_i) + \underline{\Sigma}_u^T(\omega_i)] \hat{\underline{b}}_i \tag{4.37}$$

This expression can be further simplified by noting that $\underline{\Sigma}_u(\omega)$ is a Hermitian matrix [4], giving finally

$$\nu_i = \left[\frac{\hat{\underline{c}}_i^T Q \hat{\underline{c}}_i + \omega_i^2 \dot{\hat{\underline{c}}}_i^T Q \dot{\hat{\underline{c}}}_i}{4\zeta_i \omega_i^3} \right] \hat{\underline{b}}_i^T \underline{\Sigma}_u(\omega_i) \hat{\underline{b}}_i \tag{4.38}$$

This expression is very similar to that obtained in [1] and reproduced here as (4.39). The difference between the two is the inclusion of the power spectral density of the input in (4.38).

4.2 Numerical Results

Modal Cost Analysis was performed on the 'Lazy-Z' MSAT (ZSAT) configuration modeled by Dynacon [1], which consists of five rigid body modes and eleven elastic modes, giving a state of order 32. The model has nine inputs, due to thrusters and torque actuators, and eleven outputs, which are the rigid body rotations of the vehicle, and internal displacements and rotations. A complete description of the model is contained in [1].

In [1] open-loop modal cost analysis is developed under the assumption of impulsive system excitations of the form $\underline{u} = \underline{a}\delta(t)$ where $\underline{u}$ is the input vector in (4.9) and $\delta(t)$ is the Dirac delta function. This results in the following formula for modal costs:

$$\nu_i = \left[\frac{\hat{\underline{c}}_i^T Q \hat{\underline{c}}_i + \omega_1^2 \dot{\hat{\underline{c}}}_i^T Q \dot{\hat{\underline{c}}}_i}{4\zeta_i \omega_i^3} \right] (\hat{\underline{b}}_i^T \underline{a})^2 \tag{4.39}$$

Table 4.1 displays the modal costs calculated using (4.39) with all elements of $\underline{a}$ arbitrarily chosen to be unity. These are presented in descending order with the costs corresponding to the rigid body modes being set to infinity. Alongside the costs are their mode numbers in the original system ordering. The spread of modal costs indicates that the tenth elastic mode is a factor of approximately 10^6 less important than the third elastic mode for the control problem envisioned.

The extension of Modal Cost Analysis described in Section 4.1 was used to generate Table 4.2, where now the excitation is no longer assumed uniform at all frequencies, but rather has some definite profile. The system inputs were assumed to be independent stochastic processes with power spectral density matrix arbitrarily chosen as

$$\underline{\Sigma}_u(\omega) = \mathrm{diag}[\cos^2(\tfrac{\omega}{110}),\cos^2(\tfrac{\omega}{120}),\ldots,\cos^2(\tfrac{\omega}{190})] \tag{4.40}$$

The resulting modal costs and their accompanying mode numbers are given in Table 4.2.

5. ENVIRONMENTAL DISTURBANCES FOR ZSAT

At geostationary altitude the dominant environmental influences on a spacecraft are solar radiation pressure and gravity gradient torque. The magnitudes and variation of these two effects depend on the configuration and intended mission of the vehicle in question. The 'Lazy-Z' MSAT (ZSAT) spacecraft, for example, is a communications satellite with a large Earth-tracking reflector antenna and a Sun-tracking solar array (Fig. 5.1). One would expect that the gravity gradient torque on the reflector would be approximately constant because of the latter's constant orientation with respect to the Earth, but that solar radiation pressure would vary with a daily periodicity because of the satellite's once-per-orbit rotation. Similarly, the solar radiation pressure on the array is constant,

Table 4.1

Modal Costs for ZSAT with Uniform Input Excitation

Mode No.	Modal Cost	
	0.1000D+76	Rigid Body Modes
	0.1000D+76	
	0.1000D+76	
	0.1000D+76	
	0.1000D+76	
3	0.1238D-04	Elastic Modes
1	0.9671D-05	
4	0.3148D-05	
5	0.2000D-06	
7	0.3678D-07	
2	0.3237D-07	
6	0.1284D-07	
9	0.2180D-08	
8	0.1898D-08	
11	0.3932D-10	
10	0.2088D-10	

Table 4.2

Modal Costs for ZSAT with Non-Uniform Input Excitation

Mode No.	Modal Cost	
	0.1000D+76	Rigid Body Modes
	0.1000D+76	
	0.1000D+76	
	0.1000D+76	
	0.1000D+76	
1	0.1418D-04	Elastic Modes
4	0.1075D-04	
3	0.6703D-05	
5	0.8188D-06	
2	0.6259D-06	
7	0.4010D-07	
6	0.1857D-07	
8	0.2767D-08	
9	0.1275D-08	
11	0.1966D-10	
10	0.1575D-10	

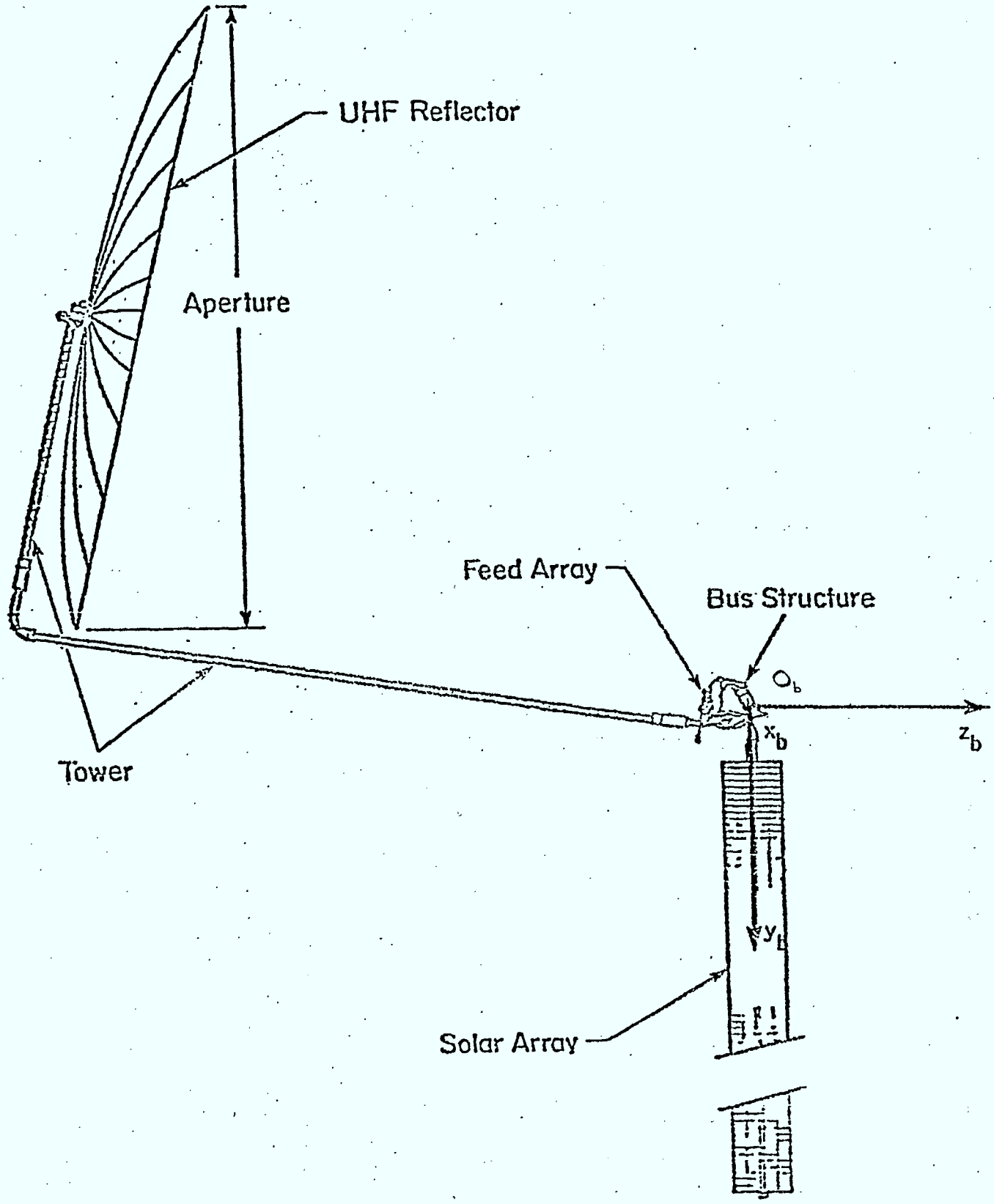

Fig: 5.1: The ZSAT Spacecraft

but the once-per-day rotation of the array with respect to the rest of the spacecraft would be expected to affect the gravity gradient torque on the spacecraft.

A description of the ZSAT spacecraft model is given in [1] and will not be repeated here. Our concern in this section is to develop additional input (disturbance) terms on the right hand side of the spacecraft motion equations to model the two environmental effects discussed above, i.e., we wish to produce $\underline{\hat{w}}_g \underline{d}_g$ and $\underline{\hat{w}}_s \underline{d}_s$ in the following equations written in terms of modal coordinates, corresponding to equation (3.9) in [1]:

$$\ddot{\underline{\eta}} + (\underline{\hat{D}} + \underline{\hat{G}})\dot{\underline{\eta}} + \underline{\Omega}^2 \underline{\eta} = \underline{\hat{B}}\underline{u} + \underline{\hat{w}}_g \underline{d}_g + \underline{\hat{w}}_s \underline{d}_s \tag{5.1}$$

where $\underline{\eta}$ is a vector of modal coordinates, $\underline{\hat{D}}$ and $\underline{\hat{G}}$ are transformed damping and gyroscopic matrices respectively, $\underline{\Omega}$ is a diagonal matrix of natural frequencies, $\underline{\hat{B}}$ is a control distribution matrix, $\underline{u}$ is a control input vector, $\underline{\hat{w}}_g$ and $\underline{\hat{w}}_s$ are gravitational and solar disturbances distribution matrices respectively, and $\underline{d}_g$ and $\underline{d}_s$ are the corresponding disturbance inputs.

5.1 Gravitational Torque

In the following discussion we treat ZSAT as though it were rigid, so that we neglect gravitational torques generated as a consequence of flexible deformations. This simplifying assumption renders the problem more tractable than it would be otherwise, but still allows for the retention of the dominant portion of the gravitational torque. We also assume a circular orbit of radius r_o = 42,164 km (geostationary altitude) with orbital frequency

$$\omega_o = \sqrt{\mu/r_o^3} \tag{5.2}$$

where $\mu = 3.9860 \times 10^5\ k_m^3/sec^2$ is the gravitational constant of the Earth. We also take the first and second moments of inertia

of ZSAT, $\underline{c}$ and $\underline{J}$, relative to O_b (Fig. 5.2) and expressed in F_b, the bus frame.

From [1] the vector of spacecraft coordinates for ZSAT, consisting of both physical and modal coordinates, is given by

$$\underline{q} = \mathrm{col}[\underline{w}_b, \underline{\theta}_b, \underline{\beta}, \underline{\delta}, \underline{\alpha}, \underline{\eta}_a, \underline{q}_i, \underline{\eta}_r] \tag{5.3}$$

where $\underline{w}_b$ and $\underline{\theta}_b$ are rigid translations and rotations of the bus, $\underline{\beta}$ consists of two gimbal angles at the reflector hub, $\underline{\delta}$ is a vector of relative displacement of tower rib to tower root, $\underline{\alpha}$ is a set of reflector rotations, and $\underline{\eta}_a$, $\underline{q}_i$, and $\underline{\eta}_r$ describe solar array, tower and reflector motions.

The gravitational torque on ZSAT is assumed here to affect only the rigid rotations of the bus, so that the gravitational disturbance vector corresponding to (5.3) with similar partitioning is

$$\underline{g}_G \triangleq \mathrm{col}[\underline{0}, \underline{g}_{G\theta}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underline{0}, \underline{0}] \tag{5.4}$$

where

$$\underline{g}_{G\theta} \triangleq \underline{g}_{G1} + \underline{g}_{G2} \tag{5.5}$$

with

$$\underline{g}_{G1} \triangleq \frac{\mu}{r_o^2} \begin{bmatrix} c_2 - c_3\theta_{1b} \\ -(c_1 + c_2\theta_{2b}) \\ c_1\theta_{1b} + c_2\theta_{2b} \end{bmatrix} \tag{5.6}$$

and

$$\underline{g}_{G2} \triangleq -3\omega_o^2 \begin{bmatrix} J_{23} + (J_{22} - J_{33})\theta_{1b} - J_{12}\theta_{2b} \\ -J_{13} + (J_{11} - J_{33})\theta_{2b} - J_{12}\theta_{1b} \\ J_{13}\theta_{1b} + J_{23}\theta_{2b} \end{bmatrix} \tag{5.7}$$

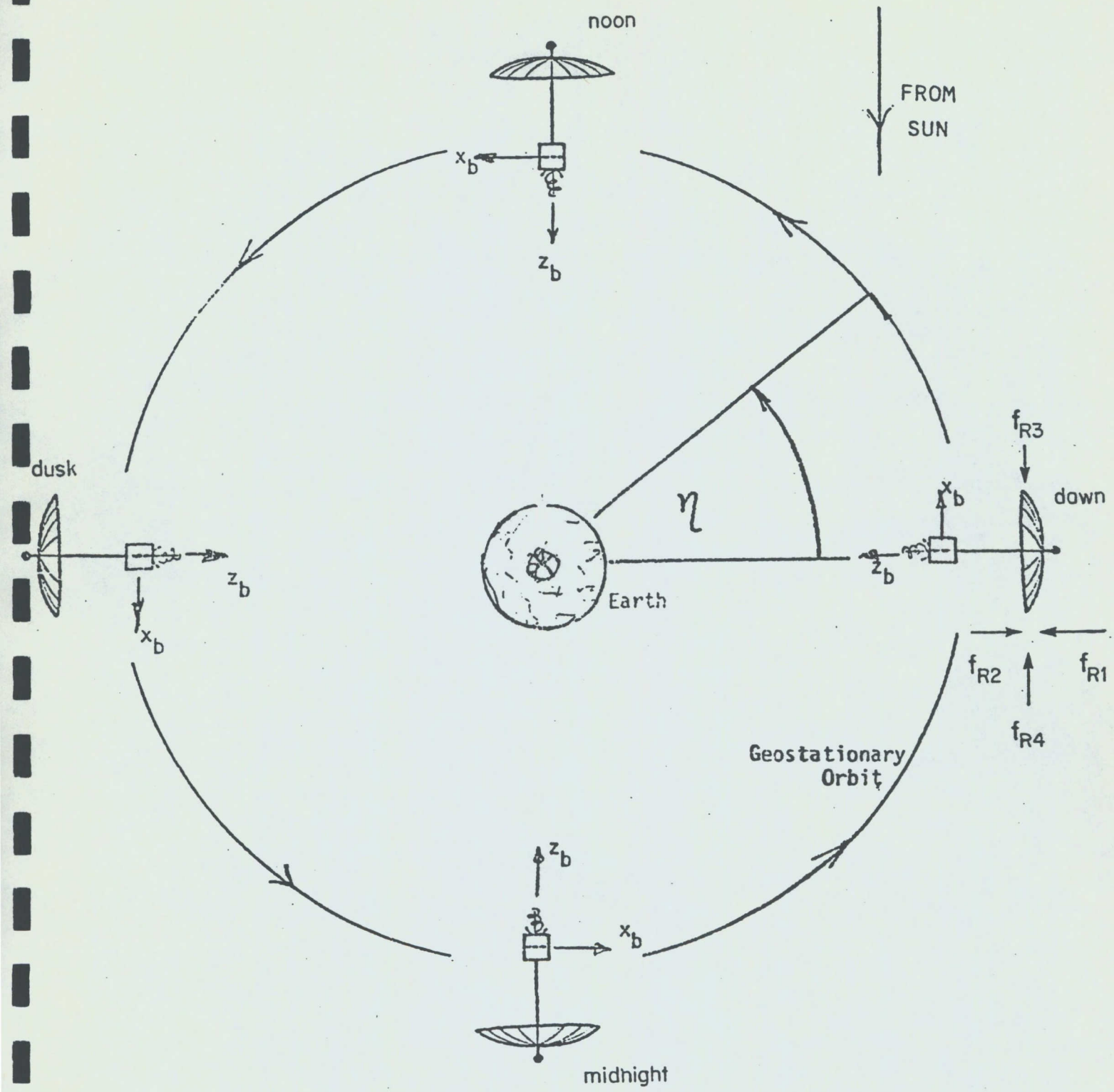

Fig. 5.2: ZSAT in Geostationary Orbit Illustrating Orbital Anomaly and Definitions of Solar Radiation Forces

Here $\underline{g}_{G1}$ depends on the first moments of inertia, c_i, for the spacecraft, while those terms related to the second moments and products of inertia, J_{ii} and J_{ij} are collected in $\underline{g}_{G2}$. The angular displacements of the spacecraft with respect to the bus axes are denoted by θ_{ib}. Let us now partition $\underline{g}_G$ according to the rigid motions of the bus and the remainder of the coordinates:

$$\underline{g}_G = \begin{bmatrix} \underline{g}_{Gr} \\ \underline{0} \end{bmatrix} \tag{5.8}$$

where

$$\underline{g}_{Gr} = \begin{bmatrix} \underline{0} \\ \underline{g}_{G\theta} \end{bmatrix} \tag{5.9}$$

To use (5.8) in (5.1) it is necessary first to premultiply it by $\underline{E}^T$, where $\underline{E}$ is the modal matrix. Partitioning $\underline{E}^T$ to correspond to that of (5.8) yields

$$\begin{aligned} \hat{\underline{g}}_G &= \underline{E}^T \underline{g}_G \\ &= \begin{bmatrix} \underline{E}_r^T & \underline{0} \\ \underline{E}_{re}^T & \underline{E}_e^T \end{bmatrix} \begin{bmatrix} \underline{g}_{Gr} \\ \underline{0} \end{bmatrix} \\ &= \begin{bmatrix} \underline{E}_r^T \underline{g}_{Gr} \\ \underline{E}_{re}^T \underline{g}_{Gr} \end{bmatrix} \end{aligned} \tag{5.10}$$

We now note that $\underline{g}_{Gr}$ can be written as the sum of a constant term and a term that varies as a consequence of purturbations in the spacecraft attitude. From (5.6) and (5.7)

$$\underline{g}_{Gr} = \underline{g}_{Go} + \underline{G}_{Gr} \underline{q}_r \tag{5.11}$$

where

$$\underline{g}_{Go} = \begin{bmatrix} \underline{0}_{3x1} \\ \hline \frac{\mu}{r_o^2} c_2 - 3\omega_o^2 J_{23} \\ \frac{-\mu}{r_o^2} c_1 + 3\omega_o^2 J_{13} \\ 0 \end{bmatrix} \tag{5.12}$$

$$\underline{G}_{Gr} = \left[\begin{array}{c|ccc} \underline{O}_{3x3} & & \underline{O}_{3x3} & \\ \hline & \frac{-\mu}{r_o^2} c_3 - 3\omega_o^2(J_{22} - J_{33}) & 3\omega_o^2 J_{12} & 0 \\ \underline{O}_{3x3} & 3\omega_o^2 J_{12} & \frac{-\mu}{r_o^2} c_s - 3\omega_o^2(J_{11} - J_{33}) & 0 \\ & \frac{\mu}{r_o^2} c_1 - 3\omega_o^2 J_{13} & \frac{\mu}{r_o^2} c_2 - 3\omega_o^2 J_{23} & 0 \end{array}\right] \tag{5.13}$$

and

$$\underline{q}_r = \begin{bmatrix} \underline{w}_b \\ \underline{\theta}_b \end{bmatrix} \tag{5.14}$$

follows from a partitioning of (5.3) consistent with that adopted in (5.8).

The modal transformation can be written as follows, where the modal matrix is partitioned as in (5.10):

$$\begin{bmatrix} \underline{q}_r \\ \underline{q}_e \end{bmatrix} = \begin{bmatrix} \underline{E}_r & \underline{E}_{re} \\ \underline{0} & \underline{E}_e \end{bmatrix} \begin{bmatrix} \underline{\eta}_r \\ \underline{\eta}_e \end{bmatrix}$$

$$= \begin{bmatrix} \underline{E}_r\underline{\eta}_r + \underline{E}_{re}\underline{\eta}_e \\ \underline{E}_e\underline{\eta}_e \end{bmatrix} \tag{5.15}$$

Substituting the expression for $\underline{q}_r$ from (5.15) into (5.11) and using the result in (5.10) gives

$$\hat{\underline{g}}_G = \begin{bmatrix} \underline{E}_r^T\underline{g}_{Go} + \underline{E}_r^T\underline{G}_{Gr}\underline{E}_r\underline{\eta}_r + \underline{E}_r^T\underline{G}_{Gr}\underline{E}_{re}\underline{\eta}_e \\ \underline{E}_{re}^T\underline{g}_{Go} + \underline{E}_{re}^T\underline{G}_{Gr}\underline{E}_r\underline{\eta}_r + \underline{E}_{re}\underline{G}_{Gr}\underline{E}_{re}\underline{\eta}_e \end{bmatrix} \tag{5.16}$$

or

$$\hat{\underline{g}}_G = \hat{\underline{g}}_{Go} + \hat{\underline{G}}_{Gr}\underline{\eta} = \hat{\underline{w}}_g\underline{d}_g \tag{5.17}$$

where

$$\hat{\underline{g}}_{Go} = \begin{bmatrix} \underline{E}_r^T \\ \underline{E}_{re}^T \end{bmatrix} \underline{g}_{Go} \tag{5.18}$$

and

$$\hat{\underline{G}}_{Gr} = \begin{bmatrix} \underline{E}_r^T \\ \underline{E}_{re}^T \end{bmatrix} \underline{G}_{Gr}[\underline{E}_r \quad \underline{E}_{re}] \tag{5.19}$$

When the equations of motion (5.1) are written in first order form with the state $\underline{x}$ defined as

$$\underline{x} = \begin{bmatrix} \underline{\eta} \\ \dot{\underline{\eta}} \end{bmatrix} \tag{5.20}$$

so that

$$\dot{\underline{x}} = \underline{Ax} + \underline{Bu} + \underline{W}_g\underline{d}_g + \underline{W}_s\underline{d}_s \tag{5.21}$$

where

$$\underline{A} = \begin{bmatrix} \underline{0}_{nxn} & \underline{1}_{nxn} \\ -\underline{\Omega}^2 & -(\hat{\underline{D}}+\hat{\underline{G}}) \end{bmatrix} \tag{5.22}$$

and

$$\underline{B} = \begin{bmatrix} \underline{0} \\ \hat{\underline{B}} \end{bmatrix} \tag{5.23}$$

Then the gravity gradient term $\underline{W}_g$ consists of a constant and a state dependent term:

$$\underline{W}_g\underline{d}_g = \begin{bmatrix} \underline{0}_{nx1} \\ \hat{\underline{g}}_{Go} \end{bmatrix} + \begin{bmatrix} \underline{0}_{nxn} & \underline{0}_{nxn} \\ \hat{\underline{G}}_{Gr} & \underline{0}_{nxn} \end{bmatrix} \underline{x} \tag{5.24}$$

The numerical values for $\hat{\underline{g}}_{Go}$ and $\hat{\underline{G}}_{Gr}$ are given in Table 5.1.

5.2 Solar Radiation Pressure

The calculation of torques on ZSAT due to solar radiation pressure is a complex task. The spacecraft contains several asymmetric substructures among which is the antenna, which has a difficult geometry and a transmissitivity to solar radiation which varies during its daily rotation. In addition, the effect of the Sun's declination and the degree-per-day march of the Earth about the Sun can introduce further complications. Here it is assumed that the Earth is stationary with respect to the Sun, and that the spacecraft is in an equatorial geostationary orbit, with all effects of

Table 5.1

```
GRAVITY GRADIENT TERMS-
CONSTANT VECTOR AND STATE DEPENDENT MATRIX

   0.0000D+00
   0.0000D+00
   0.0000D+00
  -0.7895D+00
   0.1068D-01
   0.8049D-02
   0.1276D-02
   0.4087D-01
  ~~0.5069D-02~~
   0.3147D+00
   0.8584D+00
  -0.4687D-02
  -0.2478D-02
  -0.5473D-01
  -0.1408D-01
   0.2038D-02
  -0.8378D+01
  -0.6823D-02
   0.4750D-02

   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00
   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00
   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00
   0.0000D+00   0.0000D+00   0.0000D+00   0.4768D-02  -0.2023D-05  -0.4854D-05  -0.4810D-06  -0.2469D-03  -0.4342D-05  -0.1901D-02
   0.0000D+00   0.0000D+00   0.0000D+00  -0.2023D-05   0.6320D-02   0.4425D-02   0.7304D-03   0.7803D-06  -0.3535D-02   0.2438D-04
   0.0000D+00   0.0000D+00   0.0000D+00  -0.2286D-04   0.2605D-02   0.1824D-02   0.3010D-03   0.1462D-05  -0.1457D-02   0.1883D-04
   0.0000D+00   0.0000D+00   0.0000D+00  -0.1776D-05   0.5995D-03   0.4197D-03   0.6928D-04   0.1560D-06  -0.3353D-03   0.2944D-05
   0.0000D+00   0.0000D+00   0.0000D+00  -0.2469D-03   0.5791D-06   0.5834D-06   0.7972D-07   0.1278D-04  -0.4050D-07   0.9840D-04
   0.0000D+00   0.0000D+00   0.0000D+00  -0.1474D-04  -0.4587D-02  -0.3212D-02  -0.5301D-03   0.2730D-06   0.2565D-02  -0.1123D-04
   0.0000D+00   0.0000D+00   0.0000D+00  -0.1901D-02   0.3815D-04   0.2808D-04   0.4507D-05   0.9840D-04  -0.1916D-04   0.7577D-03
   0.0000D+00   0.0000D+00   0.0000D+00  -0.5185D-02  -0.1795D-05   0.2480D-05   0.6133D-07   0.2684D-03   0.6956D-05   0.2067D-02
   0.0000D+00   0.0000D+00   0.0000D+00   0.5796D-04   0.2999D-02   0.2100D-02   0.3466D-03  -0.2680D-05  -0.1677D-02  -0.1192D-04
   0.0000D+00   0.0000D+00   0.0000D+00  -0.3533D-04  -0.5086D-02  -0.3561D-02  -0.5878D-03   0.1285D-05   0.2845D-02  -0.4887D-05
   0.0000D+00   0.0000D+00   0.0000D+00  -0.2517D-04  -0.3597D-01  -0.2519D-01  -0.4157D-02  -0.2542D-05   0.2012D-01  -0.1241D-03
   0.0000D+00   0.0000D+00   0.0000D+00  -0.6389D-04  -0.1506D-01  -0.1054D-01  -0.1740D-02   0.1698D-05   0.8423D-02  -0.3070D-04
   0.0000D+00   0.0000D+00   0.0000D+00  -0.7001D-04  -0.5835D-02  -0.4086D-02  -0.6744D-03   0.3001D-05   0.3264D-02   0.6143D-05
   0.0000D+00   0.0000D+00   0.0000D+00   0.5060D-01  -0.7450D-03  -0.5581D-03  -0.8872D-04  -0.2620D-02   0.3586D-03  -0.2017D-01
   0.0000D+00   0.0000D+00   0.0000D+00   0.1073D-04  -0.3082D-02  -0.2158D-02  -0.3562D-03  -0.8851D-06   0.1724D-02  -0.1577D-04
   0.0000D+00   0.0000D+00   0.0000D+00  -0.1053D-04   0.1837D-02   0.1286D-02   0.2123D-03   0.7413D-06  -0.1027D-02   0.1105D-04

   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00
   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00
   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00   0.0000D+00
  -0.5185D-02   0.4645D-05  -0.7377D-05  -0.2270D-04  -0.6953D-04  -0.6281D-04   0.5060D-01   0.9527D-05  -0.9721D-05
   0.3488D-05  -0.2393D-02  -0.2260D-02  -0.3572D-01  -0.1563D-01  -0.5108D-02  -0.7881D-03  -0.3204D-02   0.1918D-02
   0.2538D-04  -0.9863D-03  -0.9312D-03  -0.1472D-01  -0.6441D-02  -0.2105D-02  -0.5585D-03  -0.1321D-02   0.7907D-03
   0.2053D-05  -0.2270D-03  -0.2143D-03  -0.3388D-02  -0.1482D-02  -0.4844D-03  -0.9156D-04  -0.3039D-03   0.1820D-03
   0.2684D-03  -0.4201D-06   0.2123D-06  -0.1506D-05   0.2426D-05   0.2869D-05  -0.2620D-02  -0.7337D-06   0.6473D-06
   0.1509D-04   0.1737D-02   0.1640D-02   0.2593D-01   0.1134D-01   0.3707D-02   0.3999D-03   0.2325D-02  -0.1392D-02
   0.2067D-02  -0.1599D-04  -0.1041D-04  -0.2020D-03  -0.6463D-04  -0.5141D-05  -0.2017D-01  -0.2273D-04   0.1521D-04
   0.5637D-02  -0.3538D-05   0.9450D-05   0.4726D-04   0.8548D-04   0.7153D-04  -0.5502D-01  -0.8334D-05   0.9357D-05
  -0.6241D-04  -0.1135D-02  -0.1072D-02  -0.1695D-01  -0.7416D-02  -0.2424D-02   0.2513D-03  -0.1520D-02   0.9101D-03
   0.3738D-04   0.1926D-02   0.1818D-02   0.2875D-01   0.1258D-01   0.4111D-02   0.2421D-03   0.2578D-02  -0.1544D-02
   0.2003D-04   0.1362D-01   0.1286D-01   0.2033D+00   0.8896D-01   0.2907D-01   0.4097D-02   0.1824D-01  -0.1092D-01
   0.6640D-04   0.5701D-02   0.5384D-02   0.8512D-01   0.3724D-01   0.1217D-01   0.1149D-02   0.7634D-02  -0.4571D-02
   0.7493D-04   0.2209D-02   0.2086D-02   0.3298D-01   0.1443D-01   0.4717D-02  -0.3509D-04   0.2958D-02  -0.1771D-02
  -0.5502D-01   0.3232D-03   0.1804D-03   0.3849D-02   0.1051D-02  -0.8187D-04   0.5370D+00   0.4679D-03  -0.3228D-03
  -0.1230D-04   0.1167D-02   0.1102D-02   0.1742D-01   0.7622D-02   0.2491D-02   0.4878D-03   0.1563D-02  -0.9356D-03
   0.1182D-04  -0.6955D-03  -0.6567D-03  -0.1038D-01  -0.4542D-02  -0.1484D-02  -0.3345D-03  -0.9313D-03   0.5576D-03
```

the Earth's shadow neglected, as is the declination of the Sun. The most significant portions of the spacecraft from the viewpoint of solar radiation calculations are the reflector antenna and the solar array; only these two components are included in the following. *All torques are calculated in the spacecraft body frame* F_b.

The solar radiation pressure on the array is given by

$$\underline{g}_{SA} \triangleq \underline{\rho}_A^x \underline{f}_A \tag{5.25}$$

where $\underline{\rho}_A$ is the center of pressure of the array,

$$\underline{\rho}_A = \begin{bmatrix} 0 \\ \rho_{A2} \\ \rho_{A3} \end{bmatrix} = \begin{bmatrix} 0 \\ 21.1 \\ 1.7 \end{bmatrix} \tag{5.26}$$

and $\underline{f}_A$ is the solar radiation force on the array,

$$\underline{f}_A = -\underline{n}_{a1} P_S A_A \tag{5.27}$$

where P_S is the solar radiation pressure constant ($4.5 \times 10^{-6} N/m^2$), A_A is the area of the solar array ($152\ m^2$) and $\underline{n}_{a1}$ is a unit vector normal to the array:

$$-\underline{n}_{a1} = \begin{bmatrix} -\cos\eta \\ 0 \\ \sin\eta \end{bmatrix} \tag{5.28}$$

The orbital anomaly η, illustrated in Fig. 5.2 , is not to be confused with the modal coordinate η_i used in previous sections of this report. The skew symmetric 3 x 3 matrix $\underline{\rho}_A^x$, when post multiplied by $\underline{f}_A$, produces the components of the vector cross product of $\underline{\rho}_A$ with $\underline{f}_A$. Here we have

$$\underline{\rho}_A^x = \begin{bmatrix} 0 & -\rho_{A3} & \rho_{A2} \\ \rho_{A3} & 0 & 0 \\ -\rho_{A2} & 0 & 0 \end{bmatrix} \tag{5.29}$$

so that for the array

$$\underline{g}_{SA} = P_S A_A \underline{\rho}_A^x \underline{n}_{a1}$$

$$= \begin{bmatrix} \rho_{A2} \sin\eta \\ -\rho_{A3} \cos\eta \\ \rho_{A2} \cos\eta \end{bmatrix} P_S A_A \tag{5.30}$$

We now turn our attention to the reflector. The center of pressure of the reflector is

$$\underline{\rho}_R = \begin{bmatrix} 0 \\ \rho_{R2} \\ \rho_{R3} \end{bmatrix} = \begin{bmatrix} 0 \\ -23.7 \\ -43.7 \end{bmatrix} \tag{5.31}$$

The reflector has a plan diameter of D = 44.4m and its focal length is f = 43.7m. The projected planar area of the reflector is

$$A_{R1} \triangleq \frac{\pi D^2}{4} \tag{5.32}$$

and the projected area of the sideview (or top and bottom) can be shown to be

$$A_{R3} \triangleq \frac{D^3}{24f} \tag{5.33}$$

Fig. 5.2 specifies the labeling of the solar forces on the spacecraft in the body frame. Since the inclination of the equatorial plane with respect to the ecliptic is neglected f_5 and f_6, which

face into and out of the page, are assumed to be zero in the present model.

We can now use the solar array normal $\underline{n}_{a1}$ to specify the sun direction in the calculation of the solar forces f_1 through f_4:

$$\underline{f}_{R1} = \underline{n}_{a1}(\underline{n}_{a1}^T\underline{n}_{R1})(0.05)P_S A_{R1} \tag{5.34}$$

$$\underline{f}_{R2} = \underline{n}_{a1}(\underline{n}_{a1}^T\underline{n}_{R2})(0.05)P_S A_{R1} \tag{5.35}$$

$$\underline{f}_{R3} = \underline{n}_{a1}(\underline{n}_{a1}^T\underline{n}_{R3})P_S A_{R3} \tag{5.36}$$

$$\underline{f}_{R4} = \underline{n}_{a1}(\underline{n}_{a1}^T\underline{n}_{R4})P_S A_{R3} \tag{5.37}$$

The reflector is assumed to have a transmissitivity in the $\underline{f}_{R1}$ and $\underline{f}_{R2}$ direction of 0.05, hence the introduction of this factor into (5.34) and (5.35). The reflector is assumed opaque in the $\underline{f}_{R3}$ and $\underline{f}_{R4}$ directions. The $\underline{n}_{Ri}$ are unit normal vectors in the four directions, and in the spacecraft body frame these are

$$\underline{n}_{R1} = \begin{bmatrix} 0 \\ \sin\gamma \\ \cos\gamma \end{bmatrix}$$

$$= -\underline{n}_{R2} \tag{5.38}$$

where γ is the inclination of the reflector to the Z_b axis (14.2°), and

$$\underline{n}_{R3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= -\underline{n}_{R4} \tag{5.39}$$

We can now rewrite (5.34) through (5.37) as

$$\underline{f}_{R1} = (0.05) \begin{bmatrix} \cos\eta \\ 0 \\ -\sin\eta \end{bmatrix} (-\sin\eta\cos\gamma)P_S A_{R1} \tag{5.40}$$

$$\underline{f}_{R2} = -\underline{f}_{R1} \tag{5.41}$$

$$\underline{f}_{R3} = \begin{bmatrix} \cos\eta \\ 0 \\ -\sin\eta \end{bmatrix} (\cos\eta)P_S A_{R3} \tag{5.42}$$

$$\underline{f}_{R4} = -\underline{f}_{R3} \tag{5.43}$$

We have now found the solar pressure forces in the reflector. To calculate the resulting scalar torques we must find the cross product of these forces with the vector to the center of pressure (5.31). As only two surfaces of the reflector are exposed to the Sun at any time in ZSAT's orbit, we can most conveniently calculate the solar torques in the four quadrants of the orbit:

$\eta = 0 - \frac{\pi}{2}$:

$$\underline{g}_{SR} = \underline{\rho}_R^{\times}\underline{f}_{R2} + \underline{\rho}_R^{\times}\underline{f}_{R3} \tag{5.44}$$

$\eta = \frac{\pi}{2} - \pi$:

$$\underline{g}_{SR} = \underline{\rho}_R^{\times}\underline{f}_{R2} + \underline{\rho}_R^{\times}\underline{f}_{R4} \tag{5.45}$$

$\eta = \pi - \frac{3\pi}{2}$:

$$\underline{g}_{SR} = \underline{\rho}_R^{\times}\underline{f}_{R1} + \underline{\rho}_R^{\times}\underline{f}_{R4} \tag{5.46}$$

$\eta = \frac{3\pi}{2} - 2\pi$:

$$\underline{g}_{SR} = \underline{\rho}_R^{\times}\underline{f}_{R1} + \underline{\rho}_R^{\times}\underline{f}_{R3} \tag{5.47}$$

Substituting (5.31) and (5.40) through (5.43) into (5.44) through

(5.47) and defining

$$\underline{g}_1 \triangleq 0.05 \begin{bmatrix} -\rho_{R2}\sin^2\eta \\ \rho_{R3}\sin\eta\ \cos\eta \\ -\rho_{R2}\sin\eta\ \cos\eta \end{bmatrix} P_S A_{R1}\cos\gamma \tag{5.48}$$

$$\underline{g}_2 \triangleq \begin{bmatrix} -\rho_{R2}\sin\eta\ \cos\eta \\ \rho_{R3}\cos^2\eta \\ -\rho_{R2}\cos^2\eta \end{bmatrix} P_S A_{R3} \tag{5.49}$$

one obtains

$\eta = 0 - \frac{\pi}{2}$:

$$\underline{g}_{SR} = \underline{g}_1 + \underline{g}_2 \tag{5.50}$$

$\eta = \frac{\pi}{2} - \pi$:

$$\underline{g}_{SR} = \underline{g}_1 - \underline{g}_2 \tag{5.51}$$

$\eta = \pi - \frac{3\pi}{2}$:

$$\underline{g}_{SR} = -\underline{g}_1 - \underline{g}_2 \tag{5.52}$$

$\eta = \frac{3\pi}{2} - 2\pi$

$$\underline{g}_{SR} = -\underline{g}_1 + \underline{g}_2 \tag{5.53}$$

The solar torques experienced by the reflector and the array can now be added

$$\underline{g}_S \triangleq \underline{g}_{SA} + \underline{g}_{SR} \tag{5.54}$$

and included in the equations of motion (5.21) by premultiplying (5.54) by the appropriate partitions of the modal matrix, as was done in (5.18):

$$\hat{\underline{g}}_s \triangleq \begin{bmatrix} \underline{E}_r^T \\ \underline{E}_{re}^T \end{bmatrix} \begin{bmatrix} \underline{0}_{3x1} \\ \underline{g}_s \end{bmatrix} \tag{5.55}$$

so that

$$\hat{\underline{W}}_s \underline{d}_s = \begin{bmatrix} \underline{0}_{nx1} \\ \hat{\underline{g}}_s \end{bmatrix} \tag{5.56}$$

with $\hat{\underline{W}}_s$ a constant matrix and $\underline{d}_s$ a vector dependent on the orbital anomaly. The numerical values for $\hat{\underline{w}}_s$ (the lower half of $\hat{\underline{W}}_s$) are given in Table (5.2), with $\underline{d}_s$ given by

$$\underline{d}_s = \begin{bmatrix} \sin\eta \\ \cos\eta \\ \cos\eta \\ \begin{bmatrix} \sin^2\eta \\ \sin\eta\cos\eta \\ \sin\eta\cos\eta \end{bmatrix} \operatorname{sgn}(\sin\eta) \\ \begin{bmatrix} \sin\eta\cos\eta \\ \cos^2\eta \\ \cos^2\eta \end{bmatrix} \operatorname{sgn}(\cos\eta) \end{bmatrix} \tag{5.57}$$

Here sgn $(\cdot)$ = +1 for positive values of the argument and -1 for negative values of the argument.

6. CONCLUDING REMARKS

This report is something of a *potpourri* of topics, some of which are loose ends from [1]. The major method of model reduction in [1], cost-decoupled coordinates, is extended in this

SOLAR DISTURBANCE MATRIX

Table 5.2

0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.1312D-04	0.0000D+00	0.0000D+00	-0.4715D-06	0.0000D+00	0.0000D+00	0.8580D-05	0.0000D+00	0.0000D+00
-0.5569D-08	-0.1217D-05	0.0000D+00	0.2001D-09	0.1001D-05	0.0000D+00	-0.3641D-08	-0.1821D-04	0.0000D+00
-0.1336D-07	-0.8524D-06	0.2891D-04	0.4799D-09	0.7008D-06	-0.1038D-05	-0.8733D-08	-0.1275D-04	0.1890D-04
-0.1324D-08	-0.1407D-06	0.2080D-05	0.4756D-10	0.1157D-06	-0.7471D-07	-0.8655D-09	-0.2105D-05	0.1359D-05
-0.6795D-06	-0.1301D-09	0.3194D-08	0.2441D-07	0.1070D-09	-0.1147D-09	-0.4442D-06	-0.1947D-08	0.2088D-08
-0.1195D-07	0.6809D-06	0.1670D-04	0.4294D-09	-0.5598D-06	-0.5998D-06	-0.7813D-08	0.1019D-04	0.1092D-04
-0.5232D-05	-0.4541D-08	-0.2187D-06	0.1879D-06	0.3733D-08	0.7854D-08	-0.3420D-05	-0.6793D-07	-0.1429D-06
-0.1427D-04	-0.2480D-09	0.8388D-07	0.5126D-06	0.2039D-09	-0.3013D-08	-0.9329D-05	-0.3710D-08	0.5483D-07
0.1279D-07	0.4609D-06	-0.8561D-04	-0.4593D-09	-0.3789D-06	0.3075D-05	0.8358D-08	0.6896D-05	-0.5596D-04
-0.2031D-07	0.4352D-06	0.4488D-04	0.7294D-09	-0.3578D-06	-0.1612D-05	-0.1327D-07	0.6511D-05	0.2934D-04
-0.6248D-07	0.6881D-05	0.3963D-05	0.2244D-08	-0.5657D-05	-0.1424D-06	-0.4084D-07	0.1029D-03	0.2591D-05
-0.1914D-06	0.3010D-05	-0.9052D-05	0.6874D-08	-0.2475D-05	0.3252D-06	-0.1251D-06	0.4504D-04	-0.5917D-05
-0.1729D-06	0.9838D-06	0.1155D-04	0.6211D-08	-0.8088D-06	-0.4151D-06	-0.1130D-06	0.1472D-04	0.7553D-05
0.1393D-03	0.1477D-06	-0.6850D-06	-0.5003D-05	-0.1214D-06	0.2460D-07	0.9104D-04	0.2209D-05	-0.4477D-06
0.2622D-07	0.6172D-06	-0.1937D-05	-0.9420D-09	-0.5074D-06	0.6957D-07	0.1714D-07	0.9234D-05	-0.1266D-05
-0.2676D-07	-0.3695D-06	0.1295D-05	0.9612D-09	0.3038D-06	-0.4652D-07	-0.1749D-07	-0.5528D-05	0.8466D-06

report to discrete-time systems in Section 2. It is shown that the method is basically unchanged from the continuous-time case. A second extension of cost-decoupled coordinates performed in this report is to systems stabilized by an observer. A measure of controller quality comparable to the CQI of [1] is developed here for observer stabilized systems, and it is shown in a numerical case study using ZSAT that an observer has a deleterious effect on the extent to which a model can be reasonably reduced.

A derivation of a version of modal cost analysis is presented in Section 4. It is interesting because it uses frequency domain considerations rather than exclusively time domain procedures as in previous derivations. Of greater interest, however, is that this version of modal cost analysis is valid for 'less ideal' inputs than that in [1] which was predicated upon impulsive inputs.

In the final section of this report a discussion of the two main environmental disturbances on ZSAT is presented. These are gravity gradient torque, which is shown to consist of a constant part and an attitude dependent part, and solar radiation torque, which varies with orbital anomaly. The numerical values of these disturbances are calculated and presented as terms to be included on the right hand side of the equations of motion.

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APPENDIX A

Gregory Internal Balancing Transformation

The concept of balanced realizations and balancing transformations as proposed by Moore [5] and others is briefly reviewed in [1]. The essential idea is that it is possible to transform a system into a set of coordinates such that the contribution to controllability of each state variable is equal to its contribution to observability. Gregory [6] has found a general balancing transformation for mechanical systems in modal coordinates and has shown that a simplified transformation resulting from the assumption of light damping results in an approximately internally balanced system if the damping is in fact light. Ranking of state components due to this internal balancing is distinct from (although related to) ranking via modal cost analysis.

The system in modal coordiantes with which we deal consists of N second-order systems of the form (4.1)

$$\ddot{\eta}_i + 2\zeta_i\omega_i\dot{\eta}_i + \omega_i^2\eta_i = \underline{\hat{b}}_i^T\underline{u} \qquad i = 1,\ldots,N \tag{A.1}$$

where the quantities are defined following (4.1). These can be rewritten as N systems of the form

$$\begin{bmatrix} \dot{\eta}_i \\ \ddot{\eta}_i \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta_i\omega_i \end{bmatrix} \begin{bmatrix} \eta_i \\ \dot{\eta}_i \end{bmatrix} + \begin{bmatrix} \underline{0} \\ \underline{\hat{b}}_i^T \end{bmatrix} \underline{u} \tag{A.2}$$

Each such system can also be assigned an output equation of the form

$$\underline{y}_i = [\underline{\hat{c}}_i \quad \underline{\hat{\dot{c}}}_i] \begin{bmatrix} \eta_i \\ \dot{\eta}_i \end{bmatrix} \tag{A.3}$$

The following transformation casts each subsystem of the form (A.2), (A.3) into internally balanced coordinates $\begin{bmatrix} \delta_{1i} \\ \delta_{2i} \end{bmatrix}$:

$$\begin{bmatrix} \eta_i \\ \dot{\eta}_i \end{bmatrix} = \left[\frac{\hat{\underline{b}}_i^T \hat{\underline{b}}_i}{4\zeta_i\omega_i}\right]^{\frac{1}{2}} \begin{bmatrix} \omega_i^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_{2i} & \beta_{1i} \\ \beta_{1i} & -\beta_{2i} \end{bmatrix} \begin{bmatrix} \sigma_{1i}^{-1} & 0 \\ 0 & \sigma_{2i}^{-1} \end{bmatrix} \begin{bmatrix} \delta_{1i} \\ \delta_{2i} \end{bmatrix} \tag{A.4}$$

where

$$\sigma_{1i}^2,\ \sigma_{21}^2 \triangleq \frac{(\hat{\underline{b}}_i^T \underline{b}_i)^{\frac{1}{2}}}{4\zeta_i\omega_i} [\dot{\hat{\underline{c}}}_i^T \dot{\hat{\underline{c}}}_i + \omega_i^{-2} \hat{\underline{c}}_i^T \hat{\underline{c}}_i [1 - 2\zeta_i\gamma_i \pm 2\zeta_i(1 + \gamma_i^2)^{\frac{1}{2}}]^{\frac{1}{2}}] \qquad \text{(A.5), (A.6)}$$

$$\beta_{1i},\ \beta_{2i} = \left[\frac{1 \pm \alpha_i}{2}\right]^{\frac{1}{2}} \qquad \text{(A.7),(A.8)}$$

$$\alpha_i \triangleq \operatorname{sgn}(\gamma_i) \left[\frac{\gamma_i^2}{1 + \gamma_i^2}\right]^{\frac{1}{2}} \tag{A.9}$$

and

$$\gamma_i \triangleq \frac{\omega_i \hat{\underline{c}}_i^T \dot{\hat{\underline{c}}}_i}{\hat{\underline{c}}_i^T \hat{\underline{c}}_i} - \zeta_i \tag{A.10}$$

The resulting internally balanced version of (A.2), (A.3) is

$$\begin{bmatrix} \dot{\delta}_{1i} \\ \dot{\delta}_{2i} \end{bmatrix} = \omega_i \begin{bmatrix} (1 + 2\zeta_i\beta_{1i}\beta_{2i})\sigma_{2i}/\sigma_{1i} & -2\zeta_i\beta_{1i}^2 \\ -2\zeta_i\beta_{2i}^2 & -(1 - 2\zeta_i\beta_{1i}\beta_{2i})\sigma_{1i}/\sigma_{2i} \end{bmatrix} \begin{bmatrix} \delta_{1i} \\ \delta_{2i} \end{bmatrix} +$$

$$+ \left[\frac{4\zeta_i\omega_i}{\hat{\underline{b}}_i^T\hat{\underline{b}}_i}\right]^{\frac{1}{2}} \begin{bmatrix} -\sigma_{2i}\beta_{2i}\hat{\underline{b}}_i^T \\ \sigma_{1i}\beta_{1i}\hat{\underline{b}}_i^T \end{bmatrix} \underline{u} \tag{A.11}$$

$$y_i = \left[\frac{\hat{\underline{b}}_i^T\hat{\underline{b}}_i}{4\zeta_i\omega_i}\right]^{\frac{1}{2}} \begin{bmatrix} \dfrac{-\hat{\dot{\underline{c}}}_i\beta_{2i} + \omega_i^{-1}\underline{c}_i\beta_{1i}}{\sigma_{2i}} , & \dfrac{\hat{\dot{\underline{c}}}_i\beta_{1i} + \omega_i^{-1}\hat{\underline{c}}_i\beta_{2i}}{\sigma_{1i}} \end{bmatrix} \begin{bmatrix} \delta_{1i} \\ \delta_{2i} \end{bmatrix} \tag{A.12}$$

Now if the assumption of light damping is valid ($\zeta_i << 1$) the unwieldy balancing transformation and resulting system became far more tractable. With

$$\sigma_{1i} \doteq \sigma_{2i} \doteq \frac{(\hat{\underline{b}}_i^T\hat{\underline{b}}_i)^{\frac{1}{2}}}{4\zeta_i\omega_i} \left[\hat{\dot{\underline{c}}}_i^T\hat{\dot{\underline{c}}}_i + \omega_i^{-2}\hat{\underline{c}}_i^T\hat{\underline{c}}_i\right]^{\frac{1}{2}} \tag{A.13}$$

(A.4) becomes

$$\begin{bmatrix} \eta_i \\ \dot{\eta}_i \end{bmatrix} = \left[\frac{\hat{\underline{b}}_i^T\hat{\underline{b}}_i}{\hat{\dot{\underline{c}}}_i^T\hat{\dot{\underline{c}}}_i + \omega_i^{-2}\hat{\underline{c}}_i^T\hat{\underline{c}}_i}\right]^{\frac{1}{4}} \begin{bmatrix} \omega_i^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_{2i} & \beta_{1i} \\ \beta_{1i} & -\beta_{2i} \end{bmatrix} \begin{bmatrix} \delta_{1i} \\ \delta_{2i} \end{bmatrix} \tag{A.14}$$

and the resulting system is

$$\begin{bmatrix} \dot{\delta}_{1i} \\ \dot{\delta}_{2i} \end{bmatrix} = \omega_i \begin{bmatrix} (1 + 2\zeta_i\beta_{1i}\beta_{2i}) & -2\zeta_i\beta_{2i}^2 \\ -2\zeta_i\beta_{1i}^2 & -(1 - 2\zeta_i\beta_{1i}\beta_{2i}) \end{bmatrix} \begin{bmatrix} \delta_{1i} \\ \delta_{2i} \end{bmatrix} +$$

$$+ \left[\frac{\hat{\underline{b}}_i^T\hat{\underline{b}}_i}{\hat{\dot{\underline{c}}}_i^T\underline{c}_i + \omega_i^{-2}\hat{\underline{c}}_i^T\hat{\underline{c}}_i}\right]^{-\frac{1}{4}} \begin{bmatrix} -\beta_{2i}\hat{\underline{b}}_i^T \\ \beta_{1i}\hat{\underline{b}}_i^T \end{bmatrix} \underline{u} \tag{A.15}$$

$$y_i = \left[\frac{\hat{\underline{b}}_i^T \hat{\underline{b}}_i}{\hat{\dot{\underline{c}}}_i^T \hat{\dot{\underline{c}}}_i + \omega_i^{-2} \hat{\underline{c}}_i^T \hat{\underline{c}}_i} \right]^{\frac{1}{4}} \left[-\hat{\dot{\underline{c}}}_i \beta_{2i} + \omega_i^{-1} \hat{\underline{c}}_i \beta_{1i} \;,\; \hat{\dot{\underline{c}}}_i \beta_{1i} + \omega_i^{-1} \hat{\underline{c}}_i \beta_{2i} \right] \begin{bmatrix} \delta_{1i} \\ \delta_{2i} \end{bmatrix} \tag{A.16}$$

The main point here is that if the light damping assumption is valid, system (A.15-16) is approximately internally balanced.

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--Development of system order reduction techniques applicable to third generation spacecraft

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