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2010-2

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Bank of Canada Discussion Paper 2010-2

March 2010

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ISSN 1914-0568

Acknowledgements

We thank Jun Yang, Matt Davison, Scott Hendry, Stéphane Lavoie, Tiago Rubin, Toni Gravelle, Virginie Traclet, and Yuliya Romanyuk for comments made on an earlier version of this paper.

Abstract

Security prices contain valuable information that can be used to make a wide variety of economic decisions. To extract this information, a model is required that relates market prices to the desired information, and that ideally can be implemented using timely and low-cost methods. The authors explore two models applied to option prices to extract the risk-neutral probability density function (R-PDF) of the expected Can$/US$ exchange rate. Each of the two models extends the Black-Scholes model by using a mixture of two lognormals for the terminal distribution, instead of a single lognormal: one mixed lognormal imposes a specific stochastic process for the underlying asset, and the other does not. The contribution of the paper is to propose a simple methodology to build R-PDFs with a constant time to maturity in the absence of option prices for the maturity of interest. The authors apply this methodology and find that the two models provide similar results for the degree of uncertainty (i.e., the variance) surrounding the future level of the exchange rate, but differ on the likely direction of the exchange rate movements (i.e., the skewness).

JEL classification: C00, C02, G13
Bank classification: Exchange rates; Econometric and statistical methods; Financial markets

Résumé

Les prix des titres renferment une information très utile pouvant servir à des décisions économiques multiples. Cette information est extraite au moyen d’un modèle qui met en relation les prix du marché avec les renseignements voulus et dont la mise en œuvre repose idéalement sur des méthodes rapides et peu coûteuses. Les auteurs comparent deux modèles appliqués à des prix d’options afin de trouver la densité de probabilité neutre à l’égard du risque (DPNR) correspondant au taux de change anticipé entre le dollar canadien et le dollar américain. Chaque modèle développe l’équation de Black et Scholes en combinant deux distributions lognormales pour obtenir la distribution finale, au lieu d’employer une distribution lognormale simple. L’une des distributions combinées impose à la trajectoire de l’actif sous-jacent un processus stochastique, contrairement à l’autre. L’originalité de l’étude réside dans le fait qu’elle propose, en l’absence de prix d’options pour l’échéance considérée, une méthode d’élaboration simplifiée des DPNR dans laquelle l’intervalle avant échéance est constant. L’application de cette méthode révèle que les deux modèles fournissent des résultats similaires pour le degré d’incertitude (variance) qui entoure le taux de change futur, mais différents pour le sens probable des mouvements de change (asymétrie).

Classification JEL : C00, C02, G13
Classification de la Banque : Taux de change; Méthodes économétriques et statistiques; Marchés financiers

1 Introduction

Various agents in the economy are interested in obtaining information on the market's view of the Canada/U.S. exchange rate (Can$/US$) in the near future for different reasons. Firms exposed to currency risk can use this information to determine the level of coverage they need in their hedging programs, the central bank can use this information as an input in the formulation of monetary policy, and, in general, agents can use this information to determine their desired portfolio allocation.

There are a number of approaches to extract information from market prices. One approach is to estimate time-series models and treat their forecasts as representing the economic agents' expectations. Others use forward prices as expectations of future prices of the underlying asset (e.g., Fama 1984). And yet others use option prices to potentially extract a much richer set of information than the other two approaches by recovering the entire distribution of the underlying asset at some point in the near future. The approaches that use derivative prices have the advantage that they refer explicitly to the future outcome of the underlying asset, whereas the first approach uses historical data which, by definition, is based on the past.

In this paper, we obtain information on the market's view of the Can$/US$ exchange rate in the near future (e.g., 45 days ahead) by recovering the *risk-neutral probability density function* (R-PDF) from the price of European options. This allows us to extract the distribution of the underlying asset, since we observe a cross-section of prices for the exchange rate for a constant maturity.

We consider two models for the terminal distribution of prices for the underlying asset. The first, *a mixed lognormal model*, assumes a flexible terminal distribution for the underlying asset and makes no assumptions on its path from the time the option originated until its expiry date. The second, *a shifted mixed lognormal model*, assumes a specific stochastic process for the underlying asset that results in a mixed lognormal terminal distribution. Both of these models are more flexible than the widely used Black-Scholes model, in that their terminal distributions can better capture some of the stylized facts observed in financial data, such as leptokurtic series ("fat tails") and high skewness.

We find that both models provide similar results for the degree of uncertainty (i.e., variance) surrounding the future level of the exchange rate. However, the results with respect to the likely direction of the exchange rate movements (i.e., skewness) across models are not robust.[1] The standard mixed lognormal model exhibits more intuitive results, because it allows for periods of appreciation and

[1] Unless otherwise specified, when we refer to the mean, variance, and skewness, we refer to the model-implied risk-neutral moments, and not the moments of the real-world probability density function.

depreciation, whereas the shifted lognormal model does not allow for depreciations in the exchange rate, because it is not able to produce R-PDFs with negative skewness.

The main contribution of this paper is to propose a simple methodology to build R-PDFs with a constant time to maturity. This methodology is applied to both models and consists of interpolating the volatility curve between available contracts.

This paper is organized as follows. Section 2 briefly summarizes the theory supporting these models. Section 3 describes an application of the methodology to European options of the Can$/US$ exchange rate. Section 4 discusses the results. Section 5 concludes.

2 Theory

2.1 Standard-model approach

Perhaps the most widely used model to describe the evolution of the price S_t of a security is the geometric Brownian motion,

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1}$$

where dS_t is the small (infinitesimal) change in the price of the underlying asset S_t in a small interval of time dt, μ is the expected rate of return, σ is the volatility of the underlying asset, and dW_t is the Weiner process introduced to add "noise" or variability to the path of S_t. Using this model, we can specify many possible paths of the asset value, thus leading to a distribution of values at a set moment in time.

We can then transform Equation (1) by applying Itô's rule on $\log S_t$ to obtain

$$d\log S_t = (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t. \tag{2}$$

Since Equation (2) no longer contains the term S_t on the right-hand side, and since μ and σ are constant, we can conclude that $\log S_t$ is normally distributed and that, therefore, S_t is lognormally distributed, i.e.,

$$\log S_T \sim N(\ln(S_0) + (\mu - \frac{\sigma^2}{2})T, \sigma^2 T). \tag{3}$$

The modelling direction of the standard model presented above is as follows. First, we use a stochastic process that reasonably describes the evolution of the underlying asset. Next, we derive the risk-neutral distribution corresponding to the stochastic process (e.g., lognormal in the case of Black-Scholes). Finally, we calculate option prices as the discounted expected value of the payoff under the risk-neutral probability distribution.

STANDARD-MODEL DIRECTION: `Process` $\rightarrow$ `distribution` $\rightarrow$ `option prices`

2.2 Inverse-model approach

An alternative modelling direction, sometimes referred to as the inverse problem, can be summarized as follows. First, we observe option prices to learn about the possible shape of the implied risk-neutral distribution. Next, we use this information to determine a stochastic process that could result in the observed distribution.

INVERSE-MODEL DIRECTION: `Option prices` $\rightarrow$ `distribution` $\rightarrow$ `process`

One advantage of this approach over the standard-model approach is that it allows us to learn about the distribution and the stochastic process of the underlying asset, rather than assuming a particular form.

Our approach to capturing market expectations is based on the recent advancements of the inverse-problem literature.

2.3 Models to obtain risk-neutral distributions

There are many models that address the inverse problem and allow us to estimate the implied risk-neutral PDFs from option prices.[2] We classify these models into two broad groups. The first group encompasses the *parametric models*, which assume a specific family of distributions for the terminal distribution of the underlying asset. The second encompasses the *non-parametric models*, which interpolate either the observed option prices or the observed implied volatilities for a constant maturity, and then derive the risk-neutral PDFs as an approximation of the second derivative of the call price with respect to the strike.[3] In this paper, we focus on parametric models, since we find them to be more efficient when automating across an entire sample.[4]

A common extension of the Black-Scholes model uses a mixture of two lognormals for the terminal distribution, rather than a single lognormal. Instead of working with an entire stochastic process, this model simply uses a candidate for the terminal distribution. The mixed lognormal distribution can take many forms, which allows the call pricing model to better approximate observed option prices. We refer

[2]See Mandler (2003) for a broad survey of techniques.

[3]The relationship was discovered by Breeden and Litzenberger (1978).

[4]For papers that study non-parametric techniques, see Jackwerth and Rubinstein (1996), and Jackwerth (2000, 2004).

to this model as the *standard mixed lognormal*. A slightly different model, proposed by Brigo and Mercurio (2002), also assumes a mix of lognormals, but derives the specific stochastic process that would result in the mix distribution. It allows for changing the volatility terms σ_i in each component, but does not allow for modifications to the mean levels. We refer to this model as the *shifted mixed lognormal*. These models are explained in the following subsections.

2.3.1 *Standard mixed lognormal*

For this model, we assume that the terminal distribution is given by a mix of two lognormal distributions. To illustrate the distributional assumption, let X_1 and X_2 be lognormal random variables, and let I be a random variable that takes the value 1 with probability p, and 0 with probability $1-p$. The variable I could represent, for example, whether an interest rate cut is going to occur, or whether the exchange rate regime will change. Thus, I models two future states of the world: one where the underlying asset has value X_1, and one where it has value X_2. We can model this dependence on the future state of the world by setting $X = I \cdot X_1 + (1-I) \cdot X_2$. Thus, when $I = 1, X = X_1$, and when $I = 0$, $X = X_2$. In other words, X is equal to X_1 with probability p, and equal to X_2 with probability $1-p$. The resulting density for X is derived in Appendix C and corresponds to the following:

$$f_X(x) = pf_{X_1}(x) + (1-p)f_{X_2}(x). \tag{4}$$

One advantage of using a mixture of lognormals is that we still have analytical Black-Scholes type equations for the options prices. For example, suppose that the underlying asset given by S_t is an exchange rate. In the traditional Black-Scholes model, under the risk-neutral measure, S_t is lognormally distributed such that

$$\ln(S_T) \sim N(\ln(S_0) + (r - r_f - \frac{\sigma^2}{2})T, \sigma^2 T),$$

where r is the domestic risk-free rate of interest, r_f is the foreign risk-free rate of interest, and σ is the volatility of the exchange rate. Note that this is the same as Equation (3) with μ replaced by $r - r_f$. Let us write $\ln X_i \sim N(\ln(S_{0,i}) + (r - r_f - \frac{\sigma_i^2}{2})T, \sigma_i^2 T)$ for $i = 1, 2$, and let S_T have distribution mixed from X_1 and X_2 with parameter p. Then, we can show that, under this model, the price of the call option is given by[5]

$$C(S_0, K, r, r_f, T) = p \cdot C_{BS}(S_{0,1}, K, r, r_f, T, \sigma_1) + (1-p) \cdot C_{BS}(S_{0,2}, K, r, r_f, T, \sigma_2), \tag{5}$$

[5]See Appendix C.

where C_{BS} denotes the Black-Scholes equation for the call option price:

$$
\begin{aligned}
C_{BS}(S_{0,i}, K, r, r_f, T, \sigma_i) &= S_{0,i}N(d_1) - Ke^{-(r-r_f)T}N(d_2) \\
d_1 &= \frac{\ln(S_{0,i}/K) - ((r-r_f) - \sigma_i^2/2)T}{\sigma_i\sqrt{T}} \\
d_2 &= d_1 - \sigma_i\sqrt{T}.
\end{aligned}
$$

Under the risk-neutral measure, the discounted expected value of S_T should equal the current observed level S_0, which implies that $S_0 = pS_{0,1} + (1-p)S_{0,2}$. Given a set of observed prices $C^{obs}(K_i)$ for varying strikes K_i on a single underlying asset, we perform the following minimization:

$$
\min_{p,S_{0,1},S_{0,2},\sigma_1,\sigma_2} \sum_{i=1}^{n}[C^{th}(K_i) - C^{obs}(K_i)]^2 + [S_0 - (pS_{0,1} + (1-p)S_{0,2})]^2, \tag{6}
$$

where $C^{th}(K_i)$ is the theoretical price of the option using Equation (5). After finding the values for $p, S_{0,1}, S_{0,2}, \sigma_1, \sigma_2$ from Equation (6), we use them to obtain the risk-neutral density using Equation (4).

One critique of the above mix of lognormal models is that we have not determined a stochastic process S_t for the underlying asset. In order to apply the no-arbitrage arguments leading to the risk-neutral pricing formula $C_0 = e^{-rT}E^Q[(S_T - K)^+]$ for a risk-neutral probability measure Q, we need to be able to construct a portfolio consisting of the underlying asset and bonds that replicates the payoff of C_t. This would be possible if, for example, S_t was determined by a *local volatility* model,

$$
dS_t = (r - r_f)S_t dt + \sigma(S_t, t)S_t dW_t,
$$

with $\sigma(\cdot,\cdot)$ being a well-behaved deterministic function. Hence, we interpret the above standard mixed lognormal approach in a slightly different fashion by following Brigo and Mercurio (2002). Suppose that we have two processes, each following a geometric Brownian motion:

$$
\begin{aligned}
dS_t^1 &= \mu_1 S_t^1 dt + \sigma_1 S_t^1 dW_t, \\
dS_t^2 &= \mu_2 S_t^2 dt + \sigma_2 S_t^2 dW_t,
\end{aligned}
$$

and both starting at the same value S_0. Then, S_T^1 and S_T^2 are lognormally distributed. We wish to mix the above two processes to give risk-neutral dynamics of our underlying asset as:

$$
dS_t = (r - r_f)S_t dt + v(t, S_t)S_t dW_t,
$$

with $v(t, S_t)$ chosen so that

$$f_{S_t}(x) = p f_{S_t^1}(x) + (1-p) f_{S_t^2}(x).$$

For a fixed maturity time T, the relationship between this model and the standard mixed lognormal approach would be to set $S_{0,1} = S_0 e^{\mu_1 T}$ and $S_{0,2} = S_0 e^{\mu_2 T}$.

Having knowledge of the entire stochastic process also aids in moving between the real world and risk-neutral world. This is usually done by employing the *Girsanov theorem*, which informally says that we can adjust the Brownian motion by a drift $\lambda(t)$ to get $d\tilde{W}_t = \lambda(t)dt + dW_t$, and that $\tilde{W}_t$ will be a new Brownian motion under a new equivalent probability measure. This is the process by which we can transform the drift term μ to r in the standard Black-Scholes context, where $\lambda = \frac{\mu - r}{\sigma}$ is the *market price of risk*. Without having knowledge of a stochastic process, we cannot directly apply the Girsanov theorem, and so it is not clear how to recover a real-world distribution.[6]

2.3.2 *Shifted mixed lognormal*

This model was introduced by Brigo and Mercurio (2002). In this model, we force $S_{0,1} = S_{0,2} = S_0$ (or, equivalently, $\mu_1 = \mu_2 = \mu$ in the above alternate description), but the underlying asset price at time t, A_t, is *shifted* from S_t so that

$$A_t = A_0 \alpha e^{(r-r_f)t} + S_t \tag{7}$$

for some parameter α.

In order for Equation (7) to hold for $t = 0$, we need to have $A_0 = A_0\alpha + S_0$, and so we set $S_0 = A_0(1-\alpha)$. Here, A_0 is the current observed asset price. When $\alpha = 0$, this reduces to the standard lognormal approach with $S_{0,1} = S_{0,2} = S_0$. The purpose of the shifting in Equation (7) is to move the minimum point of the volatility smile; that is, to add a skew component. Hence, the overall density of the term A_T will be a mixture of shifted lognormals.

In this case, we can explicitly write the dynamics of S_t as

$$dS_t = (r - r_f)S_t dt + \sqrt{\frac{a\sigma_1^2 t p_t^1(S_t) + (1-a)\sigma_2^2 t p_t^2(S_t)}{p_t(S_t)}} S_t dW_t. \tag{8}$$

Here, the individual lognormal densities are

$$p_t^i(x) = \frac{1}{x\sigma_i\sqrt{2\pi t}} \exp\left\{ -\frac{1}{2\sigma_i^2 t}[\ln(x) - \ln(S_0) - ((r - r_f) - \frac{1}{2}\sigma_i^2)t]^2 \right\}, \tag{9}$$

[6] See Appendix E for a comparison of the differences of the implied distribution of the underlying asset under the risk-neutral and real-world probability measures.

and the mixed lognormal density is

$$p_t(x) = ap_t^1(x) + (1-a)p_t^2(x). \tag{10}$$

Note that $p_t(x)$ is a mixed lognormal density, but here we estimate only the terms σ_i, unlike in the previous case, which had terms $S_{0,1}$ and $S_{0,2}$. Brigo and Mercurio (2002) prove that the above stochastic process has the property that the probability density of S_t given S_0 is determined by $p_t(x)$ in Equation (10), and so the density function for A_t is $p_{A_t}(x) = p_t(x - A_0\alpha e^{(r-r_f)t})$.

Using the above distribution of A_t, Brigo and Mercurio (2002) show that, assuming $K > A_0\alpha e^{\mu t}$, call and put prices are each given by a sum of two Black-Scholes type equations:

$$C^{th} = aC_1 + (1-a)C_2, \tag{11}$$
$$P^{th} = aP_1 + (1-a)P_2, \tag{12}$$

where

$$C_i = \mathcal{A}_0 e^{-r_f T} N(d_1^i) - e^{-rT}\mathcal{K}N(d_2^i), \tag{13}$$
$$P_i = -\mathcal{A}_0 e^{-r_f T} N(-d_1^i) + e^{-rT}\mathcal{K}N(-d_2^i), \tag{14}$$

with $\mathcal{A}_0 = A_0(1-\alpha)$, $\mathcal{K} = K - A_0\alpha e^{(r-r_f)T}$ for $i = 1, 2$, and

$$\begin{aligned} d_1^i &= \frac{\ln(\mathcal{A}_0/\mathcal{K}) + (r - r_f + \frac{1}{2}\sigma_1^2)T}{\sigma_i\sqrt{T}}, \\ d_2^i &= d_1^i - \sigma_i\sqrt{T}. \end{aligned}$$

Given a set of n strikes, let $C^{th}(K_i)$ be the theoretical call price for strike K_i given by Equations (11) and (13), and let $C^{obs}(K_i)$ be the market-observed call price. Then, we minimize the sum of squared errors:

$$\min_{\alpha,\sigma_1,\sigma_2,a} \sum_{i=1}^{N} (C^{th}(K_i) - C^{obs}(K_i))^2.$$

Note that we do not have an extra term to help ensure a risk-neutral constraint, since it is already built in.

3 Application to Exchange Rate Options

3.1 Description of the data

The option price data used here are the daily bid and ask end-of-day price published by the Montreal Exchange for European options on the Can$/US$ exchange rate.

The quoted convention for Can$/US$ refers to the number of Canadian dollars required to by one U.S. dollar. This contract is called USX by the Montreal Exchange. We obtain historical data from 2 January 2007 to 18 June 2008.[7] The daily spot exchange rate is obtained from Bloomberg and corresponds to the mid-rate for the end of day.[8]

3.2 Data issues

There are three issues with the data. First, for a few days in our sample there are no prices available (i.e., data are missing). Second, the relation between put and call prices (put-call parity) does not always hold, which implies that there could be arbitrage opportunities. Third, the monotonic relation that would be expected between the price of the option and strikes does not always hold. Bliss and Panigirtzoglou (2002) also document common sources of errors in observed option prices. Similar to our findings, the authors find violations of arbitrage restrictions such as put-call parity, and of a non-monotonic relation of prices with respect to strikes. The authors point to four possible sources of errors: data-entry errors, non-synchronicity from using multiple prices that should be recorded at the same time, liquidity premia to compensate for differences between at-the-money and far out-of-the-money options, and discrepancies due to reporting in discrete increments.

For the first issue, *missing price data*, we simply do not estimate the risk-neutral PDF for days when no prices are recorded.

For the second issue, *breakdown in put-call parity*, we find the domestic and foreign risk-free rates that minimize the squared errors derived from the put-call parity relation. We do this because the breakdown in parity may be the result of not using the appropriate domestic and foreign risk-free rates. Initially, to check this relation, we used yields from U.S. and Canadian treasury bills. Although treasury bill rates seem a good proxy for the risk-free rates (it could be argued that other rates, such as LIBOR, may be appropriate), we obtained very erratic implied volatility curves when using treasury rates, and two different curves for puts and calls. To address this issue, we instead determine the implied risk-free rates from the option data itself by finding the domestic and foreign risk-free rates that minimize the put-call parity errors. That is, we perform the following minimization:

$$\min_{r_f,r_d} \sum_i (C_i - P_i - X_0 \cdot e^{-r_f T} + K_i \cdot e^{-r_d T})^2, \tag{15}$$

where X_0 represents the spot price for the Can$/US$ exchange rate; C_i and P_i represent the call and put prices, respectively; K_i the strike price; and r_f and r_d foreign

[7]Further details on the options contracts are provided in Appendix B.

[8]Historical data on the spot price are obtained from Bloomberg using the command `CADUSD Curncy HP`.

and domestic interest rates, respectively. Nevertheless, after using the implied rates from the minimization, some deviations from the put-call parity relation persist. These remaining deviations, or errors, have been found previously by other authors. For example, Kamara and Miller (1995) observe failures of put-call parity in European option prices, and find that deviations in put-call parity are systematically related to proxies for liquidity risk in the stock and option markets.

Figure 1 shows the implied risk-free rates obtained from Equation (15). The implied rates thus obtained average around 5 per cent, which seems to be reasonable for the initial time period in question, but they do not capture the rapid decline in rates observed as the crisis intensified.[9]

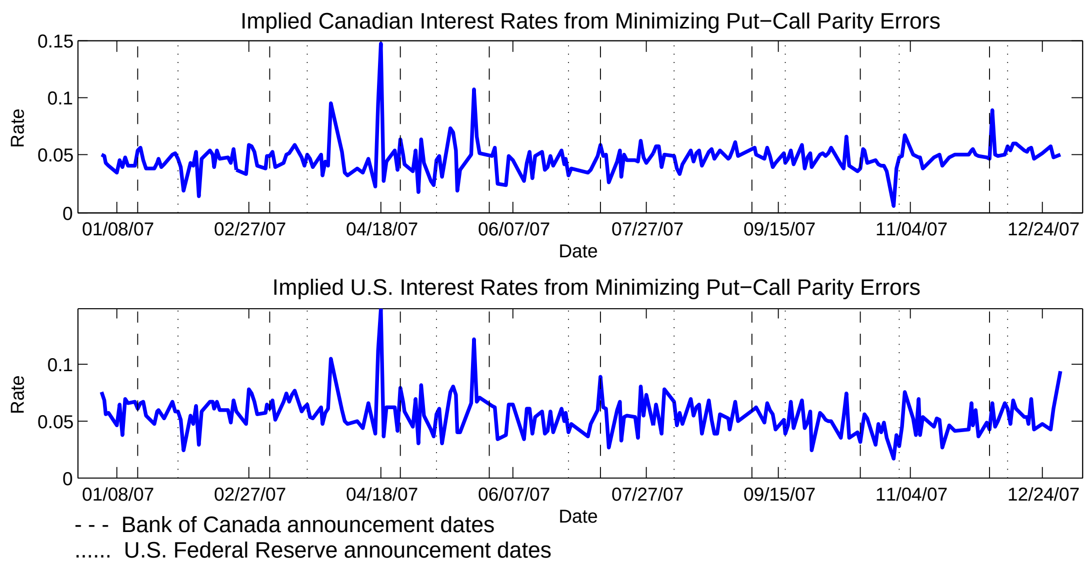

Figure 1: Implied Rates

For the third issue, *monotonic relation between the option price and strikes*, we use only prices that changed with respect to the strikes in a monotonic way. For example, for call prices, the price of the option should decrease as the strike increases, and thus when we observe prices that did not change (or increased) as strikes increased, we exclude them from the sample. By doing this, we are in fact omitting prices for far in- or out-of-the-money contracts that may be determined by the exchange rate when no price information is available.

[9]Treasury rates for both countries were around 5 per cent at the beginning of 2007 and dropped significantly at the end of 2008 as a result of the financial crisis.

3.3 Implementation

Our objective is to obtain a daily time series of risk-neutral PDFs and the associated moments for a constant maturity, where the maturity dictates the time for which the expectations are being extracted. To illustrate our results, we use a 45-day maturity. To obtain the PDFs, we follow the sequence of steps outlined below.

1. For each trading day, find the two closest contracts expiring after and before 45 days from now.

2. For these two contracts, find risk-free rates that minimize put-call parity errors by using Equation (15).

3. Find the implied volatility curves for puts and calls for the above contracts.

4. Calculate one implied volatility curve per contract by averaging across the implied volatility curve for puts and the implied volatility curve for calls.

5. Calculate the implied volatility curve corresponding to the 45-day maturity by taking a linear interpolation between the two above-mentioned implied volatility curves. In other words, suppose that we have $T_1 < T_{45} < T_2$, with T_{45} representing our forecast date 45 days from now, T_2 the expiration date of the next contract expiring after our forecast date, and T_1 the expiration date of the last contract expiring before our forecast date. Then, set

   $$a = \frac{T_2 - T_{45}}{T_2 - T_1}$$

   to obtain

   $$\tilde{\sigma}_{\text{45-day}}(K) = a\sigma_1(K) + (1-a)\sigma_2(K),$$

   where $\sigma_1(K)$ and $\sigma_2(K)$ are the implied volatility curves for the two contracts.

6. With the construction of a 45-day volatility curve, estimate call option prices for each trading day using the standard Black-Scholes formula.

7. Apply the two models explained in section 2.3 to find the R-PDF from the option prices.

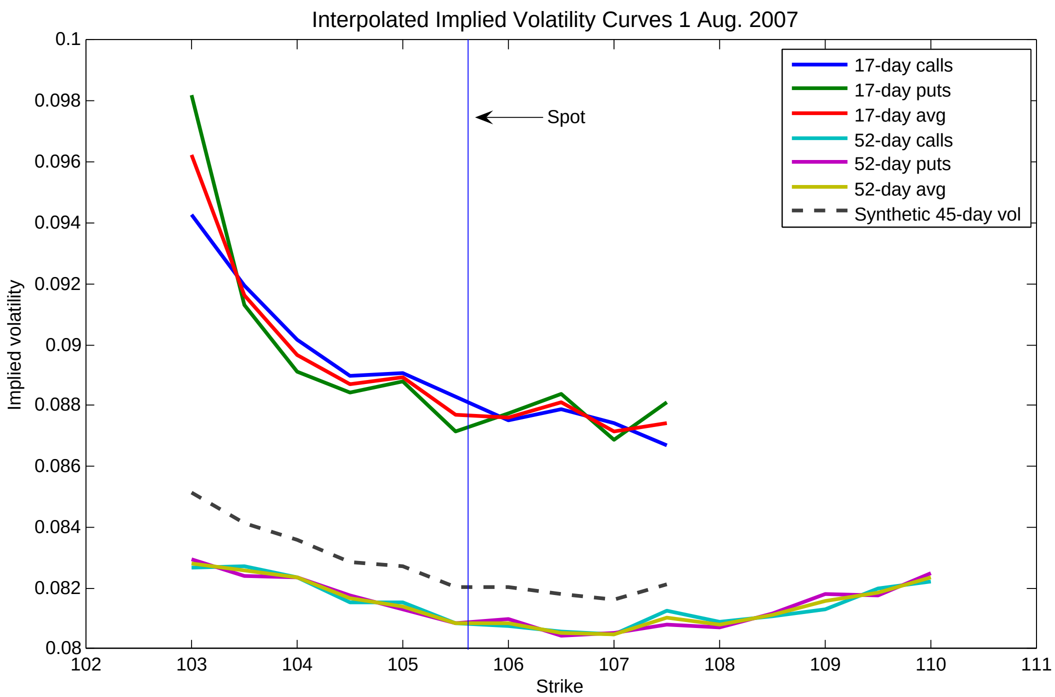

Figure 2: Example Construction of a 45-Day Volatility Curve

Figure 2 shows an example of the construction of a 45-day volatility curve. If the current day is 1 August 2007, then the next two contracts expire on 17 August 2007 and 21 September 2007, which have 17 and 52 days to maturity, respectively. We interpolate between the volatility curves of these two contracts to get a synthetic 45-day volatility curve. Once we have this curve, we can use the Black-Scholes formula to get the synthetic/implied 45-day option prices. It is important to note that the use of the implied volatility and Black-Scholes formula is just a mechanism to interpolate, and says nothing regarding the validity of the Black-Scholes formula for option pricing.

4 Results

We estimate the risk-neutral PDF for each trading day in 2007, using the two flexible parametric models described earlier, and then compare the corresponding volatility, skewness, and kurtosis measures obtained from the models. Figures 3 and 4 compare the second and third moments of the distribution.[10]

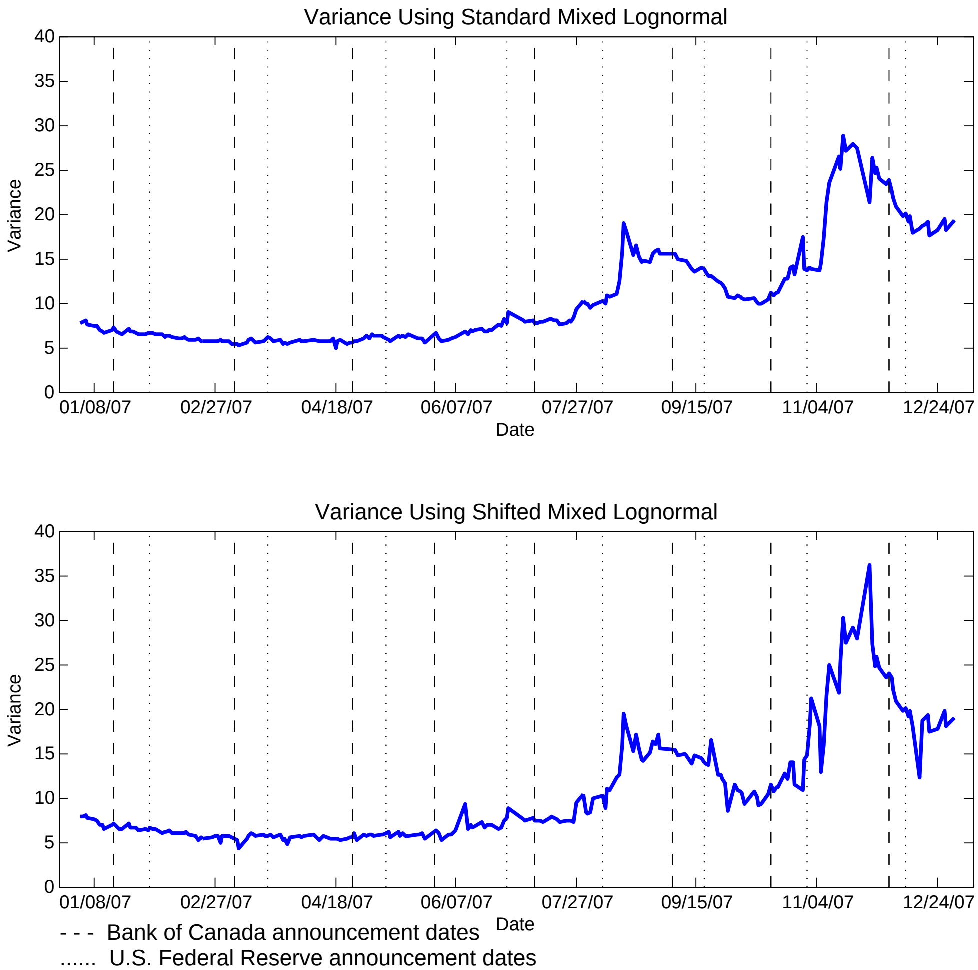

Figure 3: Variance of 45-Day Expectations

[10]The mean of the mixed lognormal corresponds to the current value of the Can$/US$ exchange rate plus an estimation error. We do not focus on the results of the mean, since they are, by construction, very close to the current value of the Can$/US$ exchange rate. The kurtosis is analyzed in Appendix A. Notice that the focus of the paper is not to explore the moments, but to show how, under the proposed methodology, we can obtain moments for a constant maturity. Therefore, we choose to illustrate the results with only the second and third moments.

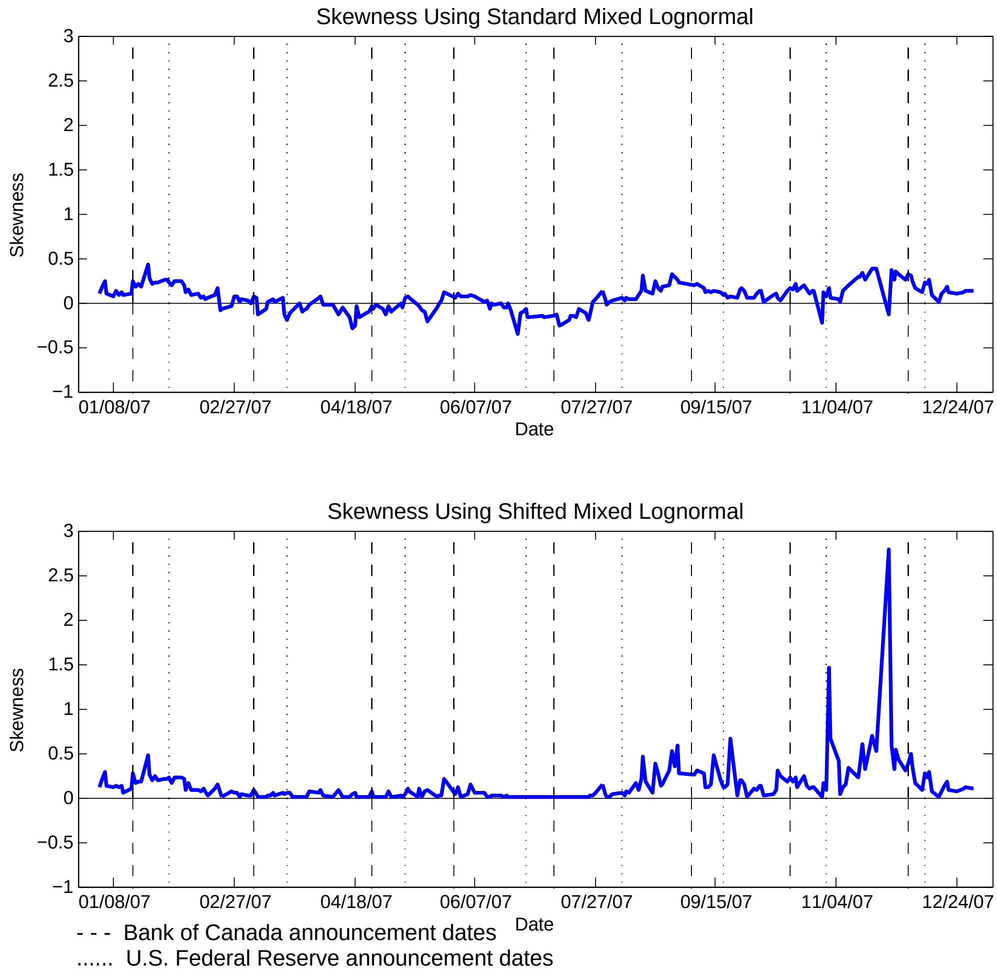

Figure 4: Skewness of 45-Day Expectations

The main results of our analysis are the following:

1. Figure 3 shows similar results for the variance of expectations for both models.

2. Both models show an increase in volatility from the end of July 2007; that is, uncertainty in the expectations surrounding the level of the Can$/US$ exchange rate increased. This result is consistent with the high degree of uncertainty in financial markets that was characteristic of the crisis and likely also affected expectations for the exchange rate. In particular, on 17 August 2007 the volatility (from the standard model) reached its first peak (11.37), which coincides with the vote by the Federal Reserve Board to reduce the primary credit rate (discount window) by 50 basis points to 5.75 per cent and

increase the maximum primary credit borrowing term to 30 days. And, in Canada, on 15 August 2007 the Bank of Canada temporarily expanded the list of collateral eligible for special purchase and resale agreement transactions. The next peak in volatility (16.90) occurred on 19 October 2007. Later, on 31 October, the U.S. Federal Open Market Committee (FOMC) reduced its target for the federal funds rate by 25 basis points to 4.50 per cent, and the Federal Reserve Board reduced the primary credit rate by 25 basis points to 5.00 per cent. All of these events reflect the increased uncertainty associated with the financial crisis.

3. The results for skewness shown in Figure 4, however, differ between the models. The *standard mixed lognormal* model exhibits more intuitive results, because it allows for periods of negative and positive skewness (appreciation and depreciation of the exchange rate), whereas the *shifted mixed lognormal* model does not allow for negative skewness.

4. The skewness results for the shifted lognormal may be due to the fact that we cannot have the same flexibility in changing the means as we do in the standard mixed lognormal. For this reason, we consider that the standard mixed lognormal model may give more reliable values for skewness.

Similarly, we can use the entire R-PDF to compare the impact of announcements on the Can$/US$ exchange rate by comparing the densities on days before, during, and after a given announcement. Figure 5 illustrates an example for Wednesday, 18 September 2007, when the FOMC decided to lower its target for the federal funds rate by 50 basis points to 4 3/4 per cent. The figure also shows the densities for the Can$/US$ exchange rate 45 days in the future observed on the Friday before and after the announcement. The FOMC announcement seems to have led to a small immediate appreciation and a wider dispersion of the expected outcome. After a few days, an expectation of further appreciation became embedded in markets, but the distributions became tighter, possibly showing a reduction in uncertainty. Although it seems reasonable to consider that the decrease in U.S. interest rates could result in an appreciation of the Canadian dollar, all else equal, these results should be interpreted with caution, because the analysis can differ from the actual behaviour of the real-world probability density. This is due to the effect of risk premiums, which suggests that, to use the information extracted from option prices for policy advice, further research is needed to find a reasonable mapping from the R-PDF into the real-world density.

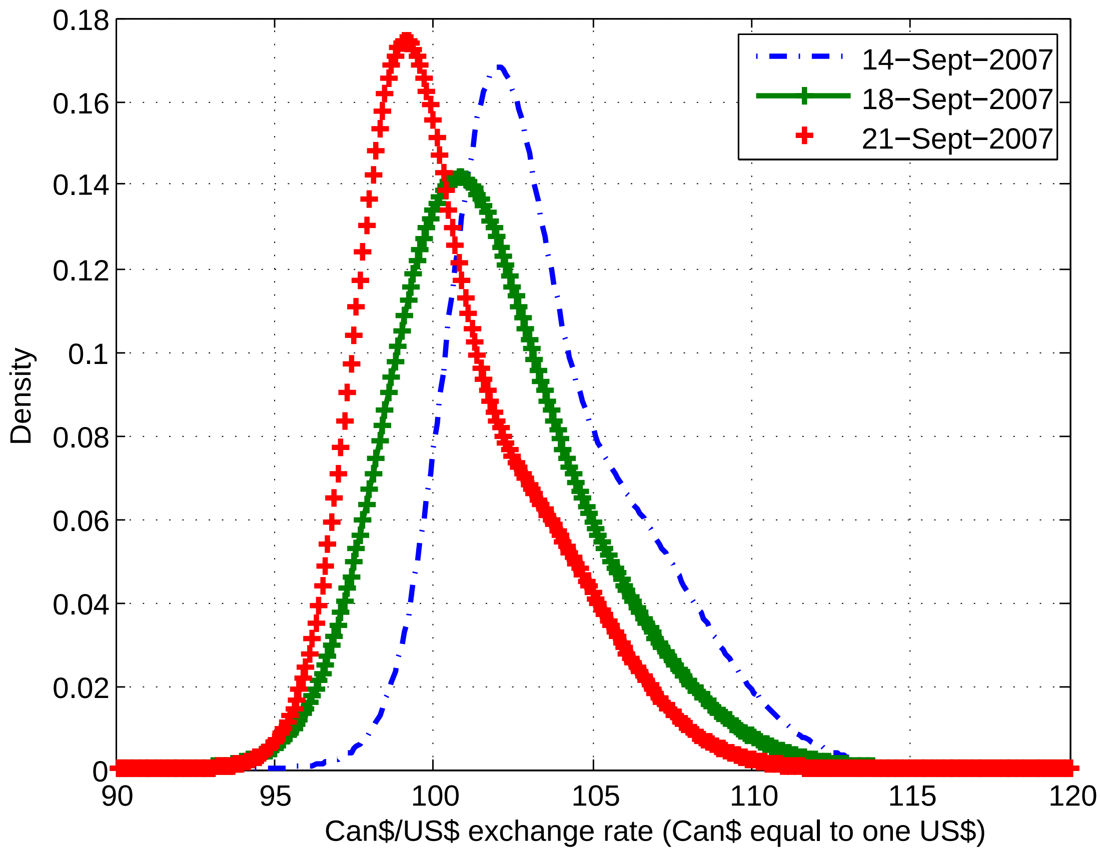

Figure 5: Probability Density Functions Before, During, and After Announcement
On 18 September 2007, the Federal Open Market Committee decided to lower its target for the federal funds rate by 50 basis points, to 4 3/4 per cent. The Friday before the announcement corresponds to 14 September 2007, and the Friday after the announcement corresponds to 21 September 2007.

5 Conclusion

Our paper's main contribution is to propose a simple methodology to build risk-neutral PDFs with a constant time to maturity in the absence of option prices for the maturity of interest. We show that, for two flexible parametric models, under reasonable assumptions, the resulting estimates for the volatility of expectations are consistent.

This paper has taken a first step at extracting information on market expectations. Future research to improve our estimates of the implied real-world distribution would include (i) understanding how to move from the risk-neutral to the real-world probabilities under less-restrictive assumptions, and (ii) comparing the results with a wider set of models for the terminal distribution.[11]

[11] Appendix E shows an example of the value of the real-world and risk-neutral distributions under the assumption of a constant mean for the real-world distribution.

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Appendix A: Analysis of Kurtosis

Figure A1 shows the kurtosis in both models. The shifted mixed lognormal approach seems to have times of much higher kurtosis. However, it is not easy to determine whether this is from numerical instability or if it is an accurate reflection of the data. The horizontal line is at 3, which is the kurtosis of a normal random variable. The standard mixed lognormal dips below 3 at times, but this does not occur for the shifted mixed lognormal.

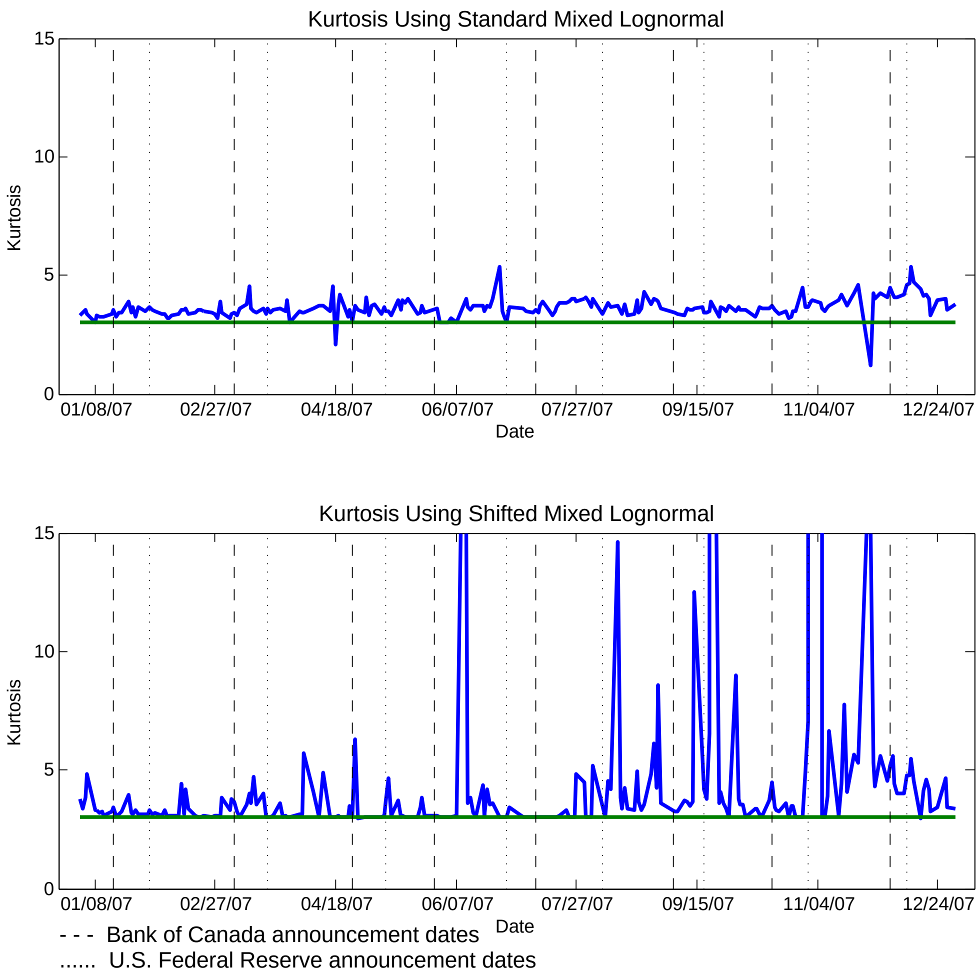

Figure A1: 45-Day Expectations - Kurtosis

Appendix B: Contract Specifications

The USX option contract on the U.S. dollar is a European-style option on the Can$/US$ exchange rate. Options are listed in the first three months plus the next two quarterly months in the March, June, September, and December cycles. USX options expire at 12:00 p.m. (Montréal time) on the third Friday of the expiry month. USX strikes are in minimum intervals of 50 cents, and quoted as the number of Canadian dollars required to buy one U.S. dollar. The trading unit per contract is US$10,000.

Appendix C: Formula Derivations

Consider a mixed lognormal random variable $X = I \cdot X_1 + (1 - I) \cdot X_2$, with X_1 and X_2 being lognormal, and I a Bernoulli(p) random variable. By the law of total probability,

$$\begin{aligned} P(X \leq x) &= P(X \leq x \mid I = 1)P(I = 1) + P(X \leq x \mid I = 0)P(I = 0) \\ &= p \cdot P(X_1 \leq x) + (1 - p) \cdot P(X_2 \leq x). \end{aligned}$$

Differentiating with respect to x gives

$$f_X(x) = pf_{X_1}(x) + (1 - p)f_{X_2}(x). \tag{16}$$

If we have that $\ln X_i \sim N(\ln(S_{0,i}) + (r - r_f - \frac{\sigma_i^2}{2})T, \sigma_i^2 T)$ for $i = 1, 2$, then $E[X_i] = e^{(r-r_f)T} S_{0,i}$ for $i = 1, 2$. As well, because of the form of the density (16), we have that

$$\begin{aligned} E[S_T] &= p \cdot E[X_1] + (1 - p) \cdot E[X_2] \\ &= pe^{(r-r_f)T} S_{0,1} + (1 - p)e^{(r-r_f)T} S_{0,2}. \end{aligned}$$

Under the risk-neutral measure, $E[S_T] = e^{(r-r_f)T} S_0$. Substituting this into the above equation for $E[S_T]$ and dividing by $e^{(r-r_f)T}$ gives us $S_0 = pS_{0,1} + (1 - p)S_{0,2}$.

Next, let us derive the call option formula in the mixed lognormal case. In order to apply risk-neutral pricing, an assumption that seems to be largely overlooked in the literature is the existence of a local volatility model,

$$dS_t = (r - r_f)S_t dt + \sigma(t, S_t)S_t dW_t,$$

with $\sigma(\cdot, \cdot)$ being a well-behaved deterministic function. This gives us a complete market, so that a contract that has payoff depending on S_T can be perfectly hedged. Even if such a process does not exist, we could assume the existence of a process that gives terminal distributions sufficiently close to our mixed lognormal. Brigo and Mercurio (2002) propose a process that allows for different terms σ_i, but we have yet to discover a process that also allows for changing terms $S_{0,1}, S_{0,2}$, even though the practice of having all four mixed lognormal parameters vary is widely used in the literature.

In any case, consider the formula $C_0 = e^{-rT}E^Q[(S_T - K)^+]$ for a risk-neutral measure Q. Recall that $S_T = X_1$ with probability p, and that $S_T = X_2$ with probability $(1-p)$, where X_1, X_2 are lognormal variables. It follows, by taking appropriate expected values, that

$$C_0 = e^{-rT}pE^Q[(X_1 - K)^+] + e^{-rT}(1-p)E^Q[(X_2 - K)^+].$$

Since, under Q, $\ln X_i \sim N(\ln(S_{0,i}) + (r - r_f - \frac{\sigma_i^2}{2})T, \sigma_i^2 T)$ for $i = 1, 2$, we can directly compute the expected values $E^Q[(X_i - K)^+]$ using the same substitution tricks when deriving the standard Black-Scholes formula to obtain the result.

Appendix D: Expressions for Moments of the PDFs

We wish to compute the standardized central moments of a mixture of lognormals. In particular, we are interested in the following:

$$\begin{aligned} \text{variance} = \sigma^2 &= E[(X-\mu)^2], \\ \text{skewness} &= E[(X-\mu)^3]/\sigma^3, \\ \text{kurtosis} &= E[(X-\mu)^4]/\sigma^4. \end{aligned}$$

Suppose that a random variable X has probability density of the form

$$f(x) = pf_1(x) + (1-p)f_2(x),$$

where $f_1(x)$ and $f_2(x)$ are two other probability density functions for random variables X_1 and X_2. Then, for any integer $k \geq 1$, we have that

$$E[X^k] = pE[X_1^k] + (1-p)E[X_2^k]. \tag{17}$$

Hence, we can easily compute the non-central moments of X given the corresponding non-central moments of X_1 and X_2. In particular, we have that $\mu = E[X] = pE[X_1] + (1-p)E[X_2]$.

We can compute the higher central moments $E[(X-\mu)^k]$ simply by expanding $(X-\mu)^k$ to express it in terms of non-central moments. In particular,

$$E[(X-\mu)^2] = E[X^2] - \mu^2, \tag{18}$$
$$E[(X-\mu)^3] = E[X^3] - 3E[X^2]\mu + 2\mu^3, \tag{19}$$
$$E[(X-\mu)^4] = E[X^4] - 4E[X^3]\mu + 6E[X^2]\mu^2 - 3\mu^4. \tag{20}$$

In the case that X_i is lognormal with $\log X_i \sim N(\mu_i, \sigma_i^2)$, then

$$E[X_i^k] = e^{k\mu_i + \frac{1}{2}k^2\sigma_i^2}. \tag{21}$$

By using equations (17) to (21), we can compute the moments of the mixed lognormal variables.

Appendix E: Implied Real-World PDFs

This appendix compares the differences between the risk-neutral and real-world distribution under the assumption of a constant mean for the stochastic process of the underlying asset.

Recall that, in Brigo and Mercurio (2002), the real-world dynamics is described by

$$dS_t = \mu S_t + \sqrt{\frac{a\sigma_1^2 t p_t^1(S_t) + (1-a)\sigma_2^2 t p_t^2(S_t)}{p_t(S_t)}} S_t dW_t, \tag{22}$$

for some drift term μ.

The risk-neutral dynamics is described by

$$dS_t = (r - r_f)S_t + \sqrt{\frac{a\sigma_1^2 t p_t^1(S_t) + (1-a)\sigma_2^2 t p_t^2(S_t)}{p_t(S_t)}} S_t dW_t. \tag{23}$$

Unfortunately, the problem of estimating μ is difficult. This is, in part, because the drift of the underlying asset will change over time. Our goal in this appendix is not to accurately determine the real-world PDF, but rather to simply determine what differences there may be between the risk-neutral PDF and the real-world PDF, and their moments. We try $\mu = 20$ per cent and $\mu = -20$ per cent to determine the difference between the real-world and risk-neutral distributions under this model.

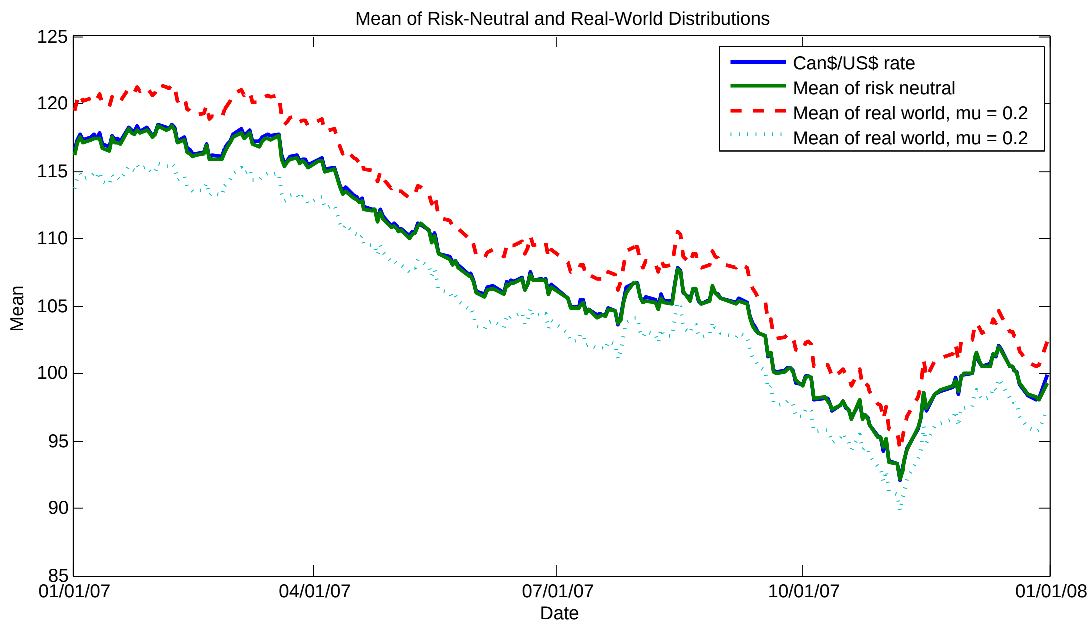

Figure E1: Means of Distributions

We see that, in Figure E1, the mean of the risk-neutral distribution is almost identical to the spot exchange rate. This is because, under the risk-neutral measure, $E[S_T] = e^{(r-r_f)T}S_0$. Recall that, in our case study, $T = 45/360$, and we usually have that $r \approx r_f$, so that $(r - r_f)T \approx 0$ and $E[S_T] \approx S_0$. In the real-world measure, the resulting means of the real-world and risk-neutral measures will differ because μ could be, say, 20 per cent, which is much larger than any reasonable value for $r - r_f$. In general, our model shows that, when μ is positive, the mean of the real-world expectations will be higher than the spot, and when μ is negative, it will be lower than the spot.

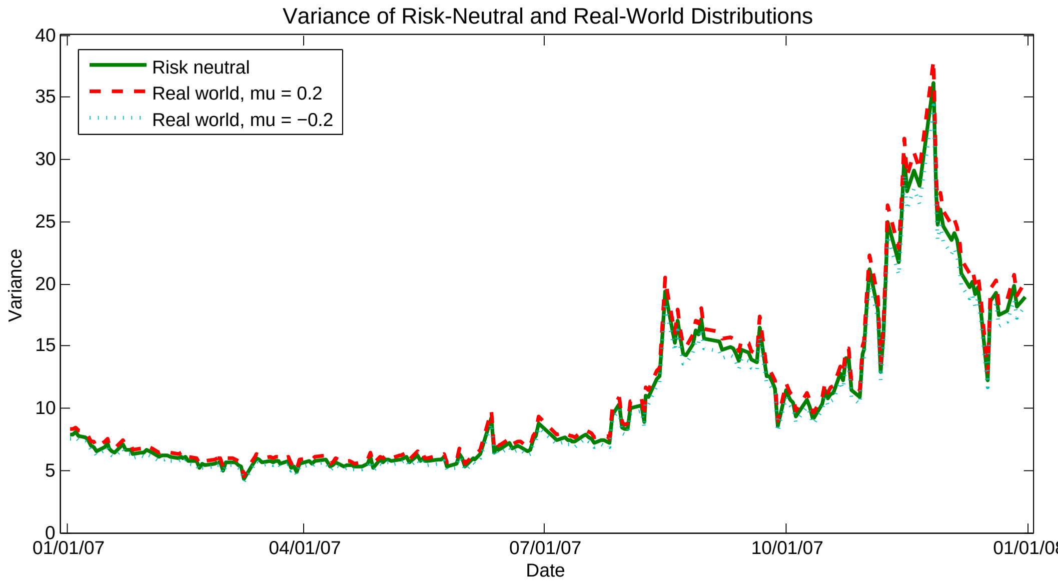

Figure E2: Variance of Distributions

In Figure E2, we see that the risk-neutral and real-world variances are very close to each other. This is because the differences between the risk-neutral and real-world distributions are mostly of level, and not of shape. Thus, even if μ were to change, it would not have a large effect on the variance. Hence, if we see a change in the risk-neutral variance, we can assume that, most likely, there is a change in the real-world variance. Changes in the R-PDF variance can thus be used to deduce information about the view of the markets.