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Small area estimation under a Fay-Herriot model with preliminary testing for the presence of random area effects

Isabel Molina, J.N.K. Rao and Gauri Sankar Datta[1]

Abstract

A popular area level model used for the estimation of small area means is the Fay-Herriot model. This model involves unobservable random effects for the areas apart from the (fixed) linear regression based on area level covariates. Empirical best linear unbiased predictors of small area means are obtained by estimating the area random effects, and they can be expressed as a weighted average of area-specific direct estimators and regression-synthetic estimators. In some cases the observed data do not support the inclusion of the area random effects in the model. Excluding these area effects leads to the regression-synthetic estimator, that is, a zero weight is attached to the direct estimator. A preliminary test estimator of a small area mean obtained after testing for the presence of area random effects is studied. On the other hand, empirical best linear unbiased predictors of small area means that always give non-zero weights to the direct estimators in all areas together with alternative estimators based on the preliminary test are also studied. The preliminary testing procedure is also used to define new mean squared error estimators of the point estimators of small area means. Results of a limited simulation study show that, for small number of areas, the preliminary testing procedure leads to mean squared error estimators with considerably smaller average absolute relative bias than the usual mean squared error estimators, especially when the variance of the area effects is small relative to the sampling variances.

Key Words: Area level model; Empirical best linear unbiased predictor; Mean squared error; Preliminary testing; Small area estimation.

1 Introduction

A basic area-level model, called the Fay-Herriot (FH) model, is often used to obtain efficient estimators of area means when the sample sizes within areas are small. This model involves unobservable area random effects, and the empirical best linear unbiased predictor (EBLUP) of a small area mean is obtained by estimating the associated random effect. The EBLUP is a weighted combination of a direct area-specific estimator and a regression-synthetic estimator that uses all the data. An estimator of the mean squared error (MSE) of the EBLUP was obtained first by Prasad and Rao (1990) using a moment estimator of the random effects variance and later by Datta and Lahiri (2000) for the restricted maximum likelihood (REML) estimator of the variance. Rao (2003, Chapter 7) gives a detailed account of EBLUPs and their MSE estimators for the FH model.

Sometimes the observed data do not support the inclusion of the area effects in the model. Excluding the area effects leads to the regression-synthetic estimator. Using this idea, Datta, Hall and Mandal (2011) proposed to do a preliminary test for the presence of the area random effects at a specified significance level, and then to define the small area estimator depending on the result of the test. If the null hypothesis of no area random effects is not rejected, the model without the area effects is considered to estimate the small area means, i.e., the regression-synthetic estimator is used. If the null hypothesis is rejected, the usual EBLUP under the FH model with area effects is used. Datta et al. (2011) remarked that the above preliminary test estimator (PTE) could lead to significant efficiency gains over the EBLUP, particularly

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when the number of small areas is only modest in size. For preliminary testing, they considered a normality-based test as well as a bootstrap test that avoids the normality assumption.

When the estimated area effects variance is zero, the EBLUP becomes automatically the regression-synthetic estimator. However, the estimated MSE obtained by Prasad and Rao (1990) or Datta and Lahiri (2000) does not reduce to the estimated MSE of the regression-synthetic estimator. Thus, the usual MSE estimators are biased for small random effects variance. For this reason, we propose MSE estimators of the EBLUP based on the preliminary testing procedure. If the random effects variance is not significant according to the test, we consider the MSE estimator of the synthetic estimator. Otherwise, we consider the usual MSE estimators of the EBLUP.

The EBLUP attaches zero weight to the direct estimates for all areas when the estimated area effects variance is zero. On the other hand, survey practitioners often prefer to attach a strictly positive weight to the direct estimates, because the latter make use of the available area-specific unit level data and also incorporate the sampling design. Li and Lahiri (2010) introduced an adjusted maximum likelihood (AML) estimator of the variance of random effects that is always positive and therefore leads to EBLUPs giving strictly positive weights to direct estimators. As we shall see, a price is paid in terms of bias when using the EBLUP based on the AML estimator. We propose here alternative small area estimators that always give a positive weight to the direct estimators but with a smaller bias.

This paper studies empirically the properties of PTEs of small area means, in comparison with the usual EBLUPs and other proposed estimators. In particular, we study the choice of the significance level for the area estimates and for the MSE estimates based on the preliminary test (PT). EBLUPs based on the AML estimator of the random effects variance of Li and Lahiri (2010), which give non-zero weights to the direct estimators in all areas, are also studied and compared to PT versions of AML (PT-AML). Different MSE estimators of these PT-AML estimators are also studied with respect to relative bias. Based on simulation results, the EBLUPs and the associated MSE estimators that performed well are recommended. Finally, coverage and length of normality-based prediction intervals, obtained using the EBLUPs and the associated MSE estimators, are examined.

The paper is organized as follows. Section 2 describes the FH model and the EBLUPs of small area means. Section 3 comments on MSE estimation. PTEs of small area means and MSE estimators based on the PT are introduced in Section 4. Section 5 describes small area estimators and associated MSE estimators under AML estimation of the area effects variance. Alternative estimators that also attach positive weights to direct estimators together with proposed MSE estimators are introduced in Section 6. Section 7 reports the results of the simulation study. Finally, Section 8 gives some concluding remarks.

2 Estimation of small area means

Consider a population partitioned into m areas and let θ_i be the mean of the variable of interest for area $i, i = 1, \ldots, m.$ We assume that a sample is drawn independently from each area. Let y_i be a design-unbiased direct estimator of θ_i obtained using survey data from the sampled area $i.$ Direct estimators are very inefficient for areas with small sample sizes. We study small area estimation under an area level model, in which the values of area level covariates are available for all areas. The basic model of this type is the Fay-Herriot model, introduced by Fay and Herriot (1979), to estimate per capita income for small places in the United States. This model consists of two parts. The first part assumes that direct estimators, y_i, of small area means, θ_i, are design unbiased, satisfying

$$y_i = \theta_i + e_i, \quad e_i \overset{\text{ind}}{\sim} N(0, D_i), \quad i = 1, \ldots, m. \tag{2.1}$$

Here, the sampling variance $D_i = \text{Var}(y_i | \theta_i)$ is assumed to be known for all areas $i = 1, \ldots, m$. In practice, the D_i's are ascertained from external sources or by smoothing the estimated sampling variances using a generalized variance function method (Fay and Herriot 1979).

In the second part, the Fay-Herriot model treats θ_i as random and assumes that a p-vector of area level covariates, $\mathbf{x}_i$, linearly related to θ_i, is available for each area i, i.e.,

$$\theta_i = \mathbf{x}_i' \boldsymbol{\beta} + v_i, \quad v_i \overset{\text{iid}}{\sim} N(0, A), \quad i = 1, \ldots, m, \tag{2.2}$$

where v_i is the random effect of area i, assumed to be independent of e_i and $A \geq 0$ is the variance of the random effects. Observe that marginally,

$$y_i \overset{\text{ind}}{\sim} N(\mathbf{x}_i' \boldsymbol{\beta}, D_i + A), \quad i = 1, \ldots, m. \tag{2.3}$$

Letting $\mathbf{y} = (y_1, \ldots, y_m)'$, $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_m)'$ and $\mathbf{D} = \text{diag}(D_1, \ldots, D_m)$, model (2.3) may be expressed in matrix notation as $\mathbf{y} \sim N\{\mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma}(A)\}$ with $\boldsymbol{\Sigma}(A) = \mathbf{D} + A\mathbf{I}_m$, where $\mathbf{I}_m$ denotes the $m \times m$ identity matrix. If A is known, the componentwise best linear unbiased predictor (BLUP) of $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)'$ is given by

$$\tilde{\boldsymbol{\theta}}(A) = \left(\tilde{\theta}_1(A), \ldots, \tilde{\theta}_m(A)\right)' = \mathbf{X}\tilde{\boldsymbol{\beta}}(A) + A\boldsymbol{\Sigma}^{-1}(A)\{\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}}(A)\}, \tag{2.4}$$

where

$$\begin{aligned} \tilde{\boldsymbol{\beta}}(A) &= \{\mathbf{X}'\boldsymbol{\Sigma}^{-1}(A)\mathbf{X}\}^{-1} \mathbf{X}'\boldsymbol{\Sigma}^{-1}(A)\mathbf{y} \\ &= \left\{\sum_{i=1}^{m} (A + D_i)^{-1} \mathbf{x}_i \mathbf{x}_i'\right\}^{-1} \sum_{i=1}^{m} (A + D_i)^{-1} \mathbf{x}_i y_i \end{aligned} \tag{2.5}$$

is the weighted least squares (WLS) estimator of $\boldsymbol{\beta}$. In practice, however, A is not known. Substituting a consistent estimator $\hat{A}$ for A in the BLUP (2.4), we get the EBLUP given by

$$\hat{\boldsymbol{\theta}} = \left(\hat{\theta}_1, \ldots, \hat{\theta}_m\right)' = \mathbf{X}\hat{\boldsymbol{\beta}} + \hat{A}\hat{\boldsymbol{\Sigma}}^{-1}\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}\right), \tag{2.6}$$

where $\hat{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}}(\hat{A})$ and $\hat{\boldsymbol{\Sigma}} = \mathbf{D} + \hat{A}\mathbf{I}_m$. For the i^{th} area, the EBLUP of θ_i can be expressed as a convex linear combination of the regression-synthetic estimator $\mathbf{x}_i'\hat{\boldsymbol{\beta}}$ and the direct estimator y_i, as

$$\hat{\theta}_i = B_i(\hat{A})\mathbf{x}_i'\hat{\boldsymbol{\beta}} + \{1 - B_i(\hat{A})\} y_i, \tag{2.7}$$

where the weight attached to the regression-synthetic estimator $\mathbf{x}_i'\hat{\boldsymbol{\beta}}$ is given by $B_i(\hat{A})$, where $B_i(A) = D_i/(A + D_i)$. Observe that the weight increases with the sampling variance D_i. Thus, when the direct estimator is not reliable, i.e., D_i is large as compared with the total variance $\hat{A} + D_i$, more weight is attached to the regression-synthetic estimator $\mathbf{x}_i'\hat{\boldsymbol{\beta}}$. On the other hand, when the direct estimator is efficient, D_i is small relative to $\hat{A} + D_i$, and then more weight is given to the direct estimator y_i.

Several estimators of A have been proposed in the literature including moment estimators without normality assumption, ML estimator and restricted (or residual) ML estimator (REML) estimator. The ML estimator of A is $\hat{A}_{\text{ML}} = \max(0, \hat{A}^*_{\text{ML}})$, where $\hat{A}^*_{\text{ML}}$ can be obtained by maximizing the profile likelihood function given by

$$L_P(A) = c\,|\boldsymbol{\Sigma}(A)|^{-1/2} \exp\left\{-\frac{1}{2}\mathbf{y}'\mathbf{P}(A)\,\mathbf{y}\right\},$$

where c denotes a generic constant and

$$\mathbf{P}(A) = \boldsymbol{\Sigma}^{-1}(A) - \boldsymbol{\Sigma}^{-1}(A)\,\mathbf{X}\{\mathbf{X}'\boldsymbol{\Sigma}^{-1}(A)\,\mathbf{X}\}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(A).$$

The REML estimator of A is $\hat{A}_{\text{RE}} = \max(0, \hat{A}^*_{\text{RE}})$, where $\hat{A}^*_{\text{RE}}$ is obtained by maximizing the restricted/residual likelihood, given by

$$L_{\text{RE}}(A) = c\,|\mathbf{X}'\boldsymbol{\Sigma}^{-1}(A)\,\mathbf{X}|^{-1/2}\,|\boldsymbol{\Sigma}(A)|^{-1/2} \exp\left\{-\frac{1}{2}\mathbf{y}'\mathbf{P}(A)\,\mathbf{y}\right\}.$$

In this paper, we focus on the REML estimator $\hat{A}_{\text{RE}}$ which is frequently used in practice, and we denote by $\hat{\boldsymbol{\theta}}_{\text{RE}} = (\hat{\theta}_{\text{RE},1}, \ldots, \hat{\theta}_{\text{RE},m})'$ the EBLUP given in (2.6) obtained with $\hat{A} = \hat{A}_{\text{RE}}$.

3 Mean squared error

Note that the BLUP $\tilde{\theta}_i(A)$ of the small area mean θ_i is a linear function of $\mathbf{y}$. Hence, its MSE can be easily calculated and it is given by the sum of two terms:

$$\text{MSE}\{\tilde{\theta}_i(A)\} = g_{1i}(A) + g_{2i}(A),$$

where $g_{1i}(A)$ is due to the estimation of the random area effect v_i and $g_{2i}(A)$ is due to the estimation of the regression parameter $\boldsymbol{\beta}$, with

$$\begin{aligned} g_{1i}(A) &= D_i\{1 - B_i(A)\}, \\ g_{2i}(A) &= B_i^2(A)\,\mathbf{x}_i'\{\mathbf{X}'\boldsymbol{\Sigma}^{-1}(A)\,\mathbf{X}\}^{-1}\mathbf{x}_i. \end{aligned}$$

However, the EBLUP $\hat{\theta}_i$ given in (2.7) is not linear in $\mathbf{y}$ due to the estimation of the random effects variance A. Using a moments estimator of A, Prasad and Rao (1990) obtained a second order correct approximation for the MSE of the EBLUP. Later, Datta and Lahiri (2000) and Das, Jiang and Rao (2004) obtained second order correct MSE approximations under ML and REML estimation of A. When using the REML estimator of A, their approximation to the MSE, for large m, is given by

$$\mathrm{MSE}\left(\hat{\theta}_{\mathrm{RE},i}\right) = g_{1i}(A) + g_{2i}(A) + g_{3i}(A) + o\left(m^{-1}\right), \tag{3.1}$$

where

$$g_{3i}(A) = B_i^2(A)\frac{V_{\mathrm{RE}}(A)}{A + D_i} \text{ and } V_{\mathrm{RE}}(A) = \frac{2}{\sum_{i=1}^{m}\left(A + D_i\right)^{-2}}.$$

Note that as $m \to \infty, g_{1i}(A) = O(1), g_{2i}(A) = O\left(m^{-1}\right)$ and $g_{3i}(A) = O\left(m^{-1}\right)$, so $g_{1i}(A)$ is the leading term in the MSE for large m. However, for small $A, g_{1i}(A)$ is approximately zero and then $g_{3i}(A)$ might be the leading term for small m. For example, taking only one covariate $(p = 1)$ with constant values $x_i = 1$ and constant sampling variances $D_i = D, i = 1, \ldots, m$ and letting $A = 0$, we obtain $g_{1i}(0) = 0, g_{2i}(0) = D/m$ and $g_{3i}(0) = 2D/m$; that is, $g_{3i}(0)$ is twice as large as $g_{2i}(0)$.

Datta and Lahiri (2000) obtained an estimator of the MSE of the EBLUP $\hat{\theta}_{\mathrm{RE},i}$ given by

$$\mathrm{mse}\left(\hat{\theta}_{\mathrm{RE},i}\right) = g_{1i}\left(\hat{A}_{\mathrm{RE}}\right) + g_{2i}\left(\hat{A}_{\mathrm{RE}}\right) + 2g_{3i}\left(\hat{A}_{\mathrm{RE}}\right). \tag{3.2}$$

The MSE estimator (3.2) is second-order unbiased in the sense that

$$E\left\{\mathrm{mse}\left(\hat{\theta}_{\mathrm{RE},i}\right)\right\} = \mathrm{MSE}\left(\hat{\theta}_{\mathrm{RE},i}\right) + o\left(m^{-1}\right).$$

In the case that $A = 0$, the BLUP $\tilde{\theta}_{\mathrm{RE},i}$ of θ_i becomes the regression-synthetic estimator $\hat{\theta}_{\mathrm{SYN},i} = \mathbf{x}_i'\tilde{\boldsymbol{\beta}}(0)$. But surprisingly, the approximation to the MSE of the EBLUP given in (3.1) can be very different from the MSE of the synthetic estimator. Note that the latter is

$$\mathrm{MSE}\left(\hat{\theta}_{\mathrm{SYN},i}\right) = g_{2i}(0) < g_{2i}(0) + g_{3i}(0),$$

because $g_{3i}(0)$ is strictly positive even for $A = 0$. In fact, in the simple example with only one covariate $(p = 1)$ with constant values $x_i = 1$ and constant sampling variances $D_i = D, i = 1, \ldots, m$, we have $\mathrm{MSE}\left(\hat{\theta}_{\mathrm{SYN},i}\right) = g_{2i}(0) = D/m$ whereas the approximation to the MSE of the EBLUP given in (3.1) with $A = 0$ gives $\mathrm{MSE}\left(\hat{\theta}_{\mathrm{RE},i}\right) \approx g_{2i}(0) + g_{3i}(0) = 3D/m$, three times larger. It turns out that (3.1) is not a good approximation of the MSE of the EBLUP when $A = 0$ and, instead, we should use $\mathrm{MSE}\left(\hat{\theta}_{\mathrm{RE},i}\right) = g_{2i}(0)$. Moreover, since for $A = 0$ this quantity does not depend on any unknown parameter, we can take it also as MSE estimator, i.e., we can take $\mathrm{mse}\left(\hat{\theta}_{\mathrm{RE},i}\right) = g_{2i}(0)$.

In practice, the true value of A is not known but we have the consistent estimator $\hat{A}_{\mathrm{RE}}$. When $\hat{A}_{\mathrm{RE}} = 0$, the EBLUP becomes the regression-synthetic estimator for all areas, that is

$$\hat{\theta}_{\mathrm{RE},i} = \hat{\theta}_{\mathrm{SYN},i} = \mathbf{x}_i'\tilde{\boldsymbol{\beta}}(0), i = 1, \ldots, m.$$

In this case, $g_{1i}\left(\hat{A}_{\mathrm{RE}}\right) = 0$ for all areas and the MSE estimator given in (3.2) reduces to

$$\text{mse}\left(\hat{\theta}_{\text{RE},i}\right) = g_{2i}(0) + 2g_{3i}(0) > g_{2i}(0) = \text{MSE}\left(\hat{\theta}_{\text{SYN},i}\right), i = 1, \ldots, m.$$

Thus, the MSE estimator given in (3.2) can be seriously overestimating the MSE for $\hat{A}_{\text{RE}} = 0.$ To reduce the overestimation, we consider a modified MSE estimator of $\hat{\theta}_{\text{RE},i}$ given by

$$\text{mse}_0\left(\hat{\theta}_{\text{RE},i}\right) = \begin{cases} g_{2i} & \text{if } \hat{A}_{\text{RE}} = 0, \\ g_{1i}\left(\hat{A}_{\text{RE}}\right) + g_{2i}\left(\hat{A}_{\text{RE}}\right) + 2g_{3i}\left(\hat{A}_{\text{RE}}\right) & \text{if } \hat{A}_{\text{RE}} > 0, \end{cases} \quad (3.3)$$

where $g_{2i} = g_{2i}(0) = \mathbf{x}_i'\left(\mathbf{X}'\mathbf{D}^{-1}\mathbf{X}\right)^{-1}\mathbf{x}_i, i = 1, \ldots, m.$

In fact, for A close to zero, it may happen that g_{2i} is closer to the true MSE than the full MSE estimator $\text{mse}\left(\hat{\theta}_{\text{RE},i}\right),$ but the question of when is A close enough to zero arises. This question motivates the use of a preliminary testing procedure of $A = 0$ to define alternative MSE estimators of the EBLUP in Section 4.

4 Preliminary test estimators

The estimator of A used in the EBLUP of θ_i introduces uncertainty, which might not be negligible for small m. Indeed, the term g_{3i} in the MSE estimator (3.2) arises due to the estimation of A. However, when the value of A is small enough relative to the sampling variances, this uncertainty could be avoided by using the regression-synthetic estimator $\mathbf{x}_i'\tilde{\boldsymbol{\beta}}(0)$ instead of the EBLUP. Datta et al. (2011) proposed a small area estimator based on a preliminary testing procedure of $H_0 : A = 0$ against $H_1 : A > 0.$ When H_0 is not rejected, the regression-synthetic estimator is taken as the estimator of θ_i; otherwise, the usual EBLUP is used. They proposed the test statistic

$$T = \left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}_{\text{PT}}\right)' \mathbf{D}^{-1}\left(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}_{\text{PT}}\right),$$

where $\hat{\boldsymbol{\beta}}_{\text{PT}} = \left(\mathbf{X}'\mathbf{D}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{D}^{-1}\mathbf{y}$ is the WLS estimator of $\boldsymbol{\beta}$ obtained assuming that $H_0 : A = 0$ is true. The test statistic T is distributed as X^2_{m-p} with $m - p$ degrees of freedom under H_0. Then, for a specified significance level $\alpha,$ the PTE of $\boldsymbol{\theta}$ defined by Datta et al. (2011) is given by

$$\hat{\boldsymbol{\theta}}_{\text{PT}} = \left(\hat{\theta}_{\text{PT},1}, \ldots, \hat{\theta}_{\text{PT},m}\right)' = \begin{cases} \mathbf{X}\hat{\boldsymbol{\beta}}_{\text{PT}} & \text{if } T \le X^2_{m-p,\alpha}; \\ \hat{\boldsymbol{\theta}}_{\text{RE}} & \text{if } T > X^2_{m-p,\alpha}, \end{cases}$$

where $X^2_{m-p,\alpha}$ is the upper α-point of X^2_{m-p}. The PTE is especially designed to handle cases with a modest number of small areas, say $m = 15.$

Here we propose to use the PT procedure for the estimation of MSE of the EBLUP, by considering only the MSE of the synthetic estimator g_{2i} whenever the null hypothesis is not rejected and the full MSE estimate otherwise. But observe that the test statistic T in the PT procedure does not depend on the estimator of A. This means that, even when H_0 is rejected, it may happen that $\hat{A}_{\text{RE}} = 0.$ Thus, here we define the PT estimator of the MSE of the EBLUP $\hat{\theta}_{\text{RE},i}$ as

$$\mathrm{mse}_{\mathrm{PT}}\left(\hat{\theta}_{\mathrm{RE},i}\right) = \begin{cases} g_{2i} & \text{if } T \le X^2_{m-p,\alpha} \quad \text{or} \quad \hat{A}_{\mathrm{RE}} = 0, \\ g_{1i}\left(\hat{A}_{\mathrm{RE}}\right) + g_{2i}\left(\hat{A}_{\mathrm{RE}}\right) + 2g_{3i}\left(\hat{A}_{\mathrm{RE}}\right) & \text{if } T > X^2_{m-p,\alpha} \quad \text{and} \quad \hat{A}_{\mathrm{RE}} > 0. \end{cases} \tag{4.1}$$

5 Adjusted maximum likelihood

The estimation methods for A described in Section 2 might produce zero estimates. In this case, the EBLUPs will give zero weight to the direct estimators in all areas, regardless of the efficiency of the direct estimator in each area. On the other hand, survey sampling practitioners often prefer to give always a strictly positive weight to direct estimators because they are based on the area-specific unit level data for the variable of interest without the assumption of any regression model. For this situation, Li and Lahiri (2010) proposed the AML estimator that delivers a strictly positive estimator of A. This estimator, denoted here $\hat{A}_{\mathrm{AML}}$, is obtained by maximizing the adjusted likelihood defined as

$$L_{\mathrm{AML}}(A) = A \times L_P(A).$$

The EBLUP given in (2.6) with $\hat{A} = \hat{A}_{\mathrm{AML}}$ will be denoted hereafter as $\hat{\boldsymbol{\theta}}_{\mathrm{AML}} = \left(\hat{\theta}_{\mathrm{AML},1}, \ldots, \hat{\theta}_{\mathrm{AML},m}\right)'$. Note that $\hat{\boldsymbol{\theta}}_{\mathrm{AML}}$ assigns strictly positive weights to direct estimators.

Li and Lahiri (2010) proposed a second order unbiased MSE estimator of $\hat{\theta}_{\mathrm{AML},i}$ given by

$$\begin{aligned} \mathrm{mse}\left(\hat{\theta}_{\mathrm{AML},i}\right) &= g_{1i}\left(\hat{A}_{\mathrm{AML}}\right) + g_{2i}\left(\hat{A}_{\mathrm{AML}}\right) + 2g_{3i}\left(\hat{A}_{\mathrm{AML}}\right) \\ &\quad - B_i^2\left(\hat{A}_{\mathrm{AML}}\right) b_{\mathrm{AML}}\left(\hat{A}_{\mathrm{AML}}\right), \end{aligned} \tag{5.1}$$

where $b_{\mathrm{AML}}(A)$ is the bias of $\hat{A}_{\mathrm{AML}}$ and it is given by

$$b_{\mathrm{AML}}(A) = \frac{\mathrm{trace}\left\{\mathbf{P}(A) - \boldsymbol{\Sigma}^{-1}(A)\right\} + 2/A}{\mathrm{trace}\left\{\boldsymbol{\Sigma}^{-2}(A)\right\}}.$$

6 Combined estimators

The strictly positive AML estimator of A has typically a larger bias than ML or REML estimators for A small relative to the D_i's. Thus, if we still wish to obtain a small area estimator that attaches a strictly positive weight to the direct estimator, to reduce the mentioned bias it will be better to use the AML estimator only when strictly necessary; that is, either when data does not provide enough evidence against $A = 0$ or when the resulting REML estimator of A is zero. This section introduces two small area estimators of $\boldsymbol{\theta}$ that give a strictly positive weight to the direct estimator, which are obtained as a combination of the EBLUP based on the AML method and the EBLUP based on REML estimation.

In the first combined proposal, the AML method is used to estimate A when the preliminary test does not reject the null hypothesis and in the second combined proposal, when the REML estimate is non positive. Specifically, the first combined estimator, called hereafter PT-AML, is defined by

$$\hat{\boldsymbol{\theta}}_{\text{PTAML}} = \begin{cases} \hat{\boldsymbol{\theta}}_{\text{AML}} & \text{if } T \le X^2_{m-p,\alpha} \quad \text{or} \quad \hat{A}_{\text{RE}} = 0, \\ \hat{\boldsymbol{\theta}}_{\text{RE}} & \text{if } T > X^2_{m-p,\alpha} \quad \text{and} \quad \hat{A}_{\text{RE}} > 0. \end{cases} \tag{6.1}$$

The second combined estimator, called REML-AML, is given by

$$\hat{\boldsymbol{\theta}}_{\text{REAML}} = \begin{cases} \hat{\boldsymbol{\theta}}_{\text{AML}} & \text{if } \hat{A}_{\text{RE}} = 0, \\ \hat{\boldsymbol{\theta}}_{\text{RE}} & \text{if } \hat{A}_{\text{RE}} > 0, \end{cases} \tag{6.2}$$

see Rubin-Bleuer and Yu (2013). For the estimation of MSE of $\hat{\boldsymbol{\theta}}_{\text{REAML}}$, these authors proposed

$$\text{mse}\left(\hat{\theta}_{\text{REAML},i}\right) = \begin{cases} \text{mse}\left(\hat{\theta}_{\text{AML},i}\right) & \text{if } \hat{A}_{\text{RE}} = 0, \\ \text{mse}\left(\hat{\theta}_{\text{RE},i}\right) & \text{if } \hat{A}_{\text{RE}} > 0. \end{cases} \tag{6.3}$$

Using $\text{mse}\left(\hat{\theta}_{\text{AML},i}\right)$ when $\hat{A}_{\text{RE}} = 0$ leads to substantial overestimation if the true value of A is small because $\hat{\theta}_{\text{AML},i}$ will be closer to the regression-synthetic estimator. Hence, we propose the alternative MSE estimator

$$\text{mse}_0\left(\hat{\theta}_{\text{REAML},i}\right) = \begin{cases} g_{2i} & \text{if } \hat{A}_{\text{RE}} = 0, \\ \text{mse}\left(\hat{\theta}_{\text{RE},i}\right) & \text{if } \hat{A}_{\text{RE}} > 0. \end{cases} \tag{6.4}$$

Again, since for small A, $\text{mse}\left(\hat{\theta}_{\text{RE},i}\right)$ might still be overestimating the true MSE of $\hat{\theta}_{\text{REAML},i}$, we consider also the following PT estimator

$$\text{mse}_{\text{PT}}\left(\hat{\theta}_{\text{REAML},i}\right) = \begin{cases} g_{2i} & \text{if } T \le X^2_{m-p,\alpha} \quad \text{or} \quad \hat{A}_{\text{RE}} = 0, \\ \text{mse}\left(\hat{\theta}_{\text{RE},i}\right) & \text{if } T > X^2_{m-p,\alpha} \quad \text{and} \quad \hat{A}_{\text{RE}} > 0. \end{cases} \tag{6.5}$$

7 Simulation experiments

A simulation study was designed with the following purposes in mind:

(a) To study the properties, in terms of bias and MSE, of the PT estimators as α varies for fixed A and as A varies for fixed α. We would like to see which values of α are adequate for a given A.

(b) To compare the PTEs with the EBLUPs based on REML and with the EBLUPs based on AML.

(c) To study the performance of the proposed MSE estimators in terms of relative bias and also in terms of coverage and length of prediction intervals.

(d) To compare the three introduced small area estimators that give strictly positive weight to the direct estimator for all areas, namely EBLUP based on AML, PT-AML and REML-AML estimators.

To accomplish the above goals, data were generated from the Fay-Herriot model given by (2.1)-(2.2) with a constant mean, that is, with $p = 1, \boldsymbol{\beta} = \mu$ and $\mathbf{x}_i = 1, i = 1, \ldots, m$. We let $\mu = 0$ without loss of generality, number of areas $m = 15$ and $D_i = 1, i = 1, \ldots, m$. The simulation study was repeated for increasing values of the model variance, $A \in \{0.01, 0.02, 0.05, 0.1, 0.2, 1\}$, and also for six significance levels of the test of $H_0 : A = 0$ against $H_0 : A > 0$, namely $\alpha = \{0.05, 0.1, 0.2, 0.3, 0.4, 0.5\}$. For each combination of A and α, the following steps were performed for each simulation run $\ell = 1, \ldots, L$ with $L = 10{,}000$ runs:

1. Generate data from the assumed model with constant zero mean; i.e.,

$$\begin{aligned} \theta_i^{(\ell)} &= v_i^{(\ell)}, \quad v_i^{(\ell)} \overset{\text{ind}}{\sim} N(0, A), \\ y_i^{(\ell)} &= \theta_i^{(\ell)} + e_i^{(\ell)}, \quad e_i^{(\ell)} \overset{\text{ind}}{\sim} N(0, D_i), \quad i = 1, \ldots, m. \end{aligned}$$

2. Calculate the following estimators of $\boldsymbol{\theta}$: the EBLUP based on REML estimation of $A, \hat{\boldsymbol{\theta}}_{\text{RE}}^{(\ell)}$, the PT estimate $\hat{\boldsymbol{\theta}}_{\text{PT}}^{(\ell)}$, the EBLUP based on AML estimation of $A, \hat{\boldsymbol{\theta}}_{\text{AML}}^{(\ell)}$, the combined PT-AML estimate $\hat{\boldsymbol{\theta}}_{\text{PTAML}}^{(\ell)}$ and the REML-AML estimate $\hat{\boldsymbol{\theta}}_{\text{REAML}}^{(\ell)}$.
3. For each area $i = 1, \ldots, m$, calculate: the three estimates of the MSE of the EBLUP $\hat{\theta}_{\text{RE},i}$ given in (3.2), (3.3) and (4.1), denoted respectively by $\text{mse}^{(\ell)}(\hat{\theta}_{\text{RE},i}), \text{mse}_0^{(\ell)}(\hat{\theta}_{\text{RE},i})$ and $\text{mse}_{\text{PT}}^{(\ell)}(\hat{\theta}_{\text{RE},i})$, and the three estimates (6.3), (6.4) and (6.5) of the MSE of the combined small area estimator $\hat{\theta}_{\text{REAML},i}$, denoted $\text{mse}^{(\ell)}(\hat{\theta}_{\text{REAML},i}), \text{mse}_0^{(\ell)}(\hat{\theta}_{\text{REAML},i})$ and $\text{mse}_{\text{PT}}^{(\ell)}(\hat{\theta}_{\text{REAML},i})$ respectively.
4. For each area $i = 1, \ldots, m$, obtain the normality-based $1 - \alpha$ prediction intervals for the small area mean θ_i based on the three considered MSE estimators of the EBLUP:

$$\begin{aligned} \text{CI}_i^{(\ell)} &= \hat{\theta}_{\text{RE},i}^{(\ell)} \mp Z_{\alpha/2} \sqrt{\text{mse}^{(\ell)}(\hat{\theta}_{\text{RE},i})}, \\ \text{CI}_{0,i}^{(\ell)} &= \hat{\theta}_{\text{RE},i}^{(\ell)} \mp Z_{\alpha/2} \sqrt{\text{mse}_0^{(\ell)}(\hat{\theta}_{\text{RE},i})}, \\ \text{CI}_{\text{PT},i}^{(\ell)} &= \hat{\theta}_{\text{RE},i}^{(\ell)} \mp Z_{\alpha/2} \sqrt{\text{mse}_{\text{PT}}^{(\ell)}(\hat{\theta}_{\text{RE},i})}, \end{aligned}$$

   where $Z_{\alpha/2}$ is the upper $\alpha/2$- point of a standard normal distribution.
5. Repeat Steps 1-4 for $\ell = 1, \ldots, L$, for $L = 10{,}000$. Then, for each small area estimator $\hat{\theta}_i \in \{\hat{\theta}_{\text{RE},i}, \hat{\theta}_{\text{PT},i}, \hat{\theta}_{\text{AML},i}, \hat{\theta}_{\text{PTAML},i}, \hat{\theta}_{\text{REAML},i}\}, i = 1, \ldots, m$, compute its empirical bias and MSE as

$$B(\hat{\theta}_i) = \frac{1}{L} \sum_{\ell=1}^{L} (\hat{\theta}_i^{(\ell)} - \theta_i^{(\ell)}), \quad \text{MSE}(\hat{\theta}_i) = \frac{1}{L} \sum_{\ell=1}^{L} (\hat{\theta}_i^{(\ell)} - \theta_i^{(\ell)})^2 .$$

   Then obtain the average over areas of absolute biases and MSEs as

$$\overline{\text{AB}}(\hat{\boldsymbol{\theta}}) = \frac{1}{m} \sum_{i=1}^{m} |B(\hat{\theta}_i)|, \quad \overline{\text{AMSE}}(\hat{\boldsymbol{\theta}}) = \frac{1}{m} \sum_{i=1}^{m} \text{MSE}(\hat{\theta}_i).$$

6. Calculate the relative bias of each MSE estimator, $\text{mse}(\hat{\theta}_i)$, as follows

$$\text{RB}\{\text{mse}(\hat{\theta}_i)\} = \left\{\frac{1}{L}\sum_{\ell=1}^{L}\text{mse}^{(\ell)}(\hat{\theta}_i) - \text{MSE}(\hat{\theta}_i)\right\} \Big/ \text{MSE}(\hat{\theta}_i).$$

Calculate the average over areas of the absolute relative biases as

$$\overline{\text{ARB}}\{\text{mse}(\hat{\boldsymbol{\theta}})\} = \frac{1}{m}\sum_{i=1}^{m}\left|\text{RB}\{\text{mse}(\hat{\theta}_i)\}\right|.$$

7. For each type of prediction interval $\text{CI}_i^{(\ell)} = (L_i^{(\ell)}, U_i^{(\ell)})$, for $\text{CI}_i^{(\ell)} \in \{\text{CI}_i^{(\ell)}, \text{CI}_{0,i}^{(\ell)}, \text{CI}_{\text{PT},i}^{(\ell)}\}$ given in Step 4, calculate the empirical coverage rate (CR) and the average length (AL) as

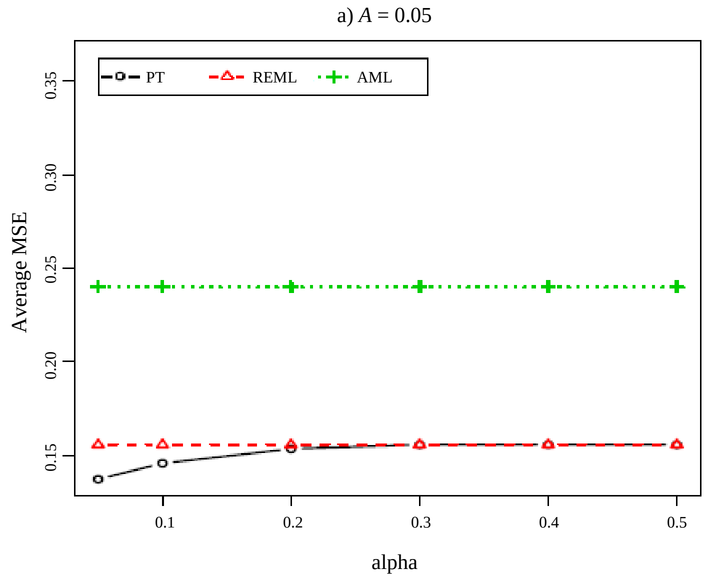
$$\text{CR}(\text{CI}_i) = \frac{\#\{\theta_i^{(\ell)} \in \text{CI}_i^{(\ell)}\}}{L}, \quad \text{AL}(\text{CI}_i) = \frac{1}{L}\sum_{\ell=1}^{L}(U_i^{(\ell)} - L_i^{(\ell)}).$$

Finally, average over areas the coverage rates and average lengths, as

$$\overline{\text{CR}}(\text{CI}) = \frac{1}{m}\sum_{i=1}^{m}\text{CR}(\text{CI}_i), \quad \overline{\text{AL}}(\text{CI}) = \frac{1}{m}\sum_{i=1}^{m}\text{AL}(\text{CI}_i).$$

Figures 7.1 and 7.2 plot the average MSEs of the PTEs for each $A \in \{0.05, 0.1, 0.2\}$, together with the average MSE of the EBLUPs based on REML and AML, against the significance level α. Note that when A is small, for large α the PT procedure is rejecting H_0 more often and therefore the PTE becomes more often the usual EBLUP, whereas for small α the PT procedure rejects H_0 less often and the regression-synthetic estimator is then more often used. In contrast, for a large value of A, the PTE becomes the EBLUP more frequently regardless of α. The absolute biases of the estimators are not shown here because they are roughly the same for all the PTEs across α values. The reason for this is that when the model holds, both components of the PTE, the synthetic estimator and the EBLUP, are unbiased for the target parameter. Note that the synthetic estimator is unbiased even when $A > 0$. The first conclusion arising from Figures 7.1 and 7.2 is that the MSE of the PTE is practically constant across $\alpha \geq 0.1$. See also that the average MSE of the PTE for a given α increases with A because the PTE reduces to the EBLUP more often as A increases and the MSE of the EBLUP increases with A. Observe also that the PTE and the EBLUP based on REML perform very similarly for $\alpha \geq 0.2$. However, for $\alpha < 0.2$, the PTE becomes more efficient than the EBLUP as soon as A moves close to the null hypothesis $(A < 0.1)$, which agrees with the remark of Datta et al. (2011).

Turning to the EBLUP based on AML, Figures 7.1 and 7.2 show that its average MSE is significantly larger than that of the other two estimators, but the differences with the other ones decrease as A increases. This is due to bias of the AML estimator of A for small A. We shall study later the combined small area estimators PT-AML and REML-AML, which use the EBLUP based on AML only when null hypothesis is not rejected or when the realized estimate of A is zero.



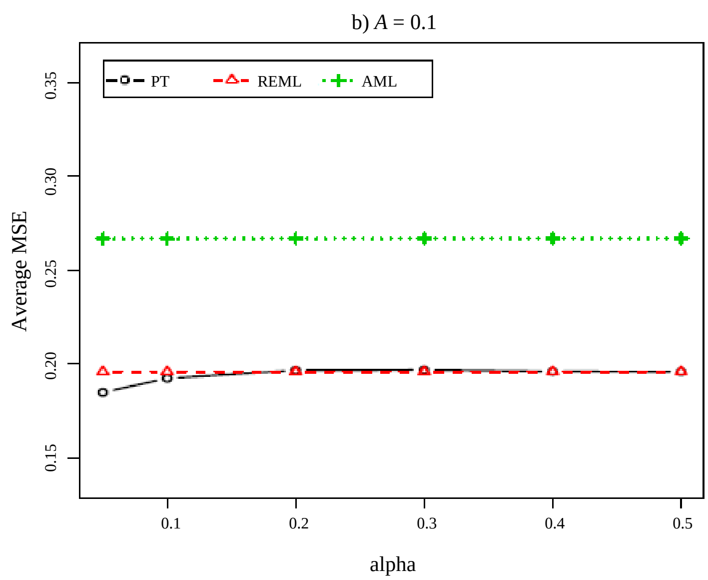


Figure 7.1 Average MSEs of PTE, EBLUP based on REML and EBLUP based on AML against α, for a) $A = 0.05$ and b) $A = 0.1$.

Datta et al. (2011, page 366) recommended $\alpha \geq 0.2$ for the PTE. Moreover, the literature on PT estimation for fixed effects models suggests that a good choice of α in terms of bias and MSE is $\alpha = 0.2$ (Bancroft 1944; Han and Bancroft 1968). But the above results suggest that for $\alpha \geq 0.2$, the PTE is practically the same as the EBLUP and therefore one might choose to always use the EBLUP.

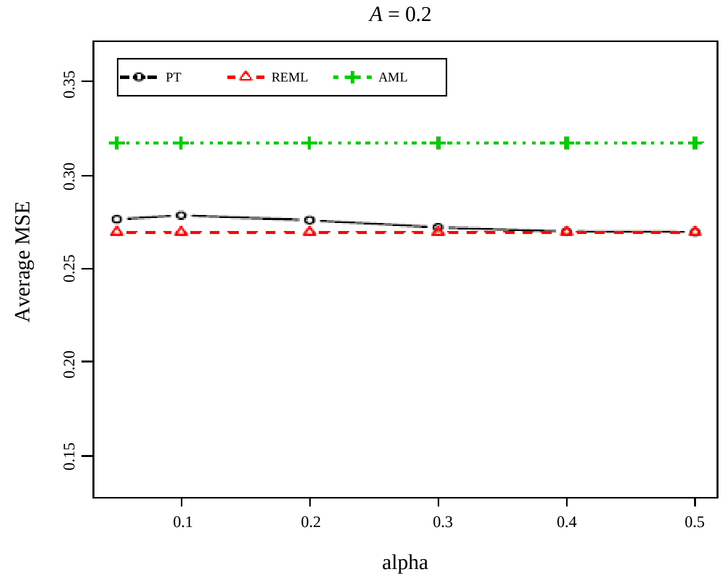


Figure 7.2 Average MSEs of PTE, EBLUP based on REML and EBLUP based on AML against α, for $A = 0.2$.

Now we study the properties of the PT for MSE estimation in terms of α. Figure 7.3 plots the average absolute relative bias of the MSE estimators $\text{mse}_{\text{PT}}(\hat{\theta}_{\text{RE},i})$ labelled PT, against the significance level α, for each value $A \in \{0.05, 0.1, 0.2, 1\}$. When α is taken very small $\alpha < 0.1$, the null hypothesis $H_0 : A = 0$ is less often rejected and $\text{mse}_{\text{PT}}(\hat{\theta}_{\text{RE},i})$ becomes often g_{2i}, which leads to underestimation. For α large $(\alpha > 0.2)$, the null hypothesis is more often rejected and $\text{mse}_{\text{PT}}(\hat{\theta}_{\text{RE},i})$ becomes the usual MSE estimator of the EBLUP, which severely overestimates the true MSE for small A. The value $\alpha = 0.2$ appears to be a good compromise choice, with an average absolute relative bias around 10% for $A \geq 0.1$ and 20% for $A = 0.05$.

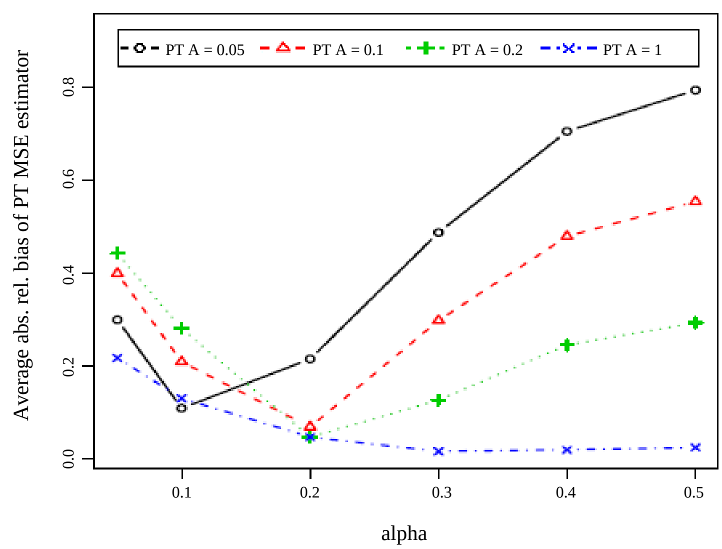
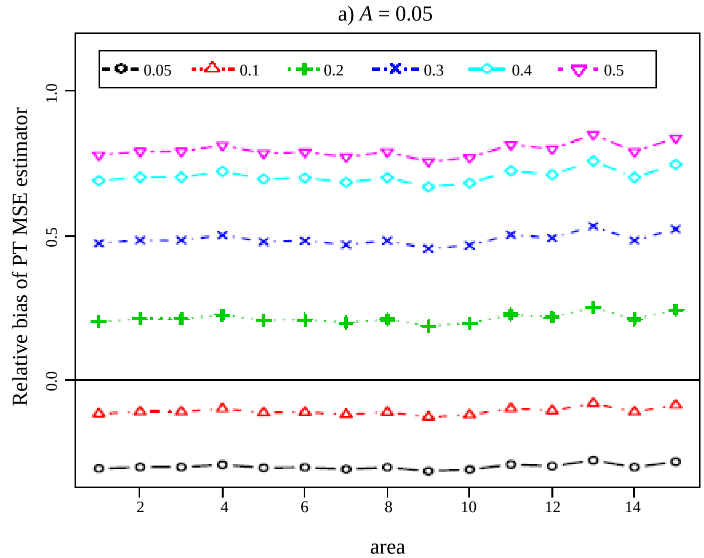


Figure 7.3 Average over areas of absolute relative biases of the MSE estimator $\text{mse}_{\text{PT}}(\hat{\theta}_{\text{RE},i})$, labelled PT, for $A \in \{0.05, 0.1, 0.2, 1\}$ against significance level α.

The above results suggest that $\alpha = 0.2$ is a good choice when using the PT procedure to estimate the MSE of the usual EBLUP. This has been more thoroughly studied by looking at the (signed) relative biases of $\text{mse}_{\text{PT}}\left(\hat{\theta}_{\text{RE},i}\right)$ for each area. These results are plotted in Figures 7.4 and 7.5 with four plots, one for each value of $A \in \{0.05, 0.1, 0.2, 1\}$. The figures appearing in the legends of these plots are the significance levels α for the PT MSE estimator $\text{mse}_{\text{PT}}\left(\hat{\theta}_{\text{RE},i}\right)$. These plots confirm our previous observations: the MSE estimator based on the PT, $\text{mse}_{\text{PT}}\left(\hat{\theta}_{\text{RE},i}\right)$, underestimates $\text{MSE}\left(\hat{\theta}_{\text{RE},i}\right)$ for small α and overestimates for large α. It turns out that $\text{mse}_{\text{PT}}\left(\hat{\theta}_{\text{RE},i}\right)$ with $\alpha = 0.2$ is a good candidate for all values of A.



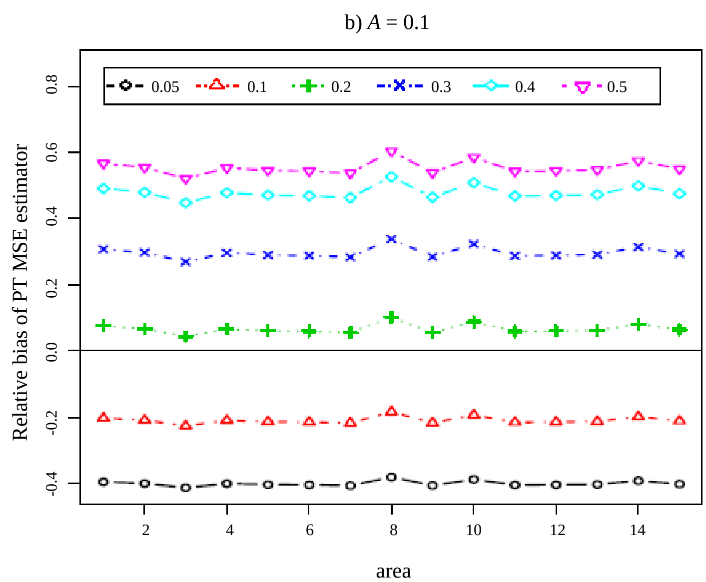


Figure 7.4 Relative biases of $\text{mse}_{\text{PT}}\left(\hat{\theta}_{\text{RE},i}\right)$, for each significance level $\alpha \in \{0.05, 0.1, 0.2, 0.3, 0.4, 0.5\}$, against area i, for a) $A = 0.05$ and b) $A = 0.1$.

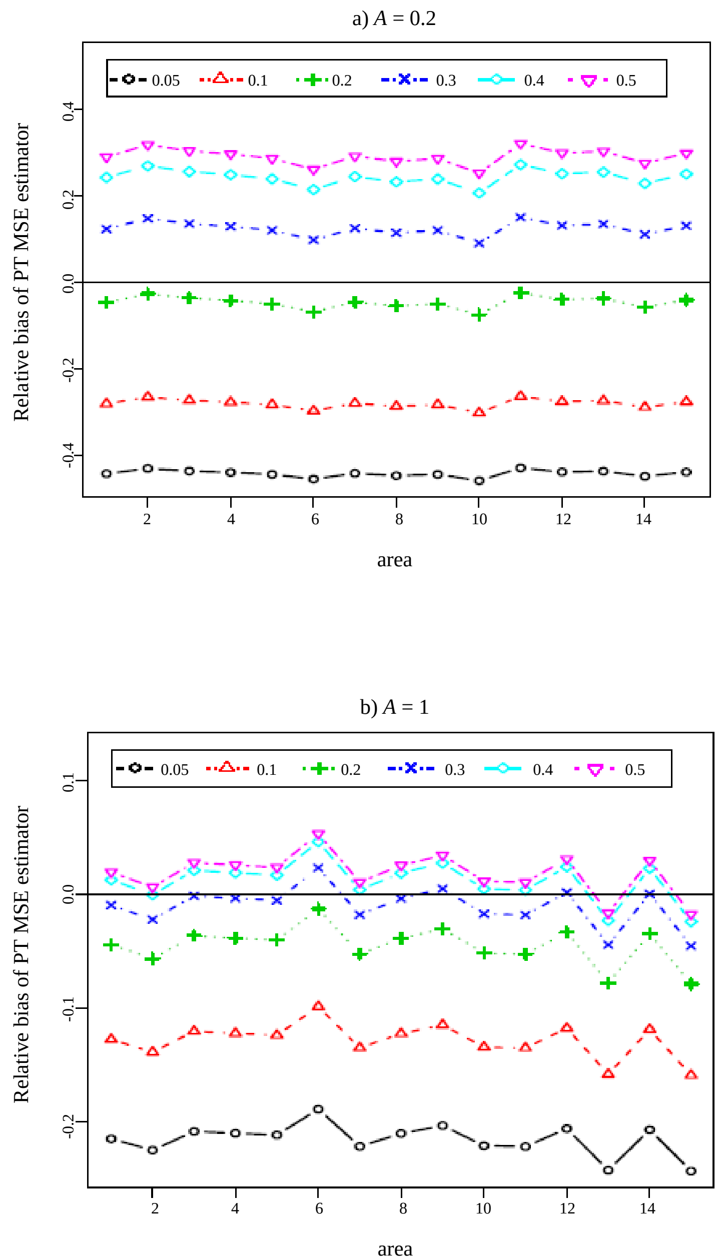


Figure 7.5 Relative biases of $\mathrm{mse}_{\mathrm{PT}}(\hat{\theta}_{\mathrm{RE},i})$, for each significance level $\alpha \in \{0.05, 0.1, 0.2, 0.3, 0.4, 0.5\}$, against area i, for a) $A = 0.2$ and b) $A = 1$.

Let us now compare $\mathrm{mse}_{\mathrm{PT}}(\hat{\theta}_{\mathrm{RE},i})$ for the selected significance level $\alpha = 0.2$ with the other two MSE estimators $\mathrm{mse}_0(\hat{\theta}_{\mathrm{RE},i})$ and $\mathrm{mse}(\hat{\theta}_{\mathrm{RE},i})$ given by (3.3) and (3.2) respectively. Figure 7.6 plots the average absolute relative biases of the three MSE estimators, labelled respectively PT, REML0 and REML. We note that $\mathrm{mse}_0(\hat{\theta}_{\mathrm{RE},i})$ performs better than $\mathrm{mse}(\hat{\theta}_{\mathrm{RE},i})$ for all areas, but still $\mathrm{mse}_{\mathrm{PT}}(\hat{\theta}_{\mathrm{RE},i})$ is better than $\mathrm{mse}_0(\hat{\theta}_{\mathrm{RE},i})$ for all considered values of A except for $A = 1$, where the differences between the three estimators are negligible. The differences decrease as A increases, but observe that the usual MSE estimator, $\mathrm{mse}(\hat{\theta}_{\mathrm{RE},i})$, can be severely biased for small A, with an average absolute relative bias over 50% for $A < 0.2$ and exponentially growing as A tends to zero. The conclusion is that, when H_0 is not rejected, even if the realized estimate of A is positive, it seems better to omit the g_{3i} term in the MSE estimator and consider only g_{2i}.

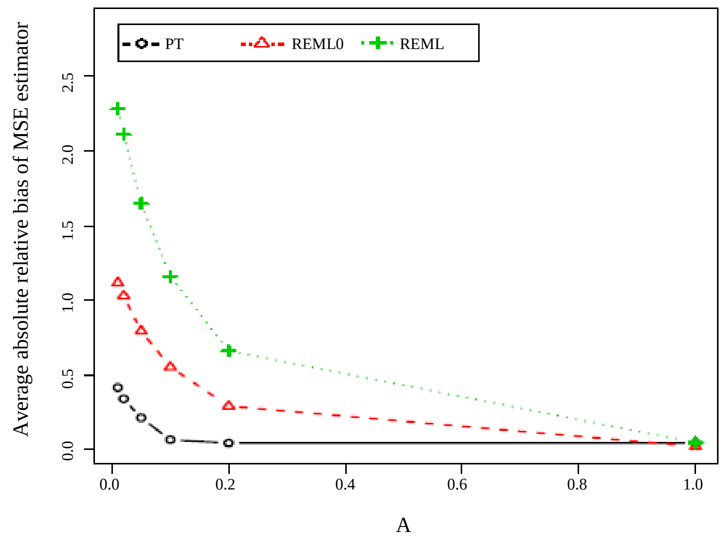


Figure 7.6 Average over areas of absolute relative biases of MSE estimators $\text{mse}_{\text{PT}}\left(\hat{\theta}_{\text{RE},i}\right)$ with $\alpha = 0.2$, labelled PT, $\text{mse}\left(\hat{\theta}_{\text{RE},i}\right)$ labelled REML and $\text{mse}_0\left(\hat{\theta}_{\text{RE},i}\right)$ labelled REML0, against A.

We now turn to the small area estimators that attach strictly positive weight to the direct estimator for all areas: EBLUP based on AML, $\hat{\boldsymbol{\theta}}_{\text{AML}}$, and the two combined estimators, PT-AML given in (6.1), and REML-AML given in (6.2). Average MSEs are plotted in Figure 7.7 for these three estimators. In this plot, $\hat{\boldsymbol{\theta}}_{\text{AML}}$ seems to be a little less efficient, followed by PT-AML. The combined estimator REML-AML seems to perform slightly better than its two counterparts for small A, although for $A \geq 0.2$ the PT-AML estimator is very close to it. For MSE estimation, we focus on REML-AML because of its better performance.

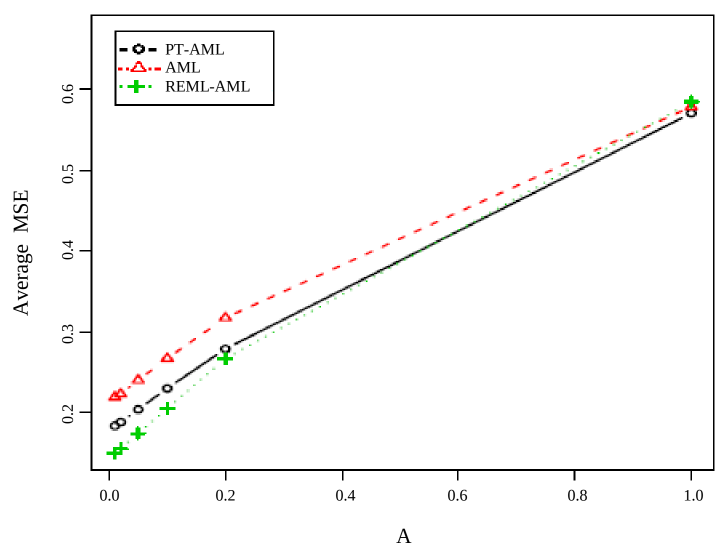


Figure 7.7 Average over areas of MSEs of PT-AML estimator with $\alpha = 0.2$, EBLUP based on AML and REML-AML estimator against A.

For the combined estimator REML-AML, Figure 7.8 shows that the MSE estimator based on the PT, $\text{mse}_{\text{PT}}(\hat{\theta}_{\text{REAML},i})$, which uses only g_{2i} whenever $\hat{A}_{\text{RE}} = 0$ or the null hypothesis is not rejected, has average absolute relative bias less than 10% for $A \geq 0.1$ and it is smaller than the corresponding values for $\text{mse}(\hat{\theta}_{\text{REAML},i})$ and $\text{mse}_0(\hat{\theta}_{\text{REAML},i})$, especially for $A \leq 0.4$.

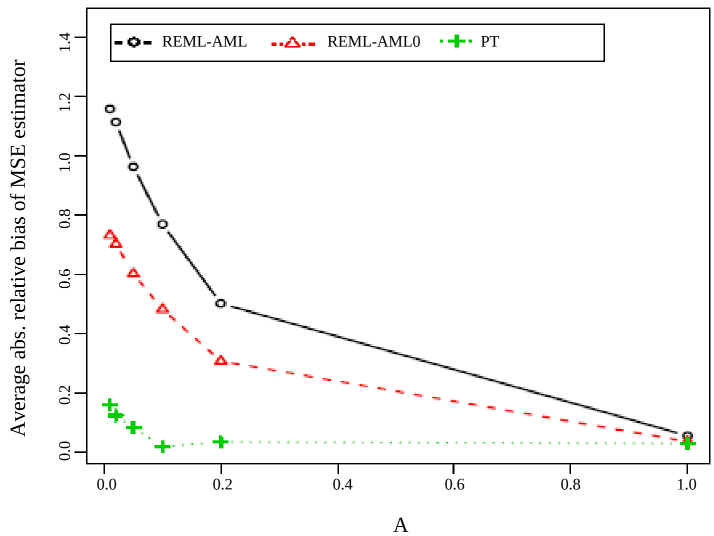


Figure 7.8 Average over areas of absolute relative biases of the MSE estimators $\text{mse}(\hat{\theta}_{\text{REAML},i})$, $\text{mse}_0(\hat{\theta}_{\text{REAML},i})$ and $\text{mse}_{\text{PT}}(\hat{\theta}_{\text{REAML},i})$, labelled respectively REML-AML, REML-AML0 and PT, against A.

Finally, we analyze the average over areas of coverage rates and average lengths of normality-based prediction intervals for the small area mean θ_i using the EBLUP based on REML as point estimate and the three different MSE estimators of the EBLUP, namely $\text{mse}(\hat{\theta}_{\text{RE},i}), \text{mse}_0(\hat{\theta}_{\text{RE},i})$ and $\text{mse}_{\text{PT}}(\hat{\theta}_{\text{RE},i})$. Figure 7.9 shows the coverage rates of these three types of intervals, where the MSE estimators based on the PT procedure were obtained taking $\alpha = 0.2, 0.3$. It seems that the good relative bias properties of the MSE estimator based on the PT, $\text{mse}_{\text{PT}}(\hat{\theta}_{\text{RE},i})$, for small A cannot be extrapolated to coverage based on normal prediction intervals, showing undercoverage especially for $A = 0.2$. In this case, taking a larger significance level $\alpha = 0.3$ reduces a little the undercoverage of the prediction intervals obtained using $\text{mse}_{\text{PT}}(\hat{\theta}_{\text{RE},i})$. Still, the coverage rates of $\text{mse}_0(\hat{\theta}_{\text{RE},i})$ are better for all values of A. As expected, the usual MSE estimator $\text{mse}(\hat{\theta}_{\text{RE},i})$ provides overcoverage for small values of A, which is due to the severe overestimation of the MSE. On the other hand, the intervals showing undercoverage also lead to shorter prediction intervals as shown by Figure 7.10.

It is worthwhile to mention that the construction of prediction intervals for θ_i based on the Fay-Herriot model with accurate coverage rates is not an obvious task. Several papers have appeared in the literature for this problem. For example, Chatterjee, Lahiri and Li (2008) proposed prediction intervals with second order correct coverage rate using only the g_{1i} term as MSE estimate and applying a bootstrap procedure to find the calibrated quantiles. Diao, Smith, Datta, Maiti and Opsomer (2014) have recently obtained prediction intervals with second order correct coverage rate avoiding the use of resampling procedures and using the full MSE estimator. Obtaining prediction intervals with accurate coverage using other MSE estimates is still a challenge and it is out of scope of this paper.

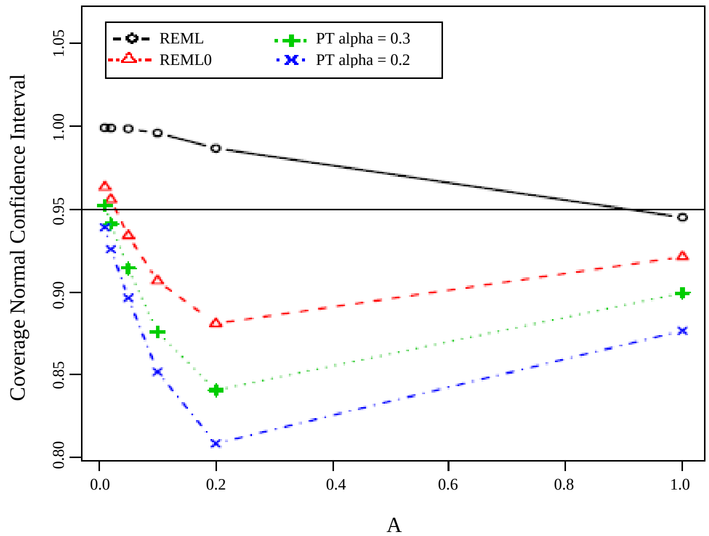


Figure 7.9 Average over areas of coverage rates of normality-based prediction intervals for θ_i using the MSE estimators $\text{mse}(\hat{\theta}_{\text{RE},i})$, $\text{mse}_0(\hat{\theta}_{\text{RE},i})$ and $\text{mse}_{\text{PT}}(\hat{\theta}_{\text{RE},i})$ with $\alpha = 0.2, 0.3,$ labelled respectively REML, REML0 and PT, against A.

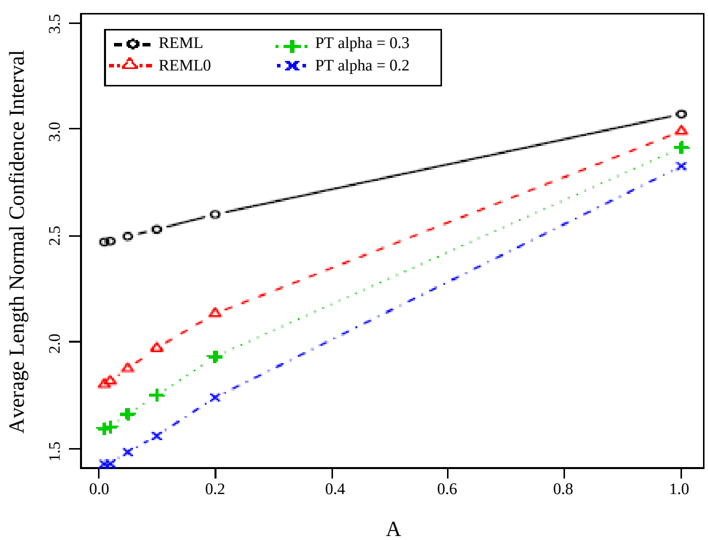


Figure 7.10 Average over areas of average lengths of normality-based intervals for θ_i using the MSE estimators $\text{mse}(\hat{\theta}_{\text{RE},i})$, $\text{mse}_0(\hat{\theta}_{\text{RE},i})$ and $\text{mse}_{\text{PT}}(\hat{\theta}_{\text{RE},i})$ with $\alpha = 0.2, 0.3,$ labelled respectively REML, REML0 and PT, against A.

This simulation study described above was repeated for several patterns of unequal sampling variances D_i. Although results are not reported here, conclusions are very similar as long as the variance pattern is not extremely uneven.

8 Conclusions

The following major conclusions may be drawn from the results of our simulation study on the estimation of small area means, based on the Fay-Herriot area-level model when the number of areas is modest in size $(\text{say } m = 15)$: 1) Under the Fay-Herriot model with a value of random effects variance, A, clearly away from zero, the PTE does not seem to noticeably improve efficiency relative to the usual EBLUP unless the significance level is taken small $(\alpha \leq 0.1$ in our simulation study). 2) Our simulation results indicate that using the PT procedure with a moderate α, in particular $\alpha = 0.2$, to estimate the MSE of the usual EBLUP leads to a reduction in bias as compared with the usual MSE estimator. Hence, we recommend the use of $\text{mse}_{\text{PT}}(\hat{\theta}_{\text{RE},i})$, given by (4.1), to estimate the MSE of the EBLUP. 3) Among the estimators that attach a strictly positive weight to the direct estimator for all areas, we recommend the combined estimator REML-AML given by (6.2), because it achieves slightly higher efficiency than the EBLUP based on AML and the PT-AML given by (6.1). 4) For estimating the MSE of the recommended REML-AML estimator, the estimator $\text{mse}_{\text{PT}}(\hat{\theta}_{\text{REAML},i})$ given by (6.5) performs better than the alternative ones. 5) Our results on prediction intervals, based on normal theory, indicate that the good performance of the proposed MSE estimators may not translate to coverage properties of these intervals. Construction of prediction intervals that lead to accurate coverages, using the proposed MSE estimates, appears to be a difficult task.

Smooth alternatives to the preliminary test estimates in the case of location parameters have been proposed in the literature using weighted means of the estimates obtained under the null and alternative hypotheses, with weights depending on the test statistic, see e.g., Saleh (2006). Mean squared error estimates of this kind have not been studied and we leave this subject for further research.

Acknowledgements

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