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Statistical tests for the assessment of single channel and imaging standoff radiation detector sensitivity

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Defence R&D Canada – Ottawa

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Abstract

Evaluating different radiation detection technologies, with the intent of either directing research and development or equipment acquisition, involves the consideration of many criteria, the suitability of the detection sensitivity being normally of prior importance. This report presents two general statistical methods allowing the evaluation of the detection sensitivity for different technologies. These methods rely on a limited set of assumptions, thus allowing more accurate sensitivity estimates in scenarios where the signal and background levels are low. One of the models also supports sensitivity estimation for imaging detectors, under the same limited assumptions.

Résumé

L'évaluation de différentes technologies pour la détection de radiations, en vue de diriger des efforts de recherche et de développement ou encore l'acquisition d'équipement, implique de considérer plusieurs critères, ce qui inclue habituellement la convenabilité de la sensibilité de détection, et ce de façon prioritaire. Ce rapport présente deux méthodes statistiques généralistes qui permettent l'évaluation de la sensibilité de détection pour différentes technologies. Ces méthodes reposent sur un ensemble limité d'hypothèses et permettent ainsi d'estimer de façon plus précise cette sensibilité pour des scénarios où les niveaux de signal et de bruit sont faibles. Un des modèles supporte aussi l'estimation de la sensibilité des détecteurs d'imagerie, en utilisant toujours les mêmes hypothèses.

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Executive summary

Statistical tests for the assessment of single channel and imaging standoff radiation detector sensitivity

Pierre-Luc Drouin; DRDC Ottawa CR 2013-123; Defence R&D Canada – Ottawa; December 2013.

Introduction and background: The Chemical, Biological and Radiological (CBR) Memorandum Of Understanding (MOU) International Task Force (ITF) – 53 on pre-event radiological standoff detection was tasked to explore options, gather relevant technological data and develop a way forward to achieve the standoff detection of radiological hazards. The evaluation of different radiation detection technologies in the context of a standoff detection application can be quite challenging due to the small signal to background ratios and absolute signal strengths that are often involved. In order to assess the relative merits of such technologies, evaluating their respective detection sensitivity is of prime importance.

Results and significance: The author of this note developed and implemented two statistical tests that allow evaluating the detection sensitivity of different technologies. These methods rely on a limited set of assumptions, thus allowing more accurate sensitivity estimates in scenarios where the signal and background levels are low. One of the models also supports the sensitivity estimation for scenarios involving imaging detectors with multiple channels, which allow to better constrain the background level in addition to improving the signal to background ratio. These tests were used to evaluate the numerous scenarios considered by ITF-53.

Sommaire

Statistical tests for the assessment of single channel and imaging standoff radiation detector sensitivity

Pierre-Luc Drouin ; DRDC Ottawa CR 2013-123 ; R & D pour la défense Canada – Ottawa ; décembre 2013.

Introduction et mise en contexte : Le groupe de travail international lié au protocole d'accord Chimique, Biologique et Radiologique (CBR) (ITF-53) sur la détection radiologique à distance pré-événement a été mandaté d'explorer les options, de rassembler les données technologiques pertinentes et de développer un plan afin de parvenir à la détection à distance de risques radiologiques. L'évaluation de différentes technologies de détection radiologiques dans le contexte d'une application de détection à distance peut être un défi, considérant les faibles rapports signal à bruit et intensités absolues des signaux qui sont sont habituellement impliqués. Afin d'estimer la valeur relative de ces technologies, l'évaluation de leur sensibilité de détection respective représente un élément clef.

Résultats et importance : L'auteur de cette note a développé ainsi qu'implémenté deux tests statistiques permettant d'évaluer la sensibilité de détection de différentes technologies. Ces méthodes reposent sur un ensemble limité d'hypothèses et permettent ainsi d'estimer de façon plus précise cette sensibilité pour des scénarios où les niveaux de signal et de bruit sont faibles. Un des modèles supporte aussi l'estimation de la sensibilité des détecteurs d'imagerie, en utilisant toujours les mêmes hypothèses. Ces tests ont été utilisés afin d'évaluer les nombreux scénarios considérés par ITF-53.

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1 Introduction

The evaluation of different radiation detection technologies in the context of a standoff detection application can be quite challenging due to the small signal to background ratios and absolute signal strength that are often involved. In order to assess the relative merits of such technologies, evaluating their respective detection sensitivity is of prime importance. There are different criteria that can be considered in the assessment of these sensitivities, that normally involve the probability of false alarm as well as the probability of alarming in the presence of a signal. This note presents a pair of algorithms that allow to evaluate the sensitivity of different types of radiation detectors. The first model consists of a "brute force" approach that directly evaluates the sensitivity of single channel detectors based on a target probability of false alarm. The other model uses a Bayesian estimator to evaluate the sensitivity of imaging-type detectors using a criterion which is strongly correlated to the probability of false alarm. Because these tests are meant to be applicable to various technologies, their models do not aim at simulating the instrumental behaviour of specific detectors. Both models rely on a common set of principles:

1. The production rate for the radiation background is assumed to be normally distributed, such that the expectation value for the detected background rate is μ_{b}, with a detection rate uncertainty due to the background fluctuation given by σ_{b}.

2. Signal and background detection are affected by Poisson statistical fluctuations that are approximately Gaussian for large detection rates, but that have significantly asymmetrical distributions when the number of counts per detection channel is low.

3. Detection efficiencies are evaluated by computing the fraction of simulation measurements above some alarm thresholds. These thresholds are determined by evaluating the minimum number of detected events which results in a false alarm probability that does not exceed a target level. A given detector configuration is assumed to be effective at detecting a radioactive source when the presence of such a source is expected to trigger the detector with a pre-established probability.

2 Single Channel Detectors

Establishing the statistical model to evaluate the sensitivity of single channel radiation detectors is relatively simple. Given the parameters μ_{b} and σ_{b} related to the background production rate such as previously defined, the probability mass function (PMF) for the detection of background events is given by

$$\begin{aligned} P(n_{\mathrm{b}}|\mu_{\mathrm{b}},\sigma_{\mathrm{b}}) &= \int_{-\infty}^{\infty} f(n_{\mathrm{b}},\nu_{\mathrm{b}}|\mu_{\mathrm{b}},\sigma_{\mathrm{b}})\,d\nu_{\mathrm{b}} \\ &= \int_{-\infty}^{\infty} P(n_{\mathrm{b}}|\nu_{\mathrm{b}})\,f(\nu_{\mathrm{b}}|\mu_{\mathrm{b}},\sigma_{\mathrm{b}})\,d\nu_{\mathrm{b}}, \end{aligned} \tag{1}$$

where ν_b is the parameter for the Poisson fluctuations of the detected number of background events. $f(n_b, \nu_b|\mu_b, \sigma_b)$ is the joint probability density function (PDF) of n_b and ν_b, given μ_b and σ_b. The second line was obtained using the definition of conditional probability and noting that $P(n_b|\nu_b, \mu_b, \sigma_b) = P(n_b|\nu_b)$. $P(n_b|\nu_b)$ is the Poisson PMF

$$P(n_b|\nu_b) = \frac{e^{-\nu_b}{\nu_b}^{n_b}}{n_b!} \tag{2}$$

and $f(\nu_b|\mu_b, \sigma_b)$ is the Gaussian PDF

$$f(\nu_b|\mu_b, \sigma_b) = \frac{1}{\sqrt{2\pi}\sigma_b} e^{-\frac{(\nu_b-\mu_b)^2}{2\sigma_b^2}}. \tag{3}$$

From Equations (1) to (3), the PMF for the detection of background events is thus given by

$$P(n_b|\mu_b, \sigma_b) \propto \int_0^\infty \frac{e^{-\nu_b}{\nu_b}^{n_b}}{n_b!} \frac{1}{\sqrt{2\pi}\sigma_b} e^{-\frac{(\nu_b-\mu_b)^2}{2\sigma_b^2}} d\nu_b, \tag{4}$$

where the proportionality symbol has been added to signify that the PMF is not exactly normalised, due to the (normally small) probability of ν_b to be negative.

To ensure a given false alarm probability, the alarm threshold for the number of detected events must be determined. This threshold corresponds to the smallest positive integer n_t that satisfies

$$P(n_b \geq n_t|\mu_b, \sigma_b) = \sum_{n_b=n_t}^{\infty} P(n_b|\mu_b, \sigma_b) \leq P(\text{false alarm}). \tag{5}$$

Because the integral of Equation (4) cannot be solved analytically, a natural method to determine n_t is to use Monte Carlo integration. At the limit where the number of generated Monte Carlo events n_{gen} tends to infinity, the method described by Equations (4) and (5) is thus equivalent to the following procedure:

1. Generate random values ν_b distributed according to $N(\mu_b, \sigma_b^2)$ until a positive value is drawn.

2. Generate a random value n_b using a Poisson distribution with parameter ν_b.

3. Store the value n_b in a histogram if the number of n_b values generated so far is smaller than $\lceil n_{gen} P(\text{false alarm}) + 1 \rceil$, or if n_b is greater or equal to the smallest value currently stored in the histogram.

4. If a new measurement was stored in the histogram, verify if the number of measurements in the first bin of the histogram is smaller or equal to the excess in the total number of measurements in the histogram compared to $\lceil n_{gen} P(\text{false alarm}) + 1 \rceil$. If this condition is met, discard the first bin of the histogram.

5. Repeat the previous step while bins keep being discarded.

6. If the number of ν_b values generated so far is smaller than n_{gen}, go back to Step 1.

7. The estimated value for n_t is given by the smallest value stored in the histogram, plus one.

8. The estimated value for the false alarm probability resulting from the selected threshold, $\hat{P}(n_b \geq n_t|\mu_b, \sigma_b)$, is given by the number of measurements contained in the whole histogram, except in the first bin, divided by n_{gen}. Its expectation value is lower than the target false alarm probability, due to the quantised n_b parameter.

9. From binomial probability theory, the uncertainty on the calculated false alarm probability for the selected threshold is approximated by

$$\sqrt{\frac{\hat{P}(n_b \geq n_t|\mu_b, \sigma_b)[1 - \hat{P}(n_b \geq n_t|\mu_b, \sigma_b)]}{n_{gen}}}. \tag{6}$$

   This uncertainty can be used to determine if the number of generated measurements n_{gen} is sufficiently large to set the alarm threshold accurately.

3 Imaging Detectors

Developing a statistical model to determine an alarm threshold is obviously more difficult in the case of an imaging detector due to the presence of multiple channels. It can be also highly beneficial for such detectors to use the data collected in the channels that are only exposed to background events to measure the current background production rate. For imaging detectors, a "brute force" Monte Carlo approach, such as the one described in the previous section, would be difficult to apply, due to the very large number of simulation measurements that would be required and also due to the very important numerical truncation errors that can easily occur when attempting to compute the probabilities of individual simulation measurements as a metric to determine an alarm threshold.

In the case of imaging detectors, the construction of a Bayesian statistical model combined with the usage of a Markov Chain Monte Carlo (MCMC) method allows to establish an alarm threshold based on a target false alarm probability. Effectively, requiring a given false alarm probability is very closely related to evaluating the probability of the signal production rate to not being strictly positive. This thus leads to an interest for the distribution of the signal and background production rate parameters, which can be described by a Bayesian statistical model and then sampled by an MCMC algorithm. Using a notation similar to the one used in the previous section, Bayes' theorem allows to write the PDF for ν_s and ν_b, which constitute the parameters for the Poisson fluctuations of the detected number of signal and background events, respectively:

$$\begin{aligned} f(\nu_s,\nu_b|n_{sb},n_b,\mu_b,\sigma_b) =& \frac{f(n_{sb},n_b,\nu_s,\nu_b|\mu_b,\sigma_b)}{P(n_{sb},n_b|\mu_b,\sigma_b)} = \frac{P(n_{sb},n_b|\nu_s,\nu_b,\mu_b,\sigma_b)f(\nu_s,\nu_b|\mu_b,\sigma_b)}{P(n_{sb},n_b|\mu_b,\sigma_b)} \\ =& P(n_{sb},n_b|\nu_s,\nu_b)f(\nu_b|\mu_b,\sigma_b)\frac{f(\nu_s)}{P(n_{sb},n_b|\mu_b,\sigma_b)}. \end{aligned} \tag{7}$$

In the above equation, n_{sb} represents the total number of detected events in the channels that are assumed to be exposed to signal events and n_b is the number of detected events in the remaining channels (which are only exposed to background events). $P(n_{sb},n_b|\nu_s,\nu_b)$ represents the likelihood function describing the current model. $f(\nu_b|\mu_b,\sigma_b)$ is a prior PDF for the parameter ν_b, which is given by Equation (3). $f(\nu_s)$ is the prior for the Poisson parameter associated to the radioactive source, which is assumed to be flat. Note that ν_s corresponds to the expectation value for the average number of detected signal events per channel among all signal channels. In this model it is assumed that all channels have the same sensitivity and that they are all equally exposed to background events. The PMF $P(n_{sb},n_b|\mu_b,\sigma_b)$ does not depend on the estimated parameters of the model, such that it represents a constant scaling factor of the PDF $f(\nu_s,\nu_b|n_{sb},n_b,\mu_b,\sigma_b)$ for a given measurement. Because of the additive property of Poisson-distributed random variables, the total number of detected events within all background-only channels follows a Poisson distribution with parameter $(n_c-n_{sc})\nu_b$, where n_c is the total number of channels for the imager, where n_{sc} is the number of signal channels and where ν_b is the parameter for the Poisson fluctuations of the background events per channel. In the case of the signal channels, the number of detected events follows a Poisson distribution with parameter $n_{sc}(\nu_s+\nu_b)$, such that the likelihood function is given by

$$\begin{aligned} P(n_{sb},n_b|\nu_s,\nu_b) =& P(n_{sb}|\nu_s+\nu_b)P(n_b|\nu_b) \\ =& \frac{e^{-n_{sc}(\nu_s+\nu_b)}[n_{sc}(\nu_s+\nu_b)]^{n_{sb}}}{n_{sb}!}\frac{e^{-(n_c-n_{sc})\nu_b}[(n_c-n_{sc})\nu_b]^{n_b}}{n_b!} \\ =& \frac{e^{-(n_{sc}\nu_s+n_c\nu_b)}[(n_c-n_{sc})\nu_b]^{n_b}[n_{sc}(\nu_s+\nu_b)]^{n_{sb}}}{n_{sb}!n_b!}. \end{aligned} \tag{8}$$

In the above model, determining the most likely production rate parameters $(\hat{\nu}_s,\hat{\nu}_b)$ associated to a given measurement can be achieved analytically. Due to the properties of the logarithm function, including its monotonicity, finding the position of the maximum for $f(\nu_s,\nu_b|n_{sb},n_b,\mu_b,\sigma_b)$ is equivalent to finding the position of the maximum for its logarithm. Expanding (7) using Equations (3) and (8), evaluating its logarithm and regrouping the terms that are constant for a given measurement leads to the expression

$$\begin{aligned} \log f(\nu_s,\nu_b|n_{sb},n_b,\mu_b,\sigma_b) =& -n_{sc}\nu_s - n_c\nu_b + n_b\log\nu_b + n_{sb}\log(\nu_s+\nu_b) - \\ & \frac{(\nu_b-\mu_b)^2}{2\sigma_b^2} + C, \end{aligned} \tag{9}$$

where C does not depend on either ν_s or ν_b. Note that the above expression does not evaluate to a real value when $\nu_b \leq 0$ or when $\nu_s + \nu_b \leq 0$. However, the third and fourth terms in the expression evaluate to 0 when $n_{sb} = 0$ and $\nu_b = 0$ or when $n_{sb} = 0$ and $\nu_s + \nu_b = 0$, because the Poisson PMFs at the origin of these terms must evaluate to 1 in these conditions. Negative values for ν_b are non-physical, but forcing positive values would lead to a positively biased signal estimator. Such negative values do not present a problem, as long as the physical region is not excluded by the estimator [1]. Enforcing non-negative ν_b values cannot be mathematically avoided, but the negative region should normally be mostly excluded by Equation (3).

The position of the maximum within the parameter space can thus now be determined:

$$\left.\frac{\partial \log f}{\partial \nu_s}\right|_{(\hat{\nu}_s,\hat{\nu}_b)} = -n_{sc} + \frac{n_{sb}}{\hat{\nu}_s + \hat{\nu}_b} = 0 \qquad \Rightarrow \hat{\nu}_s + \hat{\nu}_b = \frac{n_{sb}}{n_{sc}} \tag{10}$$

$$\left.\frac{\partial \log f}{\partial \nu_b}\right|_{(\hat{\nu}_s,\hat{\nu}_b)} = -n_c + \frac{n_b}{\hat{\nu}_b} + \frac{n_{sb}}{\hat{\nu}_s + \hat{\nu}_b} - \frac{\hat{\nu}_b - \mu_b}{\sigma_b^2} = 0 \quad \Rightarrow \frac{1}{\sigma_b^2}\hat{\nu}_b^2 + \left[(n_c - n_{sc}) - \frac{\mu_b}{\sigma_b^2}\right]\hat{\nu}_b - n_b = 0. \tag{11}$$

In the above expressions, the first equation was used to simplify the second. In Equation (11), the second order polynomial in $\hat{\nu}_b^2$ can be solved, leading to

$$\hat{\nu}_b = \left[-B + \sqrt{B^2 + 4\frac{n_b}{\sigma_b^2}}\right]\frac{\sigma_b^2}{2}, \quad B \equiv (n_c - n_{sc}) - \frac{\mu_b}{\sigma_b^2}, \tag{12}$$

after using the positive prior for $\hat{\nu}_b$.

It was thus possible to find an analytical solution that provides estimators for the Poisson parameters related to the signal and background production rates. These estimators correspond to the most likely values for the Poisson parameters, based on a given measurement. However, if one wants to estimate the probability of the signal production rate to not be strictly positive, integrating the left tail of the parameter PDF is of primary interest. Because the parameter PDF is non-Gaussian (see Equation (9)), particularly when ν_s and ν_b are small, using an MCMC method to estimate this probability appears to constitute a potentially efficient technique. Since it was possible to analytically find the position of the peak in the parameter PDF, it seems appropriate to use an MCMC proposal distribution defined around this peak, rather than being function of the previously accepted point. The proposal distribution is thus asymmetrical (unless a flat distribution is chosen) such that the more general Metropolis-Hasting [2] algorithm constitutes a natural MCMC method choice.

In order to properly sample the parameter space, MCMC methods require a proposal distribution. Since the Metropolis-Hasting algorithm does not require a symmetrical jumping function, it appears reasonable to use for this purpose a bivariate normal distribution centred around the peak of the parameter PDF:

$$q(\nu_s,\nu_b|\hat{\nu}_s,\hat{\nu}_b,\sigma_s,\sigma_b,\rho) = \frac{1}{2\pi\sigma_s\sigma_b\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(\nu_s-\hat{\nu}_s)^2}{\sigma_s^2}+\frac{(\nu_b-\hat{\nu}_b)^2}{\sigma_b^2}-\frac{2\rho(\nu_s-\hat{\nu}_s)(\nu_b-\hat{\nu}_b)}{\sigma_s\sigma_b}\right]}$$

$$\log q(\nu_s,\nu_b|\hat{\nu}_s,\hat{\nu}_b,\sigma_s,\sigma_b,\rho) = -\frac{1}{2(1-\rho^2)}\left[\frac{(\nu_s-\hat{\nu}_s)^2}{\sigma_s^2}+\frac{(\nu_b-\hat{\nu}_b)^2}{\sigma_b^2}- \frac{2\rho(\nu_s-\hat{\nu}_s)(\nu_b-\hat{\nu}_b)}{\sigma_s\sigma_b}\right]+C', \tag{13}$$

where σ_s and σ_b are the standard deviations for the signal and background Poisson parameters, respectively, where ρ is the correlation factor of the parameter PDF and C' is a constant that does not depend on either ν_s or ν_b. It is possible to estimate these three parameters based on the analytical calculation of the second order partial derivatives of the parameter PDF logarithm at the peak. From Equation (13),

$$\frac{\partial^2 \log q}{\partial \nu_s^2} = -\frac{1}{1-\rho^2}\frac{1}{\sigma_s^2} \quad \Rightarrow \quad \sigma_s^2 = -\frac{1}{1-\rho^2}\left(\frac{\partial^2 \log q}{\partial \nu_s^2}\right)^{-1} \tag{14}$$

$$\frac{\partial^2 \log q}{\partial \nu_b^2} = -\frac{1}{1-\rho^2}\frac{1}{\sigma_b^2} \quad \Rightarrow \quad \sigma_b^2 = -\frac{1}{1-\rho^2}\left(\frac{\partial^2 \log q}{\partial \nu_b^2}\right)^{-1} \tag{15}$$

$$\frac{\partial^2 \log q}{\partial \nu_s \partial \nu_b} = \frac{\rho}{1-\rho^2}\frac{1}{\sigma_s\sigma_b} \quad \Rightarrow \quad \rho = \frac{\partial^2 \log q}{\partial \nu_s \partial \nu_b}\left(\frac{\partial^2 \log q}{\partial \nu_s^2}\frac{\partial^2 \log q}{\partial \nu_b^2}\right)^{-\frac{1}{2}}. \tag{16}$$

Now using Equation (9),

$$\frac{\partial^2 \log q}{\partial \nu_s^2} \approx \left.\frac{\partial^2 \log f}{\partial \nu_s^2}\right|_{(\hat{\nu}_s,\hat{\nu}_b)} = -\frac{n_{sc}^2}{n_{sb}} \tag{17}$$

$$\frac{\partial^2 \log q}{\partial \nu_b^2} \approx \left.\frac{\partial^2 \log f}{\partial \nu_b^2}\right|_{(\hat{\nu}_s,\hat{\nu}_b)} = -\frac{n_b}{\hat{\nu}_b^2} - \frac{n_{sc}^2}{n_{sb}} - \frac{1}{\sigma_b^2} \tag{18}$$

$$\frac{\partial^2 \log q}{\partial \nu_s \partial \nu_b} \approx \left.\frac{\partial^2 \log f}{\partial \nu_s \partial \nu_b}\right|_{(\hat{\nu}_s,\hat{\nu}_b)} = -\frac{n_{sc}^2}{n_{sb}}. \tag{19}$$

The proposal distribution parameters can thus be approximated by combining Equations (14) to (19).

So far, the parameter PDF $f(\nu_s,\nu_b|n_{sb},n_b,\mu_b,\sigma_b)$ and the proposal distribution identified by $q(\nu_s,\nu_b|\hat{\nu}_s,\hat{\nu}_b,\sigma_s,\sigma_b,\rho)$ have been determined, along with the position $(\hat{\nu}_s,\hat{\nu}_b)$ of the parameter PDF peak and an estimate for the proposal distribution parameters. To complete the MCMC algorithm, a method to determine the number of burn-in steps, the number of following steps as well as the amount of thinning necessary to produce the final results is required. Because the objective of the MCMC integration is to accurately measure a quantile of the posterior ν_{s} distribution (the quantile here corresponds to the value $\nu_{\mathrm{s}}^{\mathrm{q}}$ for which $P(\nu_{\mathrm{s}} \leq \nu_{\mathrm{s}}^{\mathrm{q}} | n_{\mathrm{sb}}, n_{\mathrm{b}}, \mu_{\mathrm{b}}, \sigma_{\mathrm{b}}) \approx P(\text{false alarm})$), a method based on the `gibbsit` algorithm [3] can be used. Using the notation presented in [3], the resulting algorithm that determines if a given imaging detector event should trigger an alarm is the following:

1. Calculate the position $(\hat{\nu}_{\mathrm{s}}, \hat{\nu}_{\mathrm{b}})$ of the parameter PDF peak, using Equations (10) and (12).

2. If $\hat{\nu}_{\mathrm{s}} \leq 0$, there is no alarm and the algorithm exits.

3. Calculate initial proposal distribution parameters $(\sigma_{\mathrm{s}}^2, \sigma_{\mathrm{b}}^2, \rho)$ using Equations (14) to (19).

4. Calculate $N_{\min}$ according to [3], using for example $q = 0.01$ (the false alarm probability), $s = 0.99$ for the confidence level and $r = 0.001$ for the uncertainty on q. This provides the minimum number of MCMC steps to generate in order to reach the desired accuracy on the false alarm probability.

5. Generate the determined number of MCMC steps, starting at position $(\nu_{\mathrm{s}0}, \nu_{\mathrm{b}0}) \equiv (\hat{\nu}_{\mathrm{s}}, \hat{\nu}_{\mathrm{b}})$ and after evaluating $\log f(\nu_{\mathrm{s}0}, \nu_{\mathrm{b}0} | n_{\mathrm{sb}}, n_{\mathrm{b}}, \mu_{\mathrm{b}}, \sigma_{\mathrm{b}})$ as well as the value for $\log q(\nu_{\mathrm{s}0}, \nu_{\mathrm{b}0} | \hat{\nu}_{\mathrm{s}}, \hat{\nu}_{\mathrm{b}}, \sigma_{\mathrm{s}}, \sigma_{\mathrm{b}}, \rho)$ using Equations (9) and (13), respectively. Each step of the MCMC chain is generated using the following sub-algorithm:

   (a) Randomly generate a pair of parameters $(\nu_{\mathrm{s}i}, \nu_{\mathrm{b}i})$ according to the proposal distribution $q(\nu_{\mathrm{s}}, \nu_{\mathrm{b}} | \hat{\nu}_{\mathrm{s}}, \hat{\nu}_{\mathrm{b}}, \sigma_{\mathrm{s}}, \sigma_{\mathrm{b}}, \rho)$. Such correlated normally distributed random variables can be generated using a Cholesky decomposition.

   (b) Evaluate $\log f(\nu_{\mathrm{s}i}, \nu_{\mathrm{b}i} | n_{\mathrm{sb}}, n_{\mathrm{b}}, \mu_{\mathrm{b}}, \sigma_{\mathrm{b}})$.

   (c) Evaluate $\log q(\nu_{\mathrm{s}i}, \nu_{\mathrm{b}i} | \hat{\nu}_{\mathrm{s}}, \hat{\nu}_{\mathrm{b}}, \sigma_{\mathrm{s}}, \sigma_{\mathrm{b}}, \rho)$.

   (d) If

   $$\log f(\nu_{\mathrm{s}i}, \nu_{\mathrm{b}i} | \ldots) + \log q(\nu_{\mathrm{s}i-1}, \nu_{\mathrm{b}i-1} | \ldots) \geq \log f(\nu_{\mathrm{s}i-1}, \nu_{\mathrm{b}i-1} | \ldots) + \log q(\nu_{\mathrm{s}i}, \nu_{\mathrm{b}i} | \ldots), \quad (20)$$

   accept the current values $(\nu_{\mathrm{s}i}, \nu_{\mathrm{b}i})$. Otherwise, draw a uniform deviate in the interval $[0, 1[$. If the random value is smaller than the expression

   $$e^{\log f(\nu_{\mathrm{s}i}, \nu_{\mathrm{b}i} | \ldots) - \log f(\nu_{\mathrm{s}i-1}, \nu_{\mathrm{b}i-1} | \ldots) + \log q(\nu_{\mathrm{s}i-1}, \nu_{\mathrm{b}i-1} | \ldots) - \log q(\nu_{\mathrm{s}i}, \nu_{\mathrm{b}i} | \ldots)}, \quad (21)$$

   accept the current values $(\nu_{\mathrm{s}i}, \nu_{\mathrm{b}i})$. Otherwise, set $(\nu_{\mathrm{s}i}, \nu_{\mathrm{b}i})$ to $(\nu_{\mathrm{s}i-1}, \nu_{\mathrm{b}i-1})$.

6. Use the `gibbsit` method [3] to determine k (the thinning parameter), M (the number of MCMC steps used for burn-in) and N (the number of MCMC steps following burn-in), using the array of resulting ν_s values, the same q, r and s inputs along with $\varepsilon = r$.

7. If the value $M + N$ is smaller or equal to the previous value (or N_{min} in the case of the first iteration), go to Step 10.

8. Compute σ_s, σ_b and ρ using all generated (ν_s, ν_b) pairs. Revert to the previous parameter values if the resulting correlation factor is not strictly negative.

9. Go back to Step 5.

10. Compute I, as defined in [3]. If $I > 5$, reject the current detector event.

11. Compute the quantile ν_s^q, after discarding the burn-in events and using thinning.

12. Compute the value of q that corresponds to $\nu_s^q = 0$.

13. If ν_s^q is larger than 0, the detector event generates an alarm. Otherwise, no alarm is triggered.

Typically in the above procedure, Step 12 is unnecessary when the number of background events in the signal channels is a few thousands of events or larger, in which case requiring a given false alarm probability is equivalent to evaluating the probability of the signal production rate to not being strictly positive. However, as the background rate goes down, the quantised nature of event detection causes shifts in the effective false alarm probability when the alarm threshold is computed using $q = P(\text{false alarm})$. At the limit where the probability of background detection in the signal channels becomes smaller than the targeted false alarm probability, it becomes impossible for the effective false alarm probability to be as large as the targeted value, unless alarms are triggered without measuring any event in the signal channels. In the case where the background rate is still sufficiently large to reach the target false alarm probability when triggering on measurements having events in the signal channels, the default method can be modified to apply a threshold correction, using the following procedure:

1. Compute the effective false alarm probability of a scenario using the default procedure, but a sample containing only background events.

2. If the effective false alarm probability significantly differs from the targeted value, use the distribution of q values from Step 12 of the last procedure to determine the value that should allow to correct the false alarm probability. Otherwise, go to Step 4.

3. Go back to Step 1, after reducing the value for r and ε if necessary due to a smaller value of q. Typically both parameters can be reduced by an order of magnitude when q/r becomes smaller than 5. A decrease of the value for r significantly increases the required computation time.

4. Perform the computation described in the default procedure, using a regular sample containing signal and background events, but using the updated values for q, r and ε.

As mentioned early in this section, the statistical model for the imaging detector is based on the assumption that the events from the radioactive source are contained in a reduced number of channels, which is hopefully a single channel. This obviously constitutes an idealised scenario, which can require perfect event reconstruction from the detector. As the angular dimension of a source relative to a detector depends on the effective size of the source as well as on the distance between the source and the detector, these parameters must be taken into account when evaluating the different scenarios. This is particularly important for shielded sources where radiation does not appear to be emitted from a single point, but rather by the whole volume of the shield (albeit with a non-uniform distribution). Assuming a flat detector located at a distance d from the centre of a spherical source surrounded by a spherical shield of radius R, the angle θ_s sustained by the shield on each side of the segment linking the source to the centre of the detector surface, is given by

$$\theta_s = \arcsin\left(\frac{R}{d}\right). \tag{22}$$

Assuming that radiation can be emitted from anywhere within the spherical shield, the solid angle of the shield, as seen from the centre of the detector surface, is thus given by

$$\Omega_s = 2\pi \int_0^{\theta_s} \sin\theta d\theta = 2\pi(1-\cos\theta_s) = 2\pi\left(1-\frac{\sqrt{d^2-R^2}}{d}\right). \tag{23}$$

Because the maximum solid angle that is visible by the imaging detector is assumed to be 2π sr and that all channels have the same sensitivity, the number of signal channels for a given scenario can be approximated by the expression

$$n_{sc} \approx \left\lceil n_c\left(1-\frac{\sqrt{d^2-R^2}}{d}\right)\right\rceil. \tag{24}$$

Note that although n_{sc} could be further constrained for a specific detector geometry, the best case scenario is assumed here.

4 Example

In this section, an example of simulation for an imaging detector is presented. It consists of a scenario involving a detector with 10 000 channels, which expectation value for the number of background counts per channel is $\mu_b = 0.455$, with an uncertainty of $\sigma_b = 0.182$. A radioactive source has a composition, location and environment that are such that signal events are only detected by a single known channel of the imaging detector and that the expected number of detected signal events is $\nu_s = 2.317$. The requirements for the detection system are a false alarm probability of 1% and an alarm trigger efficiency of 99% in the presence of a signal. Consequently, this translates into an initial value of 0.01 for q. The value for both ε and r can be set to 0.001, to require resulting relative uncertainties of 10% for the false alarm probability and the parameter governing the uncertainty for the number of burn-in iterations, respectively. Also, s, the level of confidence for the uncertainty on q, is set to 99%.

Based on the above parameter values, including the background constraints, the second procedure from the previous section was performed, resulting in $q = 0.009875$. MCMC simulations where then performed using a sample containing both signal and background events. Figures 1 and 2 present distributions of generated MCMC events for two different sample events. These sample events were selected to show how asymmetrical the ν_s parameter space can be, but also how its width can vary on an event by event basis, mostly due to the involved Poissonian distributions. Also, Figure 1 presents an event that triggers an alarm under the condition $q = 0.009875$, while the event from Figure 2 does not.

After simulating 1×10^6 of these sample events, the alarm trigger efficiency of the detector was computed for the scenario. Figure 3 presents the distribution of ν_s^q as generated from these events. It shows that 64% of the generated events falls in the region defined by $\nu_s^q > 0$. For this particular scenario, the detector does thus not satisfy the required 99% trigger efficiency. The figure also shows important structures within the ν_s^q distribution that are due to the quantisation of the $(\hat{\nu}_s, \hat{\nu}_b)$ positions and of the width for the $f(\nu_s, \nu_b | n_{sb}, n_b, \mu_b, \sigma_b)$ PDF.

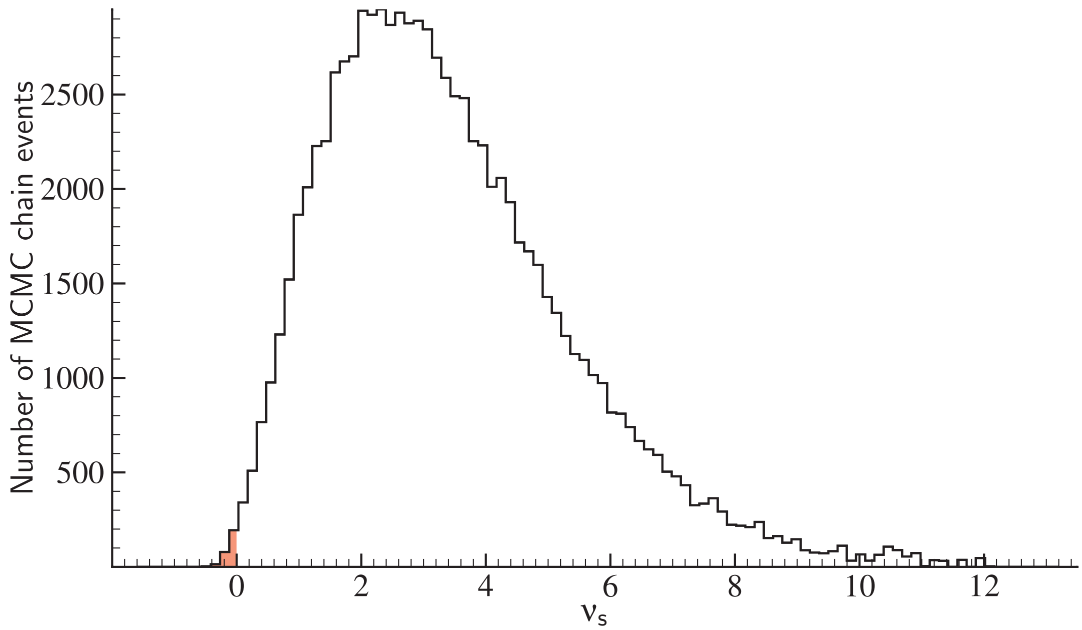

Figure 1: *Distribution of MCMC events for the* ν_s *parameter when* $n_{sb} = 3$ *and* $n_b = 5721$*, with the fraction corresponding to the absence of positive signal (0.003) shown in red.*

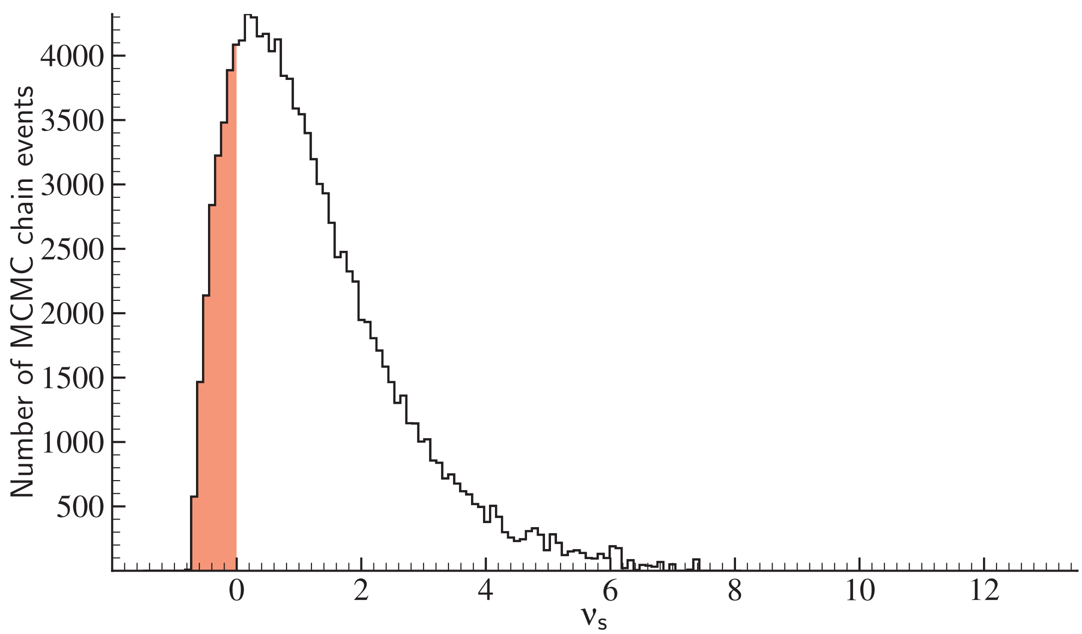

Figure 2: *Distribution of MCMC events for the* ν_s *parameter when* $n_{sb} = 1$ *and* $n_b = 7382$*, with the fraction corresponding to the absence of positive signal (0.168) shown in red.*

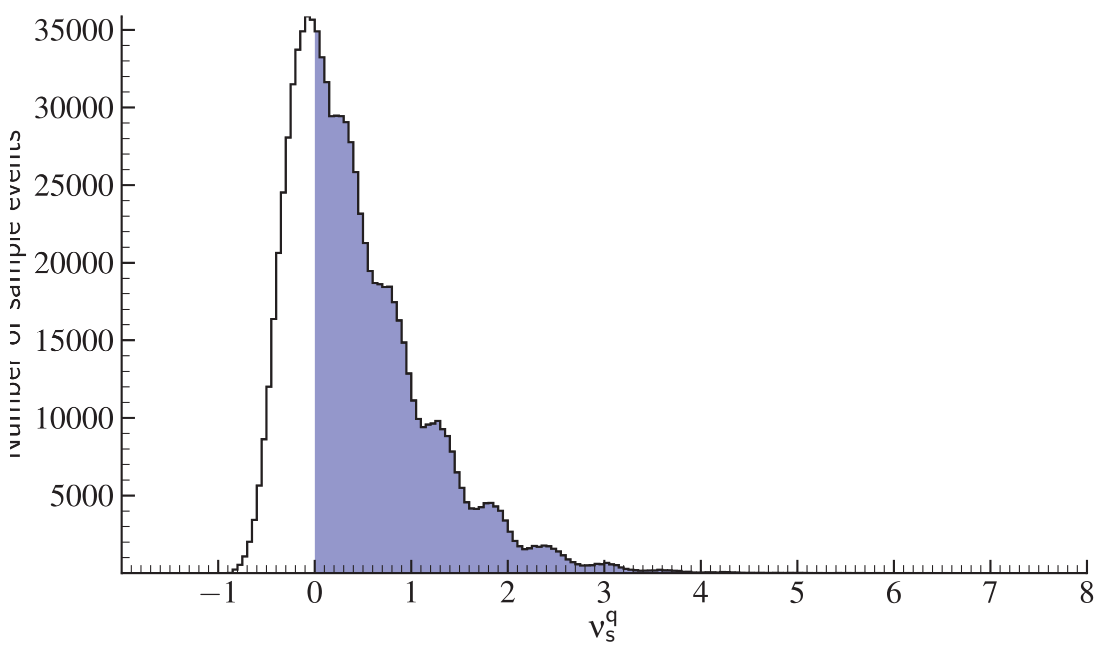

Figure 3: *Distribution of ν_s^q for the sample events generated for the described scenario, with the fraction of events triggering an alarm (0.64) shown in blue.*

5 Conclusions

This note presented two statistical models that allow to evaluate the sensitivity of radiation detectors. These models correctly handle the asymmetrical nature of the distributions associated to event detection with limited statistics. They are also general and compatible, in the sense that they are based on the same limited assumptions and that they normally lead to the same results when applied to the same scenario in the limit where sufficient statistics are generated. The first model consists of a more direct approach which is however only applicable to scenarios involving single channel detectors. The other model supports multi channel detection such as imaging detectors, but is more complex and requires more input parameters. It can also be used for single channel detection in which case the number of signal channels is the same as the total number of channels.

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Evaluating different radiation detection technologies, with the intent of either directing research and development or equipment acquisition, involves the consideration of many criteria, the suitability of the detection sensitivity being normally of prior importance. This report presents two general statistical methods allowing the evaluation of the detection sensitivity for different technologies. These methods rely on a limited set of assumptions, thus allowing more accurate sensitivity estimates in scenarios where the signal and background levels are low. One of the models also supports sensitivity estimation for imaging detectors, under the same limited assumptions.

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