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Solving the Max-Min SNR Optimization Problem of Array Antenna Signal Processing

An Iterative Approach using Convex Optimization Steps

A. Yasotharan

Defence R&D Canada – Ottawa

Technical Memorandum
DRDC Ottawa TM 2012-120
December 2012

Canada

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Abstract

The Max-Min SNR optimization problem was formulated in the author's previous report 'DRDC Ottawa TM 2011-202', in the context of using an array antenna to protect a GPS receiver from interferences. There, it was proposed to choose the array combining weights to maximize the minimum SNR of the satellites. Towards solving this problem, a convex *Min-Max Eigenvalue* problem was stated, and it was shown that: 1) the min-max eigenvalue is an upper bound on the Max-Min SNR, 2) if the min-max eigenvalue is *simple*, the upper bound is tight and the corresponding eigenvector solves the Max-Min SNR problem. A combinatorial search was proposed for the case when the min-max eigenvalue is *multiple*.

Recently, the author discovered that Sidiropoulos, Davidson, and Luo, writing in a communications context (physical-layer multicasting), had formulated two problems that are equivalent to the Max-Min SNR problem and proposed to solve them via *Semidefinite Relaxation* (SDR). This method sometimes finds optimum solutions, but in general gives suboptimum solutions.

In this report, we derive another solution method and show by simulations that it can outperform the SDR method. We formulate an auxiliary optimization problem which is equivalent to the Max-Min SNR problem and solve the auxiliary problem by an iterative process which uses convex optimization. We mathematically prove some convergence properties of the iterative process and show by simulations that by repeating the process several times, each time with a random initialization, a near-optimal solution can be found.

Résumé

Le problème d'optimisation du rapport signal/bruit maximin a été formulé dans le rapport précédent de l'auteur, le rapport DRDC Ottawa TM 2011-202, dans le contexte de l'utilisation d'une antenne réseau pour protéger un récepteur GPS contre le brouillage. Dans ce rapport, il est proposé de choisir les poids de combinaison d'un réseau de façon à maximiser le rapport signal/bruit minimal des satellites. Pour résoudre ce problème, un problème connexe de calcul de la valeur propre minimax (MMEV) a été formulé ; il a été montré : 1) que la valeur propre minimax est une borne supérieure du rapport signal/bruit maximin et 2) que si la valeur propre minimax est simple, la borne supérieure est exacte et le vecteur propre correspondant résout le problème rapport signal/bruit max-min. Une recherche combinatoire a été proposée pour le cas où la valeur propre minimax est multiple.

Récemment, l'auteur a découvert que Sidiropoulos, Davidson, et Luo, dans un article portant sur les communications (multidiffusion à la couche physique), ont formulé deux problèmes qui équivalent au problème du rapport signal/bruit maximin et ont proposé de résoudre ces problèmes par relaxation semidéfinie (SDR). Cette méthode permet parfois de trouver une solution optimale, mais elle donne en général des solutions sous optimales.

Dans le présent rapport, nous développons une autre méthode de résolution et nous montrons au moyen de simulations qu'elle peut surpasser la méthode SDR. Nous formulons un problème d'optimisation auxiliaire qui équivaut au problème du rapport signal/bruit minimax et nous résolvons ce problème par un procédé itératif qui fait appel à l'optimisation convexe. Nous prouvons mathématiquement les propriétés de convergence du procédé itératif et nous montrons au moyen de simulations qu'il est possible d'obtenir une solution quasi optimale en répétant le procédé itératif plusieurs fois en utilisant chaque fois des données initiales aléatoires.

Executive summary

Solving the Max-Min SNR Optimization Problem of Array Antenna Signal Processing

A. Yasotharan; DRDC Ottawa TM 2012-120; Defence R&D Canada – Ottawa; December 2012.

Background: The Max-Min SNR optimization problem was formulated in the author's previous report 'DRDC Ottawa TM 2011-202', in the context of using an array antenna to protect a GPS receiver from interferences. There, it was proposed to choose the array combining weights to maximize the minimum SNR of the satellites. Towards solving this problem, a related *Min-Max Eigenvalue* (MMEV) problem was also formulated and it was shown that: 1) the MMEV problem is convex (hence easily solvable), 2) the min-max eigenvalue is an upper bound on the Max-Min SNR, 3) if the min-max eigenvalue is *simple*, the upper bound is tight and the corresponding eigenvector solves the Max-Min SNR problem. A combinatorial search was proposed for the case when the min-max eigenvalue is *multiple*.

Recently, the author discovered that Sidiropoulos, Davidson, and Luo, writing in a communications context (physical-layer multicasting), had formulated two transmit-beamforming problems that are equivalent to the Max-Min SNR problem. Sidiropoulos *et.al.* proved those problems to be NP-hard and proposed to use the *Semidefinite Relaxation* (SDR) method. This method finds optimum solutions when a solved positive semidefinite (p.s.d.) matrix $\mathbf{X}$ has rank 1, but in general it gives suboptimum solutions.

Principal results: Simulations done using MATLAB and CVX show that very often the MMEV method gives a multiple eigenvalue and the SDR method gives a p.s.d. matrix $\mathbf{X}$ with rank higher than 1. We develop a new method to better handle these situations.

We formulate an auxiliary optimization problem which is equivalent to the Max-Min SNR problem, and solve the auxiliary problem by an iterative process which is repeated several times, each time with a random initialization. We mathematically prove that the minimum SNR of the satellites increases through the iterations and converges. Simulations show that, if the number of repetitions is large enough, this new method 1) attains the performance of MMEV and SDR when the latter yield verifiably optimal solutions, 2) exceeds the performance of SDR when the latter yields only suboptimal solutions. An iteration consists mainly of a convex optimization.

Significance of results: With the new method, we now have a complete suite of methods for solving the Max-Min SNR problem in all situations. It appears that the new method will be useful in adapting to small changes in the problem data.

The considered optimization problems may arise in multibeam radars as well when it is desired to 1) simultaneously illuminate targets that are in different directions, 2) simultaneously detect targets that are in different directions.

Future work: The new method must be tested in the contexts of GPS reception, physical-layer multicasting, and multibeam radars.

Sommaire

Solving the Max-Min SNR Optimization Problem of Array Antenna Signal Processing

A. Yasotharan ; DRDC Ottawa TM 2012-120 ; R & D pour la défense Canada – Ottawa ; décembre 2012.

Introduction : Le problème d'optimisation du rapport signal/bruit maximin a été formulé dans le rapport précédent de l'auteur, le rapport DRDC Ottawa TM 2011-202, dans le contexte de l'utilisation d'une antenne réseau pour protéger un récepteur GPS contre le brouillage. Dans ce rapport, il est proposé de choisir les poids de combinaison d'un réseau de façon à maximiser le rapport signal/bruit minimal des satellites. Pour résoudre ce problème, un problème connexe de calcul de la valeur propre minimax (MMEV) a aussi été formulé ; il a été démontré : 1) que le problème MMEV est convexe (et donc facile à résoudre), 2) que la valeur propre minimax est une borne supérieure du rapport signal/bruit maximin et 3) que si la valeur propre minimax est simple, la borne supérieure est exacte et le vecteur propre correspondant résout le problème du rapport signal/bruit maximin. Une recherche combinatoire a été proposée pour le cas où la valeur propre minimax est multiple.

Récemment, l'auteur a découvert que Sidiropoulos, Davidson, et Luo, dans un article portant sur les communications (multidiffusion à la couche physique), ont formulé deux problèmes de mise en forme de faisceaux d'émission qui équivalent au problème du rapport signal/bruit maximin. Sidiropoulos et ses collaborateurs ont prouvé que ces problèmes avaient une difficulté NP et ont suggéré l'utilisation de la méthode de relaxation semidéfinie (SDR). Cette méthode trouve des solutions optimales lorsque qu'une matrice résolue semidéfinie positive X a un rang 1, mais elle donne généralement des solutions sous-optimales.

Résultats : Des simulations faites au moyen de MATLAB et CVX montrent qu'il est fréquent que la méthode MMEV donne des valeurs propres multiples et que la méthode SDR donne une matrice semidéfinie positive X d'un rang supérieur à 1. Nous avons mis au point une méthode pour mieux traiter ces situations.

Nous formulons un problème d'optimisation auxiliaire qui équivaut au problème d'optimisation du rapport signal/bruit maximin et nous résolvons le problème par un procédé itératif qui est répété plusieurs fois en utilisant chaque fois des données initiales aléatoires. Nous prouvons mathématiquement que le rapport signal/bruit des satellites augmente au fil des itérations et converge. Des simulations montrent que si le nombre de répétitions est suffisamment élevé, cette nouvelle méthode 1) atteint la performance des méthodes MMEV et SDR lorsque cette dernière fournit des solutions dont l'optimalité est vérifiable et 2) dépasse la performance de la méthode SDR lorsque cette dernière ne donne que des solutions sous-optimales. Une itération consiste principalement en une optimisation convexe.

Portée : Avec cette nouvelle méthode, nous disposons maintenant d'un ensemble complet de méthodes de résolution du problème d'optimisation du rapport signal/bruit maximin

dans toutes les situations. Il semble que la nouvelle méthode s'adapte bien aux petits changements des données du problème.

Le problème d'optimisation étudié peut également apparaître dans l'optimisation de radars multifaisceaux quand il faut : 1) illuminer simultanément des cibles se trouvant dans des directions différentes et 2) détecter simultanément des cibles se trouvant dans des directions différentes.

Recherches futures : La nouvelle méthode doit être mise à l'essai dans les contextes de la réception GPS, de la multidiffusion à la couche physique et des radars multifaisceaux.

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Acknowledgements

I would like to thank Dr. Gerard Pieris for reviewing this report and providing comments that helped to improve its readability.

1 Introduction

The *Max-Min SNR* optimization problem was formulated by the present author [1] in the context of using an array antenna to protect a legacy GPS receiver[1] from interferences, where the outputs of the antenna elements are combined through a weight-and-sum operation and fed into a legacy GPS receiver. It was proposed in [1] to choose the array combining weights to maximize the minimum SNR of the satellites, that is by solving the Max-Min SNR problem which will be stated in Section 2.

The *Maximum Weighted-Average SNR* (Max-WA SNR) problem, formulated and solved in [1], gives an upper bound on the Max-Min SNR. This led to another optimization problem called the *Min-Max Eigenvalue* (MMEV) problem in terms of the same data as for Max-Min SNR and Max-WA SNR. It was shown that: 1) the MMEV problem is convex (hence easily solvable), 2) the min-max eigenvalue is an upper bound on the Max-Min SNR, 3) if the min-max eigenvalue is *simple*, the upper bound is tight and the corresponding eigenvector solves the Max-Min SNR problem. The MMEV problem will be stated in Section 3.1. For the cases where the min-max eigenvalue is multiple, a combinatorial search strategy was proposed in [1], but it is very complex.

Recently, the author discovered [2] in which two optimization problems equivalent to the Max-Min SNR problem had been formulated in a communications context, specifically physical-layer multicasting. We shall call these optimization problems as *Transmit Beamforming* (TB) problems. In [2], the authors proved that those two problems are NP-hard, i.e. they belong to a class of problems considered impossible to solve in polynomial time, and then proposed the technique of *Semidefinite Relaxation* (SDR) for obtaining approximate solutions with reasonable computational requirements. In SDR, a TB problem is restated in terms of a *positive semidefinite* (p.s.d.) matrix variable $\mathbf{X}$ as a semidefinite program, which is a convex optimization problem. If the optimal p.s.d. matrix $\mathbf{X}$ has rank 1 then it yields an optimal solution to the TB problem. If the rank of $\mathbf{X}$ is higher than 1 then heuristic procedures are used to infer an approximately optimal solution to the TB problem. The TB problems will be stated in Section 2.1 where their equivalence to the Max-Min SNR problem will become apparent. The SDR method will be briefly described in Section 3.2. See [3] for a general tutorial review.

The Max-Min SNR problem, by its equivalence to the TB problems, is also NP-hard, i.e. difficult to solve. It can be restated as either one of the two TB problems considered in [2], and hence approximately solved by SDR. As SDR does not always give the optimal solution, or even a good solution, it is necessary to develop better methods for solving the Max-Min SNR problem (and hence the TB problems by equivalence). In fact, simulations have shown that when the SDR problem of [2] is solved, the optimum p.s.d. matrix $\mathbf{X}$ often has rank higher than 1. Similarly, when the MMEV problem of [1] is solved, the min-max eigenvalue is often multiple.

A popular book on convex optimization is [4]. For general optimization theory, see [5] [6].

[1] A legacy GPS receiver has only one input port, and all satellite signals are received through that port.

1.1 Contribution of this Report

We develop a new solution method for the Max-Min SNR problem and show by simulations that 1) it attains the performance of MMEV and SDR when the latter yield verifiably optimal solutions, 2) it exceeds the performance of SDR when the latter yields suboptimal solutions. The simulations were done in MATLAB using CVX software [7] [8].

The new method is motivated by an auxiliary optimization problem *equivalent* to the Max-Min SNR problem. The auxiliary problem has an extra set of variables $\{\theta_i\}$ in addition to the weight vector variable $\mathbf{w}$ of the Max-Min SNR. It is solved using an iterative process, where in each iteration, $\{\theta_i\}$ is held fixed and $\mathbf{w}$ is optimized, and this is a convex optimization step. In the first iteration, $\{\theta_i\}$ is randomly chosen, and thereafter it is chosen based on the result of the previous iteration. We mathematically prove that the minimum SNR of the satellites increases through the iterations and converges. The simulations suggest that the variable $\mathbf{w}$ converges to a local optimum of the Max-Min SNR problem. Thus by repeating this process a few times, starting each time with a randomly chosen $\{\theta_i\}$, a near-optimal solution can be found for the Max-Min SNR problem.

The new method is particularly useful when the MMEV or SDR methods do not yield optimal solutions, that is when the min-max eigenvalue of MMEV is not simple, or when the optimum p.s.d. matrix $\mathbf{X}$ of SDR has rank higher than 1.

1.2 Layout of Report

In Section 2, we recapitulate the Max-Min SNR optimization problem of [1] and provide three other equivalent forms. The first two equivalent forms correspond to the TB problems of [2], and they motivate the SDR approach to solving them. The third equivalent form motivates the development of the new solution method of this report.

In Section 3, we recapitulate 1) the Min-Max Eigenvalue problem of [1] for solving the Max-Min SNR optimization problem, and 2) the Semidefinite Relaxation (SDR) method of [2] for solving the TB problems.

In Section 4, we formulate an auxiliary optimization problem and prove its equivalence to the Max-Min SNR problem, particularly to the third equivalent form given in 2.

In Section 5, we develop an iterative method for solving the auxiliary problem, and by equivalence the Max-Min SNR problem. We also mathematically prove some convergence properties of the iterative process.

In Section 6, we provide simulation results on the Min-Max Eigenvalue method of [1], the SDR method of [2], and the new method of this report, and discuss their relative performances.

In Section 7, we make conclusions and provide suggestions for further work.

2 The Max-Min SNR Problem and its Equivalent Forms

This section is devoted to the recapitulation of the Max-Min SNR optimization problem of [1] and, more importantly, to its restatement in three equivalent forms. The first two equivalent forms correspond to the TB problems of [2], and they are amenable to the SDR solution method. The third equivalent form facilitates the development of the new solution method of this report.

When using an array antenna to receive signals from GPS satellites, we weight-and-sum the outputs of the antenna elements and feed the sum signal into a legacy GPS receiver.[2] Under several interference mitigation paradigms considered in [1], the Signal-to-Noise Ratio, or Signal-to-Interference-plus-Noise Ratio, for the i^{th} satellite can be expressed as

$$\mathrm{SNR}_i = \frac{\left|\mathbf{v}_i^H \mathbf{w}\right|^2}{\|\mathbf{w}\|^2} \qquad\qquad \text{for } i = 1, 2, \ldots, K. \tag{1}$$

Here $\{\mathbf{v}_i\}$ and $\mathbf{w}$ are complex-valued *column* vectors whose length M is equal to the number of antenna elements. When there is no interference, $\{\mathbf{v}_i\}$ are just the array steering vectors of the satellites and $\mathbf{w}$ contains the weights applied to the antenna outputs. When there are interferences, $\{\mathbf{v}_i\}$ are transformed versions of the steering vectors and $\mathbf{w}$ is also generally a transformed version of the weights vector. See [1] for details.

It was proposed in [1] that the weights vector $\mathbf{w}$ be chosen by solving the following Max-Min SNR optimization problem

$$\max_{\mathbf{w}\neq\mathbf{0}} \min_{i\in I} \frac{\left|\mathbf{v}_i^H \mathbf{w}\right|^2}{\|\mathbf{w}\|^2} \tag{2}$$

Here $I = \{1, 2, \ldots, K\}$ is the set of all visible satellites.

The above problem can also be written in the following equivalent ways.

$$\max_{\mathbf{w}\neq\mathbf{0}} \frac{\min_{i\in I} \left|\mathbf{v}_i^H \mathbf{w}\right|^2}{\|\mathbf{w}\|^2} \tag{3}$$

$$\min_{\mathbf{w}\neq\mathbf{0}} \frac{\|\mathbf{w}\|^2}{\min_{i\in I} \left|\mathbf{v}_i^H \mathbf{w}\right|^2} \tag{4}$$

In the above problems, if $\mathbf{w}$ is optimum then so is $\alpha\mathbf{w}$ for any complex scalar $\alpha \neq 0$.

In the following, we will restate the Max-Min SNR problem in three equivalent forms which are all constrained optimization problems. Their optimum $\mathbf{w}$ solutions are related through scaling. But their SNRs are the same under all forms.

[2] All description of signal processing in this report is based on the complex-equivalent, i.e. I/Q, signals.

2.1 Equivalent Forms

2.1.1 Form 1

The problem (4) can be equivalently restated by removing $\min_{i \in I} \left|\mathbf{v}_i^H \mathbf{w}\right|^2$ from the denominator of the objective function and introducing the constraint $\min_{i \in I} \left|\mathbf{v}_i^H \mathbf{w}\right|^2 \geq 1$. Thus we have

$$\min_{\mathbf{w}} \|\mathbf{w}\|^2 \tag{5}$$

$$\text{s.}t. \qquad \min_{i \in I} \left|\mathbf{v}_i^H \mathbf{w}\right|^2 \geq 1. \tag{6}$$

This can be further restated as

$$\min_{\mathbf{w}} \|\mathbf{w}\|^2 \tag{7}$$

$$\text{s.}t. \qquad \left|\mathbf{v}_i^H \mathbf{w}\right|^2 \geq 1, \qquad \text{for } i = 1, 2, \ldots, K. \tag{8}$$

This is one of the TB problems of [2], specifically problem $\mathcal{Q}$ of page 2240. In the TB context, i is the receiver index, and the transmitted power is minimized subject to constraints on the SNR of each receiver.

In [2], the problem $\mathcal{Q}$ was proven to be NP-hard, and the SDR method for obtaining approximate solutions was presented. This involves solving the relaxed semidefinite programming (SDP) problem $\mathcal{Q}_r$ of page 2241, followed by randomization.

2.1.2 Form 2

The problem (3) can be equivalently restated by removing $\|\mathbf{w}\|^2$ from the denominator of the objective function and introducing the constraint $\|\mathbf{w}\|^2 = P$, for some arbitrary $P > 0$, or even $\|\mathbf{w}\|^2 \leq P$ because this constraint will be met with equality. Thus we have

$$\max_{\mathbf{w}} \min_{i \in I} \left|\mathbf{v}_i^H \mathbf{w}\right|^2 \tag{9}$$

$$\text{s.}t. \qquad \|\mathbf{w}\|^2 \leq P. \tag{10}$$

This is the other TB problem of [2], specifically problem $\mathcal{F}$ of page 2242 where P is the transmitted power. Again i is the receiver index, and $\mathcal{F}$ is called 'Maximize the minimum received SNR over all receivers, subject to a bound on the transmitted power'. We can assume $P = 1$ without loss of generality.

In [2], the problem $\mathcal{F}$ was shown to be equivalent to problem $\mathcal{Q}$ and hence also NP-hard. Then the SDR method for obtaining approximate solutions was presented. This involves solving the relaxed SDP problem $\mathcal{F}_r$ of page 2243, followed by randomization.

2.1.3 Form 3

This is a minor variation of (7) and (8) obtained by dropping the exponent 2 from the constraint functions (8). This facilitates the presentation of the new solution method of this report.

$$\min_{\mathbf{w}} \|\mathbf{w}\|^2 \tag{11}$$

$$\text{s.}t. \qquad \left|\mathbf{v}_i^H \mathbf{w}\right| \geq 1, \qquad \text{for } i = 1, 2, \ldots, K. \tag{12}$$

We shall see that we can even drop the exponent 2 from the objective function without any consequence.

3 Previous Solution Methods

3.1 Min-Max Eigenvalue

In [1], the following problem was posed and solved: Given a set of real-valued weights $\{\mu_i \geq 0 : \sum_i \mu_i = 1\}$, choose the array combining vector $\mathbf{w}$ so as to Maximize the Weighted-Average SNR of the satellites.

From the solution, it was deduced that the maximum eigenvalue of the matrix

$$\mathbf{U} = \sum_i \mu_i \mathbf{v}_i \mathbf{v}_i^H \tag{13}$$

is an upper bound on Max-Min SNR for every $\{\mu_i\}$. The minimization of this upper bound w.r.t. $\{\mu_i\}$ is the Min-Max Eigenvalue (MMEV) problem which is a convex problem.

Therefore the min-max eigenvalue of $\mathbf{U}$ is an upper bound on the Max-Min SNR. It was proved in [1] that when the min-max eigenvalue of $\mathbf{U}$ is *simple*: 1) the upper bound is tight (min-max eigenvalue = Max-Min SNR), 2) the corresponding eigenvector is optimal for the Max-Min SNR problem.

What happens when the min-max eigenvalue of $\mathbf{U}$ is *multiple* is yet to be investigated.

3.2 Semidefinite Relaxation

Here we briefly review the Semidefinite Relaxation (SDR) method proposed in [2] for obtaining *approximate* solutions for Forms 1 and 2 of Section 2.1. See [3] for a general overview of SDR.

First note that Forms 1 and 2 involve the functions $\|\mathbf{w}\|^2$ and $\left|\mathbf{v}_i^H \mathbf{w}\right|^2$ which can be written as

$$\begin{aligned} \|\mathbf{w}\|^2 &= \mathbf{w}^H \mathbf{w} && (14)\\ &= \operatorname{trace}\left(\mathbf{w}\mathbf{w}^H\right) && (15)\\ \left|\mathbf{v}_i^H \mathbf{w}\right|^2 &= \mathbf{v}_i^H \mathbf{w}\mathbf{w}^H \mathbf{v}_i && (16)\\ &= \operatorname{trace}\left(\mathbf{v}_i \mathbf{v}_i^H \mathbf{w}\mathbf{w}^H\right). && (17) \end{aligned}$$

Since $\mathbf{w}\mathbf{w}^H$ is a Hermitian symmetric positive semidefinite (p.s.d.) matrix of rank 1, we can restate Forms 1 and 2 of Section 2.1 in terms of a Hermitian symmetric p.s.d. matrix variable $\mathbf{X}$ together with the additional constraint $\operatorname{rank}(\mathbf{X}) = 1$. Discarding this rank constraint results in relaxed optimization problems which are convex problems of the semidefinite program (SDP) type. Therefore these semidefinite programs are called semidefinite relaxations of Forms 1 and 2. Thus using $\mathbf{w}\mathbf{w}^H = \mathbf{X}$, we have

$$\begin{aligned} \|\mathbf{w}\|^2 &= \operatorname{trace}(\mathbf{X}) && (18)\\ \left|\mathbf{v}_i^H \mathbf{w}\right|^2 &= \operatorname{trace}(\mathbf{Q}_i \mathbf{X}) && (19) \end{aligned}$$

where $\mathbf{Q}_i = \mathbf{v}_i \mathbf{v}_i^H$ is a Hermitian symmetric p.s.d. matrix.

3.2.1 Semidefinite Relaxation of Form 1 - SDR1

Using (18) and (19) in (7) and (8) respectively, we obtain the SDP

$$\min_{\mathbf{X}} \operatorname{trace}(\mathbf{X}) \tag{20}$$

$$\text{s.}t. \qquad \operatorname{trace}(\mathbf{Q}_i \mathbf{X}) \geq 1, \qquad \text{for } i = 1, 2, \ldots, K \tag{21}$$

$$\mathbf{X} \geq \mathbf{0} \tag{22}$$

which can be converted into a standard form SDP. See problem $\mathcal{Q}_r$ of [2], page 2241.

Denote by $\mathbf{X}_1$ the optimum p.s.d. matrix. Then $1/\text{trace}(\mathbf{X}_1)$ is an upper bound on the Max-Min SNR, which was verified by simulations to coincide with the min-max eigenvalue.

3.2.2 Semidefinite Relaxation of Form 2 - SDR2

Using (19) and (18) in (9) and (10), with $P = 1$, we obtain the SDP

$$\max_{\mathbf{X}} \min_{i \in I} \operatorname{trace}(\mathbf{Q}_i \mathbf{X}) \tag{23}$$

$$\text{s.}t. \qquad \operatorname{trace}(\mathbf{X}) \leq 1 \tag{24}$$

$$\mathbf{X} \geq \mathbf{0} \tag{25}$$

which can be converted into a standard form SDP. See problem $\mathcal{F}_r$ of [2], page 2243.

The maximum result of the above problem is an upper bound on the Max-Min SNR, which was verifed by simulations to coincide with the min-max eigenvalue.

3.2.3 Converting from Optimal Matrix X to Vector w

Having found an optimal $\mathbf{X}$, say $\mathbf{X}_{opt}$, for one of the above SDRs, if $\mathbf{X}_{opt}$ has rank 1, then a $\mathbf{w}$ such that $\mathbf{X}_{opt} = \mathbf{w}\mathbf{w}^H$ can be obtained by Cholesky factorization and such a $\mathbf{w}$ will be optimal for the underlying Form 1 or 2 of Section 2.1.

But what if $\mathbf{X}_{opt}$ has rank higher than 1? Then there are heuristic procedures called *randomizations* to obtain a $\mathbf{w}$ which will be *approximately* optimal for the underlying Form 1 or 2. Three such procedures - *randA, randB, randC* - are discussed in [2], Section IV. See also [3], page 24.

In *randA* and *randC*, the eigenvectors of $\mathbf{X}_{opt}$ are randomly combined to obtain a $\mathbf{w}$. In *randB*, the square roots of the diagonal elements of $\mathbf{X}_{opt}$ are randomly rotated to form the elements of a $\mathbf{w}$. In all three procedures, many random such $\mathbf{w}$ vectors are generated and the best one is used. Thus all three procedures have a parameter called the *number of randomizations*. However, it is not clear whether increasing this number indefinitely will guarantee truly optimum solutions for Forms 1 and 2 of Section 2.1.

4 An Auxiliary Problem Equivalent to the Max-Min SNR Problem

In this section we state an optimization problem which is equivalent to Form 3 of Section 2.1.3, and hence also equivalent to the Max-Min SNR Problem. An algorithm will be described in Section 5 for solving this auxiliary problem, and hence also Form 3.

The following optimization problem will be called the *auxiliary* problem:

$$\min_{\mathbf{w},\Theta} \|\mathbf{w}\|^2 \tag{26}$$

$$\text{s.}t. \qquad \Re\left(e^{-j\theta_i}\mathbf{v}_i^H\mathbf{w}\right) \geq 1, \qquad \text{for } i = 1, 2, \ldots, K \tag{27}$$

where $\Re$ denotes the *real part*. The problem data are the set of K complex-valued vectors $\{\mathbf{v}_i\}$. The variables to be optimized are the complex-valued vector $\mathbf{w}$ and the real-valued vector $\Theta = \{\theta_1, \theta_2, \ldots, \theta_K\}$.

Lemma 1 *The above auxiliary problem is equivalent to Form 3 in the sense that*

1. *If the pair* $(\mathbf{w}, \Theta)$ *is optimum for the auxiliary problem then* $\mathbf{w}$ *is optimum for Form 3.*
2. *If* $\mathbf{w}$ *is optimum for Form 3, then there exists a* Θ *such that the pair* $(\mathbf{w}, \Theta)$ *is optimum for the auxiliary problem.*

In other words, as far as the variable $\mathbf{w}$ *is concerned, the optimum sets of the two problems are identical.*

Proof: First we prove the similar statement where 'optimum' is replaced with 'feasible', i.e. the constraints are satisfied. Then, since the objective functions of both problems are identical, the Lemma follows.

We make some general observations which will be useful later as well.

1. For every $(\mathbf{w}, \Theta)$, we have

$$\left|\mathbf{v}_i^H\mathbf{w}\right| = \left|e^{-j\theta_i}\mathbf{v}_i^H\mathbf{w}\right| \geq \Re\left(e^{-j\theta_i}\mathbf{v}_i^H\mathbf{w}\right) \qquad \text{for } i = 1, 2, \ldots, K. \tag{28}$$

2. Given any $\mathbf{w}$, calculate $\Theta = \{\theta_1, \theta_2, \ldots, \theta_K\}$ as follows:

$$\theta_i = \text{angle}\left(\mathbf{v}_i^H\mathbf{w}\right) \qquad \text{for } i = 1, 2, \ldots, K. \tag{29}$$

   Then

$$\Re\left(e^{-j\theta_i}\mathbf{v}_i^H\mathbf{w}\right) = \left|\mathbf{v}_i^H\mathbf{w}\right| \qquad \text{for } i = 1, 2, \ldots, K. \tag{30}$$

Returning to the proof, suppose $(\mathbf{w}, \Theta)$ is feasible for the auxiliary problem, i.e., (27) holds. Then by (28), we see that (12) holds, i.e., $\mathbf{w}$ is feasible for Form 3.

Suppose $\mathbf{w}$ is feasible for Form 3, i.e., (12) holds. Calculate $\Theta = \{\theta_1, \theta_2, \ldots, \theta_K\}$ as in (29). Then by (30), we see that (27) holds, i.e. $(\mathbf{w}, \Theta)$ is feasible for the auxiliary problem.

We have just shown that, as far as the variable $\mathbf{w}$ is concerned, the feasible sets of the auxiliary problem and Form 3 are identical. Because the objective functions of both problems are identical as well, the Lemma follows. $\square$

5 A New Method for Solving the Max-Min SNR Problem

We showed in Section 4 that the Max-Min SNR problem, as stated in Form 3, can be solved by solving the auxiliary problem (26),(27). Towards solving the auxiliary problem, consider the following problem obtained by fixing $\Theta = \{\theta_1, \theta_2, \ldots, \theta_K\}$:

$$\min_{\mathbf{w}} \|\mathbf{w}\|^2 \tag{31}$$

$$\text{s.}t. \qquad \Re\left(e^{-j\theta_i}\mathbf{v}_i^H\mathbf{w}\right) \geq 1, \qquad \text{for } i = 1, 2, \ldots, K. \tag{32}$$

This problem is a convex problem and hence can be solved efficiently using, for example, the CVX software [7]. Note that even if the exponent 2 is dropped from the objective function, the problem remains convex. This problem is a *restriction* of Form 3 in the sense that its feasible set is a subset of the feasible set of Form 3. Therefore, its minimum cannot be lower than the minimum of Form 3. A special case of this problem where $\theta_i = 0$ for $i = 1, 2, \ldots, K$ was considered in [2], page 2248 and denoted $\mathcal{Q}_s$. The purpose there was to use its restriction property to evaluate the success of the SDR method in solving Form 1 and Form 2.

A brute-force approach to solving the auxiliary problem is to solve (31),(32) over a large number of randomly selected Θ and select the $(\mathbf{w}, \Theta)$ pair for which $\|\mathbf{w}\|$ is the lowest. In the following, we present a refined approach with appealing mathematical properties, again based on solving problems of the form (31),(32).

5.1 The Method

The method consists of many runs of iterations where each run is started by a randomly selected Θ. Thus the first iteration of a run consists of solving (31),(32) for a random Θ. The subsequent iterations of a run consist of calculating a Θ based on the result of the previous iteration and then solving (31),(32). Simulations show that each run converges to a *local minimum* of the auxiliary problem. If enough runs are conducted, then there is a high probability that one of those runs will converge to a *global minimum*. Within a run, Θ is calculated so as to relax the constraints (32) and thus allow further reduction of $\|\mathbf{w}\|$. Therefore, the method can be considered a *descent method*. Below we describe a run using mathematical notation.

At the n^{th} iteration, we solve the following problem for some predetermined $\{\theta_i(n)\}$:

$$\min_{\mathbf{w}} \|\mathbf{w}\|^2 \tag{33}$$

$$\text{s.}t. \qquad \Re\left(e^{-j\theta_i(n)}\mathbf{v}_i^H\mathbf{w}\right) \geq 1, \qquad \text{for } i = 1, 2, \ldots, K. \tag{34}$$

The optimum solution of the above problem is denoted by $\mathbf{w}(n)$.

At the first iteration, $\{\theta_i(1)\}$ are chosen randomly from independent uniform distributions over $[0, 2\pi)$. At the n^{th} iteration, for $n > 1$, $\{\theta_i(n)\}$ are chosen based on the solution found in the $(n-1)^{\text{th}}$ iteration as follows:

$$\theta_i(n) = \text{angle}\left(\mathbf{v}_i^H \mathbf{w}(n-1)\right). \tag{35}$$

The iterative process is terminated when the following condition is reached for some small $\delta > 0$:

$$|\Im\left(e^{-j\theta_i(n)}\mathbf{v}_i^H \mathbf{w}(n)\right)| < \delta, \qquad \text{for } i = 1, 2, \ldots, K \tag{36}$$

where $\Im$ denotes the *imaginary part*.

5.2 Convergence Properties

Let us denote

$$\alpha(n) = \min_i \left|\mathbf{v}_i^H \mathbf{w}(n)\right| \tag{37}$$

$$\beta(n) = \alpha(n)/\|\mathbf{w}(n)\| \tag{38}$$

$$= \min_i \sqrt{\text{SNR}_i}. \tag{39}$$

Lemma 2 *The iterative process has the following properties:*

1. $\alpha(n) \geq 1$ *for all* n.
2. *The sequence* $\|\mathbf{w}(n)\|$ *is monotonically decreasing and convergent.*
3. *If* $\alpha(n) > 1$ *then* $\|\mathbf{w}(n+1)\| < \|\mathbf{w}(n)\|$.
4. *The sequence* $\alpha(n)$ *converges to* 1 *from above.*
5. *The sequence* $\beta(n)$ *is monotonically increasing and convergent.*
6. *If* $\alpha(n+1) > 1$ *then* $\beta(n+1) > \beta(n)$.

Proof

1. The solution of the n^{th} iteration satisfies

   $$|\mathbf{v}_i^H \mathbf{w}(n)| \geq \Re\left(e^{-j\theta_i(n)}\mathbf{v}_i^H \mathbf{w}(n)\right) \geq 1, \qquad \text{for } i = 1, 2, \ldots, K. \tag{40}$$

   Taking $\min_i$ we have $\alpha(n) \geq 1$ for all n, proving Claim 1. □

2. The $\{\theta_i(n+1)\}$, chosen for the $(n+1)^{\text{th}}$ iteration, satisfies (see (35))

$$\Re\left(e^{-j\theta_i(n+1)}\mathbf{v}_i^H\mathbf{w}(n)\right) = |\mathbf{v}_i^H\mathbf{w}(n)| \qquad \text{for } i = 1, 2, \ldots, K. \tag{41}$$

   Combining (40) and (41), we get

$$\Re\left(e^{-j\theta_i(n+1)}\mathbf{v}_i^H\mathbf{w}(n)\right) \geq 1 \qquad \text{for } i = 1, 2, \ldots, K, \tag{42}$$

   which means $\mathbf{w}(n)$ is a feasible solution for the minimization in the $(n+1)^{\text{th}}$ iteration, and hence

$$\|\mathbf{w}(n+1)\| \leq \|\mathbf{w}(n)\|, \tag{43}$$

   that is, the sequence $\|\mathbf{w}(n)\|$ is monotonically decreasing. Since $\|\mathbf{w}(n)\|$ is bounded from below by the optimum $\|\mathbf{w}\|$ of the auxiliary problem, the sequence $\|\mathbf{w}(n)\|$ converges ([9], page 55, Theorem 3.14). We have proved Claim 2. $\square$

3. In (41), taking $\min_i$ on the RHS gives

$$\Re\left(e^{-j\theta_i(n+1)}\mathbf{v}_i^H\mathbf{w}(n)\right) \geq \alpha(n) \qquad \text{for } i = 1, 2, \ldots, K. \tag{44}$$

   Dividing by $\alpha(n)$ gives

$$\Re\left(e^{-j\theta_i(n+1)}\mathbf{v}_i^H\mathbf{w}(n)/\alpha(n)\right) \geq 1 \qquad \text{for } i = 1, 2, \ldots, K, \tag{45}$$

   which means $\mathbf{w}(n)/\alpha(n)$ is a feasible solution for the minimization in the $(n+1)^{\text{th}}$ iteration, and hence

$$\|\mathbf{w}(n+1)\| \leq \|\mathbf{w}(n)\|/\alpha(n). \tag{46}$$

   Therefore, if $\alpha(n) > 1$ then $\|\mathbf{w}(n+1)\| < \|\mathbf{w}(n)\|$, proving Claim 3. $\square$

4. Combining Claim 1 and (46), we get

$$1 \leq \alpha(n) \leq \|\mathbf{w}(n)\|/\|\mathbf{w}(n+1)\|. \tag{47}$$

   Since $\|\mathbf{w}(n)\|/\|\mathbf{w}(n+1)\|$ convergers to 1 ([9], page 49, Theorem 3.3), $\alpha(n)$ also converges to 1 ([9], Theorem 3.19), proving Claim 4. $\square$

5. Using the definition of $\beta(n)$ of (38) in (46), we get

$$\|\mathbf{w}(n+1)\| \leq 1/\beta(n). \tag{48}$$

   We also have from Claim 1

$$\|\mathbf{w}(n+1)\| \geq \|\mathbf{w}(n+1)\|/\alpha(n+1) = 1/\beta(n+1). \tag{49}$$

   Combining the above, we get

$$\beta(n) \leq \beta(n+1), \tag{50}$$

   that is, the sequence $\beta(n)$ is monotonically increasing. Since $\beta(n)$ is bounded from above by the square root of Max-Min SNR, it converges ([9], Theorem 3.14). We have proved Claim 5. $\square$

6. If $\alpha(n+1) > 1$, strict inequality holds in (49). In this case, we get $\beta(n) < \beta(n+1)$ in (50), proving Claim 6. $\square$

6 Simulation Results

The previously described methods were implemented in MATLAB and comparatively evaluated on simulated data. The methods evaluated are:

1. Min-Max Eigenvalue (MMEV) of Section 3.1.
2. Semidefinite Relaxations (SDR1 and SDR2) of Section 3.2.
3. The new method of Section 5.

Recall that all of the above methods make use of convex optimization. The convex optimization parts were implemented in MATLAB using the CVX software [7] [8].

6.1 Data Simulation Scenarios

For simulating the problem data $\{\mathbf{v}_i\}$, following [2], two scenarios were considered:

1. The elements of the data vectors $\{\mathbf{v}_i\}$ are random variables with independent and identical zero-mean complex circular Gaussian distributions. It follows that the magnitudes of the complex-valued elements are Rayleigh-distributed.
2. The vectors $\{\mathbf{v}_i\}$ are plane-wave steering vectors of a Uniform Linear Array. They are Vandermonde vectors of the form $\mathbf{v}_i = [1, e^{j\alpha_i}, e^{j2\alpha_i}, \ldots, e^{j(M-1)\alpha_i}]^T$ where $\{\alpha_i\}$ are independent random variables that are uniformly distributed in $[0, 2\pi)$.

Under each scenario, many sets $\{\mathbf{v}_i\}$ were independently and randomly generated, and on each $\{\mathbf{v}_i\}$, the above methods were run and the results were analyzed.

6.2 Simulation Parameters

We first introduce some new notation, while recalling some existing ones.

- M - Number of antenna elements.
- K - Number of satellites, in the context of receiving GPS, or number of receivers, in the context of transmit beamforming.
- N_{trials} - Number of sets of $\{\mathbf{v}_i\}$ on which the methods are run.
- N_{SDR} - Number of randomizations used in SDR (see Section 3.2.3).
- N_{new} - Number of iterative runs used in the new method (see Section 5.1).
- δ - Threshold of (36) for terminating iterations in the new method.

Under both scenarios, the simulation parameters were assigned the same set of values as given in Table 1.

M	K	N_{trials}	N_{SDR}	N_{new}	δ
8	8	100	1000	20	10^{-3}

Table 1: *Simulation parameters under both Scenarios*

6.3 Analysis

We now describe the common analysis that was done in both scenarios. Based on the results of the MMEV method, the trials were separated into two subsets: 1) min-max eigenvalue is *simple* 2) min-max eigenvalue is *multiple*.[3]

Within each of the above two subsets, the following quantities were compared to one another: 1) Min SNR, i.e. $\min_i \mathrm{SNR}_i$, achieved by the new method, 2) Min SNRs achieved by SDR1 and SDR2, 3) min-max eigenvalue (upper-bound to Max-Min SNR). Histograms of various ratios between these quantites were then plotted for each subset. These histograms are given in Annexes A and B. In these plots, snr_new denotes Min SNR of the new method, snr1 and snr2 denote the Min SNRs of SDR1 and SDR2 respectively, and eigVal denotes the min-max eigenvalue.[4]

The histograms in Annexes A and B show that:

1. The min-max eigenvalue is multiple with high probability. See Figures A.1 and B.1 of Annexes A and B, respectively.

2. When the min-max eigenvalue is *simple*, the new method attains the performance of SDR and MMEV. See Figures A.4 to A.6 of Annex A.1, and Figures B.4 to B.6 of Annex B.1.

3. When the min-max eigenvalue is *multiple*, the new method exceeds the performance of SDR. See Figures A.9 and A.10 of Annex A.2, and Figures B.9 and B.10 of Annex B.2.

The above dichotomy can also be described in terms of the rank of the p.s.d. matrices returned by the SDRs as shown by (52) below.

6.4 Some Observed Relations Between MMEV and SDR

Denote by $\mathbf{X}_1$ and $\mathbf{X}_2$ the optimum p.s.d. matrices returned by SDR1 and SDR2 respectively. Further denote by t the maximum result of (23). We observed in simulations that, neglecting finite precision effects,

$$1/\mathrm{trace}(\mathbf{X}_1) = \quad t \quad = \text{min-max eigenvalue} \tag{51}$$

$$\mathrm{rank}(\mathbf{X}_1) = \mathrm{rank}(\mathbf{X}_2) = \text{multiplicity of min-max eigenvalue.} \tag{52}$$

Eq. (51) shows that SDR1 and SDR2 give the same upper bound on Max-Min SNR as MMEV, as it was mentioned in Sections 3.2.1 and 3.2.2. To estimate the ranks in (52), MATLAB's rank function wasn't reliable, and hence a new method was developed.[5]

[3] Due to finite precision effects, no two eigenvalues will be exactly equal. However, we consider two eigenvalues to be multiples of the same eigenvalue if their difference is within 10^{-4}.

[4] In some of these histograms where the numbers being binned are very close to 1, the x-axis is marked 1 everywhere. This is probably due to MATLAB's inability to finely mark the x-axis.

[5] To estimate rank($\mathbf{X}$), suppose the eigenvalues of $\mathbf{X}$ are $\{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_M\}$. Assuming $\mathbf{X}$ is rank-deficient, the index i for which the ratio μ_i/μ_{i+1} is the largest is taken as rank($\mathbf{X}$).

7 Conclusion and Suggestions for Further Work

We began by recapitulating the Max-Min SNR problem of [1] and showing its equivalence to the two transmit beamforming problems of [2]. We then reviewed the solution methods derived in [1] and [2]. The Min-Max Eigenvalue (MMEV) method of [1] gives an optimum solution to the Max-Min SNR problem when the solved min-max eigenvalue is simple. The Semidefinite Relaxation (SDR) method of [2] gives an optimum solution to the Max-Min SNR problem when the solved positive semidefinite matrix $\mathbf{X}$ has rank 1. When $\text{rank}(\mathbf{X}) > 1$, the SDR method finds, through a heuristic procedure called randomization, a generally suboptimum solution to the Max-Min SNR problem. Simulations showed that, very often, the min-max eigenvalue is multiple and $\text{rank}(\mathbf{X}) > 1$. Therefore, we developed a new method to better handle the latter situation.

We formulated an auxiliary optimization problem, proved its equivalence to the Max-Min SNR problem, and then developed a method to solve the auxiliary problem. The auxiliary problem has an extra set of variables $\{\theta_i\}$ in addition to the variable $\mathbf{w}$ of the Max-Min SNR problem. The new method consists of many runs of iterations, where in each iteration, $\mathbf{w}$ is optimized for a fixed $\{\theta_i\}$ and then $\{\theta_i\}$ is adjusted based on the optimum $\mathbf{w}$. The optimization of $\mathbf{w}$ for a fixed $\{\theta_i\}$ is a convex optimization step. We mathematically proved that $\min_i \text{SNR}_i$ increases through the iterations and converges. Each run is initialized with a random $\{\theta_i\}$, and if enough runs are conducted, then there is a high probability that one of those runs will converge to an optimum solution of the Max-Min SNR problem.

We did simulations to evaluate the new method against MMEV and SDR methods and found that the new method 1) attains the performance of MMEV and SDR when the latter yield verifiably optimum solutions, 2) exceeds the performance of SDR when the latter yields suboptimum solutions. With the new method, we now have a complete suite of methods for solving the Max-Min SNR problem in all situations.

7.1 Further Work

It appears that the new method will be useful in adapting to small changes in the problem data. This aspect must be studied.

The new method must be tested in the contexts of GPS reception, physical-layer multicasting, and multibeam radars.

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Annex A: Histograms for Scenario 1

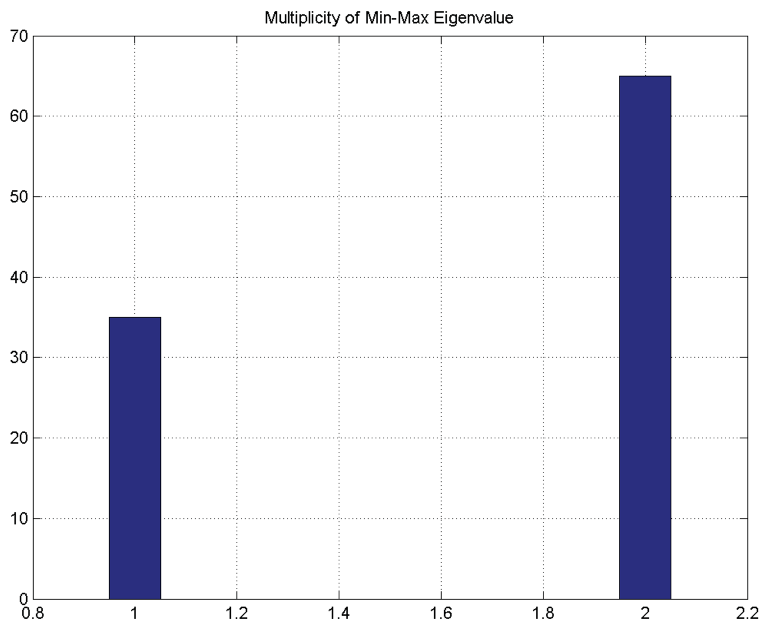

Figure A.1: *Histogram of multiplicity of min-max eigenvalue*

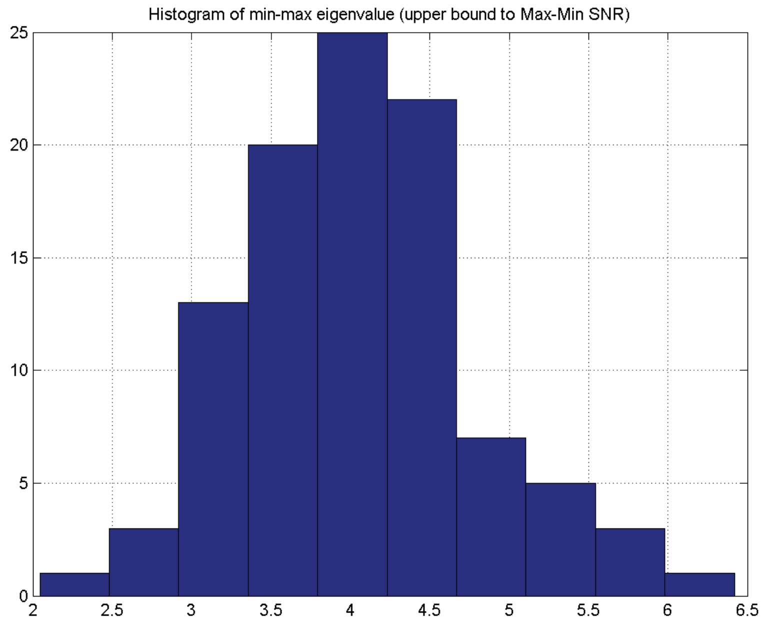

Figure A.2: *Histogram of min-max eigenvalue (upper-bound to Max-Min SNR)*

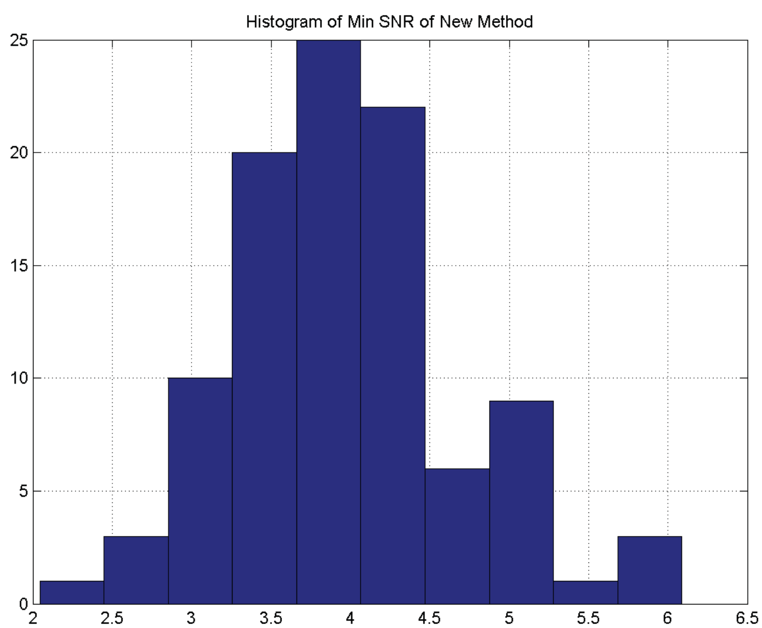

Figure A.3: *Histogram of min SNR of new method*

A.1 Min-Max Eigenvalue is Simple

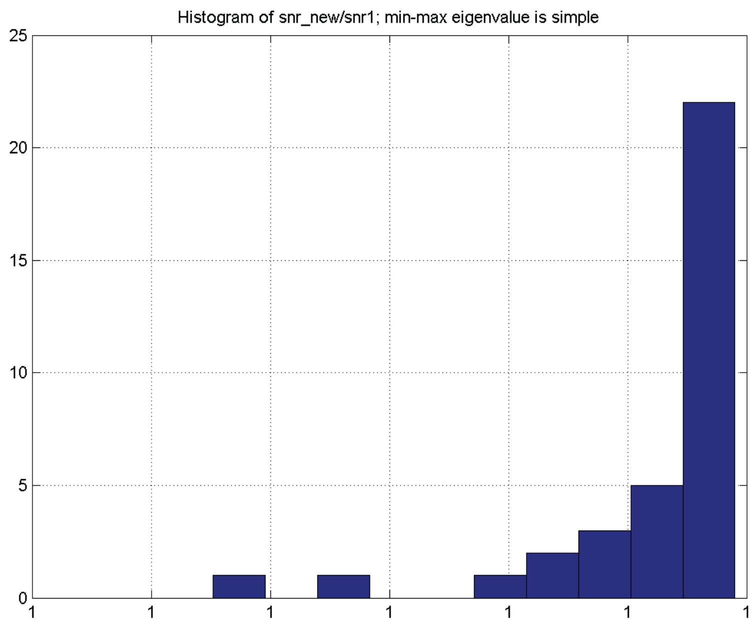

Figure A.4: *Histogram of snr_new/snr1; min-max eigenvalue is simple*

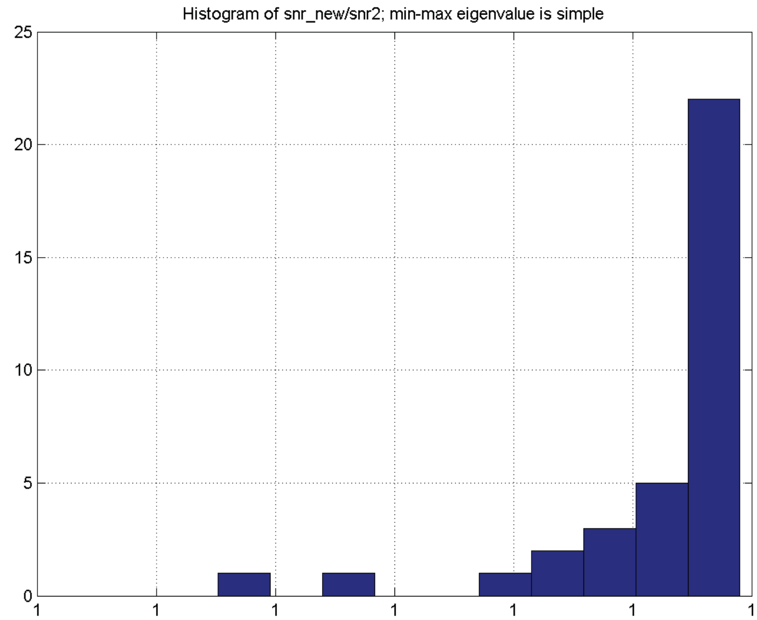

Figure A.5: *Histogram of snr_new/snr2; min-max eigenvalue is simple*

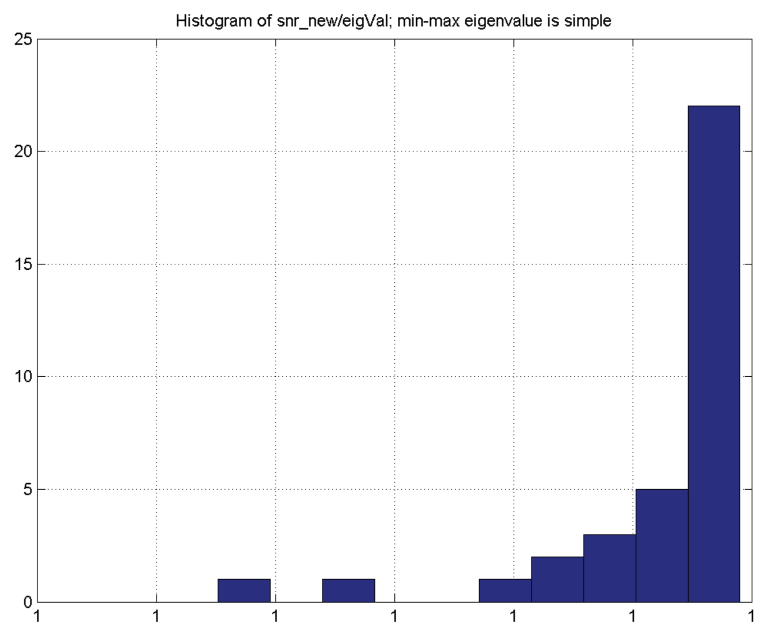

Figure A.6: *Histogram of snr_new/eigVal; min-max eigenvalue is simple*

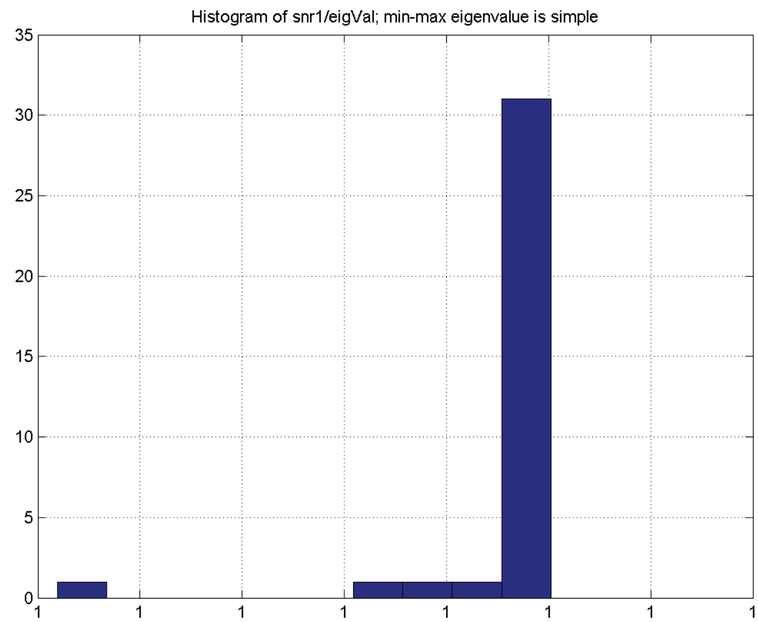

Figure A.7: *Histogram of snr1/eigVal; min-max eigenvalue is simple*

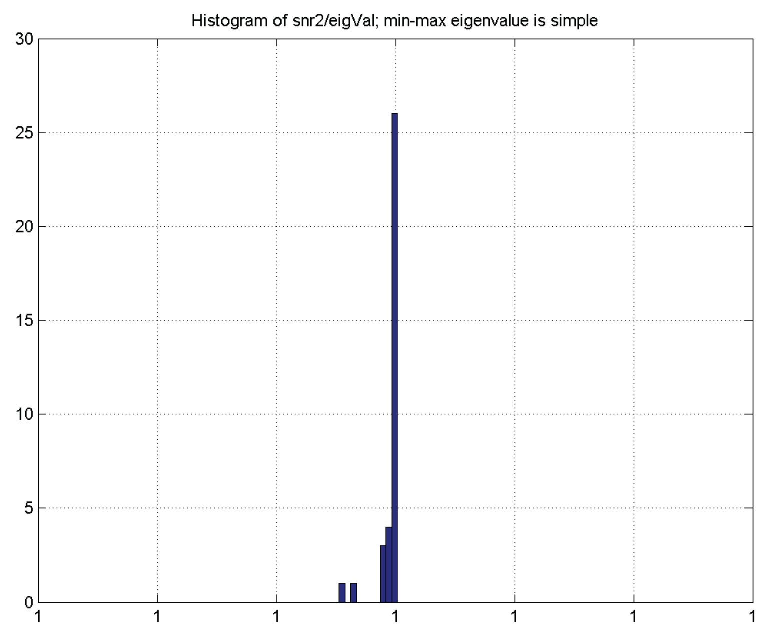

Figure A.8: *Histogram of snr2/eigVal; min-max eigenvalue is simple*

A.2 Min-Max Eigenvalue is Multiple

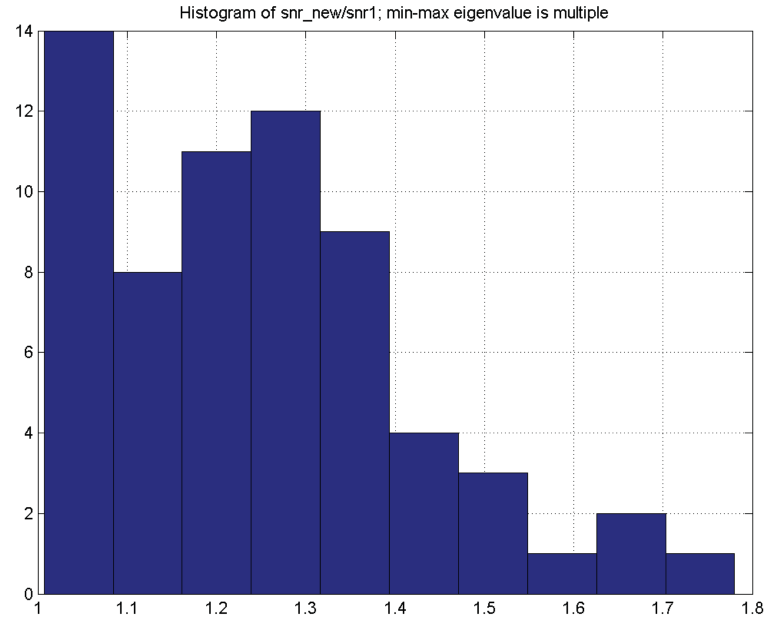

Figure A.9: *Histogram of snr_new/snr1; min-max eigenvalue is multiple*

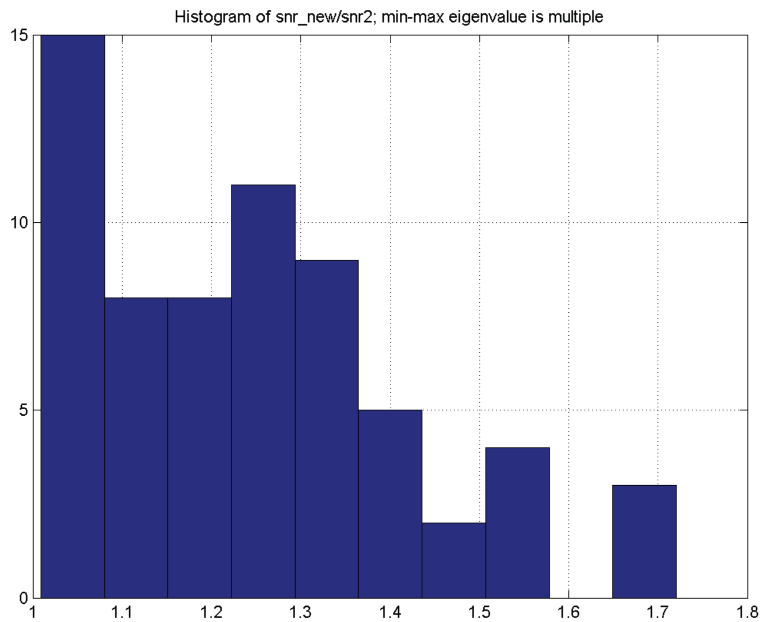

Figure A.10: *Histogram of snr_new/snr2; min-max eigenvalue is multiple*

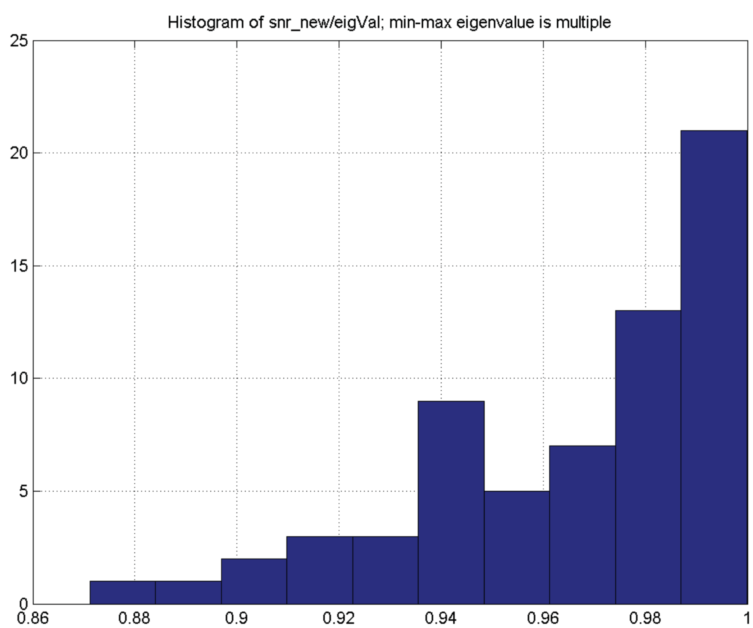

Figure A.11: *Histogram of snr_new/eigVal; min-max eigenvalue is multiple*

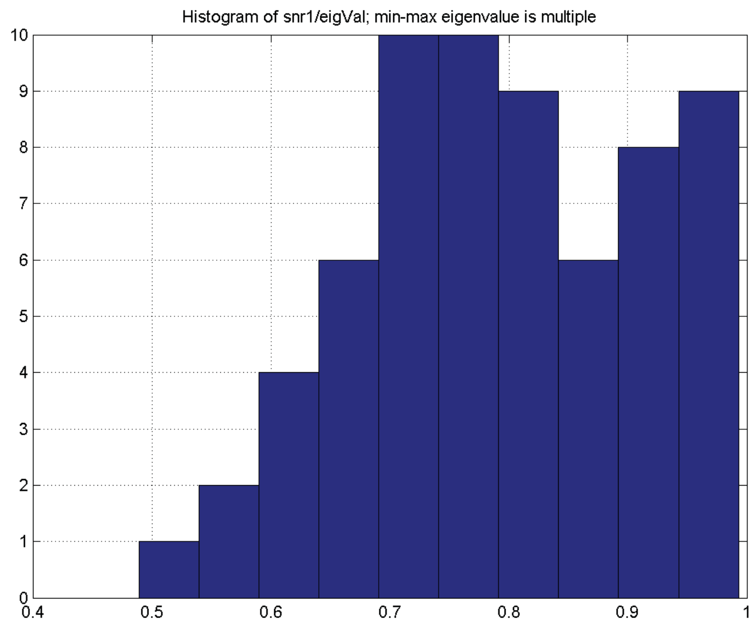

Figure A.12: *Histogram of snr1/eigVal; min-max eigenvalue is multiple*

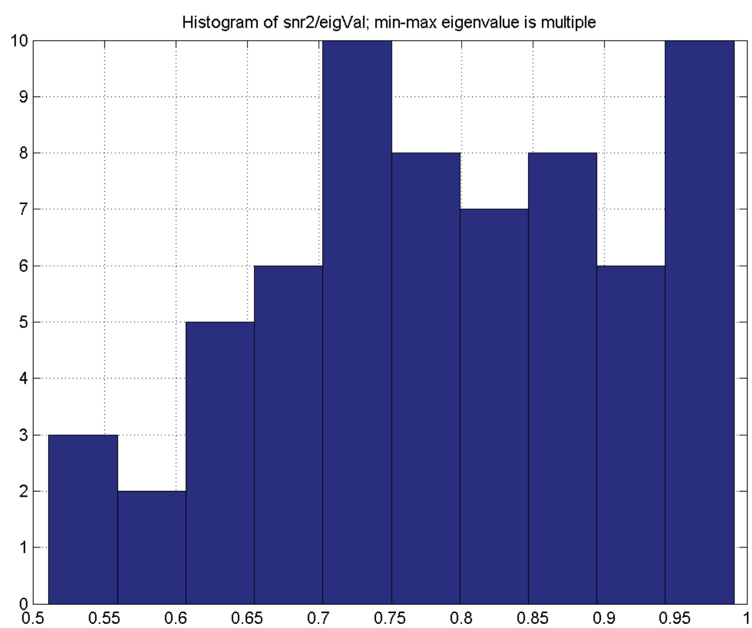

Figure A.13: *Histogram of snr2/eigVal; min-max eigenvalue is multiple*

Annex B: Histograms for Scenario 2

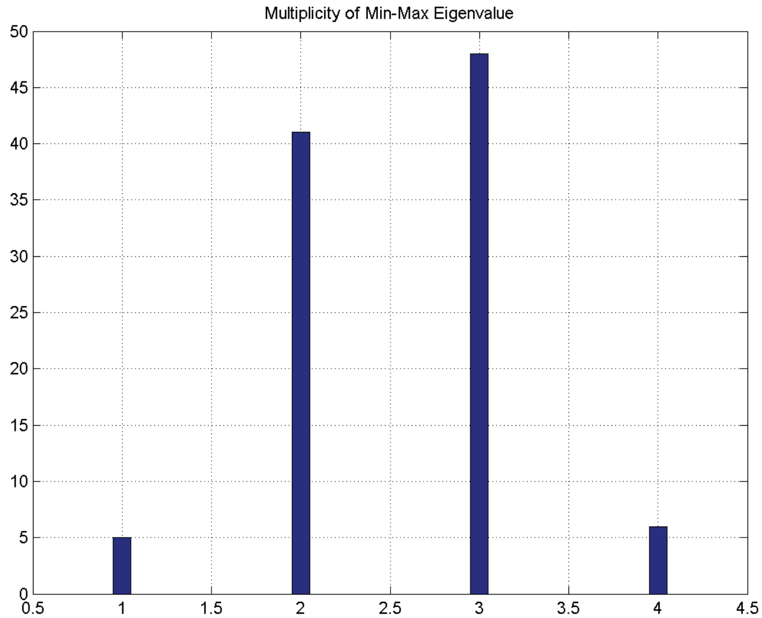

Figure B.1: *Histogram of multiplicity of min-max eigenvalue*

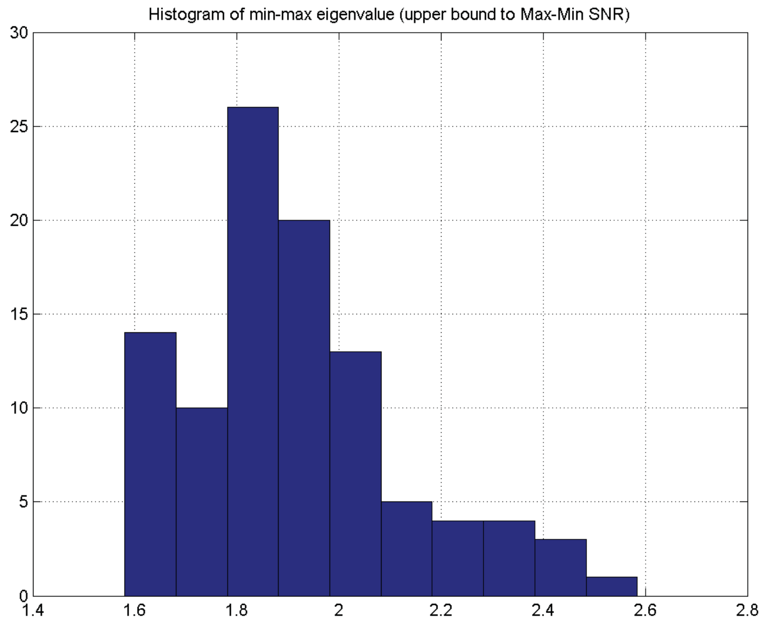

Figure B.2: *Histogram of min-max eigenvalue (upper-bound to Max-Min SNR)*

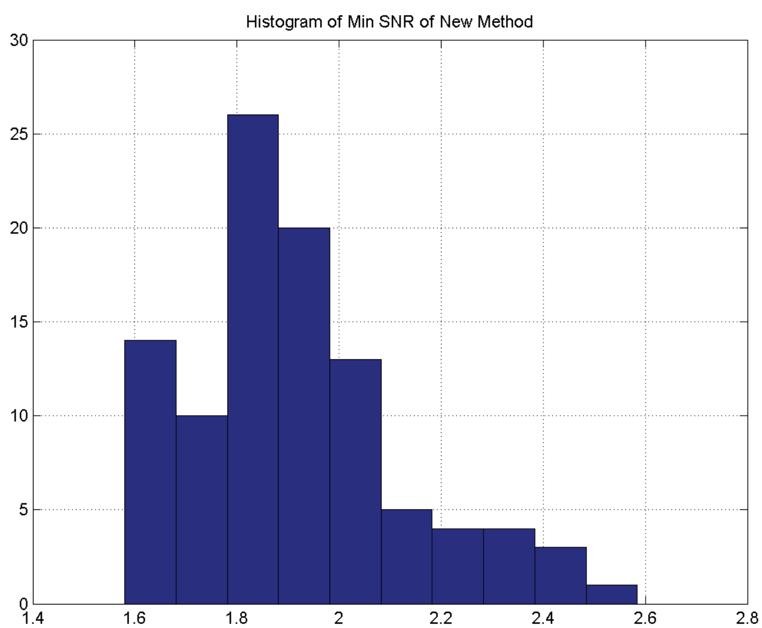

Figure B.3: *Histogram of min SNR of new method*

B.1 Min-Max Eigenvalue is Simple

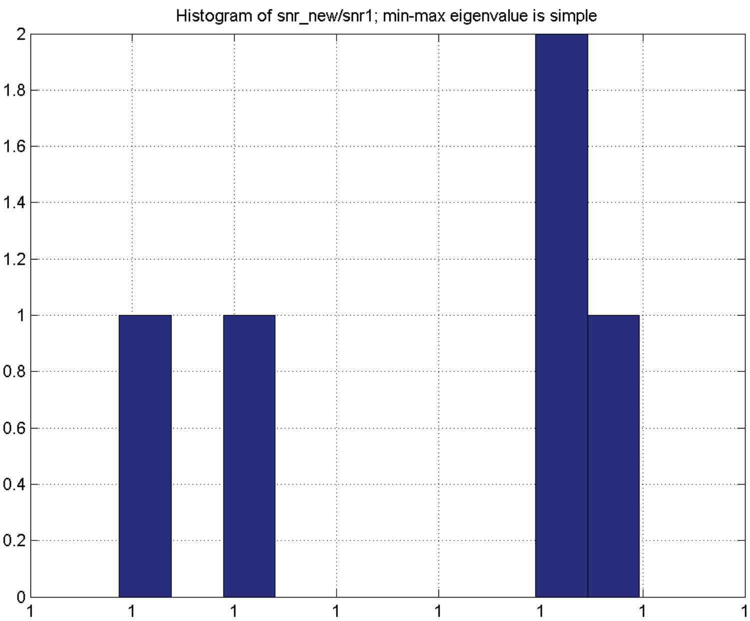

Figure B.4: *Histogram of snr_new/snr1; min-max eigenvalue is simple*

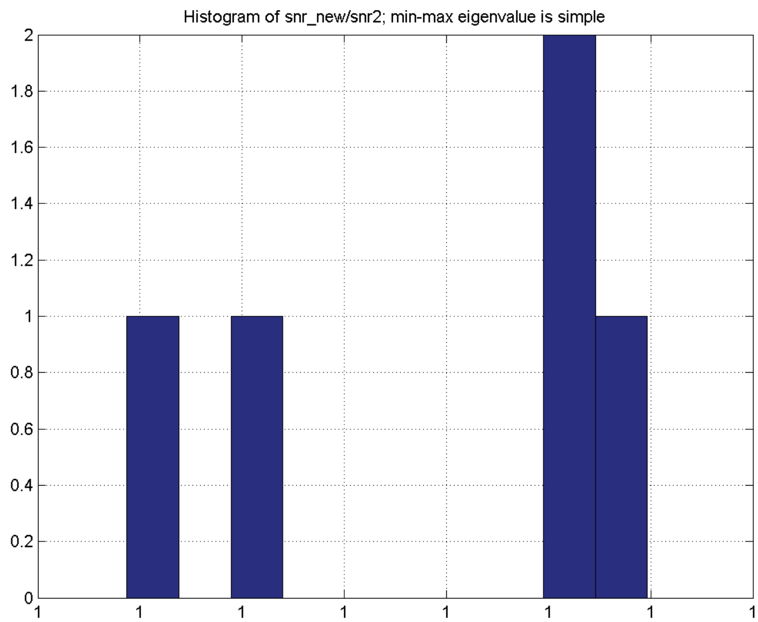

Figure B.5: *Histogram of snr_new/snr2; min-max eigenvalue is simple*

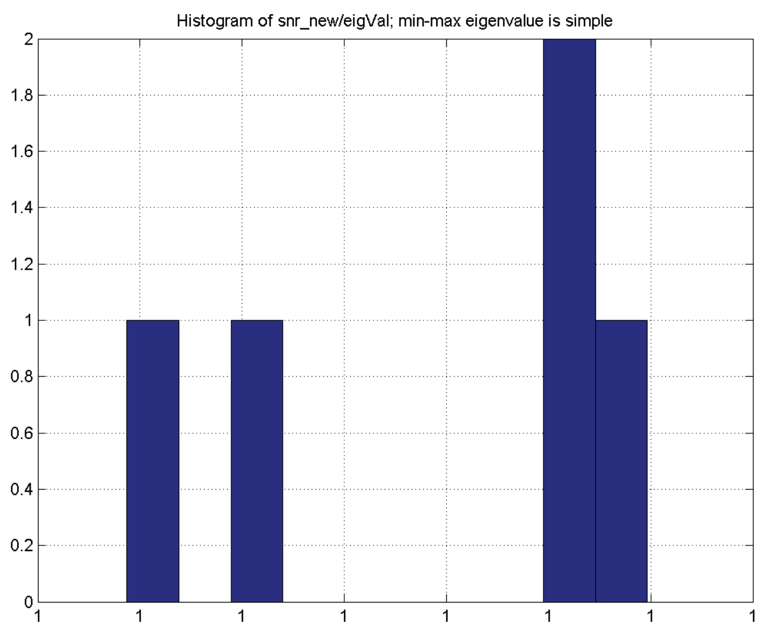

Figure B.6: *Histogram of snr_new/eigVal; min-max eigenvalue is simple*

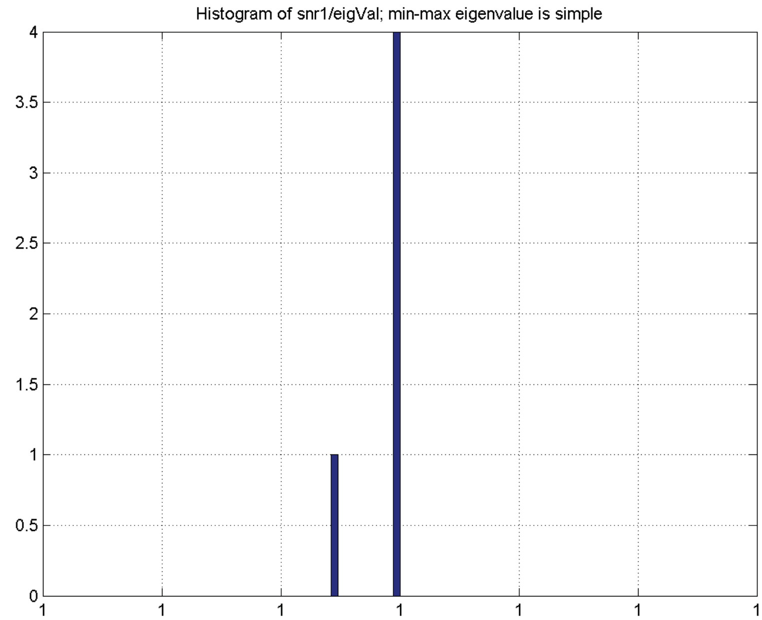

Figure B.7: *Histogram of snr1/eigVal; min-max eigenvalue is simple*

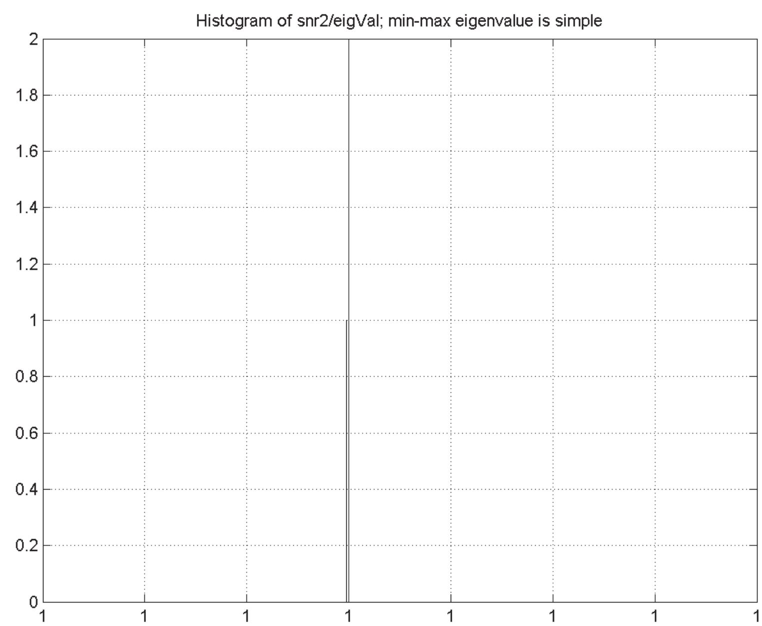

Figure B.8: *Histogram of snr2/eigVal; min-max eigenvalue is simple*

B.2 Min-Max Eigenvalue is Multiple

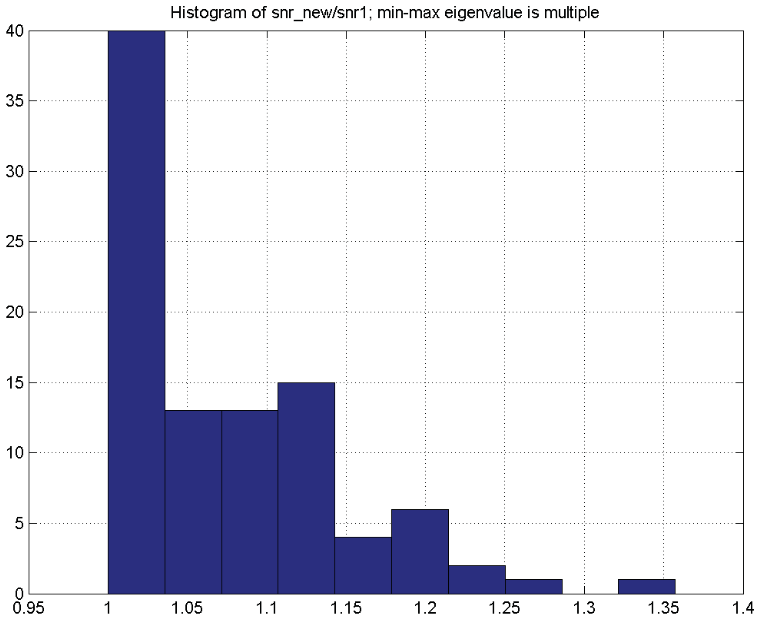

Figure B.9: *Histogram of snr_new/snr1; min-max eigenvalue is multiple*

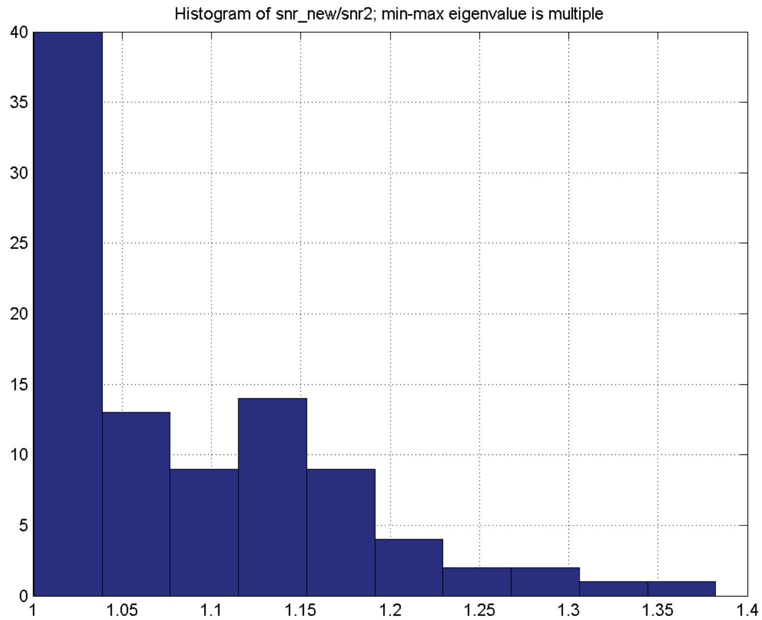

Figure B.10: *Histogram of snr_new/snr2; min-max eigenvalue is multiple*

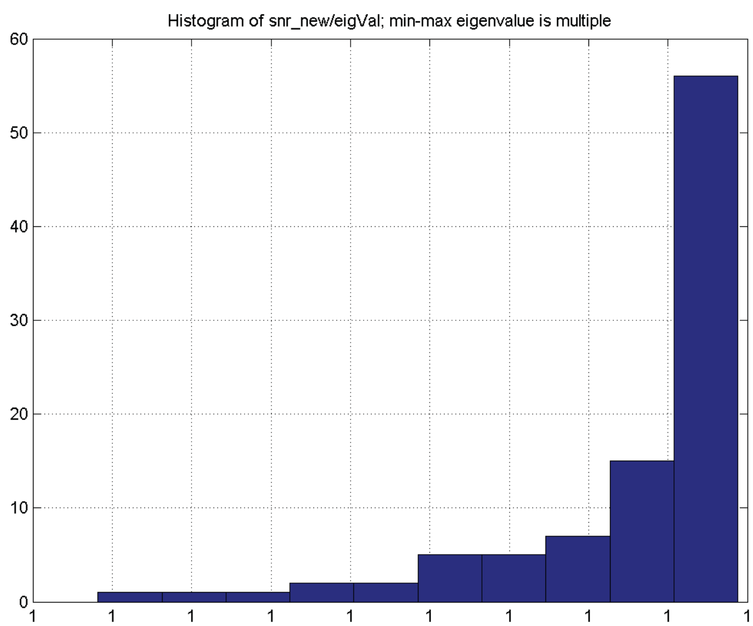

Figure B.11: *Histogram of snr_new/eigVal; min-max eigenvalue is multiple*

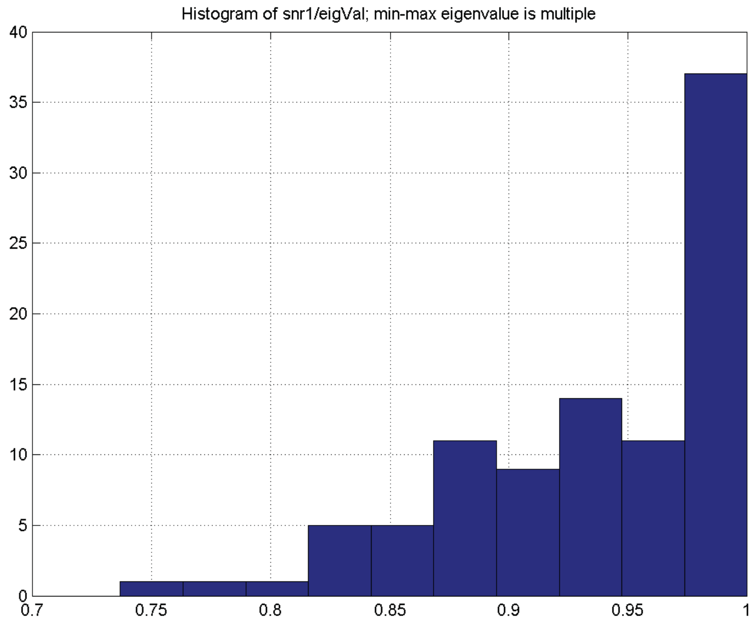

Figure B.12: *Histogram of snr1/eigVal; min-max eigenvalue is multiple*

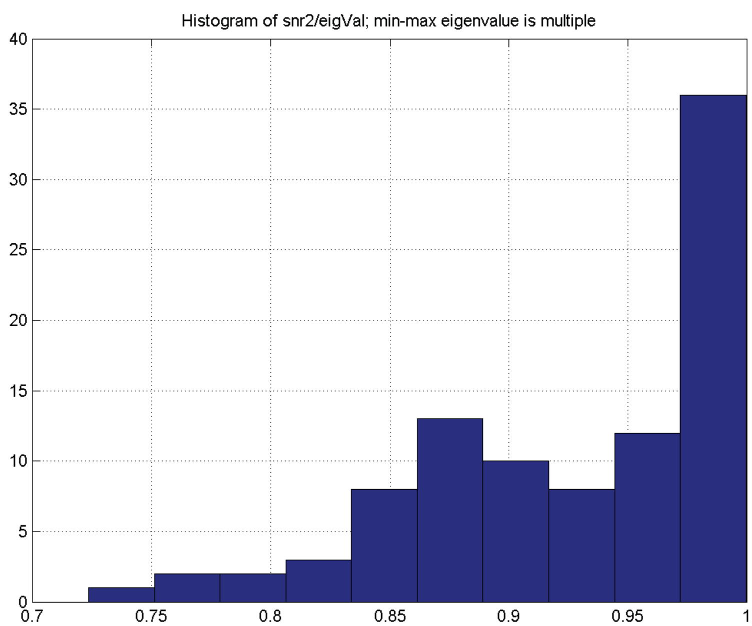

Figure B.13: *Histogram of snr2/eigVal; min-max eigenvalue is multiple*

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Solving the Max-Min SNR Optimization Problem of Array Antenna Signal Processing

4. AUTHORS (Last name, followed by initials – ranks, titles, etc. not to be used.)

Yasotharan, A.

5. DATE OF PUBLICATION (Month and year of publication of document.)	6a. NO. OF PAGES (Total containing information. Include Annexes, Appendices, etc.)	6b. NO. OF REFS (Total cited in document.)
December 2012	50	9

7. DESCRIPTIVE NOTES (The category of the document, e.g. technical report, technical note or memorandum. If appropriate, enter the type of report, e.g. interim, progress, summary, annual or final. Give the inclusive dates when a specific reporting period is covered.)

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The Max-Min SNR optimization problem was formulated in the author's previous report 'DRDC Ottawa TM 2011-202', in the context of using an array antenna to protect a GPS receiver from interferences. There, it was proposed to choose the array combining weights to maximize the minimum SNR of the satellites. Towards solving this problem, a convex *Min-Max Eigenvalue* problem was stated, and it was shown that: 1) the min-max eigenvalue is an upper bound on the Max-Min SNR, 2) if the min-max eigenvalue is *simple*, the upper bound is tight and the corresponding eigenvector solves the Max-Min SNR problem. A combinatorial search was proposed for the case when the min-max eigenvalue is *multiple*.

Recently, the author discovered that Sidiropoulos, Davidson, and Luo, writing in a communications context (physical-layer multicasting), had formulated two problems that are equivalent to the Max-Min SNR problem and proposed to solve them via *Semidefinite Relaxation* (SDR). This method sometimes finds optimum solutions, but in general gives suboptimum solutions.

In this report, we derive another solution method and show by simulations that it can outperform the SDR method. We formulate an auxiliary optimization problem which is equivalent to the Max-Min SNR problem and solve the auxiliary problem by an iterative process which uses convex optimization. We mathematically prove some convergence properties of the iterative process and show by simulations that by repeating the process several times, each time with a random initialization, a near-optimal solution can be found.

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GPS reception, physical-layer multicasting, multi-beam radar
array antenna signal processing, beamforming, optimum array weights
Max-Min SNR, Max Weighted-Average SNR
eigenvalue minimization, convex optimization
semidefinite optimization, semidefinite relaxation

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