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**THE MEAN CIRCULATION OF  
UNSTRATIFIED WATER BODIES**

by

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## Executive Summary

For practical considerations such as the movement of contaminants and other dissolved or suspended matter in large lakes, the net displacements of water masses over long periods of time are of primary importance. During the past few decades, a great deal of effort has therefore been devoted to the study of monthly, seasonal and annual mean water circulations in the Great Lakes. In these studies it has been customary to assume a linear relationship between the currents in the lake and the forcing exerted by the wind field over the lake. At mid-latitude regions such as the Great Lakes Basin, the winds are characterized by rapid fluctuations in speed and direction. While strong winds are common on a day-to-day basis, the frequent reversals of direction cause the mean wind vector to become small if the wind is averaged over a sufficiently long period. According to conventional linear models the resulting mean circulation of the Great Lakes must therefore be expected to be quite weak. However, extensive field measurements made on Lake Ontario by NWRI during the fall and winter of 1982/83 showed that the mean seasonal water circulation was very strong although the wind averaged over the season was almost negligible. Experiments with various hydrodynamic models showed that this mean

circulation was caused by nonlinear flow dynamics and that the generally-accepted linear models should be rejected. In order to support that conclusion, model calculations of mean winter circulations were made on the basis of 12 years of climatological wind records. For each year, the mean circulation obtained from the nonlinear model was typically two to three times stronger than the result of the linear model and the circulation patterns were drastically altered by the nonlinear effects. From a practical point of view it is therefore concluded that this new model must be used for all problems where the long-term displacements of water masses in the Great Lakes play an important role.

## Résumé administratifs

À des fins pratiques, il faut tenir compte du mouvement des contaminants et d'autres matières dissoutes ou en suspension dans de vastes lacs. C'est pourquoi les déplacements nets de masses d'eau sur de longues périodes sont si importants. Au cours des dernières décennies, on a déployé de nombreux efforts pour étudier les circulations moyennes mensuelles, saisonnières et annuelles dans les Grands Lacs. Dans le cadre de ces études, on avait comme habitude de prendre pour hypothèse qu'il existait une relation linéaire entre les courants du lacs et des forces exercées dans le champ de vent au-dessus dudit lac. Aux latitudes moyennes, comme c'est le cas du bassin des Grands Lacs, les vents sont caractérisés par des fluctuations de vitesse et de direction. Même s'il n'est pas chose rare d'enregistrer des vents forts tous les jours, leurs fréquents changements de direction font que le vecteur vent moyen s'en trouve réduit, si le vent est pondéré sur une période suffisamment longue. Selon les modèles linéaires contentionnels, la circulation moyenne des Grands Lacs devrait par conséquent être très faible. Toutefois, les mesures exhaustives faites sur le lac Ontario même par l'INRE pendant l'automne et l'hiver 1982-1983 ont révélé que la circulation moyenne saisonnière était très forte même si le vent qui avait soufflé en moyenne au cours de cette période était négligeable. Des expériences effectuées au moyen de divers modèles hydrodynamiques ont révélé que cette circulation moyenne résultait de la dynamique non linéaire du débit et qu'il y avait lieu de rejeter les modèles linéaires normalement acceptés. Afin d'étayer cette conclusion, on a fait des calculs modélisés des circulations moyennes enregistrées en hiver, en se fondant sur les registres climatologiques du vent de douze hivers. Pour chaque année, la circulation moyenne obtenue du modèle non linéaire était, de façon régulière, deux ou trois fois plus forte que le résultat du modèle linéaire et la circulation était radicalement modifiée par les effets non linéaires. D'un point de vue pratique, on conclut donc que ce nouveau modèle doit être utilisé chaque fois qu'on étudie des problèmes où les déplacements à long terme de masses d'eau dans le bassin des Grands Lacs jouent un rôle important.

## ABSTRACT

A critical evaluation is presented of the conventional view that long-term mean circulations of homogeneous basins may be identified with the quasi-steady response of linear models to atmospheric forcing. Based on climatological wind records for the Great Lakes region, it is shown that the mean circulation for unstratified seasons is dominated by rectified effects of nonlinear topographic wave interactions.

## RÉSUMÉ

Il s'agit d'une évaluation critique d'un point de vue conventionnel, à savoir que les circulations moyennes à long terme de bassins homogènes peuvent être identifiées par la réponse quasi-constante de modèles linéaires aux mouvements atmosphériques. D'après les registres climatologiques du vent pour la région des Grands-Lacs, on constate que la circulation moyenne pour des saisons non stratifiées est dominée par les effets rectifiées par des interactions non linéaires entre le relief et les vagues.

## INTRODUCTION

Theoretical studies of the mean circulation of unstratified lakes or shallow seas have concentrated on the steady-state or long-term averaged response of linear models to atmospheric forcing. In particular, the work by Birchfield (1967, 1972, 1973) has elucidated the characteristic properties of such circulations. The picture that emerges from these theoretical studies and the many numerical models of actual lakes is the following. Along the shores aligned with the wind the water moves in the direction of the wind and these nearshore currents are balanced by adjacent belts of return flow. In the inviscid deep portion of the basin the flow is confined to Ekman drift to the right of the wind. With increasing friction the belts of return flow tend to merge into a single broad band of upwind transport in the open lake and the overall circulation pattern appears as two counter-rotating gyres (see Simons, 1980, ch. 5 for a review).

A recent studies of the 1982/83 winter circulation of Lake Ontario (Simons, 1985) has shown that the above model may indeed be an acceptable approximation for intermediate time scales of a few weeks to a month. However, the long-term seasonal circulation did not resemble the solution of the steady-state linear model but could only be simulated by allowing for nonlinear dynamics. The reason for the failure of the linear model in the seasonal mean calculations was that the wind stress nearly vanished when averaged over the whole 1982/83

season and hence the mean linear response of the current to wind became equally small. On the other hand, the nonlinear response, while relatively small on a day-to-day basis, was highly persistent and produced a sizable seasonal-mean circulation.

For practical considerations such as the movement of dissolved or suspended matter in large lakes, the net displacements of water masses over long periods of time are of great importance. Therefore, it is of interest to investigate whether the above result is likely to occur under typical weather conditions in the Great Lakes region. This is done in the present study by comparing the linear and nonlinear current response to ten years of climatological wind data. The calculations are confined to the unstratified season from 1 November to 30 April. In addition, solutions are presented for idealized wind variations to illustrate the dependence of the circulations on forcing frequency. To facilitate comparison with earlier theoretical work, the model lake is a circular basin with parabolic depth profile.

### Model

The governing equations are the hydrostatic, vertically integrated equations of motion for a homogeneous basin. The ratio of scale length to surface wave speed is taken to be sufficiently small for the rigid-lid approximation to be valid. The vorticity equation for the vertically averaged flow is then



$$\frac{\partial \zeta}{\partial t} = \text{curl} \left( \frac{\tau}{H} \right) - \nabla \cdot \left( \frac{B}{H} \nabla \psi \right) + J \left( \frac{f}{H}, \psi \right) + J \left( \frac{\zeta}{H}, \psi \right) \quad (1)$$

where  $\zeta = \nabla \cdot (H^{-1} \nabla \psi)$ ,  $\psi$  the mass transport stream function,  $t$  is the time,  $\nabla$  the horizontal gradient operator,  $J$  the Jacobian,  $H$  the depth,  $\tau$  the wind stress,  $B$  a depth-dependent bottom friction coefficient, and  $f$  the Coriolis parameter. The dimensions of the basin are taken to be smaller than those of typical weather systems such that the wind stress is approximately uniform in space. The Coriolis parameter is treated as a constant and the bottom stress coefficient is inversely proportional to the square of the depth (Simons, 1985). The last term of Eq. (1) represents the nonlinear effect to be evaluated in the present study.

For application to a circular basin, the vorticity equation is written in polar coordinates. Since the depth is only a function of the radius, solutions may conveniently be obtained by the energy-conserving spectral methods familiar from atmospheric studies (e.g. Simons, 1972). Thus, for each azimuthal wave number, an equation is obtained wherein the radius of the basin appears as the only spatial variable. For a uniform wind the wind stress components are proportional to the sine and cosine of the azimuthal angle and hence the wind forcing appears only in the equation for the first

azimuthal wave number. The nonlinear self-interactions of the forced wave generate a circular vortex of zero wave number as well as the second azimuthal wave. In the lowest order energy-conserving system, the second wave number is discarded but the first wave itself must be allowed to change due to feed-back from the circular vortex. The next-order system includes the generation of the second azimuthal wave. To close this system on energy, the nonlinear interaction between the circular vortex and the second wave must be included as well as the interaction between the first and second wave and the self-interaction of the second wave. Again, the waves of higher wave number which would, in principle, be generated are to be ignored to make this low-order system energy conserving.

The spectral method may be contrasted with the method of computing second-order rectified effects as done by Bennett (1978) and Ou and Bennett (1979) for a stratified circular basin. In that case, the first azimuthal wave which is forced by the wind is considered a first-order quantity while the result of nonlinear self-interaction of this wave is assumed to be small of second order. In that framework the feed-back from the azimuthal-mean rectified flow is of higher order and hence can be neglected. It will be found that this approach is less accurate for the present problem than the spectral technique. However, in a qualitative sense, the results from both methods are similar and, since the former method is much simpler than the latter, it can provide a useful first approximation to the azimuthal-mean

flow. As an example, the second-order solution for the case of periodic forcing will be formulated.

For a spatially uniform wind periodic in time with frequency  $\sigma$  and amplitude  $\tau_0$  the forcing term in (1) becomes

$$\text{curl}\left(\frac{\tau}{H}\right) = -\tau_0 \frac{dH^{-1}}{dr} \sin \theta \sin \sigma t = -\frac{\tau_0}{2} \frac{dH^{-1}}{dr} \text{Re}[e^{i(\theta-\sigma t)} - e^{i(\theta+\sigma t)}] \quad (2)$$

where  $r$  is the radius and  $\theta$  the azimuthal angle measured counterclockwise from the downwind end of the basin. The first-order solution is a wave of the first azimuthal wave number of the form

$$\psi_1(r, \theta, \tau) = \frac{1}{2} \text{Re} [\psi_\alpha(r) e^{i(\theta-\sigma t)} - \psi_\beta(r) e^{i(\theta+\sigma t)}] \quad (3)$$

which must satisfy the linearized form of (1), thus

$$\frac{d}{dr} \left[ \frac{r(\sigma+iB)}{H} \frac{d\psi_\alpha}{dr} \right] - \left[ \frac{\sigma+iB}{rH} - f \frac{dH^{-1}}{dr} \right] \psi_\alpha = -i\tau_0 r \frac{dH^{-1}}{dr} \quad (4)$$

The equation for  $\psi_\beta$  is the same but with the sign of  $\sigma$  reversed.

The solution of the finite-difference form of (4) is readily obtained by direct matrix inversion. Since topographic waves propagate in counterclockwise direction around the basin, resonance can occur only for the first component of the forcing (2) and hence the first component of (3) contributes most of the current response. The present solutions are therefore also applicable to the case of a

periodic wind propagating alongshore in the same direction as a shelf wave (see Simons, 1983).

The second-order circular vortex is determined by the azimuthal mean of the equation of motion for the azimuthal current component

$$\frac{\partial \bar{V}}{\partial t} = -B\bar{V} - \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{r^2}{H} \overline{U_1 V_1} \right) \quad (5)$$

where  $U$  and  $V$  are the components of the vertically integrated current in radial and azimuthal direction, the subscript 1 denotes the first order solution (3) and the bar denotes an azimuthal average. For the case of periodic forcing, Eq. (5) may be averaged in time and (3) may be substituted to obtain the time-averaged circular vortex

$$\bar{\bar{V}} = \frac{H^2}{8br^2} \frac{d}{dr} \left[ \frac{r}{H} \text{Im} \left( \psi_\alpha \frac{d\psi_\alpha^*}{dr} + \psi_\beta \frac{d\psi_\beta^*}{dr} \right) \right] \quad (6)$$

where the double bar denotes an average over time,  $t$ , as well as azimuth,  $\theta$ , and as noted below Eq. (1) the bottom stress coefficient has been written as  $B=b/H^2$  where  $b$  is a constant. Since the term within brackets vanishes at the centre and the border of the basin, the product  $\bar{V}r^2/H^2$  must vanish when integrated over the radius and hence the circular vortex must have at least one reversal in direction.

## RESULTS

It is intuitively clear that the ratio of nonlinear to linear mean seasonal circulations will tend to be proportional to the ratio of the variance to the mean of the atmospheric forcing and will depend on the shape of the wind spectrum. With this in mind, calculations were made of wind-driven currents as a function of the frequency of periodic forcing. The results are of course affected by bottom friction and the eigenfrequencies of the topographic normal modes of the basin. A typical solution is presented below. The circular basin has a radius of 40 km and a parabolic depth profile with a maximum depth of 100 m. The rigid-lid normal modes of the first azimuthal wave number for this type of basin have periods of  $2n(n+2)+1$  times the inertial period where  $n$  is the radial mode number. The bottom stress coefficient in (1) is set at  $B = 5 \times 10^{-3} H^{-2}$  where  $H$  is expressed in meters and  $B$  has dimensions of  $s^{-1}$ . The wind stress is uniform in space and periodic in time with amplitude of  $10^{-1} Nm^{-2}$ .

Figure 1 shows the offshore profile of alongshore transport in the circular vortex as a function of forcing frequency. The results have been averaged over the forcing period and the transport units are  $m^2 s^{-1}$  with positive values representing cyclonic circulations. The normal modes of the basin are denoted by triangles at the bottom of the graph. In this case the circular vortex was calculated

from (6) as the rectified effect of the wind-driven wave (3) without permitting the feedback included in energy-conserving spectral models.

The above second-order solution is contrasted with results from low-order spectral models and two-dimensional finite difference models in Figure 2. In this example the forcing period is 15 days and all model parameters are the same as in Figure 1. The curves on the left show the time-averaged circular vortex while the curves on the right show the time-averaged alongshore transport of the second azimuthal wave number in a cross section of the basin perpendicular to the wind. The dashed curves are the second-order solutions corresponding to Figure 1 without feedback from the rectified flow to the first azimuthal wave. The second wave is seen to be an order of magnitude smaller than the azimuthal-mean circular vortex. The dotted curve shows the solution of the lowest-order energy-conserving system consisting of the circular vortex and the first wave number. As expected, the rectified transport is substantially reduced by the feedback mechanism. The solid curves are solutions of the energy-conserving two-wave model. Apparently, including the second wave has a relatively small effect on the circular vortex and the second wave itself is quite similar to that obtained from the second-order solution without feedback. Finally, the black circles show the azimuthal mean of the solution of a two-dimensional finite-difference model with a grid spacing of 2 km. In such a model the number of waves generated by nonlinear processes is limited only by the spatial resolution.

According to Figure 2 and other computations for a range of parameter values, the results of low-order spectral models are in close agreement with two-dimensional finite-difference solutions. It may also be noted that the results are remarkably similar to those produced by interaction of a planetary wave and the zonal flow in an atmospheric model (Simons, 1972). This is not surprising since the topographic vorticity tendency in the circular basin is equivalent to the beta-effect in the atmosphere.

Proceeding next to calculations for realistic wind conditions, solutions of Eq. (1) were obtained for ten consecutive unstratified seasons between 1973 and 1983. The unstratified seasons were defined to cover the six-month period from 1 November to 30 April. The winds were taken from climatological records at Toronto Island Airport which were previously found to be in close agreement with wind measurements on Lake Ontario (Simons, 1985). Figure 3 shows seasonal-mean transports computed by the low-order spectral model for the same basin parameters used in Figure 1. The solid curves represent the azimuthal-mean circular vortex while the dashed curves show the alongshore transports of the first azimuthal wave in a basin cross section perpendicular to the wind. The seasonal-mean wind is in all cases directed from left to right in Figure 3 and hence the linear solution (Birchfield, 1973) consists of wind-driven coastal transport to the right of the page balanced by an adjacent band of return flow and vanishing alongshore transport in deep water. The nonlinear

solutions for the first wave (dashed curves of Figure 3) are seen to be similar to the linear results in the nearshore zone but the alongshore transports in the middle of the basin no longer vanish. Note that the dashed curves are symmetric with respect to the top of Figure 3 while the solid curves are anti-symmetric.

#### SUMMARY AND CONCLUSIONS

Using ten years of climatological winds from the Great Lakes region to generate currents in a homogeneous circular basin with parabolic depth profile, the seasonal-mean circulation was found to be dominated by an azimuthal-mean cyclonic vortex concentrated at the centre of the basin. This flow is generated by nonlinear self-interaction of the first azimuthal wave which is excited by a uniform wind. The seasonal-mean flow pattern of the first azimuthal wave itself is a modified form of the response of linear models to the seasonal-mean wind. If a circulation index is defined as the one-way transport through a cross section of the basin, then the transport of the circular vortex is found to exceed that of the wave in all cases. Averaged over the ten years of study, the ratio of vortex to wave transport is 2.1 if the nonlinear wave solution is used while the ratio is 2.8 if the wave is computed from linear models. This indicates that conventional linear models are not suitable for computing seasonal-mean circulations of homogeneous lakes or shallow



seas. While the present calculations were made for a circular basin with spatially-uniform wind, equivalent results are obtained for shelf circulations forced by spatially-periodic winds.

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## FIGURES

Figure 1 - Rectified circular vortex flow as a function of forcing frequency. Positive values represent cyclonic circulations.

Figure 2 Comparison of second-order solutions with results from low-order spectral models and a two-dimensional model.

Figure 3 Seasonal-mean alongshore transports of circular vortex (solid) and first azimuthal wave (dashed) corresponding to climatological Lake Ontario winds for unstratified seasons from 1 November to 30 April.





