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BENCHMARKING SYSTEMS

OF SOCIO-ECONOMIC TIME SERIES

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- résumé -

Ce document propose une méthode d'étalonnage simultané de systèmes de séries socio-économiques. Chaque composante du système doit se conformer à ses jalons annuels, ce qu'accomplit déjà l'étalonnage conventionnel. Les composantes régionales (disons) doivent en plus se sommer au total national étalonné pour chaque période de temps, ce qu'accomplit la méthode d'étalonnage simultanée proposée. La méthode fonctionne également pour les systèmes de séries classifiées par régions et par industrie. En plus de satisfaire leurs jalons, les composantes doivent alors s'additionner et à leur total régional et à leur total industriel respectifs. La méthode spécifie l'étalonnage comme une regression à la Chow et Lin (1971), en y incorporant le critère de préservation de mouvement fréquemment utilisé pour l'étalonnage (Denton, 1971). La technique d'estimation par les Moindres carrés généralisés permet en effet pareille incorporation. Le traitement simultané de plusieurs séries avec contraintes d'agrégation est rendu possible par les Moindres carrés généralisés conjoints (Theil, 1971).

- abstract -

This paper presents a method for benchmarking systems of socio-economic time series. Each component of the system must comply with its benchmarks, which is already accomplished by conventional benchmarking. The regional (say) series must also add to the benchmarked national total for each period of time, which is accomplished by the simultaneous benchmarking method proposed. The method also works for systems of series classified by region and by industry. The components must then add both to their regional totals and to their industrial totals (and satisfy their benchmarks). Following the Chow and Lin (1971) approach, the method specifies benchmarking in the regression framework by incorporating the commonly used benchmarking criterion of movement preservation (Denton, 1971) into the regression. This incorporation is made possible by the Generalized Least Squares estimation technique. The simultaneous processing of several series is achieved by means of Joint Generalized Least Squares (Theil, 1971).

KEYSWORDS: Benchmarking, Interpolation, Quadratic Minimization, Iterative Proportional Fitting

1. INTRODUCTION

Many of the socio-economic time series published by statistical agencies are subject to benchmarking. Benchmarking arises when a given variable is measured at different periodicities, for instance monthly and annually, quarterly and annually, annually and quinquennially. In this paper, the more frequent periodicity will be referred to as **sub-annual**; and the less frequent, as **annual**. The "annual" series is generally more reliable than the "sub-annual" and is therefore considered as a benchmark to the sub-annual series. Benchmarking is then the process of adjusting the sub-annual series, so that it becomes consistent with the values of the corresponding annual benchmark series; namely, so that the yearly totals of the former are equal the values of the latter (in the case of flow series).

The series affected by benchmarking are generally part of a system of series, bound together by additivity (or linear) constraints. Additivity constraints typically occur when a variable is measured at different levels of geographical and/or industrial aggregation. For example, the Retail Trade series for the regions of a country should add up to the national aggregate for each period of time, both annually and sub-annually. The problem addressed by this paper is that individual (proportional) benchmarking each series generally destroys additivity at the sub-annual level. The proposed simultaneous benchmarking solution produces series which satisfy both their own annual benchmarks and additivity to the aggregate.

Section 2 illustrates the problem of benchmarking a system of regional series. Before providing a solution, Section 3 shows how a variant of the Chow and Lin (1971) interpolation method can be used to benchmark individual series, according to the movement preservation principle established by Denton (1971). The Chow and Lin formulation of benchmarking has the distinct advantage of casting the problem in the more familiar regression analysis framework. That formulation also proves very economical computationnally. This provides the feasibility basis of the simultaneous benchmarking methods proposed in Section 4 and 5, which derive from the theory of Joint Generalized Least Squares (Theil, 1971). The method presented in Section 4 makes it possible to process over 50 regional monthly series on 5-year intervals, subject to (60) aggregation constraints. Section 5 generalizes the approach to handle series available by regions and by industries, which must add up both to regional and industrial totals and satisfy their benchmarks. Section 6 discusses issues related to benchmarking and is followed by a conclusion.

The paper somewhat follows the structure recommended by Ehrenberg (1982) for technical papers: Illustrations and results are presented as early as possible, followed by methodological details and background discussion. The paper provides ample technical - as opposed to methodological - details for those potentially interrested in programming the method. The more "statistical" reader may skip some of those details.

Table 1: Original regional series $x_{r,t}$, regional benchmark values $y_{r,m}$, individually benchmarked series $z^i_{r,t}$, simultaneously benchmarked series $z_{r,t}$ accompanied by relevant statistics.

(1) $x_{1,t}$	(2) $x_{2,t}$	(3) $x_{3,t}$	(4) $y_{1,m}$	(5) $y_{2,m}$	(6) $y_{3,m}$	(7) $z^i_{1,t}$	(8) $z^i_{2,5}$	(9) $z^i_{3,t}$	(10) d_t**	(11) $z_{1,t}$	(12) $z_{2,t}$	(13) $z_{3,t}$	(14) $d_{1,t}$*	(15) $d_{2,t}$*	(16) $d_{3,t}$*	(17) t
34.4	3.9	30.5	145	30	115	34.2	3.3	31.9	-2.8	34.7	3.2	31.5	1.5	-1.5	-1.3	1
31.7	6.4	25.3				31.6	5.4	26.5	-0.8	31.8	5.3	26.4	0.5	-0.8	-0.3	2
37.6	13.4	24.3				37.7	11.4	25.3	2.5	37.2	11.5	25.7	-1.3	1.0	1.3	3
41.3	11.3	30.0				41.5	10.0	31.2	0.8	41.3	9.9	31.4	-0.4	-0.2	0.6	4
35.6	5.1	30.5	173	48	125	36.0	4.7	31.6	-0.8	36.1	4.6	31.5	0.4	-0.6	-0.4	5
36.6	8.8	27.8				37.2	8.4	28.8	0.0	37.2	8.4	28.8	0.0	-0.2	0.1	6
48.2	19.6	28.6				49.1	19.0	29.8	0.7	49.0	19.1	29.9	-0.3	0.3	0.4	7
49.7	16.6	33.1				50.7	16.0	34.8	-0.1	50.8	16.0	34.8	0.1	-0.1	0.0	8
42.5	6.5	36.0	204	54	150	43.4	6.1	38.4	-2.4	44.0	6.0	38.0	1.3	-1.2	-1.2	9
45.2	11.6	33.6				46.2	10.5	36.2	-1.1	46.4	10.4	36.1	0.6	-0.9	-0.4	10
55.7	23.3	32.4				56.8	20.8	34.8	2.2	56.2	21.0	35.2	-1.1	1.0	1.1	11
56.6	18.6	38.0				57.6	16.7	40.5	0.7	57.4	16.7	40.7	-0.4	-0.2	0.5	12
41.4	6.8	34.7	192	52	140	42.0	6.3	36.3	-1.3	42.3	6.2	36.1	0.7	-0.8	-0.6	13
43.4	10.6	32.8				43.9	10.0	34.0	-0.2	43.9	10.0	34.0	0.1	-0.3	-0.1	14
50.8	20.8	30.0				51.3	19.8	31.0	1.0	51.0	19.9	31.1	-0.5	0.5	0.5	15
54.4	16.8	37.5				54.8	16.0	38.7	0.2	54.8	16.0	38.8	-0.1	-0.1	0.2	16
46.0	6.7	39.3	212	55	157	46.4	6.3	40.9	-1.5	46.8	6.2	40.6	0.9	-1.1	-0.6	17
46.8	11.4	35.5				47.3	10.5	37.1	-0.6	47.4	10.4	37.0	0.4	-0.9	-0.1	18
56.3	22.8	33.5				56.9	20.8	35.2	1.5	56.4	21.0	35.4	-0.8	0.5	0.8	19
60.9	19.1	41.7				61.5	17.4	43.9	0.3	61.4	17.4	43.9	-0.2	0.3	0.0	20
51.2	7.3	43.9				51.7	6.6	46.2	-2.1	52.2	6.7	45.5	0.9	0.4	-1.5	21
51.3	12.2	39.1				51.8	11.1	41.1	-0.8	51.9	11.3	40.7	0.3	1.5	-1.0	22
59.6	23.5	36.1				60.1	21.4	37.9	1.4	59.8	22.0	37.8	-0.6	2.9	-0.3	23
													0.38	0.98	0.77	
													average absolute value of column			

* $d_{r,t} = (z_{r,t}/z^i_{r,t} - 1)*100$

** $d_t = (z^i_{1,t}/(z_{2,t}+z_{3,t}) - 1)*100$

2. PRESENTATION OF THE SIMULTANEOUS BENCHMARKING PROBLEM

The three first columns of Table 1 contain three regional original unbenchmarked series x_{1t}, x_{2t} and x_{3t}. The first original series is the sum of the other two, $x_{1t}=x_{2t}+x_{3t}$ for all t. Columns 4 to 6 displays the annual benchmarks y_{1m}, y_{2m} and y_{3m} of the three series (which also sum up). The original series do not add up yearly to their respective benchmarks. This is the problem solved by individual benchmarking. Columns 7 to 9 display the individually benchmarked series z^i_{1t}, z^i_{2t} and z^i_{3t}, obtained by means of the proportional variant of the Denton (1971) method modified by Cholette (1984): Each of these series z^i_{rt} is as proportional as possible to its original series x_{rt} (more details below) and yearly adds up to its benchmarks y_{rm}. The problem is now that the first series is no longer the sum of the other two, z^i_{1t} not equal to $z^i_{2t}+z^i_{3t}$ for some t.

The percentage aggregation discrepancies d_t recorded in column 10 reach as high as 2.8% (in absolute value). This is the problem addressed by the simultaneous benchmarking. Except for small rounding errors (which are not the problem addressed herein), the resulting simultaneously benchmarked series z_{rt} in column 11 to 13 add up to their benchmarks and sum over regions. They are as proportional to the original series as possible in the circumstances.

Under additive benchmarking (instead of proportional), there is generally no simultaneous benchmarking problem. Each individually benchmarked series $z^i{}_{rt}$ is equal to a linear combination of its original values x_{rt} and of its benchmarks y_{rm} (see equation (3.1.6')). The weights of that combination are also the same for all series (provided all benchmarks have same reference periods across series). The additivity of the x_{rt}'s and of the y_{rm}'s implies that of the $z^i{}_{rt}$'s.

$$
\begin{array}{lcl}
 & & Z^i{}_1 = L_x X_1 + L_y Y_1 \\
Y_1 = Y_2 + Y_3 & & \quad = L_x(X_2 + X_3) + L_y(Y_2 + Y_3) \\
 & \Rightarrow & \\
X_1 = X_2 + X_3 & & \quad = L_x X_2 + L_y Y_2 + L_x X_3 + L_y Y_3 \\
 & & \quad = Z^i{}_2 + Z^i{}_3
\end{array}
$$

where $Z^i{}_1$ is a vector containing $z^i{}_{1t}$ $(t=1,\ldots,T)$, and similarly for $Z^i{}_2$, $Z^i{}_3$, Y_1, Y_2 and Y_3 and X_1, X_2 and X_3. In other words, additive benchmarking of the sum (over regions) is equal to the sums of additively benchmarked components.

For very seasonal series like those in Table 1, additive benchmarking is not advisable and could result in negative benchmarked values for low seasons of small series. Simultaneous benchmarking is then appropriate. A variant of the Chow and Lin interpolation method lends itself to benchmarking and to simultaneous benchmarking.

3. BENCHMARKING AS A REGRESSION PROBLEM

The original purpose of the Chow and Lin (1971) method is to derive a sub-annual series z_t from its known annual values y_m and from Q related series x_{qt} known sub-annually.

3.1 The regression Model

The desired and related series obey an econometric relationship:

$$(3.1.1) \qquad z_t = \sum_{q=1}^{Q} x_{t,q}\, \gamma_q + e_t, \quad t=1,\ldots,T$$

or in matrix algebra:

$$(3.1.2) \qquad \underset{T \text{ by } 1}{Z} = \underset{T \text{ by } Q}{X} \; \underset{Q \text{ by } 1}{\Gamma} + \underset{T \text{ by } 1}{e}, \qquad E(e\, e') = \underset{T \text{ by } T}{V}$$

Matrix V is the known covariance matrix of the residuals, to be specified later. The parameters in Γ are to be estimated on the annual values of X and Z. The latter are given by

$$(3.1.3) \qquad \underset{M \text{ by } 1}{Y} = \underset{M \text{ by } T}{B} \; \underset{T \text{ by } 1}{Z}$$

where M is the number of year in the series and B is the annual sum matrix operator. For a quarterly flow series (say):

$$\underset{M \text{ by } T}{B} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \ldots & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & & . & . & . & . \\ . & . & . & . & . & . & . & . & & . & . & . & . \\ . & . & . & . & . & . & . & . & & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 1 & 1 & 1 \end{bmatrix}$$

The annual version of equation (3.1.2) accordingly writes

$$(3.1.4) \qquad B\,Z = Y - B\,X\,\Gamma + B\,e \quad \text{or}$$

$$(3.1.4') \qquad Y = X_a\,\Gamma + e_a, \quad E(e_a\,e_a') = B\,V\,B' = V_a$$

where X_a=B X stand for the annual values of X. The Generalized Least Squares (G.L.S.) estimate of Γ is then $\Gamma^* = (X_a' V_a^{-1} X_a)^{-1} X_a' V_a^{-1} Y$. Chow and Lin show that the resulting best linear unbiased sub-annual estimates are

$$(3.1.5) \qquad Z = X\,\Gamma^* + V\,B'(B\,V\,B')^{-1}\,(Y - BX\Gamma^*),$$

which are the regression "predicted" values. For the purpose of benchmarking, there is only one related series (Q=1), which is the unbenchmarked version of the desired series, and Γ is known and equal to 1. (One could formally specify that in (3.1.2), estimate with G.L.S. with extraneous (prior) information on Γ (Durbin, 1953; Alba, 1988) and arrive at (3.1.5) and (3.1.6)). Solution (3.1.5) then reduces to

$$(3.1.6) \qquad \underset{T \text{ by } 1}{Z} = \underset{T \text{ by } 1}{X} + \underset{T \text{ by } M}{V\,B'} \; \underset{M \text{ by } M}{(B\,V\,B')^{-1}} \; \underset{M \text{ by } 1}{(Y\text{-}BX)} = \underset{T \text{ by } 1}{X} + \underset{T \text{ by } M}{W} \; \underset{M \text{ by } 1}{U}$$

where U=Y-BX contains the annual discrepancies between the benchmarks and the original series and $W=VB'(BVB')^{-1}$ is the weight matrix to be applied to the discrepancies. The benchmarked series is then equal to the original plus sub-annual corrections WU interpolated between the annual discrepancies U. The solution can also be expressed as

$$(3.1.6') \qquad Z = L_x X + L_y Y.$$

where L_x=(I-WB) and L_y=W. The estimates are a linear combination of the unbenchmarked series and of the benchmarks.

Benchmarking according to equation (3.1.6) requires the inversion of a M by M matrix. In other words, the size of the inversion depends on the number of years M in the series and not on the number of observations T.

Furthermore, if V were known algebraically, so would be VB' and BVB'. These considerations matter for individual benchmarking and become crucial for simultaneous benchmarking.

3.2 Relation between Generalized Least Squares and Quadratic Minimization

The next question is then what is an appropriate covariance matrix V. It is well known that G.L.S. with a covariance matrix V is equivalent to minimizing a criterion V^{-1} on the residuals. (The covariance matrix V is precisely the inverse of the corresponding criterion matrix V^{-1}; and vice versa.) As pointed out by Bournay and Laroque (1979) and Alba (1979, 1988), estimating (3.1.2) with G.L.S. amounts to minimizing the objective function $(Z-X\Gamma)'V^{-1}(Z-X\Gamma)$. Many benchmarking methods are in fact based on this minimization (rather than on regression), subject to constraints (3.1.3). The resulting Langragian augmented objective function is (with $Q=1$, $\Gamma=1$):

$$(Z - X)'V^{-1} (Z - X) - 2 \Lambda' (B Z - Y) \tag{3.2.1}$$

This lead to the solution:

$$\begin{bmatrix} Z \\ \Lambda \end{bmatrix} = \begin{bmatrix} V^{-1} & B' \\ B & 0 \end{bmatrix}^{-1} \begin{bmatrix} V^{-1} & 0 \\ B & I \end{bmatrix} \begin{bmatrix} X \\ Y-BX \end{bmatrix} \tag{3.2.1'}$$

The solution for Z can be achieved by performing the matrix inversion by parts, if V^{-1} is invertible. The resulting expression for Z is then exactly (3.1.6). This establishes the G.L.S. regression as a particular case of quadratic minimization. (If V^{-1} is not invertable, solution (3.2.1') is valid, except the inversion cannot be performed by part.)

A widely accepted benchmarking criterion V^{-1} is that of movement preservation established by Denton (1971). This principle is popular with time series builders, because - as will be seen - it is easy to understand and explain.

The additive variant of movement preservation

For the additive variant of benchmarking, V^{-1} is the quadratic first (or second) difference operator:

$$V^{-1} = D'D, \qquad \underset{(T-1) \text{ by } T}{D} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & \ldots & 0 & 0 \\ . & . & . & & . & . \\ . & . & . & & . & . \\ . & . & . & & . & . \\ 0 & 0 & 0 & \ldots & 1 & -1 \end{bmatrix} \tag{3.2.2}$$

With that criterion matrix V^{-1}, the objective function (3.2.1) specifies that the benchmarked series Z is as parallel as possible to the original X. In other words, the movements in Z are close as possible to those in X, which is the definition of movement preservation.

As stated however, matrix V^{-1} is singular, so that one cannot obtain the corresponding V required for the G.L.S. solution (3.1.6). The following redefinition of D entails an invertible V^{-1}:

$$
(3.2.3) \qquad \underset{(T-1) \text{ by } T}{D} = \begin{bmatrix} 1 & -\alpha & 0 & 0 & 0 & 0 \\ 0 & 1 & -\alpha & \dots & 0 & 0 \\ . & . & . & & . & . \\ . & . & . & & . & . \\ . & . & . & & . & . \\ 0 & 0 & 0 & \dots & 1 & -\alpha \end{bmatrix}
$$

where α is chosen lower but very close to 1.0 (e.g. 0.9999999). Such a value of α causes an infinitesimal sacrifice to the movement preservation criterion (3.2.2). Furthermore matrix V is now known algebraically:

$$
(3.2.4) \qquad \underset{T \text{ by } T}{V} = \begin{bmatrix} 1 & \alpha & \alpha^2 & \dots & \alpha^{T-1} \\ \alpha & 1 & \alpha & \dots & \alpha^{T-2} \\ \alpha^2 & \alpha & 1 & \dots & \alpha^{T-3} \\ . & . & . & & . \\ . & . & . & & . \\ . & . & . & & . \\ \alpha^{T-1} & \alpha^{T-2} & \alpha^{T-3} & \dots & 1 \end{bmatrix} 1 / (1-\alpha^2)
$$

The scalar $1 / (1-\alpha^2)$ will be ignored because it cancels out in (3.1.6). The availability of a matrix V (without $1/(1-\alpha^2)$) corresponding to V^{-1} makes solution (3.1.6) applicable. Furthermore, as α tends towards 1, the G.L.S. solution (3.1.6) tends to the quadratic minimization solution (3.2.1′) with V^{-1} as in (3.2.2) (Bournay and Laroque, 1979 p. 21).

The weights W of solution (3.1.6) - and L_x and L_y of (3.1.6′) - are independent of the series X and Y considered. For additive benchmarking the same weights may be applied to any series (interval) of the same length (provided the reference periods of the benchmarks are homogeneous, see Section 3.3.)

The proportional variant of movement preservation

For the proportional variant of movement preservation (and of benchmarking), the criterion is normally

$$
(3.2.5) \qquad V^{-1} = X^{-1} D'D\, X^{-1}, \quad X^{-1} = \begin{bmatrix} 1/x_1 & 0 & \dots \\ 0 & 1/x_2 & \dots \\ . & . & \\ . & . & \\ . & . & \end{bmatrix}
$$

with D as in (3.2.2). Under that definition of V^{-1}, the objective function (3.2.1) reduces to $Z'V^{-1}Z$, which specifies that the proportion of Z to X is as constant as possible. Although justifiable per se, this criterion may be seen as an approximation of the growth rate movement preservation $\Sigma(z_t/z_{t-1} - x_t/x_{t-1})^2$ (Smith, 1977; Monsour and Trager, 1979). Redefining D as in (3.2.3) results into a non-singular V^{-1} and a readily known covariance matrix:

$$
(3.2.6) \qquad \underset{T \text{ by } T}{V} = \begin{bmatrix} \xi_{11} & \xi_{12}\alpha & \xi_{13}\alpha^2 & \dots & \xi_{1T}\alpha^{T-1} \\ \xi_{21}\alpha & \xi_{22} & \xi_{23}\alpha & \dots & \xi_{2T}\alpha^{T-2} \\ \xi_{31}\alpha^2 & \xi_{32}\alpha & \xi_{33} & \dots & \xi_{3T}\alpha^{T-3} \\ . & . & . & & . \\ . & . & . & & . \\ . & . & . & & . \end{bmatrix}
$$

where $\xi_{ij} = x_i\, x_j \;/\; \bar{x}$. In simultaneous benchmarking, dividing by the average $\bar{x}$ of the x_t's is essential for relative calibration purposes. The knowledge of V makes solution (3.1.6) applicable. As obvious from (3.2.6), the weights of proportional benchmarking depend on each series interval considered.

3.3 Generalization for Benchmarks with Varying Reference Periods

For many applications of benchmarking, it is convenient to redefine matrix B in a more general manner:

$$
\text{(3.3.1)} \quad \underset{M \text{ by } T}{B_y} =
\begin{array}{l}
\text{columns:} \quad \quad \quad \tau_1 \quad \quad \quad \rho_1 \quad \quad \quad \tau_2 \quad \quad \quad \rho_2 \\
\left[\begin{array}{cccccccccccccccccccc}
0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots \\
. & . & & . & . & . & & . & . & \ldots & . & . & & . & . & . & \\
. & . & & . & . & . & & . & . & \ldots & . & . & & . & . & . & \\
. & . & & . & . & . & & . & . & \ldots & . & . & & . & . & . & \\
& & & & & & & & & & & & & & & &
\end{array}\right]
\end{array}
$$

where τ_m and ρ_m are the reference periods of the benchmarks. This reformulation allows for benchmarks referring to the calendar year (e.g. τ_m=1,13,25,...; $\rho_m=\tau_m+11$); for benchmarks referring to the fiscal years (e.g. τ_m=4,16,28,...; $\rho=\tau_m+11$); for benchmarks of stock series (e.g. τ_m=12,24,36,...; $\rho=\tau_m$); for benchmarks referring to more or less than 12 consecutive months; for benchmarks not available every year; etc. As a result, M now stands for the number of benchmarks (instead of years) in the series. For index series, whose annual values correspond to the average of the year, one should simply multiply the benchmarks by 12 or 4.

3.4 Statistical Properties of the benchmarked series

One by-product of regression analysis is usually an estimate of the covariance of the predicted (interpolated) values of in terms of the covariance of the residuals. This assumes however that the regressor variables are deterministic, that is not subject to error. In other words, the errors originate from the regressand (and from omitted variables). For benchmarking, the regressand and the regressor variables are respectively Z and X - more precisely their annual values Y and X_a - so that the assumption is grossly untenable. Indeed the benchmarks Y are supposed to be more reliable than the original series X. The opposite assumption would be more appropriate: the error originates from the regressor X and not from the regressand Y.

Bournay and Laroque (1979) consider both polar cases. Using the notation adopted in this document, the first case is:

$$
\text{(3.3.1)} \quad \underset{M \text{ by } 1}{Y} = \underset{M \text{ by } 1}{X_a} \; \underset{1 \text{ by } 1}{\Gamma} + e_a, \qquad E(e_a|X) = 0, \quad E(e_a\, e_a') = B\, V\, B' = V_a
$$

which is the case examined in Section 3.1, where the errors originate from the benchmarks. The second case is:

$$
\text{(3.3.2)} \quad X_a = Y \quad \delta + e_a, \qquad E(e_a|Y) = 0
$$

$$\underset{\text{M by 1}}{} \quad \underset{\text{M by 1}}{} \quad \underset{\text{1 by 1}}{} \qquad E(e_a \, e_a') = B \, V \, B' = V_a$$

where the errors originate from the original series. In (3.3.2), the regressors are the benchmarks Y instead of the unbenchmarked data X. In the first case, the authors find the Chow and Lin solution (3.1.5); and in the second,

$$(3.3.3) \qquad \begin{aligned} Z &= (1/\delta^*) \, [X - VB'(BVB')^{-1} \, (BX - Y\delta^*) \\ &= (1/\delta^*) \, [X + VB'(BVB')^{-1} \, (Y\delta^* - BX) \end{aligned}$$

Since for benchmarking, according to the movement preservation principle, both Γ and δ equal 1, both (3.1.5) and (3.3.3) reduce to (3.1.6). We conclude that (in the trivial regression case considered) both polar cases yield the same solution. The interpretation of solution (3.1.6) can then be that the error originates either from Y or from X (or from both). We select the second, - only acceptable -interpretation. This statistically legitimizes casting the benchmarking problem in the regression framework.

This conclusion also entails - in the additive case at least - that (3.1.6') can be used to derive the covariance Σ_z of the benchmarked values from the covariance Σ_x of the unbenchmarked series

$$(3.3.4) \qquad \Sigma_z = L_x \, \Sigma_x \, L_x' + L_y \, \Sigma_y \, L_y'$$

where the covariance Σ_y of Y is presumably equal to zero.

This section showed how benchmarking can be casted in the regression framework. The resulting solution (3.1.6) proves economical in terms of calculations and required computer memory.

4. BENCHMARKING A SYSTEM OF "REGIONAL" SERIES

Based on the developments of the Section 3, a feasible solution to the problem of benchmarking a system of R "regional" time series is proposed. For each period of time, the regional benchmarked series z_{rt} (r>2) must sum to the aggregate series z_{1t}.

$$(4.1) \qquad z_{1,t} - \sum_{r=2}^{R} z_{r,t} = 0, \quad t=1,\ldots,T.$$

The first region is defined to be the sum of the others. The simultaneous benchmarking problem will be seen as one of Joint Generalized Least Squares J.G.L.S. (Theil, 1971).

4.1 Derivation of the Solution

In the J.G.L.S. framework, the Chow and Lin model of equation (3.1.2) becomes

$$\begin{bmatrix} Z_1 \\ Z_2 \\ . \end{bmatrix} \quad \begin{bmatrix} X_1 \\ X_2 \\ . \end{bmatrix} \quad \begin{bmatrix} e_1 \\ e_2 \\ . \end{bmatrix}$$

$$
(4.2)\qquad \underset{(R+1)T \text{ by } 1}{\begin{bmatrix} \cdot \\ \cdot \\ Z_R \\ -X_1+X_2+\ldots+X_R \end{bmatrix}} = \begin{bmatrix} \cdot \\ \cdot \\ X_R \\ 0 \end{bmatrix} \Gamma + \begin{bmatrix} \cdot \\ \cdot \\ e_R \\ e_1-e_2-\ldots-e_R \end{bmatrix}
$$

where the X_r's are the original regional series and e_r are the corrections made to them in order to obtain the desired benchmarked series Z_r. The bottom partitions of (4.2) specify that the corrections are such that additivity (4.1) is satisfied. If the original series already satisfy additivity, then $d=-X_1+X_2+\ldots+X_R$ is equal to zero. The partitions then means that the correction made to the aggregate series X_1 to obtain Z_1 are equal to the sum of the corrections made for the component series X_2 to X_R, for each t. In the standard benchmarking situation d is usually equal to zero. This is not the case however if the benchmarking problem arises from seasonally adjusting a system of series.

Assuming independance across regions, the covariance matrix of the errors in (4.2) is:

$$
(4.3)\qquad \underset{(R+1)T \text{ by } (R+1)T}{V} = \begin{bmatrix} V_1 & 0 & \ldots & 0 & V_1 \\ 0 & V_2 & \ldots & 0 & -V_2 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \ldots & V_R & -V_R \\ V_1 & -V_2 & \ldots & -V_R & V. \end{bmatrix}
$$

where V_r is the calibrated covariance matrix of region r defined in (3.2.6) and $V.=V_1+V_2+\ldots+V_R$. This specification of V assumes that the dependances accros regions can only occur through additivity. (If this is deemed insufficient, one could replace the 0 partitions by appropriate interregional covariance matrices. However, this would tremendously increase the scale of the required computations.) The appropriate matrix B becomes:

$$
(4.4)\qquad \underset{(RM+T-M) \text{ by } (R+1)T}{B} = \begin{bmatrix} B_1 & 0 & \ldots & 0 & 0 \\ 0 & B_2 & \ldots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \ldots & B_R & 0 \\ 0 & 0 & \ldots & 0 & J \end{bmatrix}
$$

where matrix J is a T-M by T operator which eliminates one aggregation constraint per year (for which there is a benchmark) in (4.2). For instance, for 2-year quarterly series starting in the first quarter, J would be:

$$
\underset{6 \text{ by } 8}{J} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
$$

Indeed, one can show that one such constraint per year is redundant. (The dimensioning of (4.4) assumes that each series has the same number M of benchmarks). The resulting vector of discrepancies U is then

$$(4.5) \qquad \underset{}{U'} = [\underset{1 \text{ by } M}{U_1'} \quad \underset{1 \text{ by } M}{U_2'} \quad \ldots \quad \underset{1 \text{ by } M}{U_R'} \quad \underset{1 \text{ by } T\text{-}M}{U.'}]$$

where U_r stand for the "annual" discrepancies $Y_r - B_rX_r$ and $U. = J(-X_1+X_2+ \ldots +X_R)$. The matrix to be inverted is

$$(4.6) \qquad \underset{(RM+T-M) \text{ by } (RM+T-M)}{B\ V\ B'} = \begin{bmatrix} B_1V_1B_1' & 0 & \ldots & 0 & B_1V_1J' \\ 0 & B_2V_2B_2' & \ldots & 0 & -B_2V_2J' \\ . & . & & . & . \\ . & . & & . & . \\ . & . & & . & . \\ 0 & 0 & \ldots & B_RV_RB_R' & -B_RV_RJ' \\ JV_1B_1' & -JV_2B_2' & \ldots & -JV_RB_R' & JV.J' \end{bmatrix}$$

The solution corresponding to (3.1.6) is then

$$(4.7) \qquad \underset{(R+1)T \text{ by } 1}{Z} = \underset{(R+1)T \text{ by } 1}{X} + \underset{(R+1)T \text{ by } (RM+T-M)}{V\ B'} \quad \underset{(RM+T-M) \text{ by } (RM+T-M)}{(B\ V\ B')^{-1}} \quad \underset{(RM+T-M) \text{ by } 1}{U}$$

For computational purposes, it is convenient to reexpress (4.7) as:

$$(4.8) \qquad Z = X + VB' \ H$$

where

$$(4.9) \qquad H = (BVB')^{-1}\ U, \qquad H' = [H_1'\ H_2'\ \ldots\ H_R'\ H.']$$

with H partionned like U in (4.5). On noting the content of V and B, (4.8) reduces to:

$$(4.10) \qquad \begin{aligned} Z_1 &= X_1 + V_1\ B_1'\ H_1 + V_1\ J\ H. = X_1 + V_1(B_1'H_1 + J\ H.) \\ Z_r &= X_r + V_r(B_r'H_r - J\ H.), \quad r > 1 \end{aligned}$$

For 50 monthly series over 5-year intervals (R=50, T=60), the dimension of BVB' are 305 by 305. That matrix is invertible on mainframe computers and standard software (e.g. SAS) now available to statistical agencies. (Furthermore, it is possible to invert that matrix by parts.) The bigger matrices V and B (3000 by 3000 and 305 by 3000) do not have to be stored. Only their non-zero partitions need to be generated and monentarily stored as their need arises, that is in building BVB' in (4.6) for inversion in (4.9) and in doing the calculations in (4.10).

The proposed order of calculations also provides an opportunity (since X_r, V_r and B_r are available in (4.10)) to obtain the individually benchmarked series

$$(4.11) \qquad Z^i{}_r = X_r + V_rB_r'(B_rV_rB_r')^{-1}\ U_r, \quad r=1,\ldots,R$$

These series are usefull to compare each simultaneously benchmarked to the corresponding individually benchmarked series. Table 1 also displays the percentage discrepancies d_{rt} between the two.

4.2 Alterability Coefficients and Alternative Benchmarking Criteria

In a system of series, some components - e.g. the aggregate and the bigger regions - are more reliable than others. The Statistician may then want the more reliable components to be closer to the values they would obtain under individual benchmarking. In other words, the more reliable series should absorb less of the adjustment required by regional additivity (4.1); and the less reliable, more of the adjustment. This can be achieved by means of relative alterability coefficients a_r, $0 < a_r \leq 1$ (Federal Reserve Board, 1962). These coefficients are the inverse of the weights given to each component or "cell" in the litterature.

Alterability coefficients are incorporated into simultaneous benchmarking by merely multiplying each covariance matrix V_r by a_r. Choosing $a_1=a_2=...=a_R$ for instance, would tend to yield series z_{rt} which equally depart from their $z^i{}_{rt}$. In other words, the adjustment required by regional additivity would tend to be equally distributed among series. The simultaneously benchmarked series of Table 1 were obtained with alterability coefficients, $a_1=0.5$ and $a_2=a_3=1.0$. Thus the aggregate series tend to absorb half as much of the additivity adjustment, as testified by the percentage discrepancies between z_{rt} and $z^i{}_{rt}$ (columns 14 to 16).

The alterability coefficients thus make it possible to virtually "impose" the values of the individually benchmarked aggregate series on the components, by chosing a_1 relatively small (e.g. 0.001); or, to specify the benchmarked aggregate as the resultant of the benchmarked components, by chosing a_1 relatively high and a_r relatively small for $r>2$; or, any variation between those two poles. (The second pole is analogous to "indirect" seasonal adjustment of an aggregate, see Dagum 1980). The higher reliability of some components (whether aggregate or not) is likely to improve that of the less reliable components. Indeed, the alterability coefficients combine to the regional additivity constraints (4.1) operate as a cross-validation tool.

In the context of simultaneous benchmarking, other criteria besides movement preservation are justifiable. For a series z_{rt} with unreliable movement for instance, one may prefer a proximity criterion. The appropriate covariance matrix V_r is then a diagonal matrix with diagonal elements $\xi_{rt} = x_{rt}{}^2/\bar{x}_r$, $\bar{x}_r$ being the average of x_{rt} (and the identity matrix for additive benchmarking). Proximity specifies the benchmarked series z_{rt} to be more or less close (depending on the alterability coefficient a_r chosen) to the original series x_{rt}, with no specific concern about movement preservation. In individual benchmarking, the proximity criterion is reputed to cause discontinuity between years. This is somewhat prevented in simultaneous benchmarking, because the movement preserved for the other series will cause some kind of preservation for z_{rt} through the additivity constraints. In other words, the movement of z_{rt} is governed by addivity alone, instead of by movement preservation and additivity. The proximity criterion is used by the Federal Reserve Board (1962) and the

Bank of England (1972) and used by Statistics Canada (Taillon, 1988) to rebalance flow of funds series after seasonal adjustment.

4.3 Preliminary Benchmarking

Table 1 also displays preliminary benchmarked values for the sixth year of the series. That year is incomplete and has consequently no annual benchmark. Preliminary benchmarking is designed to avoid movement discontinuity between the years with benchmarks and the current year. Failure to perform preliminary benchmarking is likely to embarass the statistical agency and to complicate decision making by the users of the series.

Definition (3.3.1) of matrix B_r allows for preliminary benchmarking. Indeed, for $\rho_{rM}<T$, there is simply no benchmark for periods from $\rho_{rM}+1$ to T. For those periods, the benchmarked series z_{rt} is uniquely determined by the movement preservation principle and additivity. In order to calculate those values, it is not necessary to recalculate (4.9) and (4.10) each time a new observation becomes available. It is sufficient to specify the last benchmarked value $z_{rt'}$, for the last year with a benchmark, as a unique and temporary benchmark y_{r1} referring to period t' ($\rho_1=\tau_1=t'$). This reduces solution (4.7) to

$$(4.7') \quad \underset{\substack{(R+1)T' \\ \text{by } 1}}{Z} = \underset{\substack{(R+1)T' \\ \text{by } 1}}{X} + \underset{\substack{(R+1)T' \\ \text{by } (R+T'-1)}}{V\,B'} \quad \underset{\substack{(R+T') \\ \text{by } (R+T'-1)}}{(B\,V\,B')^{-1}} \quad \underset{\substack{(R+T'-1) \\ \text{by } 1}}{U}$$

where T' is the number of current observations plus 1, and M of (4.7) was replaced by 1. In other words, premilinary benchmarking requires much less calculations than normal benchmarking. Solution (4.7') should also be implemented by means of (4.9) and (4.10).

For individual benchmarking, the simplifications are still greater: The preliminary benchmarked values are uniquely determined by the movement preservation principle. This results into a z_t totally proportional to x_t (since there are no other constraints). The preliminary benchmarking factor simply repeat the last one calculated for the last current year:

$$z_t = z_{t'}/x_{t'} * x_t, \quad t=t'+1,t'+2,...$$

(For additive benchmarking the formula is $z_t = (z_{t'}-x_{t'}) + x_t$.)

Preliminary benchmarking and alterability coefficients are also relevant when benchmarking a two-dimensional system of series.

5. BENCHMARKING A SYSTEM OF SERIES BY REGION AND BY INDUSTRY

Benchmarking also arises in situations where the series are classified in R "regions" and N "industries". Each series must satisfy its own benchmarks. The sum over the industries for a given region r must also add to the regional aggregates (totals) z_{r1t}:

$$z_{r,1,t} - \sum_{n=2}^{N} z_{r,n,t} = 0, \quad r=1,\dots,R; \; t=1,\dots,T; \tag{5.1}$$

Similarly, the sum over the regions for a given industry n must add to the industrial aggregates z_{1nt}:

$$z_{1,n,t} - \sum_{r=2}^{R} z_{r,n,t} = 0, \quad n=1,\dots,N; \; t=1,\dots,T; \tag{5.2}$$

Again the first region is defined as the sum of the other regions; and similarly for the first industry.

The grand total z_{11t} can indifferently be expressed as the sum over regional totals z_{21t} to z_{R1t} or the sum over the industrial totals z_{12t} to z_{1Nt}. Constraints (5.1) and (5.2) express it both ways and are therefore redundant. That redundancy can be eliminated by actually dropping the constraints pertaining to anyone of the regions or of the industries. The Nth set of industrial constraints in (5.2) is dropped. This reduces the total number of constraints to (R+N-1)T.

Section 5.1 considers the problem of benchmarking the whole system containing the R*N series. Section 5.2 considers a scaled down and problably more practical version of the problem, where only the (R+N-1) regional and industrial aggregates $z_{11t}, z_{21t}, \dots, z_{R1t}, z_{12t}, z_{13t}, \dots, z_{1Nt}$ are calculated. Section 5.3 enquires into adjusting the remaining series x_{rnt} (r>1 and n>1) to conform to their respective regional and industrial aggregates $z_{11t}, \dots, z_{R1t}, z_{12t}, \dots, z_{1Nt}$ without resorting to simultaneous benchmarking.

5.1 Benchmarking the Whole System of R*N Series

Because of the size of the matrices involved, the developments are based on only 3 regions and 3 industries (R=3, N=3). The regression model of equation (4.2) becomes:

$$\underset{\substack{(RN+R+N-1)T \\ \text{by } 1}}{\begin{bmatrix} Z_{11} \\ Z_{21} \\ Z_{31} \\ Z_{12} \\ Z_{22} \\ Z_{32} \\ Z_{13} \\ Z_{23} \\ Z_{33} \\ -X_{11}+X_{21}+X_{31} \\ -X_{12}+X_{22}+X_{32} \end{bmatrix}} = \begin{bmatrix} X_{11} \\ X_{21} \\ X_{31} \\ X_{12} \\ X_{22} \\ X_{32} \\ X_{13} \\ X_{23} \\ X_{33} \\ 0 \\ 0 \end{bmatrix} \Gamma + \begin{bmatrix} e_{11} \\ e_{21} \\ e_{31} \\ e_{12} \\ e_{22} \\ e_{32} \\ e_{13} \\ e_{23} \\ e_{33} \\ e_{11}-e_{21}-e_{31} \\ e_{12}-e_{22}-e_{32} \end{bmatrix} \tag{5.1.1}$$

$$
\begin{bmatrix} -X_{13}+X_{23}+X_{33} \\ -X_{11}+X_{12}+X_{13} \\ -X_{21}+X_{22}+X_{23} \end{bmatrix}
\qquad
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\qquad
\begin{bmatrix} e_{13}-e_{23}-e_{33} \\ e_{11}-e_{12}-e_{13} \\ e_{21}-e_{22}-e_{23} \end{bmatrix}
$$

Assuming independence, the covariance matrix of the error terms in (5.1.1) is

$$
\text{(5.1.2)}\ V = \underset{(RN+R+N-1)T \text{ by } (RN+R+N-1)T}{\begin{bmatrix}
V_{11} & & & & & & & & & V_{11} & & & V_{11} & \\
 & V_{21} & & & & & & & & -V_{21} & & & & V_{21} \\
 & & V_{31} & & & & & & & -V_{31} & & & & \\
 & & & V_{12} & & & & & & & V_{12} & & V_{12} & \\
 & & & & V_{22} & & & & & & -V_{22} & & & -V_{22} \\
 & & & & & V_{32} & & & & & -V_{32} & & & \\
 & & & & & & V_{13} & & & & & V_{13} & V_{13} & \\
 & & & & & & & V_{23} & & & & -V_{23} & & -V_{23} \\
 & & & & & & & & V_{33} & & & -V_{33} & & \\
V_{11} & -V_{21} & -V_{31} & & & & & & & V_{\cdot 1} & & & V_{11} & -V_{21} \\
 & & & V_{12} & -V_{22} & -V_{32} & & & & & V_{\cdot 2} & & -V_{12} & V_{22} \\
 & & & & & & V_{13} & -V_{23} & -V_{33} & & & V_{\cdot 3} & -V_{13} & V_{23} \\
V_{11} & & & V_{12} & & & V_{13} & & & V_{11} & -V_{12} & V_{13} & V_{1\cdot} & \\
V_{21} & & & -V_{22} & & & -V_{23} & & & -V_{21} & V_{22} & V_{23} & & V_{2\cdot}
\end{bmatrix}}
$$

where only the non-zero partitions are indicated. Matrix V_{rn} is the calibrated covariance of series z_{rnt}, defined in (3.2.6) and possibly multiplied by an alterabilty coefficient; and $V_{\cdot n}$ and $V_{r\cdot}$ are respectively $V_{1n}+V_{2n}+V_{3n}$ and $V_{r1}+V_{r2}+V_{r3}$. The appropriate matrix B becomes:

$$
\text{(5.1.3)}\ B = \underset{(RN)M+(R+N-1)T \text{ by } (RN)T+(R+N-1)(T-M)}{\begin{bmatrix}
B_{11} & & & & & & & & & & & & & \\
 & B_{21} & & & & & & & & & & & & \\
 & & B_{31} & & & & & & & & & & & \\
 & & & B_{12} & & & & & & & & & & \\
 & & & & B_{22} & & & & & & & & & \\
 & & & & & B_{32} & & & & & & & & \\
 & & & & & & B_{13} & & & & & & & \\
 & & & & & & & B_{23} & & & & & & \\
 & & & & & & & & B_{33} & & & & & \\
 & & & & & & & & & J & & & & \\
 & & & & & & & & & & J & & & \\
 & & & & & & & & & & & J & & \\
 & & & & & & & & & & & & J & \\
 & & & & & & & & & & & & & J
\end{bmatrix}}
$$

where the B_{rn}'s have dimension M by T (assuming equal number of benchmarks per series); and the matrix J, T-M by T. Matrix J was defined in Section 4.1. The resulting vector of discrepancies U is then

$$
\text{(5.1.4)}\qquad U' = [\ U_{11}'\ U_{12}'\ \ldots\ U_{32}'\ U_{33}'\ U_{\cdot 1}'\ U_{\cdot 2}\ U_{\cdot 3}'\ U_{1\cdot}'\ U_{2\cdot}'\,]
$$

where U_{rn} stand for the "annual" discrepancies $Y_{rn}-B_{rn}X_{rn}$; and $U_{\cdot n}=J(-X_{1n}+X_{2n}+X_{3n})$ and $U_{r\cdot}=J(-X_{r1}+X_{r2}+X_{r3})$. The matrix to be inverted is

$$
\begin{bmatrix}
BVB'_{11} & & & & & & & & & BV_{11}J' & & & BV_{11}J' & \\
 & BVB'_{21} & & & & & & & & -BV_{21}J' & & & & BV_{21}J'
\end{bmatrix}
$$

$$
BVB' = \begin{bmatrix}
BVB'_{31} & & & & & & & -BV_{31}J' & & & & \\
& BVB'_{12} & & & & & & & BV_{12}J' & & BV_{12}J' & \\
& & BVB'_{22} & & & & & & -BV_{22}J' & & & -BV_{22}J' \\
& & & BVB'_{32} & & & & & -BV_{32}J' & & & \\
& & & & BVB'_{13} & & & & & BV_{13}J' & BV_{13}J' & \\
& & & & & BVB'_{23} & & & & -BV_{23}J' & & -BV_{23}J' \\
& & & & & & BVB'_{33} & & & -BV_{33}J' & & \\
JBV_{11}'-JBV_{21}'-JBV_{31}' & & & & & & & JV_{.1}J' & & & JV_{11}J' & -JV_{21}J' \\
& JBV_{12}'-JBV_{22}'-JBV_{32}' & & & & & & & JV_{.2}J' & & -JV_{12}J' & JV_{22}J' \\
& & & & JBV_{13}'-JBV_{23}'-JBV_{33}' & & & & & JV_{.3}J' & -JV_{13}J' & JV_{23}J' \\
JBV_{11}' & JBV_{12}' & & & JBV_{13}' & & & JV_{11}J' & -JV_{12}J' & -JV_{13}J' & JV_{1.}J' & \\
JBV_{21}' & & -JBV_{22}' & & & -JVB_{23}' & & -JV_{21} & JV_{22} & JV_{23} & & JV_{2.}J'
\end{bmatrix}
$$

(5.1.5) $(RN)M+(R+N-1)(T-M)$ by $(RN)M+(R+N-1)(T-M)$

where BVB_{rn} and BV_{rn} stand for $B_{rn}V_{rn}B_{rn}'$ and $B_{rn}V_{rn}$ respectively. The solution corresponding to (4.7) is then

$$
\underset{\substack{(RN+R+N-1)T \\ \text{by } 1}}{Z} = X + V\,B' \underset{\substack{RNM+(R+N-1)(T-M) \\ \text{by } RNM+(R+N-1)(T-M)}}{(B\,V\,B')^{-1}} U \qquad (5.1.6)
$$

For computational purposes, it is convenient to reexpress (5.1.6) as:

$$
Z = X + VB'\ H \qquad (5.1.7)
$$

where

$$
H=(BVB')^{-1}\ U \qquad (5.1.8)
$$

is partionned like U: $H' = [H_{11}'\ H_{12}'\ \ldots\ H_{32}'\ \ H_{33}'\ H_{.1}'\ H_{.2}'\ H_{.3}'\ H_{1.}'\ H_{2.}']$ On noting the content of V and B, (5.1.7) reduces to:

$$
\begin{aligned}
Z_{rn} &= X_{rn} + V_{rn}(B_{rn}'H_{rn} + J\,(H_{.n} + H_{r.})) \quad r=1,\ n=1,\ldots,N \\
Z_{rn} &= X_{rn} + V_{rn}(B_{rn}'H_{rn} - J\,(H_{.n} + H_{r.})) \quad r>1,\ n=1 \\
Z_{rn} &= X_{rn} + V_{rn}(B_{rn}'H_{rn} - J\,(H_{.n} - H_{r.})) \quad r>1,\ 1<n<N \\
Z_{rn} &= X_{rn} + V_{rn}(B_{rn}'H_{rn} - J\,H_{.n}) \quad r>1,\ n=N
\end{aligned}
\qquad (5.1.9)
$$

For 9 monthly 5-year series pertaining to 3 regions and 3 industries, the dimension of the matrix BVB' to invert in (5.1.5) is 320 by 320; and 300 by 300, for 30 quarterly 5-year series, pertaining to 5 regions and 6 industries. Such calculations are feasible. Their usefulness however is questionable - especially for monthly series. Indeed, the ability to process such 3 regions and 3 industries may be insufficient, especially when considering that one of the regions is the sum of the other (and similarly for the industries). Real situations involve more than 10 regions and 10 industries, in which case the dimension of BVB' would be prohibitive: more than 1545 by 1545 for 5-year monthly series; and 785 by 785 for 5-year quarterly series. Examining BVB'

$$(5.1.10) \qquad BVB' = \begin{bmatrix} \begin{array}{c} P_{11} \\ (RN)M \\ \text{by } (RN)M \end{array} & \begin{array}{c} P_{12} \\ (RM)M \\ \text{by } (R+N-1)(T-M) \end{array} \\ & \\ P_{12}' & \begin{array}{c} P_{22} \\ (R+N-1)T \\ \text{by } (R+N-1)(T-M) \end{array} \end{bmatrix}$$

where P_{11} is the block-diagonal (and non-singular) part of (5.1.5), reveals that the size of the matrix originates mainly from P_{22} (and P_{12}). The size of P_{22} depends heavily on the number of observations T of the series, whereas that of P_{11} depends on the number M of benchmarks (years).

One solution to that computational problem is to collapse regions and industries into fewer and larger groups, to carry out simultaneous benchmarking at that level, and then to repeat the process within each group. One could also collapse monthly series into quarterly series, carry out benchmarking at the quarterly level, and then benchmark the monthly data to the quarterly benchmarked values obtained. One could also combine both the regional-industrial and the temporal collapsing strategies.

5.2 Benchmarking the R+N-1 Regional and Industrial Aggregates

As mentionned in the introduction to this section, a more economical solution to benchmark a system of series classified by region and industry is the following. First benchmark only the regional and the industrial totals (z_{r1}, $r=1,...,R$ and z_{1n}, $n=2,...,N$) together. Second, somehow process the remaining series to be consistent with those totals (and their benchmarks). This second step is examined in Section 5.3. This sub-section examines the first step, where the number of series to simultaneously process is reduced to R+N-1, instead of RN in Section 5.1. The approach now presented is of course also relevant when the statistical agency publishes only the regional and the industrial aggregates. (The other series may be available to the agency but not released - as time series at least - to the public.)

The regression model of equation (4.2) then becomes:

$$(5.2.1) \qquad \begin{bmatrix} Z_{11} \\ Z_{21} \\ . \\ . \\ . \\ Z_{R1} \\ Z_{12} \\ Z_{13} \\ . \\ . \\ . \\ Z_{1N} \\ -X_{11}+X_{21}+...+X_{R1} \\ -X_{11}+X_{12}+...+X_{1N} \end{bmatrix} = \begin{bmatrix} X_{11} \\ X_{21} \\ . \\ . \\ . \\ X_{R1} \\ X_{12} \\ X_{13} \\ . \\ . \\ . \\ X_{1N} \\ 0 \\ 0 \end{bmatrix} \Gamma + \begin{bmatrix} e_{11} \\ e_{21} \\ . \\ . \\ . \\ e_{R1} \\ e_{12} \\ e_{13} \\ . \\ . \\ . \\ e_{1N} \\ e_{11}-e_{21}-...-e_{R1} \\ e_{11}-e_{12}-...-e_{1N} \end{bmatrix}$$

$$(R+N+1)T \text{ by } 1$$

where Z_{11} is the grand total over regions and industries. The covariance of the errors in (5.2.1) is then:

(5.2.2)

(R+N+1)T by (R+N+1)T

$$
\begin{bmatrix}
V_{11} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & V_{11} & V_{11} \\
0 & V_{21} & \cdots & 0 & 0 & 0 & \cdots & 0 & -V_{21} & 0 \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
0 & 0 & \cdots & V_{R1} & 0 & 0 & \cdots & 0 & -V_{R1} & 0 \\
0 & 0 & \cdots & 0 & V_{12} & 0 & \cdots & 0 & 0 & -V_{12} \\
0 & 0 & \cdots & 0 & 0 & V_{13} & \cdots & 0 & 0 & -V_{13} \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & V_{1N} & 0 & -V_{1N} \\
V_{11} & -V_{21} & \cdots & -V_{R1} & 0 & 0 & \cdots & 0 & V_{\cdot 1} & V_{11} \\
0 & 0 & \cdots & 0 & -V_{12} & -V_{13} & \cdots & -V_{1N} & V_{11} & V_{1\cdot}
\end{bmatrix}
$$

where all the partitions were defined in Section 5.1. Matrix B becomes:

(5.2.3)

(R+N-1)M+2T by (R+N-1)+2(T-M)

$$
\begin{bmatrix}
B_{11} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & B_{21} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
0 & 0 & \cdots & B_{R1} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & B_{12} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & B_{13} & \cdots & 0 & 0 & 0 \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & B_{1N} & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & J & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & J
\end{bmatrix}.
$$

where the matrix J has dimensions T-M by T and is defined in Section 4. The resulting vector of discrepancies U is

(5.2.4) $$U' = [\ U_{11}' \cdots U_{R1}'\ U_{12}' \ \cdots\ U_{1N}'\ U_{\cdot 1}'\ \ U_{1\cdot}'\ ]$$

where U_{rn} stand for the "annual" discrepancies $Y_{rn}-B_{rn}X_{rn}$; and $U_{\cdot n}= J(-X_{1n}+X_{2n}+X_{3n})$ and $U_{r\cdot}=J(-X_{r1}+X_{r2}+X_{r3})$. The matrix to be inverted is

(5.2.5)

$$
BVB' =
\begin{bmatrix}
BVB_{11} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & BV_{11}J' & BV_{11}J' \\
0 & BVB_{21} & \cdots & 0 & 0 & 0 & \cdots & 0 & -BV_{21}J' & 0 \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
0 & 0 & \cdots & BVB_{R1} & 0 & 0 & \cdots & 0 & -BV_{R1}J' & 0 \\
0 & 0 & \cdots & 0 & BVB_{12} & 0 & \cdots & 0 & 0 & -BV_{12}J' \\
0 & 0 & \cdots & 0 & 0 & BVB_{13} & \cdots & 0 & 0 & -BV_{13}J' \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\
\cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot
\end{bmatrix}
$$

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & BVB_{1N} & 0 & -BV_{1N}J' \\ JBV_{11}' & -JBV_{21}' & \cdots & -JBV_{R1}' & 0 & 0 & \cdots & 0 & JV_{.1}J' & JV_{11}J' \\ JBV_{11}' & 0 & \cdots & 0 & -JBV_{12}' & -JBV_{13}' & \cdots & -JBV_{1N}' & JV_{11}J' & JV_{1.}J' \end{bmatrix}$$

$(R+N-1)M+2(T-M)$ by $(R+N-1)M+2(T-M)$

where the matrices are defined in Section 5.1. The solution corresponding to (4.7) is then

$$\text{(5.2.6)} \qquad \underset{\substack{(R+N+1)T \\ \text{by } 1}}{Z} = X + V B' \underset{\substack{(R+N-1)M+2(T-M) \\ \text{by } (R+N-1)M+2(T-M)}}{(B V B')^{-1}} U$$

which is conveniently rewritten as:

$$\text{(5.2.7)} \qquad Z = X + VB' H$$

where

$$\text{(5.2.8)} \qquad H=(BVB')^{-1} U$$

is partionned like U: $H' = [\, H_{11}' \ldots H_{R1}' \; H_{12}' \ldots \; H_{1N}' \; H_{.1}' \; H_{1.}' \,]$ On noting the content of V and B, (5.2.7) reduces to:

$$\text{(5.2.9)} \qquad \begin{aligned} z_{11} &= x_{11} + V_{11}(B_{11}'H_{11} + J\,(H_{.1} + H_{1.})) && (r=n=1) \\ z_{r1} &= x_{r1} + V_{r1}(B_{r1}'H_{r1} - J\,H_{.1}) && r>1 \quad (n=1) \\ z_{1n} &= x_{1n} + V_{1n}(B_{1n}'H_{1n} - J\,H_{1.}) && n>1 \quad (r=1) \end{aligned}$$

For 45 monthly 5-year series pertaining to 20 regions and 25 industries, the dimension of the matrix BVB' to invert is 330 by 330; and for the corresponding quarterly series, 250 by 250. Such calculations are feasible and begin to be usefull, especially when combined with the regional-industrial collapsing strategies described in Section 5.1. The bigger matrices V and B do not have to be stored. Only their non-zero partitions need to be generated and momentarily stored as their need arises, in building BVB' for inversion in (5.2.8) and in doing the calculations in (5.2.9).

If one intends, by means of alterability coefficients, to "impose" the grand total to the regional and to the industrial series (which is likely to happen), one could apply the method of Section 4 to the R regional aggregates with low a_1 and then to the N industrial aggregates with low a_1. This would enable one to process a larger number of series. Under the latter method or that presented in this sub-section, Table 2, is obtained, where the aggregates series in the margins (first row and column) add to the grand total z_{11t}.

Table 2: Contingency table containing the simultaneously benchmarked aggregate regional series z_{r1t} and aggregate industrial series z_{1nt} and the unbenchmarked component series x_{rnt}

	Totals	Industries			
	$z_{1,1,t}$	$z_{1,2,t}$	$z_{1,3,t}$	. . .	$z_{1,N,t}$
R	$z_{2,1,t}$	$x_{2,2,t}$	$x_{2,3,t}$	. . .	$x_{2,N,t}$
e					
g	$z_{3,1,t}$	$x_{3,2,t}$	$x_{3,3,t}$	. . .	$x_{3,N,t}$
i					
o	.	.	.		.
n	.	.	.		.
s	.	.	.		.
	$z_{R,1,t}$	$x_{R,2,t}$	$x_{R,3,t}$	. . .	$x_{R,N,t}$

5.3 Iterative Proportional Fitting Coupled With Individual Benchmarking

The problem with Table 2 is that the component series x_{rnt}, $r>1$ and $n>1$, do not add up their corresponding row and column aggregate series z_{r1t} and z_{1nt}. Additivity can be restored by means of Iterative Proportional Fitting or "raking" (Bishop, Fienberg and Holland, 1975; Brackstone and Rao, 1979). In the context of Table 2 raking consists of adjusting each column n to the its total, by multiplying it by an adjustment factor f_n, equal to the ratio of the corresponding desired and actual totals; of adjusting each row to its total in the same manner; of re-adjusting the columns; of re-adjusting the rows; and so on. This method is known to converge.

In the context of benchmarking, there are problems with the 2-dimensional raking just described. First, it is not likely to preserve the period-to-period movement of the component original series x_{rnt}, because each time period, i.e. each of the T tables like Table 2, is processed separately. Second, the 2-dimensional raking just described will not satisfy the benhmarks y_{rnm} of the components series x_{rnt}. However this second problem may be corrected by 3-dimensional raking, processing one year (of tables like Table 2) at the time. The two first dimensions consist of the regional and the industrial totals z_{1nt} and z_{r1t}; and the third dimension, of the benchmarks y_{rnm} for a given year. The problem remaining is the possibility of movement discontinuities between years, since each year of values x_{rnt} is processed separetaly. However, this method may represent an acceptable trade-off between quality and feasibility in many situations.

A more satisfactory solution consists of the following steps.

1) First individually benchmark the component series x_{rnt} ($r>1$, $n>1$) of Table 2.

2) Tablulate the resulting benchmarked series z^{i}_{rnt} in a table like

Table 2, with the simultaneously benchmarked series z_{r1t} and z_{1nt} in the margins, and perform 2-dimensional raking. The resulting series add up to the margins but do not comply with their benchmarks.

3) Re-benchmark the raked component series individually and re-tabulate.

4) Re-rake.

Repeat steps 3) and 4) until convergence.

Our experience indicates that this sequence of individual benchmarking with raking does converge to a consistent system of benchmarked series. Alterability factors can also be built into the raking process by modifying the raking factors as follows. If f_r is the factor for row r, the modified factor is $f'_r(n) = 1+(f_r-1)*a_{rn}$ $(0 \leq a_{rn} \leq 1)$. Convergence takes longer but does takes place.

The raking approach (with or without alterability coefficients) also works for a system of regional series. Raking can therefore also be considered as a substitute to the simultaneous benchmarking method of Section 4.

6. DISCUSSION

The sequence of individual benchmarking and raking, just outlined, and the simultaneous benchmarking, presented in Sections 4 and 5, displays strikingly similar results, both in terms of movement and level of the benchmarked series. That raking approach could therefore be considered as an approximation to simultaneous benchmarking. The latter may be used to provide a justification for, and a standard against which alternative and simpler approaches may be assessed. However, that comparison and raking, which certainly deserves more attention, are not the subject of this paper.

Perhaps because simpler or "ad hoc" methods have been used to restore the consistency of systems of benchmarked series, there is to our knowledge no published litterature on the subject, apart from the Federal Reserve Board (1962) and the Bank of England (1972) which include very little mathematical details. A more technical unpublished reference is Taylor (1963). A forthcoming - and probably the most comprehensive reference to date (apart from this document) - is Taillon (1988). Both these authors view the problem as a constrained quadratic minimization programme and are based on the proximity criterion. This paper views the problem through the framework of regression analysis, which is - as explained in Section 3 - a particular but more familiar case of quadratic minimization.

An issue is starting to emerge in the literature: whether the benchmarks should be considered as fully reliable or not. For individual series, Hillmer and Trabelsi (1988) proposed a method (based on ARIMA modelling) in which the benchmarked series does not necessarily have to comply with the benchmarks. These non-binding benchmarks are merely extra observations from which to derive the sub-annual estimates. In our opinion, in many situations, the benchmarks are in some respects less reliable than the unbenchmarked series (Cholette, 1987a, 1987b). If this paper has considered them as binding, it is simply assumed that, prior to simultaneous benchmarking, a consistent set of reliable annual benchmarks were established for the system of series considered. It is possible to devise a simultaneous method in which the benchmarks would not necessarily be binding. Using quadratic minimization Cholette (1979, 1987b) developped such a method. However, it does no lend itself to application to large enough systems of series.

7. SUMMARY AND CONCLUSIONS

This document proposed formal methods to benchmark systems of socio-economic time series, in which the components series must add to aggregate series and comply to their benchmarks. The methods incorporate the accepted benchmarking principle of movement preservation (Denton, 1971) into the regression framework. The "regression" is solved by Joint Generalized Least Squares (Theil, 1971). Apart from the methodological familiarity, one advantage of the resulting approach is the dramatic reduction in the scale of the calculations made possible by the Chow and Lin (1971) particular solution.

With the methods presented, and with the computers and the standard software now available to statistical agencies, one can (for instance) simultaneously benchark a system including over 50 regional monthly series over 5-year intervals; a system including over 30 quarterly series cross-classified in 5 regions and 6 industries (over 5-year intervals); a system including over 45 monthly series representing 20 regional totals and 25 industral totals which have to sum to a common grand total. With some expertise in matrix algebra, it may be possible to expand that capacity.

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