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BENCHMARKING SOCIO-ECONOMIC TIME SERIES DATA: A UNIFIED APPROACH

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- abstract -

This paper presents a unified approach for benchmarking socio-economic time series data. Benchmarking (e.g. Denton, 1971; Hillmer and Trabelsi, 1987) consists of estimating time series when measurements of the target variable are available at differing frequencies, e.g. monthly and annually. Most benchmarking methods can be seen as particular cases of the proposed unified approach. Furthermore, the statistical properties available with the general model carry over to the previous methods, which were often developed "ad hoc". The unified approach, based on Generalized Least Squares with stochastic parameters, also encompasses interpolation and temporal disaggregation methods (e.g. Boot, Feibes and Lisman, 1967; Chow-Lin, 1971). However, the paper deals mainly with benchmarking.

KEYWORDS: Regression, Temporal Dis-Aggregation, ARIMA processes, Quadratic minimization, Proxy variables, Missing observations, Survey errors, Data revision

APPROCHE UNIFIÉE

À L'ÉTALONNAGE DES CHRONIQUES SOCIO-ÉCONOMIQUES

par

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- résumé -

Ce travail présente un cadre unifié pour l'étalonnage des séries chronologiques. L'étalonnage (ex. Denton, 1971; Hillmer et Trabelsi, 1987) consiste à estimer des chroniques en présence de mesures de la variable cible, disponibles à différentes fréquences, ex. mensuelle et annuelle. La plupart des méthodes d'étalonnage apparaissent comme des cas particuliers de l'approche unifiée. Les propriétés statistiques disponibles pour cette dernière se rapportent en outre sur les premières, souvent développées de manière empiriques. Le cadre unifiée, basé sur la théorie des Moindres carrés généralisés à paramètres stochastiques, englobe également les méthodes d'interpolation (ex. Boot, Feibes et Lisman, 1967; Chow et Lin, 1971). Cependant l'attention portera avant tout sur l'étalonnage.

1. INTRODUCTION

Statistical agencies are concerned with the accuracy, consistency, timeliness and usefulness of their socio-economic time series data. Benchmarking is the process of making the sub-annual values of a variable consistent with the corresponding annual values, while also improving both accuracy and timeliness, that is their ability to reflect the recent evolution of the socio-economic variables.

Classic benchmarking situations arise when the data for a variable originate from two sources with different periodicities; monthly versus annual, quarterly versus annual, annual versus quinquenial, and most often sub-annual versus annual. For instance the Canadian monthly Retail Trade data originate from a survey while the corresponding annual values are obtained from a census. The resulting monthly and annual series are not perfectly consistent: the annual totals of the former do not equal the corresponding values of the latter. Traditionally, benchmarking has consisted of adjusting sub-annual series to make them consistent with annual benchmarks. In this paper, benchmarking will be defined more generally as optimally combining sub-annual and annual data to obtain more accurate sub-annual series. In order words, benchmarks may simply be treated as auxiliary observations from which to estimate the sub-annual series; and thus the traditional view of benchmarking emerges as a particular case of this more general view.

Less classic benchmarking situations arise when the sub-annual data are not measurements of the variable, but an indicator or a proxy for the behaviour of the variable; or when the sub-annual indicator values are calculated from variables related to the variable of interest. These indicator values are then benchmarked to the annual values. Strictly speaking, the problem becomes one of **interpolation** (we use this term for flow, stock and index series), which is necessary when the "sub-annual" data for a target variable are not available. Interpolation is the process of generating sub-annual values from annual data avaliable for the variable, or from annual data and related sub-annual series. Statistical agencies generate large segments of their National Accounts series by interpolation. For example, at Statistics Canada (1975, p. 357), the "overwhelmingly greater part - about 75% - of the total quarterly wages and salaries estimate is interpolated between annual totals... on the basis of the movement of payroll indexes which measure month-to-month changes ...". (This was the case until 1975. A variant of this method is now in use.)

Benchmarking - and interpolation - also arise in conjunction with the **calendarization** of socio-economic data. Calendarization is the process of converting "fiscal year" data into "calendar year" values, fiscal quarter data into calendar quarter values, or bundles of weekly data into monthly values. The lack of calendarization reduces the usefullness of the data. For example, yearly data for different industries can be confusing if the data reflect different fiscal years. In some situations, calendarization can be achieved as a by-product of benchmarking. For instance, if a variable is benchmarked to fiscal year data, from the second quarter of a year to the first quarter of the following year, then calendar year values are the annual sums of the benchmarked series. When there is no series to benchmark, calendarization can be seen as an interpolation problem: the fiscal year data are disaggregated into monthly or quarterly values, and these are simply aggregated into calendar year values. Similarly, fiscal quarter data and weekly bundles of data are disaggregated into monthly and daily values respectively and then aggregated into calendar quarters and months.

Although the **general model** presented in Section 2 applies to both benchmarking and interpolation (and therefore to calendarization), attention is given mainly to benchmarking. Benchmarking and interpolation methods based on Quadratic Minimization (Boot, Feibes and Lisman, 1967; Denton, 1971; Cohen, Müller and Padberg, 1971; and others), reviewed by Alba (1979) and by Sanz (1981), will appear as particular cases of the general model. This will also be the case of the interpolation methods based on regression analysis (Chow and Lin, 1971; Fernandez, 1981 and Alba, 1988). One important outcome of the general model is to provide the variance-covariance matrix of the estimates obtained under the Quadratic Minimization and regression methods. Those properties were lacking in the first case and dubious in the second.

Section 3 presents the recent Hillmer and Trabelsi (1987) ARIMA model-based benchmarking method in the framework of the general model and discusses its implications. Section 4 examines the wide-spread benchmarking methods of the Denton family (Denton, 1971; Helfand, Monsour and Trager, 1977; Cholette, 1984) and proposes an extended Denton method to accomodate fiscal years and volatile (unreliable) benchmarks. This extended method also solves some implemention problems: preliminary benchmarking of current observations, which improves their timeliness, and revision of past estimates, which makes more useful. Finally, Section 5 discusses the suitability of the various benchmarking methods for certain situations and for mass application in statistical agencies.

2. THE UNIFIED APPROACH

This section introduces a general model for benchmarking and interpolation, based on a regression estimated by Generalized Least Squares with stochastic parameters. Most existing benchmarking and interpolation methods can be shown to be particular cases of the proposed model. The general model consists of the following system of equations:

(2.1a) $Y_s = \Gamma + e,$ $E(e)=0,\ E(e\ e')=V_e$;

(2.1b) $Y_a = B\ \Gamma + \epsilon,$ $E(\epsilon)=0,\ E(\epsilon\ \epsilon')=V_\epsilon$;

(2.1c) $\Pi(\Gamma - Y_i) = \eta,$ $E(\eta) = 0,\ E(\eta\ \eta') = V_\eta = \sigma_\eta{}^2 I$;

(2.1d) $\Gamma = Y_r + \nu,$ $E(\nu)=0,\ E(\nu\ \nu')=V_\nu$.

Vector Γ denotes the "true" sub-annual values of the socio-economic variable of interest. The estimator Γ^* of Γ will be the benchmarked series. Vectors Y_s and Y_a denote the original sub-annual and the "annual" measurements (observations) of vector Γ. The annual observations Y_a are also known as "benchmarks". Y_i is a vector of values associated with intervention effects in an ARIMA model assumed for Γ. The ARIMA model is specified by the autoregressive matrix operator Π defined later. Y_r is a vector of values corresponding to auxiliary information used in estimating Γ. Matrix B is a generalized annual sum matrix operator defined later. Random vectors e, ϵ, η and ν are mutually independent with respective covariance matrices V_e, V_ϵ, V_η and V_ν.

The first two equations apply to real observations, Y_s and Y_a, of the variable; and the last two equations refer to prior information about its behaviour. Each of the four equations of system (2.1) will now be analysed.

The first equation (2.1.a) states that the T sub-annual observations Y_s are equal to the "true" values, Γ, of the variable under question plus an error e. In this study, e follows an autoregressive process of the form,

$$(2.2) \qquad D\, e = u, \quad E(u) = 0,\ E(u\,u') = \sigma_u^2\, I,$$

where D is an autoregressive matrix operator and u is an independently distributed random variable with mean zero and known variance σ_u^2. As shown in Appendix A, given (2.2), the variance V_e of e is $\sigma_u^2\, [D'D]^{-1}$; and the corresponding criterion matrix $C_e = V_e^{-1}$ is $\sigma_u^{-2}\, D'D$.

If D is equal to the following first order autoregressive operator,

$$(2.3) \qquad \underset{(T-1)\text{ by }T}{D} = \begin{bmatrix} -\rho & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -\rho & 1 & 0 & \dots & 0 & 0 & 0 \\ . & . & . & . & & . & . & . \\ . & . & . & . & & . & . & . \\ . & . & . & . & & . & . & . \\ 0 & 0 & 0 & 0 & \dots & 0 & -\rho & 1 \end{bmatrix},$$

where ρ is given, equation (2.2) can be written as:

$$(2.4) \qquad e_t = \rho\, e_{t-1} + u_t, \quad t=2,\dots,T,$$

$$E(u_t) = 0, \quad E(u_t\, u_{t-k}) = \begin{cases} \sigma_u^2 & \text{for } k = 0, \\ 0 & \text{for } k \neq 0. \end{cases}$$

where σ_u^2 is known. When $\rho=1$, equation (2.4) specifies the **movement preservation principle** encountered in the widely used benchmarking methods of the Denton family described in detail in Section 4. The corrections e_t made to the corresponding original sub-annual value $y_{s,t}$ to obtain the benchmarked values change as little as possible from one period to the next, resulting in estimates of Γ_t as parallel as possible to the sub-annual values $y_{s,t}$. In the ARIMA model-based benchmarking procedure (Hillmer and Trabelsi, 1987) described in detail in Section 3, matrix D of (2.3) is a low order autoregressive operator. The resulting equation (2.4) specifies the behaviour of the sampling error e_t. When applicable, the elements of V_e can be specified directly as a by-product of the survey.

A high value of σ_u^2 specifies that the sub-annual measurements $y_{s,t}$ of vector Γ are inherently inaccurate; and a low value, inherently accurate (once the possible transient bias entailed by (2.4) has been taken into account).

The second equation (2.1.b) of system (2.1) states that the M benchmark values $y_{a,m}$ are equal to the appropriate sums of the Γ_t's, plus an error term ϵ_m. If ϵ_m is independently distributed, the covariance matrix V_ϵ is σ_ϵ^2 I, and equation (2.1b) can be written as

$$y_{a,m} = (\sum_{t=\tau_m}^{\kappa_m} \Gamma_t) + \epsilon_m, \quad m=1,\ldots,M, \; (\tau_1 \geq 1 \; \kappa_M \leq T), \tag{2.5}$$

where $E(\epsilon_t) = 0, \quad E(\epsilon_t \epsilon_{t-k}) = \begin{cases} \sigma_\epsilon^2 & \text{for } k = 0, \\ 0 & \text{for } k \neq 0. \end{cases}$

where σ_ϵ^2 is known. However, like V_e, V_ϵ need not be diagonal.

The symbols τ_m and κ_m in (2.5) are the **reference periods** of the benchmarks. Benchmarks covering calendar years of a monthly flow series have reference periods $\tau_1=1$, $\kappa_1=12$, $\tau_2=13$, $\kappa_2=24$, etc. Benchmarks covering calendar years of a monthly stock series (e.g. inventories) have reference periods $\tau_1=\kappa_1=12$, $\tau_2=\kappa_2=24$, etc. This notation allows for fiscal year benchmarks (e.g. $\tau_1=5$, $\kappa_1=16$, $\tau_2=17$, $\kappa_2=28$); for benchmarks with any kind of reference periods (e.g. $\tau_1=\kappa_1=1$, $\tau_2=2$, $\kappa_2=13$, $\tau_3=14$, $\kappa_3=25$); for benchmarks with overlapping reference periods; for several benchmarks for the same reference period; and for annual as well as sub-annual benchmarks. Consequently, the generalized annual sum matrix operator B of (2.1b) is defined as

(2.6)
$$\underset{M \text{ by } T}{B} = \begin{bmatrix} 0 & 0 & \ldots & 0 & \overset{\tau_1}{1} & 1 & \ldots & \overset{\kappa_1}{1} & 0 & \ldots & \overset{\tau_2}{0} & 0 & \ldots & \overset{\kappa_2}{0} & 0 & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdots & \cdot & \cdot & & \cdot & \cdot & \cdot & \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdots & \cdot & \cdot & & \cdot & \cdot & \cdot & \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdots & \cdot & \cdot & & \cdot & \cdot & \cdot & \end{bmatrix},$$

(columns: τ_1, κ_1, τ_2, κ_2)

where the reference periods τ_m and κ_m indicate the columns and m the rows to place the ones. The annual sums of the sub-annual observations are then $B\ Y_s$; and the **"annual" discrepancies** are $Y_a - B\ Y_s$.

If V_ϵ^2 of (2.1b) equals 0 (which implies $\epsilon_m=0$), the benchmarks are deterministic (i.e. fully reliable), and we shall refer to them as **binding**. (Indeed, equation (2.1b) then represents strict benchmarking constraints.) If V_ϵ^2 is not zero, the benchmarks are not strictly deterministic, and we shall refer to them as **non-binding**.

Since the Γ_t's are the "true" sub-annual values of the variable under question, σ_ϵ in (2.5) can be considered the standard deviation of the errors in the benchmarks. In practice, σ_ϵ^2 can be made to vary to reflect the fact that some benchmarks are more reliable than others. This also holds for all the other variances. To keep the notation simple, we refrain from such a generalization.

The third equation (2.1c) of system (2.1) specifies an expanded ARIMA model for the socio-economic variable Γ. If Y_i is equal to zero or to the mean of the series, the values of Γ_t behave as a pure (as opposed to an expanded) ARIMA model of the Box and Jenkins type (1970). For quarterly first order seasonal autoregressive behaviour, matrix Π would be the following autoregressive operator

$$(2.7) \quad \underset{(T-5) \text{ by } T}{\Pi} = \begin{bmatrix} \pi_1\pi_4 & -\pi_4 & 0 & 0 & -\pi_1 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & . \\ 0 & \pi_1\pi_4 & -\pi_4 & 0 & 0 & -\pi_1 & 1 & \dots & 0 & 0 & 0 & 0 & 0 & . \\ . & . & . & . & . & . & . & \dots & . & . & . & . & . & . \\ . & . & . & . & . & . & . & \dots & . & . & . & . & . & . \\ . & . & . & . & . & . & . & \dots & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \pi_1\pi_4 & -\pi_4 & 0 & 0 & -\pi_1 & 1 \end{bmatrix},$$

where π_1 and π_4 are the known regular and seasonal autoregressive parameters. With $\pi_1=1$, $\pi_4=1$ (and $Y_i=0$), the estimates of Γ tend to display a linear trend and a stable seasonal pattern; with $\pi_1=1$ and $0<\pi_4<1$, a decreasing seasonal pattern.

In many cases, Γ is subject to effects which are not well described by pure ARIMA models, e.g. trading-day and other calendar variations, structural changes and other deterministic effects. An "intervention" (Box and Tiao, 1975) is then desirable in the model. For this, Y_i should be set equal to that effect. This results in an expanded ARIMA model. One can, for instance, impose an additive seasonal pattern S by specifying $Y_i=S$. Similarly, one can impose an additive trading-day pattern by specifying $Y_i=R$, where the elements r_t of R are the monthly sums of the daily weights in month t. Daily weights have a weekly average equal to zero; they can either be constant (repeat week after week) or evolving. (More details about trading-day variations can be found in Young, 1965, and Bell and Hillmer, 1983.)

In both pure and expanded ARIMA models, the departures of Γ_t from the behaviour implied by the model are the innovations η_t of the model. The variance $V_\eta=\sigma_\eta^2 I$ of η is known. Low values of σ_η^2 specify that the vector Γ follows the chosen ARIMA pattern very closely; and large values, very loosely. Thus, low values are appropriate for socio-economic variables which behave in a predictable and smooth manner (e.g. population); and high values, for less predictable and more volatile variables (e.g. exports). In other words, high σ_η^2 specifies that the variable is not well described by ARIMA models.

The fourth equation (2.1d) of system (2.1) is generally a substitute for the third equation (2.1c). Equation (2.1d) specifies that Γ behaves according to auxiliary information available in Y_r plus an error ν. Like e in (2.1a), ν follows an autoregressive process of the form

$$(2.8) \qquad G\,\nu = \xi, \qquad E(\xi) = 0,\ E(\xi\,\xi') = \sigma_\xi^2\, I,$$

where G is an autoregressive matrix operator and ξ is an independently distributed random variable with mean zero and known variance $\sigma_\xi^2 I$. The variance σ_ξ^2 reflects the degree of accuracy of Y_r. As shown in Appendix A, given (2.8), the variance V_ν of ν is $\sigma_\xi^2\,[G'G]^{-1}$; and the corresponding criterion matrix $C_\nu = V_\nu^{-1}$ is $\sigma_\xi^{-2}\ G'G$. In this paper, G is equal to a low order autoregressive operator.

If, for example, G is the first order autoregressive operator (2.3), equation (2.8) can be written as:

$$(2.9) \qquad \nu_t = \rho\,\nu_{t-1} + \xi_t, \quad t=2,\ldots,T,$$

$$E(\xi_t) = 0, \quad E(\xi_t\,\xi_{t-k}) = \begin{cases} \sigma_\xi^2 & \text{for } k = 0, \\ 0 & \text{for } k \neq 0. \end{cases}$$

With ρ close to 1, (2.9) specifies that Γ_t is parallel to the auxiliary values $y_{r,t}$.

A straight line behaviour can thus be imposed on Γ by setting $y_{r,t}$ equal to a+bt, with a and b pre-determined. The behaviour of related variables (Chow and Lin, 1971; Nasse, 1973; Alba, 1988) can be imposed by setting $y_{r,t}$ equal to $a_0 + a_1 z_{1,t} + \ldots + a_Q z_{Q,t}$, where the a_q's are pre-determined and $z_{q,t}$ are regressors (hence subsript r for Y_r) of the variable considered. A constant or evolving seasonal pattern S or trading-day pattern R (or both) can be imposed by selecting Y_r equal to S or R.

The four equations of system (2.1) jointly specify that the desired estimate of Γ maximizes four criteria: (1) Depending on the criterion matrix C_e chosen, Γ is parallel to the original values Y_s. (2) The annual sums of Γ are close to the benchmarks Y_a. (3) Γ behaves according to the ARIMA model chosen. (4) Γ also behaves consistent with the auxiliary information Y_r. The degree to which each criterion is satisfied depends on the relative values of the inverse of their corresponding variances: i.e. σ_u^{-2}, σ_ϵ^{-2}, σ_η^{-2} and σ_ξ^{-2}.

System (2.1) can be written as

$$(2.10) \qquad Y = X\Gamma + U, \quad E(U)=0, \quad E(U U')=V,$$

where $Y' = [Y_s' \; Y_a' \; (\Pi Y_i)' \; Y_r']$, $X' = [I \; B' \; \Pi' \; I]$, $U' = [e' \; \epsilon' \; \eta' \; \nu']$

$$\text{and} \quad V = \begin{bmatrix} V_e & 0 & 0 & 0 \\ 0 & V_\epsilon & 0 & 0 \\ 0 & 0 & V_\eta & 0 \\ 0 & 0 & 0 & V_\nu \end{bmatrix} = C^{-1} = \begin{bmatrix} C_e^{-1} & 0 & 0 & 0 \\ 0 & C_\epsilon^{-1} & 0 & 0 \\ 0 & 0 & C_\eta^{-1} & 0 \\ 0 & 0 & 0 & C_\nu^{-1} \end{bmatrix}.$$

Equation (2.10) is a case of Generalized Least Squares (G.L.S.). The G.L.S. estimator of Γ and its variance-covariance matrix are then respectively: $\Gamma^* = (X'V^{-1}X)^{-1} X'V^{-1} Y$ and var $\Gamma^* = (X'V^{-1}X)^{-1} = \Omega$. Given the partitions of X, Y and $V^{-1}=C$, Γ^* and Ω can be written as

$$(2.11) \quad \begin{aligned} \Gamma^* &= \Omega \, [V_e{}^{-1} Y_s + B'V_\epsilon{}^{-1} Y_a + \Pi' V_\eta{}^{-1} \Pi Y_i + V_\nu{}^{-1} Y_r], \\ &= \Omega \, [C_e Y_s + B C_\epsilon Y_a + \Pi' C_\eta \Pi Y_i + C_\nu Y_r], \end{aligned}$$

$$(2.12) \quad \begin{aligned} \text{var } \Gamma^* = \Omega &= [V_e{}^{-1} + B'V_\epsilon{}^{-1} B + \Pi' V_\eta{}^{-1} \Pi + V_\nu{}^{-1}]^{-1}, \\ &= [C_e + B'C_\epsilon B + \Pi' C_\eta \Pi + C_\nu]^{-1}, \end{aligned}$$

where Γ^* is the benchmarked series and C_e, C_ϵ, C_η and C_ν are the known criteria matrices corresponding to each variance matrix. The criteria matrices, e.g. $C_e = V_e{}^{-1}$, can be obtained directly. It is not necessary to generate V_e, which does not always exist.

Using matrix algebra identities (e.g. Hillmer and Trabelsi, 1987), the estimator Γ^* can be calculated in two steps:

$$(2.13) \quad \begin{aligned} \Gamma_0 &= [C_e + \Pi' C_\eta \Pi + C_\nu]^{-1} \, [C_e Y_s + \Pi' C_\eta \Pi Y_i + C_\nu Y_r] \\ &= \Omega_0 \, [C_e Y_s + \Pi' C_\eta \Pi Y_i + C_\nu Y_r], \end{aligned}$$

$$(2.14) \quad \Gamma^* = \Gamma_0 + \Omega_0 \, B' \, [B \, \Omega_0 \, B' + V_\epsilon]^{-1} \, [Y_a - B \, \Gamma_0].$$

The first step (2.13) yields the estimator Γ_0 of Γ when the benchmarks Y_a are ignored. The second step (second term of (2.14)) modifies Γ_0 to take the benchmarks into account. The factor $[\Omega_0 \, B'[B \, \Omega_0 \, B' + V_\epsilon]^{-1}]$ is then the actual benchmarking operator. This two-step solution for Γ^* is feasible only if [C_e

$+ \quad \Pi' C_\eta \Pi + C_\nu]$ is non-singular, or if its inverse Ω_0 is obtainable directly. In the latter case, the two-step solution dramatically reduces the calculations in comparison with (2.11). Indeed, (2.11) requires the inversion of a T by T matrix, while (2.14) requires only the inversion of a M by M matrix. Another advantage of (2.14) is the possibility to set V_ϵ equal to zero, in which case Y_a is deterministic.

3. THE ARIMA MODEL-BASED BENCHMARKING METHOD

Unlike most of their predecessors, Hillmer and Trabelsi (1987) viewed benchmarking as an **explicitly** statistical model, as opposed to a numerical adjustment problem (e.g. Quadratic Minimization).

3.1 The model

The Hillmer and Trabelsi model is a particular case of the general model of Section 2, under the following conditions:

(1) Only the three first equations of system (2.1) are considered.

(2) The criterion $C_e = V_e^{-1}$ is equal to σ_u^{-2} D'D, where D is an autoregressive operator, which specifies the behaviour of the sampling error.

(3) The covariance of the benchmarks is equal to $V_\epsilon = \sigma_\epsilon^2$ I, where σ_ϵ^2 may be different from zero. This specifies non-binding benchmarks.

(4) The criterion $C_\eta = V_\eta^{-1}$ is equal to σ_η^{-2} I.

(5) Matrix B is the calendar-year annual-sum matrix operator.

Under conditons (1) to (5), the general model (2.1) reduces to

(3.1a) $$Y_s = \Gamma + e, \qquad E(e)=0,\ E(e\ e')=V_e \ ;$$

(3.1b) $$Y_a = B\ \Gamma + \epsilon, \qquad E(\epsilon)=0,\ E(\epsilon\ \epsilon')=V_\epsilon=\sigma_\epsilon I \ ;$$

(3.1c) $$\Pi\ (\Gamma - Y_i) = \eta, \qquad E(\eta)=0,\ E(\eta\ \eta')=V_\eta=\sigma_\eta^2 I \ .$$

where e, ϵ and η are mutually independent random vectors. Substituting $C_e = V_e^{-1} = \sigma_u^{-2}$ D'D, $V_\epsilon = \sigma_\epsilon I$ and $C_\eta = V_\eta^{-1} = \sigma_\eta^{-2}$ I in (2.11) and (2.12) yields the following benchmarked series Γ^* and variance-covariance matrix Ω

(3.2) $$\Gamma^* = \Omega\,[\sigma_u^{-2}\,D'D\,Y_s + \sigma_\epsilon^{-2}\,B'\,Y_a + \sigma_\eta^{-2}\Pi'\Pi\,Y_i],$$

(3.3) $$\text{var}\,\Gamma^* = \Omega = [\sigma_u^{-2}\,D'D + \sigma_\epsilon^{-2}\,B'B + \sigma_\eta^{-2}\Pi'\Pi]^{-1}.$$

Making the same substitution in (2.13) and (2.14) gives the two-step solution for Γ^*

(3.4) $$\Gamma_0 = [\sigma_u^{-2}\,D'D + \sigma_\eta^{-2}\,\Pi'\Pi]^{-1}\,[\sigma_u^{-2}\,D'D\,Y_s + \sigma_\eta^{-2}\,\Pi'\Pi\,Y_i] = \Omega_0\,[\sigma_u^{-2}\,D'D\,Y_s + \sigma_\eta^{-2}\Pi'\Pi\,Y_i],$$

(3.5) $$\Gamma^* = \Gamma_0 + \Omega_0\,B'\,[B\,\Omega_0\,B' + \sigma_\epsilon^2\,I]^{-1}\,[Y_a - B\,\Gamma_0].$$

The philosophy underlying the ARIMA model-based approach to benchmarking is as follows. The sub-annual values Γ of the socio-economic variable are observed through the sub-annual observations Y_s of (3.1a), which contain a sampling error e; similarly, the annual values B Γ of Γ are observed through the annual observations Y_a, which contains an error ϵ. The annual discrepancies between the annual sums B Y_s of Y_s and the benchmarks Y_a of (3.1b) originate from e and from the error ϵ in the benchmarks. In other words, if both the sub-annual and the annual observations, Y_s and Y_a, had no error, there would be no annual discrepancies and no benchmarking problem. One approach is thus to estimate the most likely sub-annual values, on the basis

(1) As the variance σ_ϵ^2 of the benchmarks Y_a tends to 0 (as the benchmarks become deterministic) the weight σ_ϵ^{-2} given to Y_a increases in (3.2), and (ceteris paribus) the benchmarks become binding. Conversely, as σ_ϵ^2 tends to infinity, the weight tends to zero, and the benchmarks become non-binding and are in the limit totally ignored.

To examine the relationship between σ_u^2 and σ_η^2, we assume that σ_ϵ^2 is infinite, so that the benchmarks are totally ignored. The problem is then to find the values which maximize the ARIMA criterion and the criterion adopted for the sampling error e.

(2) If σ_u^2 and σ_η^2 of equation (3.2) are both very small, the sub-annual measurements Y_s of the true values Γ are accurate and Γ behaves according to the chosen ARIMA model. In principle this would result in smooth and accurate benchmarked values Γ^*.

(3) If σ_u^2 is small and σ_η^2 is large, Y_s is accurate and the true values Γ of the socio-economic variable under study are inherently irregular. Little weight σ_η^{-2} is then given to the ARIMA criterion in (3.2), resulting in accurate but irregular benchmarked values very close to the Y_s. This occurs because the method can distinguish between the irregular nature of the socio-economic variable and the accuracy of its measurements.

of both Y_s and Y_a. Most benchmarking methods attempt to do this assuming a first difference behaviour of the errors e. In the ARIMA model-based approach, the model assumed for e may be more complex; and, more importantly, an ARIMA model is also assumed for the true sub-annual values Γ of the socio-economic variable (the signal). Usually, the ARIMA model assumed for e is simple and depends on the characteristics of the survey design (e.g. rotating panels). The ARIMA model is identified from the autocovariances of e, which can be obtained from the survey, and is embodied in the covariance V_e. The model for the signal Γ is first estimated on Y_s and then modified on the basis of the ARIMA model for e. The modified model is then embodied in matrix Π and in V_η-σ_η^2 I of equation (3.1c).

3.2 Some Implications of the ARIMA Model-Based Approach

The implications of the ARIMA model-based approach to benchmarking are now discussed. Some of them hold for the general model and for other benchmarking and interpolation methods.

The behaviour of the benchmarked values Γ^* is described as a function of the basic variances σ_u^2, σ_ϵ^2 and σ_η^2, which determine the weights σ_u^{-2}, σ_ϵ^{-2} and σ_η^{-2} given to each type of observation Y_s, Y_a and Y_i in (3.2).

(4) If σ_u^2 is large and σ_η^2 is small , Y_s is not accurate but Γ closely follows the ARIMA model chosen and is therefore smooth. Small weight σ_u^{-2} is given to Y_s; and a large weight σ_η^{-2}, to the ARIMA criterion in (3.2). If the ARIMA model is correctly chosen, the benchmarked values are smoother and more accurate than the sub-annual measurements Y_s. Again, this is because the method distinguishes between the irregular nature of the variable and the accuracy of its sub-annual measurements.

(5) If σ_u^2 and σ_η^2 are both large, Y_s is not accurate and Γ is inherently irregular, resulting in inaccurate and irregular benchmarked values.

The implications of the ARIMA criterion per se will now be discussed.

(6) The assumption that the signal Γ follows an ARIMA model can be verified only for series with relatively high signal-to-noise ratios. This condition is required to separate the signal Γ from the noise e. In most cases, such series result from high levels of aggregation, where improving the accuracy of the sub-annual observations is often not crucial.

(7) Even with a high signal-to-noise ratio (small e), the assumption that the signal Γ follows a pure (as opposed to expanded) ARIMA model implies that the true sub-annual values Γ are not affected by trading-day variations and other deterministic and/or exogeneous effects. Pure ARIMA models do not capture trading-day variations and other deterministic effects leaving them as residuals (Bell and Hillmer, 1983).

The example chosen by Hillmer and Trabelsi (1987) for their ARIMA model-based benchmarking method is strongly affected by trading-day variations, but the sampling error is very small, i.e. σ_u^2 is much smaller than σ_η^2, so that the behaviour "is dictated entirely by the time series model for the survey [sampling] errors" (Ibid., p. 1070). Furthermore, this model is a first order autoregressive process with $\rho=0.8$. As in the Denton-type methods described in Section 4, such a model tends to preserve the various types of movement in the original series. However, the series may have large sampling errors, and then the weights σ_u^{-2} and σ_η^{-2} given to the sub-annual observations and to ARIMA criterion in (3.2) can then be nearly equal. Consequently, the benchmarked series will have reduced trading-day variations, because these are not incorporated in pure ARIMA models.

(8) In most benchmarking applications, e_t is not just the sampling error, but also embodies non-sampling errors, due to frame deterioration through time, non-response, imputation,

reporting errors, etc. These errors are characterized by biases and/or autocorrelation processes possibly different from that assumed for the sampling error. In such cases, the ARIMA model-based approach will produce distorted results. Thus, if the survey design entails autocorrelation in the sampling error and if benchmarking is used as a technique to correct for it, the series must first be corrected for bias due to non-sampling error. Conversely, if benchmarking is used to correct for bias due to non-sampling error (as in the Denton-type methods), one should first correct for autocorrelation due to sampling error.

(9) Finally estimating ARIMA models for both the sampling errors and for the original series and deriving the ARIMA model for the benchmarked series requires a high level of expertise.

4. THE BENCHMARKING METHODS OF THE DENTON FAMILY

Most of the benchmarking methods used by statistical agencies assume that the benchmarks are fully reliable and therefore binding. Furthermore, they are based on the minimization of an objective function, and that function specifies the principle of movement preservation formally introduced by Denton (1971). According to this principle, the benchmarked series should preserve as much as possible of the consecutive period-to-period (e.g. month-to-month) movement of the original sub-annual series - including the movement from one year to the next. Movement preservation takes at least two forms: preserving the period-to-period 1) arithmetic change and 2) proportional change. These two forms yield the additive and proportional variants of benchmarking examined in Sections 4.1 and 4.2 respectively. Section 4.3 presents an extention of the Denton approach applicable to both variants.

4.1 The Additive Variant of Movement Preservation

The additive variant of the Denton method, modified by Helfand, Monsour and Trager (1977) and by Cholette (1979), minimizes the following objective function

(4.1.1) $$F(\Gamma) = (\Gamma - Y_s)' C_e (\Gamma - Y_s),$$

subject to the benchmarking constraints

(4.1.2) $$Y_a = B\,\Gamma\,,$$

where $C_e = D'D$ and D is the first difference operator. The model for the residuals $e=\Gamma-Y_s$ in (4.1.1) is a random walk, i.e. $e_t = e_{t-1} + u_t$, where u_t is an independently distributed random variable with zero mean.

Generalized Least Squares (G.L.S.) regression also implies minimizing an objective function $eV_e^{-1}e$ on the residuals e. The additive variant of the (modified) Denton method reduces to a particular case of the general model, under the following conditions:

(1) Only the two first equations of system (2.1) are considered.

(2) The criterion matrix $C_e=V_e^{-1}$ is equal to σ_u^{-2} D'D, where D is the first difference operator. This specifies movement preservation (see equation (2.4)).

(3) The covariance matrix of the benchmarks, V_ϵ, is set to zero (hence $\sigma_\epsilon^2=0$). This specifies binding benchmarks.

(4) Matrix B is the calendar year annual-sum matrix operator.

The system of equations (2.1) then reduces to

(4.1.3a) $$Y_s = \Gamma + e, \qquad E(e)=0,\ E(e\ e')=V_e,$$

(4.1.3b) $$Y_a = B\ \Gamma.$$

Substituting $C_e=\sigma_u^{-2}$ D'D and $V_\epsilon=0$ (and $C_\eta=C_\nu=0$) in (2.13) and (2.14) yields the following benchmarked series Γ^*

(4.1.4) $\Gamma^* = Y_s + (\sigma_u^{-2} D'D)^{-1} B' [B(\sigma_u^{-2} D'D)^{-1} B']^{-1} [Y_a - B Y_s]$.

In (4.1.4), σ_u^{-2} cancels out, and the G.L.S. estimator of the benchmarked series is then identical to Denton's (1971). The covariance matrix can be obtained by substituting C_e into (2.12) and by letting σ_ϵ^2 tend to zero:

(4.1.5) $$\text{var } \Gamma^* = \lim_{\sigma_\epsilon^2 \to 0} (\sigma_u^{-2} D'D + \sigma_\epsilon^{-2} B'B)^{-1}.$$

However, with D equal to the strict first difference operator σ_u^{-2} D'D is singular, so Γ^* cannot be calculated with (4.1.4) (but (2.11) could be used by letting V_ϵ tend to 0). To avoid singularity, we redefine D as the following classical quasi-first difference operator (Harvey, 1981, p. 190)

(4.1.6) $$\underset{T \text{ by } T}{D} = \begin{bmatrix} (1-\rho^2)^{1/2} & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ -\rho & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -\rho & 1 & 0 & \dots & 0 & 0 & 0 \\ . & . & . & . & & . & . & . \\ . & . & . & . & & . & . & . \\ . & . & . & . & & . & . & . \\ 0 & 0 & 0 & 0 & \dots & 0 & -\rho & 1 \end{bmatrix},$$

where ρ is very close to 1.0 (e.g. 0.999999). This operator causes minimal distortion to the movement preservation principle, because the first element is negligible. Furthermore, $(\sigma_u^{-2} D'D)^{-1}$ is now known algebraically:

(4.1.7) $$\underset{T \text{ by } T}{(\sigma_u^{-2} D'D)^{-1}} = \begin{bmatrix} 1 & \rho & \rho^2 & \dots & \rho^{T-1} \\ \rho & 1 & \rho & \dots & \rho^{T-2} \\ \rho^2 & \rho & 1 & \dots & \rho^{T-3} \\ . & . & . & & . \\ . & . & . & & . \\ . & . & . & & . \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \dots & 1 \end{bmatrix} \sigma_u^2 / (1-\rho^2).$$

Bournay and Laroque (1979, p. 21) show that, as ρ tends to 1, the solution obtained with the quasi first difference operator tends in the limit towards the solution of the strict first difference operator. (As ρ tends to 1, the denominator in (4.1.7) tends to zero, but that zero cancels out in (4.1.4).)

Both the additive and the proportional variants of the original Denton method had a first difference operator D given by (4.1.6) with ρ=1, but with the first element equal to 1. This operator implies an invertible D'D and a known inverse. However, it causes a distortion of movement preservation at the start of the series, which also affects subsequent years in a decreasing manner (Cholette, 1979, 1984). The distortion is major if the absolute magnitude of the first annual discrepancy is large. Denton seemed to be concerned with long series that have small annual discrepancies Y_a - B Y_s. The time series published by statistical agencies become shorter and display relatively large annual discrepancies. Helfand, Monsour and Trager (1977), for instance, report discrepancies of 4.8% and 7.8% for total sales series of the U.S. Retail Trade and Wholesale Trade series. In such situations, the serious distortions introduced by the initial Denton operator can easily be avoided in the manner described above.

With ρ not close to 1, equations (4.1.6) and (4.1.7) cause distortions both at the start and at the end of series if the first and/or the last discrepancies are large.

With an appropriate first difference operator, all variants of benchmarking presented in this section successfully cope with systematic and large discrepancies: The level of the sub-annual series Y_s adopts that of the benchmarks Y_a. This would seem to imply that Y_s is biased and that E(e) is not zero, in violation of the assumption of the general model. However, since these benchmarking methods assume that $e_t = e_{t-1} + u_t$, where $E(u_t)=0$, then $E(e_t)$ equals 0.

4.2 The Proportional Variant of Movement Preservation

The proportional variant of movement preservation was also formally introduced by Denton (1971). According to this principle, the benchmarked series Γ^* is as proportional as possible to the original sub-annual series Y_s. Large values of Y_s are then corrected in absolute terms more than small values. This is appropriate for series, in which seasonally low observations cannot possibly account for as much of the annual discrepancy as the seasonally large observations do. With negative annual discrepancies, this specification also tends to avoid negative benchmarked values. Defendable per se, proportionality is also a linear approximation of growth rate preservation. Indeed, any two series z_t and x_t proportional to each other have identical growth rates: $z_t/z_{t-1}=x_t/x_{t-1}$.

The proportional variant of the Denton method, modified by Helfand, Monsour and Trager (1977) and by Cholette (1979), also minimizes objective function (4.1.1) (with a different C_e) subject to (4.1.2). For the same reasons given for the additive variant, the proportional variant can be seen as a particular case of the general model the following conditions:

(1) Only the first two equations of system (2.1) are considered.

(2) The criterion $C_e = V_e^{-1}$ is equal to σ_u^{-2} diag$(Y_s)^{-1}$ D'D diag$(Y_s)^{-1}$, where D is the first difference operator. This specifies proportional movement preservation.

(3) The covariance of the benchmarks, V_ϵ, is set to zero. This specifies binding benchmarks.

(4) Matrix B is the calendar year annual-sum matrix operator.

The second condition implies the following behaviour of the errors:

$$e_t/y_{s,t} = e_{t-1}/y_{s,t-1} + u_t, \quad t=2,\ldots,T,$$

$$(4.2.1) \qquad E(u_t) = 0, \quad E(u_t\, u_{t-k}) = \begin{cases} \sigma_u^2 & \text{for } k = 0, \\ 0 & \text{for } k \neq 0. \end{cases}$$

Appendix A shows that the autoregressive (and heteroscedastic) behaviour of e_t implies $C_e = V_e^{-1}$ equal to σ_u^{-2} diag$(Y_s)^{-1}$ D'D diag$(Y_s)^{-1}$. Equation (4.2.1) specifies the

proportional movement preservation principle: The ratio of each correction e_t to the corresponding original sub-annual value $y_{s,t}$ changes as little as possible from one period to the next, resulting in estimates of Γ_t which are as proportional as possible to the sub-annual observations $Y_{s,t}$.

The system of equations (2.1) then reduces to system (4.1.3) (except for a different $V_e = C_e^{-1}$). Substituting $C_e = \sigma_u^{-2}\, diag(Y_s)^{-1}\, D'D\, diag(Y_s)^{-1}$ and $V_\epsilon = 0$ (and $C_\eta = C_\nu = 0$) in (2.13) and (2.14) yields the following benchmarked series Γ^*

$$\Gamma^* = Y_s + [\sigma_u^{-2}\, diag(Y_s)^{-1}\, D'D\, diag(Y_s)^{-1}\,]^{-1}\, B' \{B[\sigma_u^{-2}\, diag(Y_s)^{-1}\, D'D\, diag(Y_s)^{-1}\,]^{-1} B'\}^{-1}\, [Y_a - B\, Y_s]. \qquad (4.2.2)$$

Since σ_u^{-2} cancels out in (4.2.2), it is not required in practice, and this estimator of the benchmarked series is identical to Denton's (1971).

As with the additive variant, $[\sigma_u^{-2}\, diag(Y_s)^{-1}\, D'D\, diag(Y_s)^{-1}]$ of (4.2.2) is singular if D is the strict first difference operator. Again, redefining D as the quasi first difference operator of (4.1.6) entails an invertable matrix and a known T by T inverse:

$1/\hat{\Gamma}_t$. Under a Gamma distribution the resulting estimator is asymptotically efficient. For proportional benchmarking, the Gamma distribution would in fact be appropriate, because it excludes negative observations.

For the proportional variant, estimator (4.2.2) is then like Denton's numerical solution. If the mixed variant just described is not used, the covariance matrix of the estimates is undetermined.

4.3 Extending the Denton Method

The modified Denton approach to benchmarking discussed in Section 4.1 can be adapted to many applications including: non-fully reliable benchmarks, benchmarks referring to fiscal years, preliminary benchmarking and calendarization.

4.3.1 Non-Binding Benchmarks

In practice, the full reliability of the annual benchmarks are often not fully reliable (Cholette, 1988; Cholette and Higginson, 1987). Both variants of the modified Denton method easily adapted to this.

The Additive Variant

For the additive variant, the general model of Section 2 is modified in the following manner:

(1) Only the two first equations of system (2.1) are considered.

$$[\sigma_u^{-2}\ \mathrm{diag}(Y_s)^{-1}\ D'D\ \mathrm{diag}(Y_s)^{-1}]^{-1} =$$

$$\begin{bmatrix} \xi_{11} & \xi_{12}\rho & \xi_{13}\rho^2 & \cdots & \xi_{1T}\rho^{T-1} \\ \xi_{21}\rho & \xi_{22} & \xi_{23}\rho & \cdots & \xi_{2T}\rho^{T-2} \\ \xi_{31}\rho^2 & \xi_{32}\rho & \xi_{33} & \cdots & \xi_{3T}\rho^{T-3} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \end{bmatrix} \sigma_u^2 / (1-\rho^2), \qquad (4.2.3)$$

where $\xi_{ij} = y_{s,i}\ y_{s,j}$.

The solution (4.2.2) of the proportional variant works well numerically. It is identical to Denton's (1971) solution developed in the framework of Quadratic Minimization by minimizing (4.1.1) subject to (4.1.2). Statistically however, there is a problem. The inverse of the criterion matrix cannot be interpreted as the variance of e, because the weights $A = D\ \mathrm{diag}(Y_s)^{-1}$ in $A\ e = u$ are not fixed. This problem is akin to what is konw as dependent **variable heteroscedasticity**. The variance of the observations $y_{s,t}$ (ignoring $y_{a,m}$ and assuming D=I for the moment) is assumed to be equal to $\sigma^2 y_{s,t}^2$. The G.L.S. estimator (4.2.2) in then inefficient. Harvey (1981, Chapt. 3.4) describes a slightly different specification where the variance of the dependent variable is proportional to the square of its expectation, $\sigma^2 E(\Gamma_t)^2$, and G.L.S is usable. The technique yields estimates which are neither proportional nor parallel to Y_s, but **mixed**. The technique consists of two steps. First apply Ordinary Least Squares (O.L.S.) ignoring heteroscedasticity and calculate the O.L.S. estimates $\hat{\Gamma}_t$ of Γ_t (i.e. apply additive benchmarking). Second, apply G.L.S. using $\sigma^2 \hat{\Gamma}_t^2$ as the variance of the observations, therby weighing the observations by

$$[\sigma_u^{-2}\ \mathrm{diag}(Y_s)^{-1}\ D'D\ \mathrm{diag}(Y_s)^{-1}]^{-1} =$$

$$(4.2.3)\qquad \begin{bmatrix} \xi_{11} & \xi_{12}\rho & \xi_{13}\rho^2 & \cdots & \xi_{1T}\rho^{T-1} \\ \xi_{21}\rho & \xi_{22} & \xi_{23}\rho & \cdots & \xi_{2T}\rho^{T-2} \\ \xi_{31}\rho^2 & \xi_{32}\rho & \xi_{33} & \cdots & \xi_{3T}\rho^{T-3} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \end{bmatrix} \sigma_u^2 / (1-\rho^2),$$

where $\xi_{ij} = y_{s,i}\ y_{s,j}$.

The solution (4.2.2) of the proportional variant works well numerically. It is identical to Denton's (1971) solution developed in the framework of Quadratic Minimization by minimizing (4.1.1) subject to (4.1.2). Statistically however, there is a problem. The inverse of the criterion matrix cannot be interpreted as the variance of e, because the weights $A = D\ \mathrm{diag}(Y_s)^{-1}$ in $A\ e = u$ are not fixed. This problem is akin to what is konw as **dependent variable heteroscedasticity**. The variance of the observations $y_{s,t}$ (ignoring $y_{a,m}$ and assuming $D=I$ for the moment) is assumed to be equal to $\sigma^2 y_{s,t}^2$. The G.L.S. estimator (4.2.2) in then inefficient. Harvey (1981, Chapt. 3.4) describes a slightly different specification where the variance of the dependent variable is proportional to the square of its expectation, $\sigma^2 E(\Gamma_t)^2$, and G.L.S is usable. The technique yields estimates which are neither proportional nor parallel to Y_s, but **mixed**. The technique consists of two steps. First apply Ordinary Least Squares (O.L.S.) ignoring heteroscedasticity and calculate the O.L.S. estimates $\hat{\Gamma}_t$ of Γ_t (i.e. apply additive benchmarking). Second, apply G.L.S. using $\sigma^2 \hat{\Gamma}_t^2$ as the variance of the observations, therby weighing the observations by

(2) The criterion C_e is set equal to σ_u^{-2} D'D, where D is the (quasi) first difference operator, which specifies movement preservation.

(3) The covariance of the benchmarks, V_ϵ, is set to σ_ϵ^2 I (for instance), which specifies the benchmarks are non-binding (see eq. (2.5)).

System of equations (2.1) reduces to

$$\text{(4.3.1a)} \qquad Y_s = \Gamma + e, \qquad E(e)=0,\ E(e\ e')=V_e,$$

$$\text{(4.3.1b)} \qquad Y_a = B\,\Gamma + \epsilon, \qquad E(\epsilon)=0,\ E(\epsilon\ \epsilon')=V_\epsilon,$$

where e and ϵ are mutually independent random vectors. Substituting C_e and V_ϵ and making $C_\eta = C_\nu = 0$ in (2.13) and (2.14) yields the following benchmarked series Γ^*

$$\text{(4.3.2)} \quad \begin{aligned} \Gamma^* &= Y_s + C_e^{-1} B' \{B\, C_e^{-1} B' + V_\epsilon\}^{-1} [Y_a - B\, Y_s)], \\ &= Y_s + [\sigma_u^{-2} D'D]^{-1} B' \\ &\qquad \{B\, [\sigma_u^{-2} D'D]^{-1} B' + V_\epsilon\}^{-1} [Y_a - B\, Y_s)]. \end{aligned}$$

Making the same substitutions in (2.12) yields the covariance matrix:

$$\text{(4.3.3)} \qquad \text{var}\ \Gamma^* = [C_e + B'C_\epsilon\ B]^{-1} = [\sigma_u^{-2}\ D'D + B'V_\epsilon\ B]^{-1}.$$

The Proportional Variant

The proportional variant of the extended Denton benchmarking method is obtained by replacing conditions (2) and (3) of the extended additive variant with the following:

(2') Matrix C_e is set equal to $[(\sigma_u^{-2}\ \underline{Y}_s)$

$\text{diag}(Y_s)^{-1}$ D'D $\text{diag}(Y_s)^{-1}$], which specifies proportional movement preservation (see eq. (4.2.1)). The average $\underline{Y}_s$ of Y_s is needed for calibration.

(3') The covariance of the benchmarks, V_ϵ, is redefined as σ_ζ^2 diag(Y_a). The error term ϵ_m is then assummed to behave in the following heteroscedastic manner:

$$(4.3.4) \quad \epsilon_m/\surd y_{a,m} = \zeta_m \quad \text{or} \quad \epsilon_m = \zeta_m \surd y_{a,m},$$

where
$$E(\epsilon_t) = 0, \quad E(\epsilon_t \, \epsilon_{t-k}) = \begin{cases} \sigma_\zeta^2 \, y_{a,m} & \text{for } k = 0, \\ 0 & \text{for } k \neq 0. \end{cases}$$

where ζ_m is an independently distributed random variable with mean zero and known variance σ_ζ^2. σ_ζ is the proportional standard deviation of the benchmarks.

However, as was mentioned in Section 4.2, the solution for the proportional variant should be considered numerical, with undetermined covariance.

4.3.2 Fiscal Years and Sub-Annual Benchmarks

In real applications, yearly benchmarks often refer to fiscal years and may not be fully reliable (Cholette and Higginson, 1987). Both the additive and proportional variants discussed above are easily adapted to this by letting matrix B be the generalized annual-sum operator (2.10). This operator also handles sub-annual benchmarks. The resulting extended Denton method can be used for a variety of purposes, such as preliminary benchmarking, calendarization of fiscal year data, imposition of "historical" benchmarked values and time series linkage.

Preliminary Benchmarking

Yearly benchmarks are not available until well after the year is over, but the extended Denton approach (and the general model) process current as well as past observations. Preliminary benchmarking automatically occurs whenever the last reference period κ_M of the most recent benchmark $y_{a,M}$ is smaller than the number T of observations in the series. This built-in process is the same as repeating the last additive or proportional correction calculated for the most recent time period with a benchmark; that correction is then applied to the current observations. (One then needs recompute benchmarking only when a new benchmark becomes available.) As suggested by Laniel (1986), the built-in preliminary benchmarking process can sometimes be improved by forecasting the sub-annual and the benchmark series and by benchmarking the artificially extended series thus obtained. This alternative entails recomputing benchmarking each time a new current observation becomes available and is thus much more expensive than the built-in preliminary benchmarking process.

If annual discrepancies are large, preliminary benchmarking must be performed to avoid movement discontinuities between past benchmarked values and current values (Cholette, 1979, 1984).

Calendarization

With fiscal year benchmarks, calendar year values can be obtained as a by-product of benchmarking, by taking the annual sums of the benchmarked series. Similarly, calendar quarter values can be obtained - whether the benchmarks reflect fiscal years or fiscal quarters. In the case study considered by Helfand et al. (1977), the benchmarks were actually quinquennial; yearly values can be generated in a similar manner. In the absence of sub-annual observations Y_s, calendarization can be viewed as a problem of interpolation (temporal disaggregation) between the fiscal data; or as a problem of benchmarking an indicator of the variable of interest.

Historical Estimates

The extended Denton method can also be used in such a way that the benchmarked series is not changed before a certain date. The benchmarked series starts from the historical benchmarked value. That value is simply specified as a sub-annual benchmark $y_{a,1}$ with $\tau_1 = \kappa_1 = 1$, and $y_{s,1}$ is set equal to the corresponding unbenchmarked value. This feature is very easy to implement: A statistical agency can decide on a maximum number of complete previous years during which revisions (originating from benchmarking) are allowed (e.g. two years), and implement accordingly. The feature also reduces calculations: only the revisable part of the series needs to be processed when an annual benchmark becomes available.

With the extended Denton method, the accuracy of the estimates increases as they become more central in the series. In our experience, revisions should be allowed for a minimum of two complete years. Feibes (1968) finds that with the Boot, Feibes and Lisman (1967) interpolation technique, the estimates become insensitive to incorporation of new yearly benchmarks after three years. (The Boot et al. technique is a particular case of the additive variant of the extended Denton method if $V_\epsilon=0$, $Y_S=0$ and B is the calendar year sum operator.)

5. CONCLUSIONS

This paper discussed existing benchmarking methods as particular cases of a general model. Earlier empirical ad-hoc procedures based on Bassie (1958), Friedman (1962), Lisman and Sandee (1964), Glejser (1966) and Boot and Feibes (1967) were not included in this analysis.

For the most general and complex variant of the general model presented in Section 2, it is possible to have an ARIMA model for the benchmarked series, an ARIMA model for the survey errors, non-binding reliable benchmarks and fiscal year benchmarks.

The ARIMA model-based benchmarking method imposes an ARIMA structure on the benchmarked series, which then has both reduced trading-day and irregular variations. The problem with trading-day variations can be corrected with an expanded ARIMA model that includes an intervention variable Y_i containing an additive trading-day component. No such solution exists for irregular fluctuations, which may contain socio-economic information.

ARIMA modelling is feasible only when the series have a high signal-to-noise ratio. This generally means that modelling is appropriate only at higher levels of aggregation, with series which are not volatile. However, some series are essentially volatile (even at high levels of aggregation) or contain fluctuations which are not well fitted by ARIMA models. This further reduces the applicability of the ARIMA model-based method.

The general method (and the ARIMA model-based method) makes it possible to correct for sampling error, by means of an ARIMA model. This raises the issue of whether such correction should be done with benchmarking or separately with specific methods. The specific methods surveyed by Jones (1980) and by Binder and Hidiroglou (1988), for instance, allow a more sophisticated modelling of the sampling error. Another related consideration is the complexity of the modelling process in the context of benchmarking a large number of series. In that context, it is preferable to apply the Denton-type variants, where a very simple random walk model is pre-selected to maximize the parallelism between the benchmarked and the original series. The main objective of the random walk model is to preserve the seasonal signal in the sub-annual series.

The Denton type variants also cope with systematic annual discrepancies observed between the benchmarks and the corresponding annual sums of the original series. If a series has already been corrected for sampling errors, benchmarking then corrects for the remaining survey error, that is for non-sampling error due to frame deterioration through time, undercoverage, imputation, reporting errors, etc.

This paper also discussed the accuracy of annual benchmarks and noted that treating them as non-binding is useful. Indeed specifying volatile benchmarks as binding distorts the movement of the sub-annual series. More reliable annual values can then be obtained as through non-binding benchmarking by taking the annual sums of the benchmarked series.

Benchmarks which refer to fiscal instead of calendar years were also considered. The general model and the extended Denton method make it easy to specify the actual reference periods of the benchmarks, provided that all the individuals covered by the surveys have common fiscal years. Fiscal year or fiscal quarter benchmark data become legitimately usable as such. If needed, calendarized annual and quarterly values can then be obtained by taking the appropriate sums of the benchmarked series.

APPENDIX A: Relationship Between the Covariance and the Autoregressive Behaviour of a Stochastic Variable

Let a T-dimensional random vector e follow an autoregressive process of the form:

(A.1) $A\ e = u, \qquad E(u) = 0,\ E(u\ u') = V_u = \sigma_u^2\ I,$

where A is any known autoregressive operator of dimension T' by T (T'≤T) and where the covariance of u is known and equal to $\sigma_u^2 I$. The problem is now to derive the covariance V_e of e in terms of that of u. This requires solving e as a function of u. Thus

(A.2) $A\ e = u, \Rightarrow A'A\ e = A'u, \Rightarrow e = (A'A)^{-1}A'u,$

which is the required linear combination. Then:

(A.4) $$V_e = E(e\ e') = \sigma_u^2 \underset{T \text{ by } T}{(A'A)^{-1}}$$

If A'A is singular, as it must be if T'<T, then V_e in (A.4) cannot be obtained. However, it is always possible to calculate the inverse C_e of V_e directly: $C_e = (\sigma_u^2(A'A)^{-1})^{-1} = \sigma_u^{-2}(A'A)$. The inverse of covariance matrices are those actually needed in the solution of Generalized Least Squares regressions. We shall call the known matrix C_e the criterion matrix, because it entails a criterion or behaviour (e.g. autoregressive) maximized by e.

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