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BARTLETT TYPE MODIFIED TESTS FOR MOVING SEASONALITIES

by

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Summary

The adjustment of economic and social time series for seasonal variation has been and continues to be the subject of much attention. As a first step towards seasonally adjusting a series, it is essential to test for the presence of stable as well as moving seasonalities. Under the assumption that the seasonal pattern is constant over time, recently Sutradhar, Dagum and Solomon (1991) discuss a modification to the standard F-test for testing the significance of stable (constant) seasonalities. The present paper examines the presence of changing seasonal pattern versus a constant seasonal pattern over time. It is shown that the test for the presence of moving seasonalities is equivalent to test for the homogeniety of variances of several correlated groups where observations in each group are also correlated. A modification to the standard Bartlett test is proposed which accounts for the autocorrelations. The corrected Bartlett test statistic has asymptotically chi-square distribution with suitable degrees of freedom.

KEY WORDS: Constant and changing seasonal pattern; Two-way correlations; Homogeneity of variances; Bartlett test; Correction for autocorrelations; Chi-square approximations.

RÉSUMÉ

La désaisonnalisation des séries économiques et sociales est toujours un sujet d'étude de première importance. La première étape du processus de désaisonnalisation consiste à déterminer s'il y a ou non présence de saisonnalité stable ou mobile. Suthradar, Dagum et Solomon (1991) font d'abord l'hypothèse que la saisonnalité a un profil constant à travers le temps puis présentent une modification à apporter au test de F afin de tester le seuil de signification d'un tel profil. Cet article met en présence et examine un profil saisonnier constant en regard d'un profil changeant. Il en ressort que le test d'identification de la présence de saisonnalité mobile est équivalent à un test d'homogenéité des variances de plusieurs groupes d'observations correlés entre eux et dont les observations de chaque groupe sont sont correlées entre elles. Les auteurs proposent une modification du test de Bartlett afin de prendre en compte ces autocorrélations. Le test de Bartlett ainsi modifié obéit à une loi de distribution asymptotique de chi-carré dont le nombre de degrés de liberté est suffisant.

MOTS CLÉS: Profil saisonnier constant et changeant; test d'hypothèse de l'existence ou non d'autocorrélations; homogenéité des variances; test de Bartlett; correction prenant en compte les autocorrélations; approximation de la loi de chi-carré.

1 INTRODUCTION

Socio-economic time series are often presented in seasonally adjusted form so that the underlying short-term trend can be more easily analysed and current socio-economic conditions can be assessed. It is, however, customary to test for the presence of significant seasonalities in a time series before making seasonal adjustments. For the situations where the seasonal pattern is stable over time possibly at different levels (due to annual shifts), recently Sutradhar, Dagum and Solomon (1991) discuss a modification to the standard F-test for testing the presence of significant stable seasonality. There has been some attention to the other situations where the seasonal pattern may be changing over time. Suppose in a detrended series, the ith observation ($i = 1, 2, \ldots, k$) in the jth year ($j = 1, \ldots, n$), $Z_i(j)$, is expressed as

$$Z_i(j) = S_i(j) + U_i(j), \tag{1.1}$$

where $S_i(j)$ and $U_i(j)$ are the seasonal and error components respectively, corresponding to $Z_i(j)$. Under the assumption that

$$\sum_{i=1}^{k} S_i(j) = \xi_j \sim N(0, \sigma_\xi^2),$$

Franzini and Harvey (1983), among other things, test for the presence of constant seasonal pattern (i.e. $H_0 : \sigma_\xi^2 = 0$) against the alternative hypothesis ($H_1 : \sigma_\xi^2 > 0$) that the seasonal pattern is changing (moving) over the years. Their test is based on the theory of invariance due to Lehmann (1959, Chapter 5), but, the test requires a specific value of σ_ξ^2 under the alternative hypothesis, which is a major limitation.

In the present paper, more specifically in Section 2, we demonstrate that the test for the presence of a constant seasonal pattern is equivalent to test for the homogeneity of variances of n correlated groups (years) where observations in each group are also correlated. A modification to the standard Bartlett test is proposed which accounts for the autocorrelations.

The construction procedure for the test statistic is discussed in Section 3. The test statistic is easy to compute and it has asymptotically χ^2 distribution with $n-1$ degrees of freedom, n being the number of years (groups) in the series.

2 THE MODEL

Consider a stationary seasonal time series $\{Z_t\}$, given by

$$Z_t = S_t + U_t, \tag{2.1}$$

where Z_t is the observed series at time t, S_t is the seasonal component, and U_t the irregulars. If the time series contains a trend, which is most likely, it is assumed that a suitable detrending technique will yield the model (2.1).

Next, suppose that there are k seasons in a year and there are kn observations in a time series of n years. Writing $Z(t)$ for Z_t, the stationary series $\{Z_t\}$ may be expressed as

$$Z\{(i-1)n+j\} = S\{(i-1)n+j\} + U\{(i-1)n+j\}, \tag{2.2}$$

where $Z\{(i-1)n+j\}$ is the jth $(j=1,\ldots,n)$ observation under the ith season $(i=1,\ldots,k)$. Further, using the identity $Z\{(i-1)n+j\} \equiv Z_i(j)$, the equation (2.2) may be re-written as

$$Z_i(j) = S_i(j) + U_i(j). \tag{2.3}$$

which is of the form (1.1). For $i=1,\ldots,k$ and $j=1,\ldots,n$, $U_i(j)$ are assumed to follow the white noise process with mean zero and variance σ_u^2. The interpretation of the seasonal components $S_i(j)$ in (2.3) depends on the nature of the seasonalities of the series. In a deterministic seasonal model, $S_i(j)$ may be expressed as

$$S_i(j) = \mu + \beta_j + \alpha_{i(j)} \tag{2.4}$$

where μ is the mean effect, β_j is the jth annual seasonal shift and $\alpha_{i(j)}$ is the ith seasonal effect under the jth year. It may be assumed that $\sum_{j=1}^{n} \beta_j = 0$, but $\sum_{i=1}^{k} \alpha_i(j) = 0$ always for all $j = 1, \ldots, n$. If the seasonal pattern remains constant over the years, then $\alpha_i(j) = \alpha_i$ (say) for all $j = 1, \ldots, n$, with $\sum_{i=1}^{k} \alpha_i = 0$. Otherwise, the series contains moving seasonalities. For the case when $\alpha_i(j) = \alpha_i$, one may further need to test the significance of stable seasonality which may be done by testing $\alpha_i = 0$ for all $i = 1, \ldots, k$ [cf. Sutradhar, Dagum and Solomon (1991)]. In this set-up, the testing for constant seasonal pattern against the moving seasonality is equivalent to test $\alpha_i(j) = \alpha_i$ for all $i = 1, \ldots, k$ and $j = 1, \ldots, n$. If the seasonal model is stochastic, then it may be assumed that

$$S(j) \sim N(1_k \otimes (\mu + \beta_j),\ \sigma_j^2 \Sigma^{(0)}), \tag{2.5}$$

where $S(j) = [S_1(j), \ldots, S_i(j), \ldots, S_k(j)]'$ is the $k \times 1$ vector of seasonal components, $1_k = (1, \ldots, 1)'$ of order $k \times 1$, $\sigma_j^2 > 0$ is a scalar, and $\Sigma^{(0)}$ is a $k \times k$ scale matrix. One may, however, use $\mu = 0$ in (2.5) without any loss of generality. Then (2.5) reduces to

$$S(j) \sim N(m_j, \sigma_j^2 \Sigma^{(0)}), \tag{2.6}$$

where $m_j = 1_k \otimes \beta_j$ is the $k \times 1$ vector. Further, it will be assumed that the covariance between $S(j)$ and $S(j')$ $(j' = 1, \ldots, n)$ is given by

$$\mathrm{cov}(S(j), S(j')) = \sigma_j \sigma_{j'} \Sigma^{(|j-j'|)}. \tag{2.7}$$

Notice from (2.6) and (2.7) that irrespective of the situations whether σ_j^2 for $j = 1, \ldots, n$ are same or different, the $kn \times kn$ correlation matrix of the kn-dimensional vector $S = [S'(1), \ldots, S'(j), \ldots, S'(n)]'$ remains the same. Thus the introduction of different variance components for the seasonal components under different years allows to explain the moving seasonalities without changing the correlation structure of the seasonal components. If $\sigma_1^2 =$

$\ldots = \sigma_n^2 = \sigma^2 > 0$ (say) in (2.6), then the seasonal pattern is constant over the years. Thus, for testing the presence of constant seasonal pattern against the moving seasonality, we test

$$\begin{aligned} H_0 : \sigma_j^2 &= \sigma^2, \quad \text{for all} \quad j = 1, \ldots, n \\ \text{against} \quad H_1 : \sigma_j^2 &\neq \sigma^2, \quad \text{for at least one } j. \end{aligned} \tag{2.8}$$

3 CONSTRUCTION OF TEST STATISTIC

Let $Z = [Z'(1), \ldots, Z'(j), \ldots, Z'(n)]'$ be the $kn \times 1$ vector, where $Z'(j) = [Z_1(j), \ldots, Z_i(j), \ldots, Z_k(j)]$ and where $Z_i(j)$ is given in (2.3). Since $U_i(j) \sim N(0, \sigma_u^2)$, and $U_i(j)$ and $S_i(j)$ are assumed to be independent for all $i = 1, \ldots, k$, $j = 1, \ldots, n$, it follows from (2.6) and (2.7) that Z has the kn-dimensional normal distribution with mean vector m and covariance matrix Σ^{**} given by

$$m = (m_1', \ldots, m_j', \ldots, m_n')',$$

and

$$\Sigma^{**} = \sigma_u^2 (I_{kn} + \Sigma^*), \tag{3.1}$$

respectively, where I_{kn} is the $kn \times kn$ identity matrix, and Σ^* is defined as

$$\Sigma^* = \begin{bmatrix} \delta_1^2 \Sigma^{(0)} & \delta_1 \delta_2 \Sigma^{(1)} & \cdots & \delta_1 \delta_n \Sigma^{(n-1)} \\ & \delta_2^2 \Sigma^{(0)} & \cdots & \delta_2 \delta_n \Sigma^{(n-2)} \\ & & & \vdots \\ & & & \delta_n^2 \Sigma^{(0)} \end{bmatrix},$$

with $\delta_j^2 = \sigma_j^2 / \sigma_u^2$, and

$$\Sigma^{(h)} = \begin{bmatrix} \gamma_{hk} & \gamma_{hk+1} & \cdots & \gamma_{hk+k-1} \\ & \gamma_{hk} & \cdots & \gamma_{nk+k-2} \\ & & & \vdots \\ & & & \gamma_{hk} \end{bmatrix},$$

γ_{hk+m} being the correlation ($\sigma_u^2\delta_j\delta_h\gamma_{hk+v}$ being the covariance) between the ith seasonal component of the jth year and the ℓth seasonal component of the j'th year when seasons are separated by $v = |i-\ell|$, $i,\ell = 1,\ldots,k$ and the years are separated by $h = |j-j'|$, $j,j' = 1,\ldots,n$.

For the independent case, that is, when seasonal components are assumed to be independent of each other, Σ^* reduces to

$$\Sigma^* = \begin{bmatrix} \delta_1^2 I_k & 0 & \cdots & 0 \\ 0 & \delta_2^2 I_k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \delta_n^2 I_k \end{bmatrix}, \tag{3.2}$$

Then the test for $\sigma_1^2 = \ldots, \sigma_n^2 = \sigma^2$ (say) is equivalent to the test for $\sigma_u^2(1+\delta_j^2) = \sigma_u^2 + \sigma^2$ for all $j = 1,\ldots,n$, which may be performed by using the Bartlett's modified test statistic given by

$$B_M = -2\log\ell / \left\{1 + \frac{1}{3(k-1)}\left(1+\frac{1}{n}\right)\right\}, \tag{3.3}$$

where

$$-2\log\ell = (k-1)n\log s^2 - (k-1)\sum_{j=1}^{n} s_j^2,$$

with $s_j^2 = \sum_{i=1}^{k}(Z_i(j) - \bar{Z}.(j))^2/(k-1)$, $s^2 = \sum_{j=1}^{n} s_j^2/n$, and $\bar{Z}.(j) = \sum_{i=1}^{k} Z_i(j)/k$, $Z_i(j)$ being the ith detrended observation corresponding to the jth ($j = 1, \ldots, n$) year for $i = 1, \ldots, k$. The test statistic B_M has asymptotically chi-square distribution with $n-1$ degrees of freedom and the chi-square approximation is quite satisfactory even for small k. This test, however, is not robust against the departure from normality. See, for example, Box (1953), Layard (1973). The B_M test is also not robust against the departure from the assumption of independence. In the present set-up the data are correlated in two-ways. The k observations in each of the n groups (years) are correlated. Also, the groups are correlated.

Recently, Sutradhar and Dagum (1991) have proposed a modification to the standard Bartlett test for testing the equality of variances of n groups of errors of a linear regression model. Their test accounts for the two-way autocorrelations. Following Sutradhar and Dagum (1991), we now modify the Bartlett's test statistics B_M (3.3), so that the modified test statistic accounts for the autocorrelations embodied in the covariance structure (3.1).

3.1 Gamma Approximations to s_j^2 and s^2

Write $s_j^2 = Z'(j)AZ(j)/(k-1)$ for $j = 1, \ldots, n$, and $s^2 = Z'BZ/n(k-1)$, where $A = I_k - k^{-1}U_k$, $B = I_n \otimes A$, and where U_k is the $k \times k$ unit matrix and $\otimes$ denotes the Kronecker product. For the case when observations are independent, the statistics

$$q_j = (k-1)s_j^2/2\sigma_u^2(1+\delta^2), \quad \text{and} \quad q = n(k-1)s^2/2\sigma_u^2(1+\delta^2)$$

have gamma distributions $G\left(1, \frac{k-1}{2}\right)$ and $G\left(1, \frac{n(k-1)}{2}\right)$ respectively under the null hypothesis, where the gamma distribution $G(a,b)$, $a > 0$, $b > 0$, of the random variable X means that X has the probability density function given by $f(x) = e^{-ax}x^{b-1}/a^{-b}\Gamma b$, $x > 0$.

In the present set-up, the observations are correlated. Consequently, the exact distributions of q_j and q are complicated. To overcome this distributional problem, we follow Box (1954), Andersen, Jensen and Schou (1981) and approximate the distributions of q_j and q

by gamma distributions with first and second moments equal to the corresponding moments of q_j and q. Under the H_0, the first two moments of q_j $(j = 1, \ldots, n)$ and q are given by

$$\begin{aligned} E(q_j) &= \frac{1}{2}(1+\delta^2)^{-1}[\text{trace } AV + \text{ trace } m_j{}'Am_j/\sigma_u^2], \qquad (3.4) \\ E(q) &= \frac{1}{2}(1+\delta^2)^{-1}[\text{trace } \{B(I_{kn}+\delta^2\Sigma)\} + \text{ trace } m'Bm/\sigma_u^2], \\ \text{var}(q_j) &= \frac{1}{2}(1+\delta^2)^{-2}[\text{trace } AVAV + 2 \text{ trace } m_j{}'AVAm_j/\sigma_u^2], \end{aligned}$$

and

$$\begin{aligned} \text{var}(q) &= \frac{1}{2}(1+\delta^2)^{-1}[\text{trace}\{B(I_{kn}+\delta^2\Sigma)B(I_{kn}+\delta^2\Sigma)\} \\ &\quad +2 \text{ trace } m'B(I_{kn}+\delta^2\Sigma)Bm], \end{aligned}$$

where $V = I_k + \delta^2\Sigma^{(0)}$, $\delta^2 = \sigma^2/\sigma_u^2$, and Σ is the $kn \times kn$ matrix given by

$$\Sigma = \begin{bmatrix} \Sigma^{(0)} & \Sigma^{(1)} & \ldots & \Sigma^{(n-1)} \\ & \Sigma^{(0)} & \ldots & \Sigma^{(n-2)} \\ & & & \vdots \\ & & & \Sigma^{(0)} \end{bmatrix}. \qquad (3.5)$$

Next suppose the distributions of q_j $(j = 1, \ldots, n)$ and q are approximated by $G(a_j, h_j)$ and $G(a, h)$ respectively. One then obtains

$$\begin{aligned} E(q_j) &= h_j/a_j, \quad \text{var}(q_j) = h_j/a_j^2, \quad j = 1, \ldots, n, \qquad (3.6) \\ E(q) &= h/a, \quad \text{var}(q) = h/a^2. \end{aligned}$$

It now follows from (3.4) and (3.6) that for $j = 1, \ldots, n$,

$$\begin{aligned} a_j &= (1+\delta^2)G_j/L_j \\ h_j &= \frac{1}{2}G_j^2/L_j, \end{aligned} \tag{3.7}$$

with

$$\begin{aligned} G_j &= \operatorname{trace}[AV + m_j{}'Am_j/\sigma_u^2], \\ L_j &= \operatorname{trace}[AVAV + 2m_j{}'AVAm_j/\sigma_u^2], \end{aligned}$$

and

$$\begin{aligned} a &= (1+\delta^2)G/L, \\ h &= \frac{1}{2}G^2/L, \end{aligned} \tag{3.8}$$

with

$$\begin{aligned} G &= \operatorname{trace}[B(I_{kn} + \delta^2\Sigma) + m'Bm/\sigma_u^2], \\ L &= \operatorname{trace}[B(I_{kn} + \delta^2\Sigma)B(I_{kn} + \delta^2\Sigma) \\ &\quad +2m'B(I_{kn} + \delta^2\Sigma)Bm/\sigma_u^2]. \end{aligned}$$

3.2 Bartlett Type Modified Test

Re-express $-2\log\ell$ from (3.3) as

$$-2\log\ell = (k-1)n\log(dq/n) - (k-1)\sum_{j=1}^{n}\log(dq_j), \tag{3.9}$$

where $d = 2\sigma_u^2(1+\delta^2)/(k-1)$. Now by taking direct expectation over (3.9), one obtains:

$$\begin{aligned} E(-2\log\ell) &= (k-1)[n\log(dh/na) - \sum_{j=1}^{n}\log(dhj/aj)] \\ &\quad +(k-1)\left\{\sum_{j=1}^{n} h_j^{-1} - nh^{-1}\right\}/2 \\ &\quad +(k-1)\left\{\sum_{j=1}^{n} h_j^{-2} - nh^{-2}\right\}/12 + O\{(k-1)^{-3}\}, \end{aligned} \tag{3.10}$$

where for $j = 1, \ldots, n$, a_j and h_j are given by (3.7), and a and h are defined by (3.8). Following Bartlett's (1937) modification, we now propose the test statistic, $B_M(c)$ [Bartlett modified statistic corrected for autocorrelations], given by

$$B_M(c) = -2\log\ell\left(1-\frac{1}{n}\right)/(k-1)\{\hat{D}_1 + \hat{D}_2/2 + \hat{D}_3/12\}, \tag{3.11}$$

for testing the null hypothesis $H_0 : \sigma_j^2 + \sigma_u^2 = \sigma^2 + \sigma_u^2$. In (3.11),

$$\hat{D}_1 = n\log(\hat{h}/n\hat{a}) - \sum_{j=1}^{n}\log(\hat{h}_j/\hat{a}_j),$$

$$\hat{D}_2 = \sum_{j=1}^{n}\hat{h}_j^{-1} - n\hat{h}^{-1},$$

and

$$\hat{D}_3 = \sum_{j=1}^{n}\hat{h}_j^{-2} - n\hat{h}^{-2},$$

where $\hat{h}_j$, $\hat{a}_j$, $\hat{h}$ and $\hat{a}$ are the suitable estimates of h_j, a_j, h and a respectively and they are computed by using the suitable estimates for σ_u^2, σ^2, m and Σ in the formulas for h_j, a_j.

h and a given in (3.7) and (3.8). The test statistic $B_M(c)$ may be treated as having a χ^2 distribution with $n-1$ degrees of freedom.

Notice that if the seasonal components over the years have the same mean level, that is, if $m_j = m^*$ (say) for all $j = 1, \ldots, n$, then the $B_M(c)$ test statistic reduces to

$$B^*_M(c) = -2\log\ell\left(1 - \frac{1}{n}\right)/(k-1)(\hat{D}^*_1 + \hat{D}^*_2/3), \tag{3.12}$$

where

$$D^*_1 = \frac{1}{2}(h^{*^{-1}} - h^{-1}), \quad \text{and} \quad D^*_2 = \frac{1}{2}D^*_1(h^{*^{-1}} + h^{-1}),$$

with $h_j = h^* = \frac{1}{2}G^{*^2}/L^*$, G^* and L^* are being given by

$$G^* \;=\; \mathrm{trace}[AV + m^{*\prime}Am^*/\sigma^2_u],$$

and

$$L^* \;=\; \mathrm{trace}[AVAV + 2m^{*\prime}AVAm^*/\sigma^2_u].$$

Furthermore, if the seasonal components are independent, which is unlikely, then $V = (1 + \delta^2)I_k$, and $\Sigma = I_{kn}$ yielding $h^* = (k-1)/2$ and $h = \frac{1}{2}n(k-1)$. Then $B^*_M(c)$ in (3.12) reduces to Bartlett's test statistic B_M given in (3.3).

4 ESTIMATION OF THE PARAMETERS

To begin with we note that the modified test statistic B_M in (3.3) is constructed under the assumption that the covariance matrix Σ^{**}, of Z is

$$\Sigma^{**} = \mathrm{diag}[(\sigma^2_u + \sigma^2_1)I_k, \ldots, (\sigma^2_u + \sigma^2_n)I_k]. \tag{4.1}$$

The corrected statistic $B_M(c)$ in (3.11), however, takes the actual correlation structure (3.1) of the data into account. In order to compute $\hat{D}_1$, $\hat{D}_2$ and $\hat{D}_3$ in (3.11), one requires to estimate m, σ_u^2, Σ and σ^2.

4.1 Estimation of the Mean Level

The kn-dimensional mean vector m may be consistently estimated by

$$\hat{m} = (\hat{m}_1 1_k', \ldots, \hat{m}_j 1_k', \ldots, \hat{m}_n 1_k')', \tag{4.2}$$

where $\hat{m}_j = \bar{Z}_.(j) = \sum_{i=1}^{k} Z_i(j)/k$, for all $j = 1, \ldots, n$.

4.2 Estimation of σ_u^2 and Σ

The estimation of σ_u^2 and Σ (3.5) will be carried out as follows.

Step 1. Initial estimates: To obtain an initial estimate of Σ, we need the initial estimates of σ_u^2 and σ_j^2 $(j = 1, \ldots, n)$. Assuming that

$$\Sigma^{**} = \operatorname{diag}[\sigma_u^2 I_k + \sigma_1^2 U_k, \ldots, \sigma_u^2 I_k + \sigma_n^2 U_k], \tag{4.3}$$

where U_k is the $k \times k$ unit matrix, one may compute the initial estimates of σ_u^2 and σ_j^2 $(j = 1, \ldots, n)$ as

$$\hat{\sigma}_u^2(0) = \sum_{j=1}^{n} \sum_{i=1}^{k} (Z_i(j) - \bar{Z}_.(j))^2 / n(k-1), \tag{4.4}$$

$$\hat{\sigma}_j^2(0) = \frac{1}{k} \left[\sum_{i=1}^{k} \sum_{i'=1}^{k} (Z_i(j) - \bar{Z}_.(j))(Z_{i'}(j) - \bar{Z}_.(j))/k - \hat{\sigma}_u^2(0) \right]. \tag{4.5}$$

We then generate a stable series $Z^* = (Z_{11}^*, \ldots, Z_{ij}^*, \ldots, Z_{kn}^*)'$ from Z, with multiplying Z_{ij} by $\{(\hat{\sigma}_j^2(0) + \hat{\sigma}_u^2(0))/\hat{\sigma}_j^2(0)\}^{\frac{1}{2}}$ for all $i = 1, \ldots, k$; $j = 1, \ldots, n$. The Z^* series has the correlation matrix Σ as in (3.5), which may be estimated by exploiting the sample autocorrelation functions of the series. Denote this estimate by $\hat{\Sigma}(0)$.

Step 2. It now follows under the true correlation structure that

$$E(\hat{\sigma}_u^2(0)) = [\sigma_u^2 \text{ trace}\{B(I_{kn} + \Sigma^*)\} + \text{ trace } m'Bm]/n(k-1), \tag{4.6}$$

where Σ^* is given in (3.1). Since trace $B = n(k-1)$, one obtains from (4.6) that

$$E(\hat{\sigma}_u^2(0)) = \sigma_u^2 + \text{ trace } B\Lambda^*/n(k-1) + \text{ trace } m'Bm/n(k-1),$$

where

$$\Lambda^* = \begin{bmatrix} \sigma_1^2\Sigma^{(0)} & \sigma_1\sigma_2\Sigma^{(1)} & \cdots & \sigma_1\sigma_n\Sigma^{(n-1)} \\ & \sigma_2^2\Sigma^{(0)} & \cdots & \sigma_2\sigma_n\Sigma^{(n-2)} \\ & & & \vdots \\ & & & \sigma_n^2\Sigma^{(0)} \end{bmatrix},$$

yielding the unbiased estimator of σ_n^2 given by

$$\sigma_u^2(1) = \hat{\sigma}_u^2(0) - \text{ trace}\{B\Lambda^* + m'Bm\}/n(k-1), \tag{4.7}$$

provided Λ^* and m are known. Putting $\Lambda^* = \Lambda^*(0)$ and $m = \hat{m}$ in (4.7), we obtain an improved (as compared to $\hat{\sigma}_u^2(0)$ in (4.4)) first step estimator for σ_u^2 given by

$$\hat{\sigma}_u^2(1) = \hat{\sigma}_u^2(0) - \text{ trace}\{B\hat{\Lambda}^*(0) + \hat{m}'B\hat{m}\}/n(k-1), \tag{4.8}$$

where $\hat{m}$ is given by (4.2) and $\hat{\Lambda}^*(0)$ is computed from Λ^* with replacing σ_j $(j = 1, \ldots, n)$ by $\hat{\sigma}_j(0)$, and $\Sigma^{(h)}$ by $\hat{\Sigma}^{(h)}(0)$ from Step 1.

Next, because the true covariance matrix of Z vector is given by $E(Z-m)(Z-m)' = \sigma_u^2(I_{kn} + \Sigma^*)$, one may obtain an improved estimator for σ_j^2 $(j = 1, \ldots, n)$ by equating the sum of the elements of the $k \times k$ matrix $\sigma_u^2(I_k + \delta_j^2\Sigma^{(0)})$ in (3.1) to its sample counterpart $\sum_{i=1}^{k}\sum_{i'=1}^{k}(Z_i(j) - \bar{Z}.(j))(Z_{i'}(j) - \bar{Z}.(j))$. The improved estimator for σ_j^2 is

$$\begin{aligned}\hat{\sigma}_j^2(1) &= \left[\sum_{i=1}^{k}\sum_{i'=1}^{k}(Z_i(j) - \bar{Z}.(j))(Z_{i'}(j) - \bar{Z}.(j))\right.\\ &\left. -k\hat{\sigma}_u^2(1)\right] / s(\hat{\Sigma}^{(0)}(0)),\end{aligned} \tag{4.9}$$

where $s(\hat{\Sigma}^{(0)}(0))$ denotes the sum of the elements of the $k \times k$ matrix $\hat{\Sigma}^{(0)}(0)$, $\hat{\Sigma}^{(0)}(0)$ being the initial estimate of $\Sigma^{(0)}$ obtained from Step 1.

Now, as in Step 1, we generate a stable series Z^* from Z, with multiplying Z_{ij} by $\{(\hat{\sigma}_j^2(1)+\hat{\sigma}_u^2(1))/\hat{\sigma}_j^2(1)\}^{\frac{1}{2}}$ for all $i = 1, \ldots, k$; $j = 1, \ldots, n$, and obtain the first step estimator of Σ by exploiting the sample autocorrelation functions of the new series Z^*. Call this estimator $\hat{\Sigma}(1)$.

The first step estimator $\hat{\Sigma}(1)$ may then be used in (4.8) and (4.9) to obtain the second step estimators $\hat{\sigma}_u^2(2)$ and $\hat{\sigma}_j^2(2)$ for σ_u^2 and σ_j^2 ($j = 1, \ldots, n$) respectively. These second step estimators for σ_u^2 and σ_j^2 may further be used to produce $\hat{\Sigma}(2)$ as in the above. Continue this cyclical operation until the estimators for σ_u^2, σ_j^2 ($j = 1, \ldots, n$), and Σ stablize. Denote these stable estimators by $\hat{\sigma}_u^2$, $\hat{\sigma}_j^2$ ($j = 1, \ldots, n$), and $\hat{\Sigma}$, among which $\hat{\sigma}_u^2$ and $\hat{\Sigma}$ may be used to compute the $B_M(c)$ statistic in (3.11).

Step 3. Estimation of σ^2. For the estimation of σ^2 (where $\sigma_j^2 = \sigma^2$ for all $j = 1, \ldots, n$, under the H_0), we begin with an initial estimate

$$\hat{\sigma}^2(0) = \left\{k\sum_{j=1}^{n}(\bar{Z}.(j) - \bar{Z}..)^2/(n-1) - \hat{\sigma}_u^2(0)\right\}/k, \tag{4.10}$$

based on Σ^{**} in (4.3). In (4.10), $\bar{Z}.. = \sum_{j=1}^{n}\bar{Z}.(j)/n$, and $\hat{\sigma}_u^2(0)$ is given by (4.4). Rewrite (4.10) as

$$\hat{\sigma}^2(0) = \{Z'PZ/k(n-1) - \hat{\sigma}_u^2(0)\}/k,$$

where $P = (I_n - n^{-1}U_n) \otimes U_k$. One then easily shows under the $H_0 : \sigma_j^2 = \sigma^2$ $(j = 1, \ldots, n)$ that

$$\begin{aligned} E(\hat{\sigma}^2(0)) &= [\{\sigma_u^2 \text{ trace } P + \sigma^2 \text{ trace } P\Sigma + \text{ trace } m'Pm\}/k(n-1) \\ &\quad -\{\sigma_u^2 \text{ trace } B + \sigma^2 \text{ trace } B\Sigma + \text{ trace } m'Bm\}/n(k-1)]/k. \end{aligned}$$

Since trace $P = k(n-1)$, and trace $B = n(k-1)$, it then follows that

$$\begin{aligned} E(\hat{\sigma}^2(0)) &= [\sigma^2\{\text{trace } P\Sigma/k(n-1) - \text{ trace } B\Sigma/n(k-1)\} \\ &\quad +\{\text{trace } m'Pm,/k(n-1) - \text{ trace } m'Bm/n(k-1)\}]/k, \end{aligned}$$

yielding the unbiased estimator for σ^2 as

$$\hat{\sigma}^2(1) = \frac{[k\hat{\sigma}^2(0) - \{\text{trace } m'Pm/k(n-1) - \text{ trace } m'Bm/n(k-1)\}]}{\text{trace } P\Sigma/k(n-1) - \text{ trace } B\Sigma/n(k-1)\}},$$

provided m and Σ are known. For unknown m and Σ, we use their estimates $\hat{m}$ from Section 4.1, and $\hat{\Sigma}$ from Step 2. The estimate of σ^2 is then given by

$$\hat{\sigma}^2 = \frac{[k\hat{\sigma}^2(0) - \{\text{trace } \hat{m}'P\hat{m}/k(n-1) - \text{ trace } \hat{m}'B\hat{m}/n(k-1)\}]}{\text{trace } P\hat{\Sigma}/k(n-1) - \text{ trace } B\hat{\Sigma}/n(k-1)\}}. \tag{4.11}$$

REMARKS

The large as well as small sample performance of the proposed Bartlett type modified test for testing the presence of constant seasonal pattern will be examined through a simulation study in the second part. Also, the testing methodology developed in this paper will be applied to real life data.

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