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WORKING PAPER NO. BSMD-94-006E

METHODOLOGY BRANCH

CAHIER DE TRAVAIL NO. BSMD-94-006E

DIRECTION DE LA MÉTHODOLOGIE

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LONG WAVE AND BUSINESS CYCLES IN THE GDP

by

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September 1994

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Abstract

Economists have developed their theory of long-wave cycles since the 1920s. However, the theory is not well-accepted in modern macroeconomics as these cycles are hardly posited to the real data and consist of too few episodes to be testable, especially in production. The stochastic modelling or nonparametric approaches are currently used to detrend the series to obtain business-cycle movements.

Using newly developed approaches of detecting and estimating hidden periodicities in time series analysis, this paper proposes a new way to deal with the hidden long-wave cycles in macroeconomic series such as the gross domestic product (GDP) series. The analysis shows strong evidence of the existence of these cycles. The detrending approach offered by this paper has many advantages for analysing economic situations and for forecasting.

Résumé

Les économistes ont developé la théorie des cycles de longue durée depuis 1920. Cette théorie n'est pas acceptée dans la macroéconomie moderne parce que ces cycles ne se produisent pas beaucoup dans les séries réelles et il y a peu d'examples disponibles pour tester leur présence. La modélisation stochastique et les approches nonparamétriques sont présentement utilisés pour enlever la tendance d'une série et obtenir les mouvements du cycle d'économique.

En utilisant les approches developpées récemment pour identifier et estimer les périodicités cachées en série temporelles, cet article propose une nouvelle façon de traiter les cycles de longue durée des séries macroéconomiques commes la série de produit intérieur brut(PIB). L'analyse montre une forte évidence de l'existence de ces cycles. L'approche d'enlever la tendance dans une série, développé dans cet article, offre plusieurs avantages pour analyser les situations économiques et pour faire de la prévision.

1 Introduction

In macroeconomics, problems of the business cycle (with respect to durations, turning points, amplitudes, patterns, etc.) and forecasting (in terms of model identification, estimation, adjustment, etc.) have caught the attention of economists and statisticians for a long time. However, these problems interact with the *trend*, on which the salient business cycles stand. Different detrending approaches may result in different business cycle movements and the forecasts of a series depend on the forecasts of the trend and of the detrended series (namely the business cycle-movements). Meaningful trends may indicate the situation of long-term developments in the economy; inappropriate trends could be very misleading. Therefore, proper modelling of trends is of primary interest in macroeconomics. The term "trend" is commonly understood but difficult to define in general. Every approach offers trends with its own properties and features. Some literal definitions of "trend" are discussed in Harvey (1989, Section 6.1.1).

Assume that a series of observations Y_t can be decomposed as

$$Y_t = T_t + C_t, \tag{1.1}$$

where series $\{C_t\}$ represents the business-cycle movements which is usually assumed to be a stochastic series, say, follows an ARMA model. Series $\{T_t\}$ represents the trend.

If T_t is a deterministic function of time, and $\{C_t\}$ is a stationary series, Nelson and Plosser (1982) called (1.1) the *trend-stationary* model and is known as the TS model. The most simple deterministic functions for trends are linear or polynomials of higher degree. However in most practical cases, such trends are far from satisfactory and most current research does not follow this direction.

Various stochastic models have been used for modelling the trends of macroeconomic series in the current literature such as the unobserved component models also known as the UC models (see Watson 1986), the structural models (e.g., see Crafts et al. 1989), etc. were proposed. Additional references are cited by the papers mentioned above and there are more in Zarnowitz (1992). A general introduction to stochastic trend-modelling may be found in Harvey (1989). Once the models of series $\{T_t\}$ and $\{C_t\}$ in (1.1) are specified, T_t and C_t may be estimated by a filter or a signal extraction procedure.

Alternatives to stochastic modelling of T_t are some moving average procedures (nonparametric smoothing). The National Bureau of Economic Research (NBER) in the United States developed a procedure called the phase-average method which combines moving averages and interpolations between mean values of the series in its successive business-cycle phases. This method has been adopted by the Organization for Economic Co-operation and Development (OECD) to produce trends of data for the major industrialized countries in their monthly published journal *Main Economic Indicators*.

An application of the linear trend, the UC trend and the phase-average trend with the logarithm of real GNP data of USA are compared in Zarnowitz (1992). A copy of his Fig.6.1 and Fig.6.2 is attached at the end of this paper for the reader's convenience. The linear trend is certainly not satisfactory and is used only for comparison purposes, to explain the advantages of modern stochastic trends. The UC trend follows the data closely, but it is too irregular and too flexible. As Zarnowitz (1992, p.187) discussed, "the stochastic trend flattens in several recessions (1953-54, 1973-75, 1981-82) and even declines in some (1957-58, 1960)." Phase-average performs the best in keeping the right sizes and patterns of the business cycles. Unfortunately there are no analytical formulae or built-in models to describe the trend, so from this trend it is difficult to produce some general and concise conclusions concerning long-term developments in the economy and it cannot be used directly for forecasting.

Diverging from the main stream of current research techniques which obtain stochastic trends, this paper advocates a regression model and uses Canadian GDP series as an example to show that the regression residuals may represent the business-cycle movements and prove to be satisfactory in comparing with the phase-average detrended series. The advantage of this approach is that it offers a trend described by an analytical function and allows us to investigate the existence of long-wave cycles.

In fact, early in the 1920s, the Russian economist Kondratieff (1984, translation) discovered the existence of long-wave cycles of 50-60 years duration in the western economy. His theory was recognized by western economists and further developments are summarized in Stocken (1978). However, the developments along this direction are basically in the domain of economics and lack convincing statistical analysis. As Zarnowitz (1992) pointed out, the Kondratieff's swing of 50-60 years are hardly general as posited, for they show up mainly in prices and not production, and consist of too few episodes to be testable. The proposed long-wave hypothesis is not supported by the data in any clear and consistent way. Some growth rates in the United States between 1840 and 1914 show 15- to 20-year fluctuations known as Kuznets cycles are now recognized as belonging to history. Commonly used statistical methods do not seem to easily verify the existence of long-wave cycles, so the research on long-wave cycles is currently in decline. Using the regression strategy offered by this paper, this "ancient" conclusion may be restored for some macroeconomic series. As a model is provided, these cycles now have new implications and the results are supported by data with statistical analysis.

The specific model is

$$Y_t = a_0 + a_1 t + a_2 t^2 + \rho\cos(\omega_0 t + \phi) + C_t \tag{1.2}$$

In fact, this is a model of the TS type where the trend consists of a quadratic function and a sinusoid (amplitude ρ, frequency ω_0 and phase ϕ are all constants). The substantial difference of (1.2) from other TS models discussed in the literature is the introduction of the sinusoid term

$$\rho\cos(\omega_0 t + \phi) = \rho_c \cos\omega_0 t + \rho_s \sin\omega_0 t \tag{1.3}$$

where

$$\rho_c = \rho\cos\phi, \qquad \rho_s = -\rho\sin\phi. \tag{1.4}$$

This sinusoid term, if it exists, claims the existences of long-wave cycles.

This paper will study three series of Canadian GDP data by fitting the model (1.2) and then compares the results with those in the literature. The data source is the CANSIM database of Statistics Canada. The three quarterly seasonally adjusted series are GDP at current prices, GDP at constant prices (which is referred to as real GDP) and implicit price index. Each series contains 186 observations ranging from the first quarter of 1947 to the second quarter of 1993. Using *quarter/year* to indicate the actual quarter in a year, the data start from 1/1947 and end by 2/1993. The following correspondence between the numbering of the time axis (from 1 to 186) and the actual quarter will be usefel for reading the plots in this paper.

20	60	100	140	180
4/1951	4/1961	4/1971	4/1981	4/1991

The other sections are arranged as follows. In Section 2, we discuss data transformation. In Section 3, the concept of hidden periodicities is introduced, as well as a recently developed approach in detecting and estimating hidden periodicities which will be used in this paper. Section 4 explains how to transform model (1.2) to the form where the methods in Section 3 are applicable. Section 5 presents the results of the regression study with Canadian GDP data mentioned above, and the existence of long-wave cycles is confirmed. Section 6 analyses the business cycles from the residuals of the regressions for the above-mentioned series. All the figures and tables are attached after the references.

Some similar results for GDP data of USA and some results of intermediate cycles which may further improve the appearance of the business-cycle movements as well as other relevant results were obtained but are not presented here.

2 Data Transformation

Data transformation is sometimes the key step in solving a practical problem. By changing the scale, we may change the properties of the data. Therefore, a proper choice of transformation can provide advantages. For example, the distribution of the transformed data may become normal or close to normal, so that many statistical inference approaches can be applied. For time series analysis, a suitable transformation can make the data stationary or close to stationary.

The most common transformation in macroeconomics is the logarithmic transformation $Y_t = \log X_t$, where X_t are the original observations of the series under investigation. Often the distribution of the data is much closer to normal after this transformation and even more meaningful.

Denote the forward differencing operator by Δ, then

$$\Delta Y_t = Y_{t+1} - Y_t = \log X_{t+1} - \log X_t = \log(1 + \frac{X_{t+1} - X_t}{X_t}).$$

In practice, for cases such as GDP, $X_{t+1} - X_t$ is much smaller than X_t. Using Taylor's expansion, we have

$$\Delta Y_t \simeq (X_{t+1} - X_t)/X_t, \tag{2.1}$$

i.e., the difference of the logarithm of X_t is approximately the percentage change of X_t.

There is a family of transformations called Box-Cox transformations, defined by

$$Y_t = \begin{cases} (X_t^\lambda - 1)/\lambda, & 0 < \lambda \leq 1; \\ \\ \log X_t & \lambda = 0. \end{cases} \tag{2.2}$$

When the logarithmic transformation is not satisfactory, i.e. the transformed data are still far from being normal or stationary, then we may use other transformations in this family.

In fact, when $\lambda > 0$, instead of (2.2) we may simply use

$$Y_t = X_t^\lambda, \tag{2.3}$$

then in the case $\lambda = 1$, the original data are unchanged. (2.2) is motivated by the need to satisfy the continuity in λ that is $(X_t^\lambda - 1)/\lambda \to \log X_t$ when $\lambda \to 0$.

The optimal choice of λ depends on its purpose. Here, stationarity is our major concern. The theory to be used for detecting and estimating the hidden periodicities (in Section 3) is based on the assumption of stationarity. Moreover, once the trend has been removed, we would like to fit the residuals with stochastic models for the purpose of forecasting, where stationarity is again a good feature for modelling.

If a series (apart from a linear trend) is stationary, then the differenced series is also stationary. Therefore, it is equivalent to trying λ on the differenced series. In the following, we will also see that this aids our visual judgement.

Now we discuss transformations of three Canadian GDP series mentioned in Section 1. Each series has been transformed with 3 different λ and then differenced. Fig.2.1, 2.2 and 2.3 are the plots of these differenced series. The top plots of these figures are for $\lambda = 1$, i.e., the differences of the original series X_t; the bottom ones are for $\lambda = 0$, i.e., the differences of $Y_t = \log X_t$; the middle ones are the differences of $Y_t = X_t^\lambda$ with "optimal" choice of λ. We obtained these "optimal" values of λ by using the *maximum entropy* criterion, which we will explain later in detail. Note that the three plots in each figure are of three different scales.

From Fig.2.1 for GDP at current prices, we see in the top plot that the stochastic fluctuation increases dramatically; that means $\{\Delta X_t\}$ is strongly nonstationary. Such an increase is due to combined effects of expanding volume and inflation. The top plot of Fig.2.2 for GDP at constant prices shows a mild tendency of increasing stochastic fluctuation, where no inflation is involved. This series is still nonstationary but not serious.

Both bottom plots of Fig.2.1 and Fig.2.2 show that, for GDP at current prices and at constant prices, $\Delta Y_t = \Delta \log X_t$ show a decreasing tendency of fluctuation. This is conceivable as the size of the economy becomes large, more "inertia" is acquired, and hence the percentage change becomes more stable, i.e. less effected by impacts of stochastic shocks. Both plots suggest mild nonstationarity.

The middle plots of Fig.2.1 and Fig.2.2 are obtained by choosing $\lambda = 0.35$ for GDP at current prices and $\lambda = 0.68$ for GDP at constant prices. These two show satisfactory stable fluctuation which suggests that, data transformed in this way are stationary (excluding, perhaps, a slight trend). Fig.2.3 shows similar results for the series of implicit price index, where the middle one is obtained by choosing $\lambda = 0.5$.

An elaboration of the maximum entropy criterion follows. For a finite discrete probability distribution family $\{p_1(\lambda), \ldots, p_k(\lambda)\}$, λ is the parameter of the distribution and $\sum_{j=1}^{k} p_j(\lambda) = 1$. The entropy is defined by

$$\mathcal{E}(\lambda) = -\sum_{j=1}^{k} p_j(\lambda) \log p_j(\lambda). \tag{2.4}$$

If the uniform distribution belongs to the family, i.e. there is a λ, such that $p_1(\lambda) = \ldots = p_k(\lambda) = 1/k$, then the maximum of $\mathcal{E}(\lambda)$ is achieved at this λ. If the uniform distribution does not belong to this family, then the maximum of $\mathcal{E}(\lambda)$ is achieved at a λ, such that $\{p_1(\lambda), \ldots, p_k(\lambda)\}$ is the "closest" to the uniform distribution.

Now suppose the observations are $X_1, \ldots, X_N$. For a fixed λ, $0 \le \lambda \le 1$, let Y_t be defined by (2.2) or (2.3). Partition the time axis into k intervals of almost the same length with end points $i_0 = 1 < i_1 < \ldots < i_k = N-1$. Consider the maximization of (2.4) with

$$p_j(\lambda) = v_j(\lambda)/v(\lambda), \quad v(\lambda) = \sum_{j=1}^{k} v_j \tag{2.5}$$

and

$$v_j(\lambda) = \frac{1}{i_j - i_{j-1} - 1} \sum_{t=i_{j-1}}^{i_j - 1} (\Delta Y_{t+1} - \Delta Y_t)^2, \quad j = 1, \ldots, k, \tag{2.6}$$

that is the average of squared variations of ΔY_t in the jth segment. Then the maximization of (2.4) suggests a value of λ which makes the variation of ΔY_t [Y_t are given by (2.2) or (2.3)] the most uniform in the whole period. Fig.2.4 is the plot of $\mathcal{E}(\lambda)$ for GDP at current prices of Canada with $N = 186$ and $k = 7$. The maximum of $\mathcal{E}(\lambda)$ at $\lambda = 0.35$ is 1.92916, which is very close to 1.94591, the entropy of the uniform distribution of $k = 7$.

There are some other ways to define $v_j(\lambda)$ in (2.6), such as using the square root of the right hand side of (2.6); or in (2.6), $(\Delta Y_{t+1} - \Delta Y_t)^2$ can be replaced by $\mid \Delta Y_{t+1} - \Delta Y_t \mid$. Our calculation shows that there is not much difference in using these alternative measures. For example, for GDP at current prices, by using $\mid \Delta Y_{t+1} - \Delta Y_t \mid$, $\mathcal{E}(\lambda)$ achieves a maximum of 1.94044 at $\lambda = 0.38$. Moreover, the number k is not crucial. A comparison of the case $k = 3$ and 7 shows the results are very close.

Notice that $\mathcal{E}(\lambda)$ is designed to measure the uniformity of stochastic variation, but is possibly affected by a hidden smooth trend in ΔY_t, though very slightly. If instead of ΔY_t, we directly use Y_t in (2.6), then because Y_t contain a strong linearly increasing trend which is the main portion of $Y_{t+1} - Y_t$, the resolution will be poor.

3 Detecting and Estimating Hidden Periodicities

Suppose a time series $\{W_t\}$, where t takes integer values, follows model

$$W_t = \rho\cos(\omega_0 t + \phi) + U_t, \tag{3.1}$$

where ρ, ω_0, ϕ are constants and $\{U_t\}$ is a stationary series (more strictly, a linear series, see e.g., Chen 1988a,b) with mean zero, i.e., $\{W_t\}$ is a sinusoid interrupted by noise $\{U_t\}$. The sinusoid completes the periodic movement in the period of $2\pi/\omega_0$, but we cannot usually see it without the help of statistical tools, so we say that $\{W_t\}$ has a *hidden periodicity*. Correspondingly, ω_0 is called a *hidden frequency* and $\rho\cos(\omega_0 t + \phi)$ is a *hidden sinusoid*. For example, in a series of quarterly observations there is a hidden sinusoid of periodicity 120 (120/4=30 years), so $\omega_0 = 2\pi/120 = 0.05236$, since

$$\cos(0.05236(t+120)+\phi) = \cos(0.05236t + 2\pi + \phi) = \cos(0.05236t + \phi).$$

In fact, our general theory may cope with multiple hidden frequencies, i.e., several terms of sinusoids in (3.1) with different values of ρ, ω_0 and ϕ. Here we mention only the simplest case.

Statistical theory and approaches developed for testing the existence of hidden periodicities have been around a long time. The traditional way in time series analysis is based on obtaining suitable test statistics from the *periodogram ordinates.*

Suppose $W_1, \ldots, W_n$ are observations, then we may define their *periodogram* by

$$I(\omega) = \frac{1}{2\pi}\{(\sum_{t=1}^{N} W_t\cos\omega t)^2 + (\sum_{t=1}^{N} W_t\sin\omega t)^2\}. \tag{3.2}$$

Then the set of statistics, $I(\omega)$ at $\omega = 2\pi j/N$, $j = 1, \ldots, [N/2]$, are defined as the periodogram ordinates. $[N/2]$ denotes the integer part of $N/2$. If W_t follow (3.1) and ρ is relatively large (the amplitude of the sinusoid is not too small compared with the variation of $U(t)$), then $I(\omega)$ should have a peak around ω_0 (in physics, this is the resonance phenomenon). Hence one of the periodogram ordinates with ω closest to ω_0 will take a larger value than others. How large is this value to suggest the existence of a hidden frequency ω_0 in the vicinity of this $2\pi j_0/N$? When $\{U_t\}$ is a normal white noise, Fisher (1929) derived the distribution of

$$g = I(2\pi j_0/N)/\{\sum_{l=1}^{[N/2]} I(2\pi l/N)\} \tag{3.3}$$

under the null hypothesis $\rho = 0$ (no hidden periodicity). Then, g exceeding the critical value of the test, say 0.05 upper percentile of the distribution, suggests the existence of a hidden frequency close to $2\pi j_0/N$.

In practice, $\{U_t\}$ is rarely normal white noise but more likely a series following an ARMA model. A general introduction to the further development along Fisher's approach may be found in Priestley (1980). The disadvantages of these type of procedures are discussed by Chen (1988a,b).

Based on the result of the divergence rate of $\max_{0\leq\omega\leq\pi} I(\omega)$ (An et al. 1983), Chen (1988a) proposed the following set of statistics

$$\hat{Z}_j = I(\pi j/N)/\{\hat{f}(\pi j/N)\log N\}, \qquad j = 1, 2, \ldots, N. \tag{3.4}$$

Where $\hat{f}(\pi j/N)$ is a specially designed estimate of the spectral density $f(\omega)$ of $\{U(t)\}$ at $\omega = \pi j/N$. This estimator has the ability to eliminate the effect of a potential hidden frequency existing in the vicinity of $\pi j/N$. The advantages of using this set of statistics and the convergence or divergence properties are discussed in Chen (1988a); here we explain these in a pragmatic way and show how to use them.

When j runs from 1 to N, $\pi j/N$ runs from nearly 0 to π and uniformly scatters along $(0, \pi]$. The larger the N, the denser the $\pi j/N$ and the more $\pi j/N$ fall in a given neighbourhood of ω_0. Then $\hat{Z}_j$ have the following properties: some $\hat{Z}_j$ with $\pi j/N$ inside a neighbourhood of ω_0 must be large; the one or two with $\pi j/N$ being closest to ω_0 could be as large as we wish provided that N is large enough. On the other hand, those $\hat{Z}_j$ with $\pi j/N$ outside the neighbourhood of ω_0 mentioned above are dominated by $1 + \epsilon$ provided that N is large enough; where $\epsilon > 0$ is any small number given a priori. That means, if N is large enough, we may always detect a neighbourhood of ω_0 by using $1 + \epsilon$ as a "threshold". For more details of the procedure which has been programmed, see Chen (1988a).

Now consider the following example. The top plot of Fig.3.1 is a sinusoid, as the first term on the right of (3.1) has $\rho = 1/2$, $\omega_0 = 0.125664$, $\phi = 0$. The middle plot is a stochastic series following an MA(2) model: $U_t = e_t - 0.75e_{t-1} + 0.125e_{t-2}$, where $\{e_t\}$ is a normal white noise series of mean zero and variance 1. The bottom plot is $\{W_t\}$, the sum of the top two, from which it is difficult to see the hidden sinusoid with the naked eye.

The upper plot of Fig.3.2 is the periodogram of $\{W_t\}$ (plot of $I(\omega)$ only at $\omega = \pi j/N$, $N = 200$, $j = 1, 2, \ldots, 200$). There is a peak at $j = 8$ which is created by the mentioned sinusoid (ω_0 is in the vicinity of $\pi 8/200$). But there is an even higher peak at $j = 154$ due to the irregularity of the periodogram of $\{U_t\}$ (in this example, the spectral density of $\{U_t\}$ is a higher level at high frequencies, so the periodogram is also more irregular on the high frequency side). Therefore, using statistics like (3.3), we may not reach the right conclusion.

The lower plot of Fig.3.2 is the $\hat{Z}_j$-statistics defined in (3.4). Only $\hat{Z}_8 = 7.55$ and $\hat{Z}_9 = 2.16$ are much larger than 1, all other $\hat{Z}_j$ are less than or at most about 1. If we had chosen, say $\epsilon = 0.1$, i.e. a "threshold" of 1.1, then we may conclude that there is a hidden frequency in the vicinity of $\pi 8/200$, and use $\pi 8/200$ as an initial estimate of ω_0.

Having located a neighbourhood (i.e., the vicinity of $\pi j_0/N$) of an apparent frequency, there are some methods which can offer more precise final estimates of the hidden frequency than the initial estimate which will be used in this paper later. One is by *maximizing periodogram* $I(\omega)$ in this neighbourhood, i.e., calculate $I(\omega)$ in a much finer lattice, then find the ω for the maximum and make this ω the final estimate. This method will be referred to as MP in this paper. Another method for obtaining a final estimate is called *secondary analysis* (denoted as SA), which is quite an old method and has a hidden fault in the original procedure. Chen (1988b) revised the procedure. It was proved by Hannan and Mackisack (1986) and by Chen (1988b), that for either MP or SA, the precision could reach the order of $(N^{-3} \log\log N)^{1/2}$. In practice, $(\log\log N)^{1/2}$ is just slightly greater than 1 (say, even $N = 1000$, $(\log\log N)^{1/2}$ is just 1.39). So basically, the precision of these final estimates is of order $N^{-3/2}$. This could be a great improvement over the initial estimate $\pi j_0/N$ which only guarantees an order of N^{-1} (usually the error does not go beyond π/N). For example, if $N = 100$, then MP or SA may offer estimates with an error as small as a few multiples of 0.001 while the initial estimate usually has errors not beyond 0.01π.

4 The Model for Fitting the data

As it was mentioned, we try to use model (1.2) to fit GDP data. It is true that, the regression on a specified family of analytical functions can often lead to worse results than stochastic modelling or smoothing procedures if the specification does not capture the right pattern of the trend or there is no pattern for the trend. This danger is mentioned by many authors. We certainly cannot guarantee that (1.2) works well for all macroeconomic data and for any given time duration. However, we will see that it does work well for the GDP data in our study. Once the regression fitting is good, the analytical function reveals the intrinsic structures and properties of long term movements of the series in a clear and simple manner which may be easily used in describing the economic laws and in forecasting.

The major problem in fitting data with (1.2) is detecting the sinusoid term $\rho\cos(\omega_0 t + \phi)$ and estimating ω_0 after this sinusoid has been detected. We cannot directly apply the methods in Section 3 to (1.2) because there is an extra quadratic term $a_0 + a_1 t + a_2 t^2$ in (1.2), while in (3.1) the deterministic part is only a sinusoid. This indicates that some preliminary treatment is necessary for Y_t.

Again let Δ denote the forward-differencing operator and apply it to (1.2). First consider the sinusoid term. Using $\mathcal{R}\{x\}$ to denote the real part of a complex number x, we have

$$\Delta\{\rho\cos(\omega_0 t + \phi)\} = \mathcal{R}\{\Delta(\rho e^{i(\omega_0 t+\phi)})\} = \mathcal{R}\{\rho e^{i(\omega_0 t+\phi)}(e^{i\omega_0} - 1)\}, \qquad (4.1)$$

formally, which can be written as

$$\Delta\{\rho\cos(\omega_0 t + \phi)\} = \rho'\cos(\omega_0 t + \phi'), \qquad (4.2)$$

where ρ' and ϕ' are constants, $\rho' > 0$, $-\pi < \phi' \leq \pi$. We see that the differencing operator changes the amplitude and the phase, but keeps the same frequency. If there is a hidden periodicity of frequency ω_0 in a series, then a hidden periodicity of the same frequency ω_0 exists in the differenced series.

When ω_0 is close to 0, then $e^{i\omega_0} - 1 \simeq i\omega_0 = \omega_0 e^{i\pi/2}$, and from (4.1) we have

$$\Delta\{\rho\cos(\omega_0 t + \phi)\} \simeq \omega_0\rho\cos(\omega_0 t + \phi + \pi/2). \qquad (4.3)$$

In this case the amplitude ($\rho' \simeq \omega_0 \rho$) of the sinusoid is dramatically reduced by differencing and hence if we can detect a hidden frequency ω_0 close to 0 in the differenced series, then we are very confident of the existence of a hidden periodicity (of the same frequency ω_0) in the original series and ρ should be much larger than the variation of the noise. Next, we will see the real reason of investigating the hidden frequency with $\{\Delta Y_t\}$ rather than with $\{Y_t\}$ itself.

Sometimes there is a situation where we cannot detect the hidden frequency ω_0 close to zero in the differenced series because $\rho' \simeq \omega_0 \rho$ becomes too small. For Canadian GDP data we will also face such a situation and a method of how to deal with it will be provided.

Differencing the quadratic term in (1.2) with $\Delta t^2 = (t+1)^2 - t^2 = 2t + 1$ in mind, we obtain

$$\Delta(a_0 + a_1 t + a_2 t^2) = (a_1 + a_2) + 2a_2 t. \tag{4.4}$$

Then (1.2), (4.2) and (4.4) give

$$\Delta Y_t = (a_1 + a_2) + 2a_2 t + \rho' \cos(\omega_0 t + \phi') + \Delta C_t. \tag{4.5}$$

Let

$$\overline{\Delta Y} = \frac{1}{N-1} \sum_{t=1}^{N-1} \Delta Y_t, \qquad W_t = \Delta Y_t - \overline{\Delta Y}, \tag{4.6}$$

then $a_1 + a_2$ is eliminated from ΔY_t. We may assume ΔC_t has mean zero, hence

$$\frac{1}{N-1} \sum_{t=1}^{N-1} \Delta C_t \simeq 0, \tag{4.7}$$

and W_t obtained in this way can be approximately expressed as

$$W_t = \mu + 2a_2 t + \rho' \cos(\omega_0 t + \phi') + \Delta C_t, \tag{4.8}$$

where

$$\mu = -\frac{\rho'}{N-1} \sum_{t=1}^{N-1} \cos(\omega_0 t + \phi') - a_2 N. \tag{4.9}$$

Comparing (4.9) with model (3.1), we see that there is still an extra linear trend $\mu + 2a_2 t$ in (4.8). Certainly $\mu + 2a_2 t$ may have some effect on the sensitivity in detecting ω_0 and on its final estimate when we use the methods described in Section 3. Therefore one may try to reduce it further. Using the differencing operator to the right of (4.8), i.e., differencing Y_t two times, and ignoring $2a_2$ (since it is usually very small), we get $\rho'' \cos(\omega_0 t + \phi'') + \Delta^2 C_t$, which is exactly like (3.1). Unfortunately, $\rho'' \simeq \omega_0^2 \rho$ is too small, thus we cannot detect the hidden periodicity.

It is difficult to transform (1.2) exactly to the form of (3.1) and also to be efficient in detecting and estimating the hidden periodicity. However in the next section we will see that by simply using W_t in (4.6), i.e., ignoring $\mu + 2a_2 t$ in (4.8), we can get satisfactory results for GDP at current prices and for implicit price index. The error from estimating ω_0 in this way does not affect the final results of the regression fitting and the residuals.

For GDP at constant prices, we cannot detect any hidden periodicity from the differenced data ΔY_t, because ρ is already small and hence $\rho' = \omega_0 \rho$ is almost 0. As $a_0 + a_1 t + a_2 t^2$ is now the dominant component of the trend (the sinusoid component is small), then initially remove this part by quadratic regression (denote the estimated parameters by $\hat{a}_j$, $j = 0, 1, 2$). Let

$$W_t = Y_t - (\hat{a}_0 + \hat{a}_1 t + \hat{a}_2 t^2), \tag{4.10}$$

and apply the methods in Section 3 to W_t to estimate ω_0. Now

$$W_t = \eta_t + \rho \cos(\omega_0 t + \phi) + C_t \tag{4.11}$$

and

$$\eta_t = (a_0 - \hat{a}_0) + (a_1 - \hat{a}_1)t + (a_2 - \hat{a}_2)t^2 \tag{4.12}$$

are small, like $\mu + a_2 t$ in (4.8), and the effect of η_t on estimating ω_0 is negligible.

It is clear that when an estimate of ω_0 has been obtained, then Y_t can be fitted by the regression model (1.2) using a linear regression procedure, where the sinusoid term in (1.2) should be expressed as (1.3) in order to avoid the nonlinear parameter ϕ.

5 Long-Wave Cycles in Canadian GDP

Now we demonstrate the results of applying the previously mentioned methods to GDP for Canada. The details and the source of the data have been explained in

Section 1. First, let us look at what happens if the sinusoid term is not introduced. Fig.5.1 (a) is a fitted quadratic trend, i.e., $a_0 + a_1 t + a_2 t^2$, to the logarithm of GDP at constant prices. At a glance, this trend appears acceptable, but if we observe the residuals of the regression Fig.5.1 (b), i.e., the deviation of the data from this trend, we find that the level drifts. It would be very misleading if we attempted to use Fig.5.1 (b) to investigate the business cycles: a "very severe" recession appears around the late 50s, but many really severe recessions are flattened. Certainly, the figure does not demonstrate the right pattern and amplitude of the business cycles.

For simplicity of notation, from now on, we use GDPC to denote GDP at current prices, reserve GDP for GDP at constant prices (the real GDP) and use IPI to denote implicit price index. Then at time t, the relationship between them is

$$(GDP)_t \times (IPI)_t/100 = (GDPC)_t. \tag{5.1}$$

The range of the data and the correspondence between t, $(t = 1, 2, \ldots, 186)$ and actual *quarter/year* have been explained in Section 1. $(IPI)_t$ is 100 in the middle of 1986. $(GDP)_t$ and $(GDPC)_t$ are measured in millions of dollars.

It is conceivable that the hidden periodicities as well as the quadratic functions depend on λ, the parameter of the transformation (2.2) or (2.3), as the transformations dramatically change the plotting patterns of the data. Table 5.1 are the results using the methods introduced in Section 3 to $\Delta Y_t = Y_{t+1} - Y_t$, where $Y_t = X_t^\lambda$ for $\lambda = 1$ or the "optimal" choices referred to in Section 2, or $Y_t = \log X_t$ for $\lambda = 0$, and X_t are the original GDPC or IPI series. As was pointed out in Section 4, since the hidden periodicities of such low frequencies are found in the differenced series with maximum values $\hat{Z}_j$ significantly greater than 1, we are extremely confident of the existence of hidden periodicities in all these cases.

As a function of j, $\hat{Z}_j$ reaches its maximum at $j_0 = 2$ (when $\lambda \neq 0$) and $j_0 = 3$ (when $\lambda = 0$), which gives the initial estimate of ω_0, i.e. $\pi 2/185 = 0.0340$ and $\pi 3/185 = 0.0509$ respectively (the corresponding periods are 46.20 and 30.86 years respectively). The columns of $M = 3$, $M = 2$ and MP in Table 5.1 list the different final estimates of ω_0 and the corresponding periods p (in years) by using different methods. Those under $M = 3$ (or $M = 2$) are the results of using SA and choosing 3 (or 2) for parameter M in the procedure, where M is an integer not less than 2 (see Chen 1988b) and small values are preferred. Those under MP are the results of using MP.

From Table 5.1 we see that periodicities are not the same for different λ, since data transformation changes the shape of trends. The periods become shorter as $\lambda \to 0$. In the cases of $\lambda = 0$ or λ being "optimal", different methods give closer final estimates, while in the case of $\lambda = 1$, the discrepancies between the estimates from using different methods increase, especially for GDPC. This was not a surprise, as from the top plot of Fig.2.1 we see that in this case ΔY_t are far from being stationary, and hence all the estimation procedures work under "difficult conditions". Nevertheless, we will see that in any case, the discrepancy between these estimates of ω_0 does not affect the final regression results and business cycles.

Having an estimate for ω_0, the estimates of linear parameters a_0, a_1, a_2, ρ_c and ρ_s in model (1.2) with relation (1.3) can be obtained. The software package SAS was used to obtain OLS estimates for these parameters. Table 5.2 shows these estimates and further results of analysis which we will explain later. Here, all the estimates of ω_0 are obtained by using SA with $M = 3$, except for GDPC where the results based on estimating ω_0 by SA with $M = 2$ are also listed for the purpose of comparison.

The OLS estimation procedure gives ρ_c and ρ_s by using SAS, but instead of listing them, the more meaningful $\rho = (\rho_c^2 + \rho_s^2)^{1/2}$ and $\phi = \arctan(-\rho_s/\rho_c)$ (see (1.4)) are listed in Table 5.2. σ is the standard error of the regression which is given directly from SAS. If we use ρ/σ as "signal to noise ratio" to measure the size of the long-wave movement compared with the size of business-cycle movement, then these numbers for GDPC and for IPI are very large; this offers strong evidence for the existence of long-wave cycles in these series.

As we use the quadratic function $a_0 + a_1 t + a_2 t^2$ to model the monotonically increasing part of the trend, then $a_2 t^2$ represents the non-linearity of this part. Notice that the divergence of $a_2 t^2$ from the linearity should be $a_2 t^2 - (a_0' + a_1' t)$ for a line $a_1' t + a_0'$ which is chosen "as close as possible" to $a_2 t^2$ in the range of t from 0 to N (where $N = 186$). Defining the distance of $a_0' + a_1' t$ from $a_2 t^2$ by $\max_{0 \le t \le N} \mid a_2 t^2 - (a_0' + a_1' t) \mid$, then the best a_0' and a_1' may be obtained by minimizing this distance. A simple and approximate solution for that is "half way" between a_2t^2 and the straight line which joins $(0, 0)$ and (N, a_2N^2), the two endpoints of $a_2t^2, 0 \leq t \leq N$. As this joint line has slope $(a_2N^2 - 0)/(N - 0) = a_2N$, i.e., the straight line is a_2Nt, then "half way" can be defined as

$$q = \max_{0 \leq t \leq N} \mid a_2t^2 - a_2Nt \mid /2 \tag{5.2}$$

The rows of q in Table 5.2 give these values. In fact, $a_2t^2 - (a_0' + a_1't)$ can also be regarded as a pattern modification to the sinusoid, i.e., we model the trend of the series as a line plus a long wave which has a pattern of sinusoid but modified by $a_2t^2 - (a_0' + a_1't)$.

Using q to measure the scale of this modification, for GDPC and IPI, q is always much smaller than ρ, when $\lambda = 0$, q is negligible. This indicates that in this case the pattern of long wave cycles is almost exactly sinusoid.

Fig.5.2 (a), (c) and (e), which correspond to $\lambda = 1$, 0.35 and 0, are the results of using model (1.2) to fit GDPC; while (b), (d) and (f) are the residuals to these fits (amplified tiny discrepancies of data from their regression curves); they are business-cycle movements. The parameters for these regression fits are in the columns GDPC ($M = 3$) of Table 5.2. To show how the sinusoid terms $\rho\cos(\omega_0t+\phi)$ and the quadratic terms a_2t^2 play a role in the regression, they are also overlayed in (a), (c) and (e) which can be easily identified.

Fig.5.2 (a) and (b) show that, in the case of $\lambda = 1$, the fit is not satisfactory. When t goes from 1 to about 100 (before 1970), we hardly see business-cycle movements from (b), and in (a) the data do not tightly twine around the regression curve as they do for t after about 100. This is because the variation of ΔX_t is very small for t from 1 to about 100 (see the top plot of Fig.2.1). In other words, the movement of X_t in this range is too smooth (compared with the movement later on) to show any evident peaks and troughs. A reason for not being tightly twined is that ordinary least squares regression treats all regression errors with the same weight; it does not pay attention to whether "the smoother part should fit better". This is a good example which shows the importance of data transformation. For $\lambda = 0.35$ and $\lambda = 0$, the regression fits the data well and from Fig.5.2 (d) and (f) we can clearly see the business-cycle movements over the whole period.

Fig.5.3 shows similar plots to those in Fig.5.2 but for IPI series with λ equal to 1, 0.5 and 0. The same situation occurs for $\lambda = 1$ though it is not as serious as GDPC case. Again this is because of the large increase of the variation of ΔX_t (see top plot of Fig.2.3). For $\lambda = 0.5$ and 0, the regression fitting is excellent. Again, in the case of $\lambda = 0$, $a_2 t^2$ almost becomes a straight line, so the long-wave cycle is almost exactly a sinusoid.

Comparing the corresponding items of GDPC ($M = 2$) with those of GDPC ($M = 3$), we see that for $\lambda = 1$ and 0.35, many estimated parameters are quite different, e.g. a_2 goes from negative to positive. However Fig.5.5 shows that despite the difference between these individual terms, the final regression fitting is almost the same as those in Fig.5.2 (a) and (c). If we graph the residuals corresponding to the plots in Fig.5.5, we obtain curves almost identical to Fig.5.2 (b) and (d). This demonstrates the important role of the "pattern modification" of $a_2 t^2 - (a'_0 + a'_1 t)$ to a sinusoid: for all reasonable estimates of ω_0, the corresponding sinusoids can be modified to almost the same long-wave cycles.

For GDP, when applying $\hat{Z}_j$-statistics to the differenced series (Fig.2.2), no hidden periodicity is detected whether λ equals to 1, 0.68 or 0. So, as was mentioned in the last section, first we fit Y_t with the quadratic regression model $Y_t = a_0 + a_1 t + a_2 t^2 + \tilde{C}_t$, obtain preliminary estimates of a_j ($j = 0, 1, 2$) as well as the residuals $\tilde{C}_t$ (for $\lambda = 0$, Fig.5.1 (b) is the plot), and then apply $\hat{Z}_j$-statistics to $\tilde{C}_t$. In this way, the hidden periodicity is detected for each λ. The maximum values of $\hat{Z}_j$, reached at $j_0 = 3$ for all $\lambda = 1$, 0.68 and 0, are 1.24, 1.82 and 1.84 respectively. They are greater than 1, but not as significant as those for GDPC. We substitute a final estimate ω_0 (again by SA with $M = 3$) in model (1.2) and then use it to fit Y_t to get other parameters. The results are listed in the columns of GDP ($M = 3$) in Table 5.2.

Fig.5.4 are the plots of the regression fitting and the residuals of transformed GDP with $\lambda = 1$, 0.68 and 0, where in (e), the plot of the regression fitting (not the curves of $a_2 t^2$ and the sinusoids) has been rescaled downward by 12 for better appearance; the original range is between 11 and 14.

A major difference between the regression results of GDP and GDPC is that the sinusoid components are much weaker. For different λ, the value of ρ/σ ranges from only 1.38 to 1.48; the q value is no longer much less than ρ, even larger than ρ in two cases. However by comparing Fig.5.4 (f) with Fig.5.1 (b), we can see a substantial improvement of the residuals since this sinusoid term is introduced in the regression model. We see that the drift of levels in Fig.5.1 has disappeared in Fig.5.4.

Another difference between the regression results of GDP and GDPC is that now the periodicity p and the phase ϕ of the sinusoid have almost the same values for different λ. (i.e. p around 32 years and ϕ around 0). The non-linear pattern change for different λ is taken care of by a_2t^2, which from a convex curve changes to a concave curve as λ changes from 1 to 0.

From the above discussion, we have the following conclusions: *strong long-wave cycles exist in the series of implicit price index; the real GDP series shows weak long-wave cycle movements; hence, the main source of the strong long-wave cycles in the series of GDP at current prices is implicit price index.*

6 Business Cycles

We do not intend to be involved in arguing the general definition of business cycles; in this paper it only means, ignoring small fluctuations, the movement from one trough to the next trough in the plot of a detrended series of GDP. In our case, a detrended series is the regression residuals of GDP data (after a transformation). GDPC's plots may be used as references. We mainly consider the logarithm of original data ($\lambda = 0$), which most economists prefer, but here there is a more important reason: using $\lambda = 0$ for all three series, a nice relationship, (6.1) below, exists. This will lead to a similar relationship for their residuals (see Fig.6.1), and from that we may get some interesting conclusions for business cycles. Also, for both GDP and GDPC, $\lambda = 0$ gives a better resolution of peaks and troughs over the whole period of data than using those "optimal" λ.

Four curves in Fig.6.1, from top to bottom, are as follows: residuals of GDP from Fig.5.4 (f) and of IPI from Fig.5.3 (f), then the sum of the two above and residuals of GDPC from 5.2 (f). In fact, all horizontal lines should be at level 0.0, but they are rescaled in Fig.6.1 for clarity by moving them up by 0.5, 0.3, 0.1 and 0.0 (no change for GDPC's) respectively. Ignoring the constant ($-\log 100$) which will not affect our discussion at all, the logarithm of (5.1) gives

$$\log(GDP)_t + \log(IPI)_t = \log(GDPC)_t. \tag{6.1}$$

Three of the plotted curves in Fig.6.1 are the residuals of the series in (6.1) after regression on a very smooth function (quadratic + sinusoid of long periodicity). From Fig.6.1 we see that the patterns of the GDPC's cycle movements are almost the same as the sum of GDP's and IPI's. Then we may analyse how the peaks and troughs of GDPC's cycles are generated from those of GDP's and IPI's as we will discuss later. This is one of the advantages of using such smooth trends in all these series. Stochastic trends cannot guarantee this property.

The vertical lines in Fig.6.1 indicate the locations of the troughs of the cyclical movement of GDP (the top curve); the dates (*quarter/year*) are marked on each line. Table 6.1 offers a clearer indication of the relationship of dates and phases between GDP, IPI and GDPC around the troughs of these cycles. The GDP columns and GDPC columns of Table 6.1 show the dates (*quarter/year*) of troughs of cycle movements. The corresponding numbers on the time axis of Fig.6.1 are listed to the right of the dates. Those *quarter/year* in brackets indicate that they are not deep troughs of the cycle movements according to the plots, so these should not be the troughs of business cycles (at least not serious) but only of the growth cycles (i.e. a slow down in the growth).

The middle column in Table 6.1 describes the phases of IPI's cycle movement around the corresponding troughs of GDP's cycles. An evident conclusion is that IPI's cycles do not coincide with the business cycles (or growth cycles), since the indicated phases could be very different.

Many occurrences of GDPC's troughs have moved from their corresponding GDP's troughs by IPI's deep troughs, perhaps quite far away (such as 1/49 and 1/71), and have been marked by "m" in Table 6.1.

When a trough of GDP's cycle is in a top area of IPI's cycles (such as 4/51, 1/68, 2/75 and 4/82), the corresponding trough of GDPC flattens. A fast downward phase of IPI's cycle may pull down the right side (say, 1/68) and a fast upward phase may then pull down the left side (say, 3/80) of a GDP's trough. These flattened troughs are indicated by "f" in Table 6.1.

Not many troughs of IPI's cycles coincide with troughs of GDP's cycles. Only 4/86 (a growth cycle in the GDP) is deepened, which in appearance is a "serious recession" from the GDPC's cycles (and is marked by "d" in Table 6.1). Certainly, some troughs of GDP's cycles are deepened and also moved by IPI's troughs (marked by "dm" in Table 6.1).

From the middle 50s to the late 60s, IPI slowly goes up with mild fluctuations. During this period only, GDPC shows a similar pattern of cycle movement as those of GDP. Perhaps, this is an indication of a good economic period.

Even though the plots in Zarnowitz (1992, Fig.6.2) are of the detrended series of the real GNP of USA, the GNP and GDP are quite similar (the difference consists of net interest and dividends paid abroad) and the economic situations in USA and in Canada are similar, therefore we may refer to those plots. From Fig.6.1 we can see except for the troughs at 1/58 and 2/75, they are not as deep as we expected, whereas all other above mentioned recessions were picked up in the right size. In general, it is superior to the UC detrended series.

A major difference of the top plot in Fig.6.1 from the plot of the phase-average detrended series in Zarnowitz, occurred during the 60s. In Fig.6.1, it drifted up and hence several growth cycles are flatter than the phase-average detrended series. But perhaps one may explain it as indicating that there were no recessions during this period.

Fig.6.2 shows the dates of peaks and troughs of business and growth cycles of Canada. The first rows of *month:year* which indicate either peaks or troughs in this chart are from Zarnowitz (1992, Fig.7.1), while the second rows of *quarter/year* are from our regression residuals of GDP. We see that they match quite well. Notice that a peak (trough) month nearby its corresponding peak (trough) quarter should still be recognised as "a good match".

A remark should be made that for GDPC, IPI and GDP, the "optimal" choice of λ gives the most uniform fluctuations of ΔY_t (see Fig.2.1, 2.2 and 2.3), but from the plots of the residuals in Fig.5.2, 5.3, and 5.4, it appears $\lambda = 0$ (not the "optimal" λ) gives the most uniform amplitude of the business cycles of Y_t. This phenomenon is due to the change of the correlation structure in ΔY_t. The positive correlation has become stronger in recent decades. So the fluctuations of the same levels in ΔY_t (with "optimal" λ) produce larger cycles of Y_t in recent decades. It seems that for GDP this skill of data transformation can help only in dealing with the stationarity of the variance of ΔY_t; when the correlation structure is involved, we should be careful. For example, it may be better to use only the residuals of recent decades in fitting a stochastic model for forecasting instead of using all the residuals. We do not pursue this topic further in this paper.

7 Conclusions

In this paper, we use quadratic-sinusoid function to model trends of Canadian GDP data, where the sinusoid components are detected and estimated by newly developed statistical approaches and the regression coefficients are estimated by ordinary least squares estimator. The results show that strong long wave cycles of periods of several decades exist in GDP at current prices and in implicit price index. Similar long wave cycles also exist in real GDP but they are much weaker. The periods of these cycles also depend on the data transformtion.

Using regression residuals of real GDP to represent the business-cycle movement for indicating recessions and booms in the economy, the results match the conclusion of economists well, and are competitive with the phase-average method which is currently producing trends for economic indexes by an international statistical agency.

The methodological advantage of the approach offered by this paper is that analytical smooth trends are obtained which can easily be used for describing the long-term situation of economic development and for forecasting. Also, we may analyse how implicit price index affects GDP at current prices through this approach.

Acknowledgements

Special thanks to M.Pascal who helped to identify the series from CANSIM data base of Statistics Canada. Many thanks to M.Morry and K.Findlay for valuable comments and suggestions. Thanks also to J.Higginson, P.Cholette, B.Quenneville and other colleagues for their help and support.

References

An, H.Z., Chen, Z.G. and Hannan, E.J. (1983). The maximum of periodogram. *J. of Multivariate Aalysis*, **13**, 383-400.

Chen, Z.G. (1988a). An alternative consistent procedure for detecting hidden frequencies. *J. of Time Series Analysis*, **9**, 301-317.

Chen, Z.G. (1988b). Consistent estimates for hidden frequencies in a linear process. *Adv. Appl. Prob.*, **20**, 295-314.

Crafts, N.F.R., Leybourne, S.J. and Mills, T.C. (1989). Trends and Cycles in British Industrial Production, 1700-1913. *J.Roy. Statist. Soc.*, **A 125**, 43-60.

Fisher, A.R. (1929). Test of significance in harmonic analysis. *Proc. Roy. Soc. London* **A**, 54-59.

Hannan, E.J. and Mackisack, M. (1986). A law of the iterated logarithm for an estimate of frequency. *Stoch. Proc. Appl.*, **22**, 103-109.

Harvey, A.C. (1989). *Forecasting, Structural Time Series Models and Kalman Filter*. Combridge University Press, Cambridge.

Kondratieff, N. (1984). *The Long Wave Cycle*. Translated by G.Daniels, introduced by J.M. Synder. Richardson & Snyder.

Nelson, C.R. and Plosser, C.I. (1982). Trends and random walks in macroeconomic time series. *J. of Monetary Economics*. **10**, 139-162.

Priestley, M.B. (1981). *Spectral Analysis and Time Series. Vol.1: Univariate Series*. Academic Press, London.

Stoken, D.A. (1978). *Cycles: What we are, What they mean, How to profit by them*. McGraw-Hill Book Company. New York.

Watson, M.W. (1986). Univariate detrending methods with Stochastic trends. *Journal of Monetary Economics*. **18**, 49-75.

Zarnowitz, V. (1992). *Business Cycles — Theory, History, Indicators, and Forecasting*. The University of Chicago, Chicago and London.

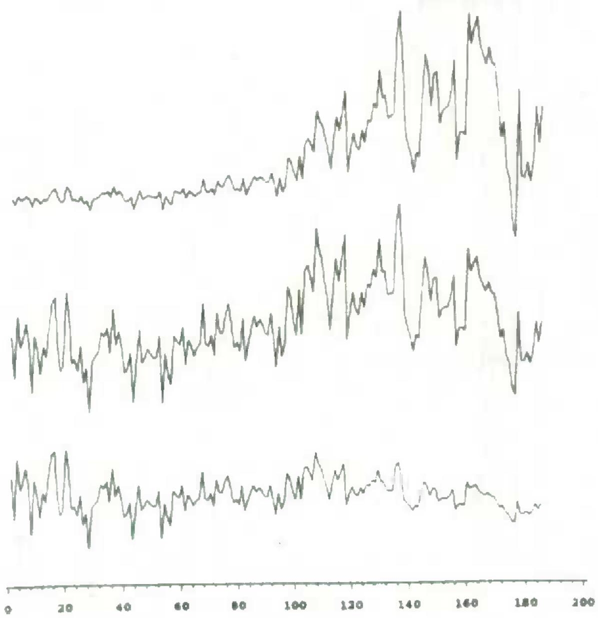


Fig.2.1 ΔY_t: GDP at current prices

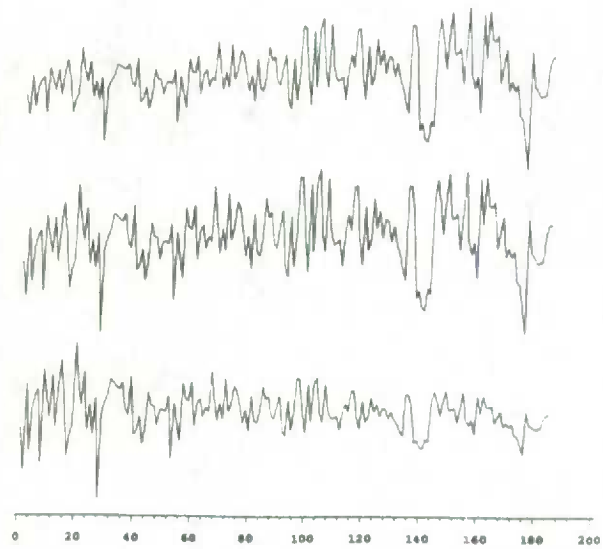


Fig.2.2 ΔY_t: GDP at constant prices

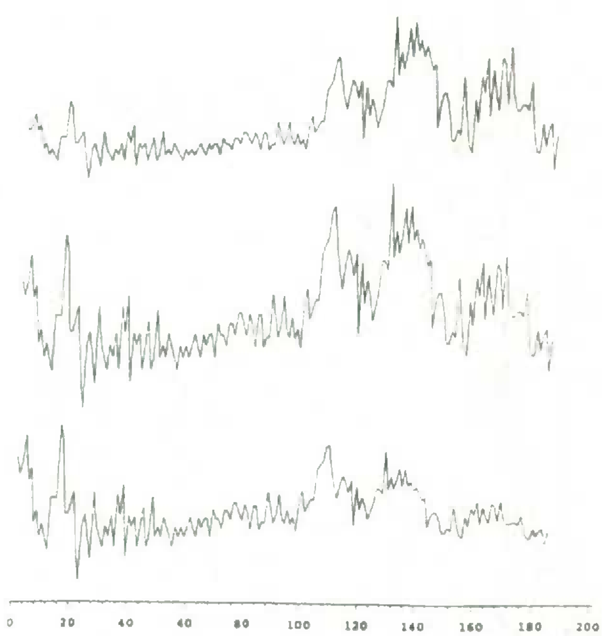


Fig.2.3 ΔY_t: implicit price intex

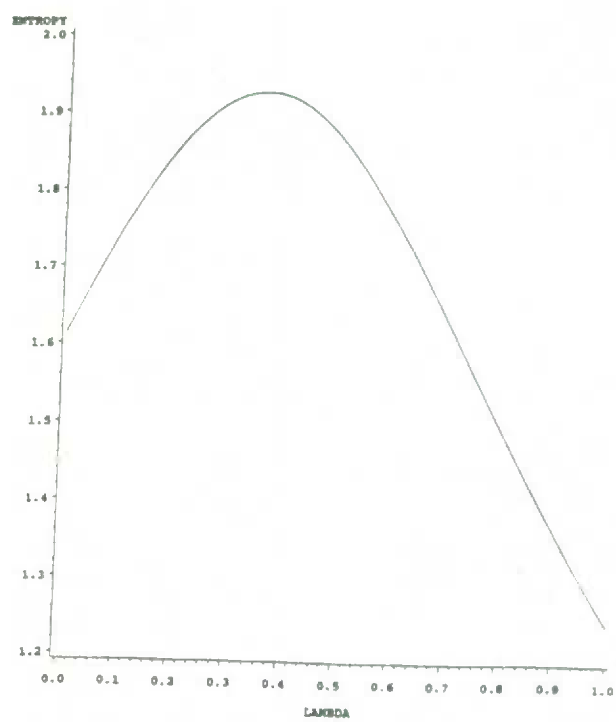


Fig.2.4 Entropy

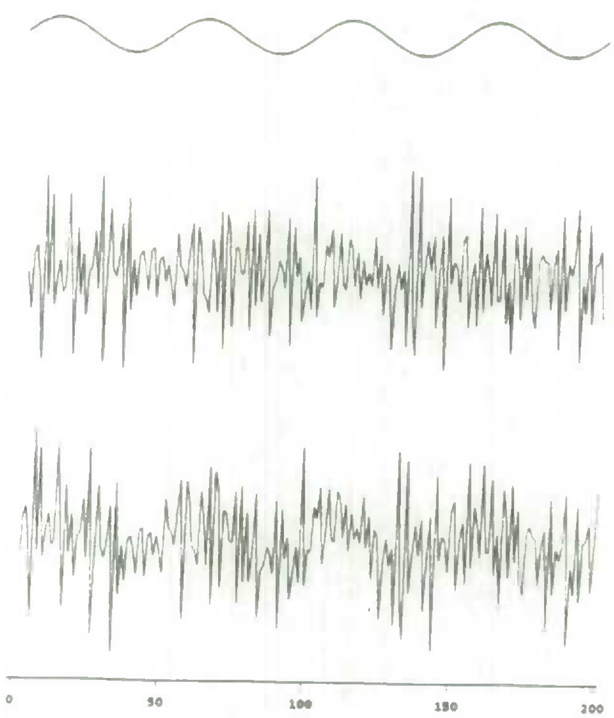


Fig.3.1 Sinusoid, noise and their sum

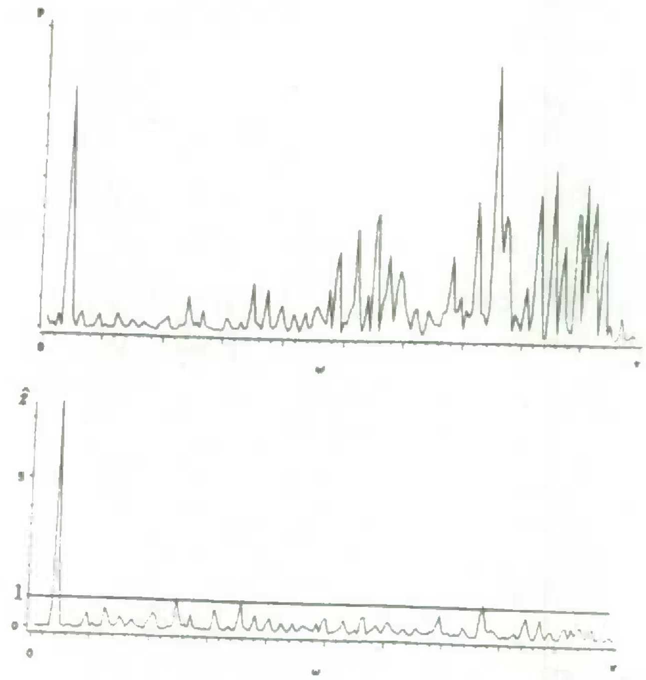
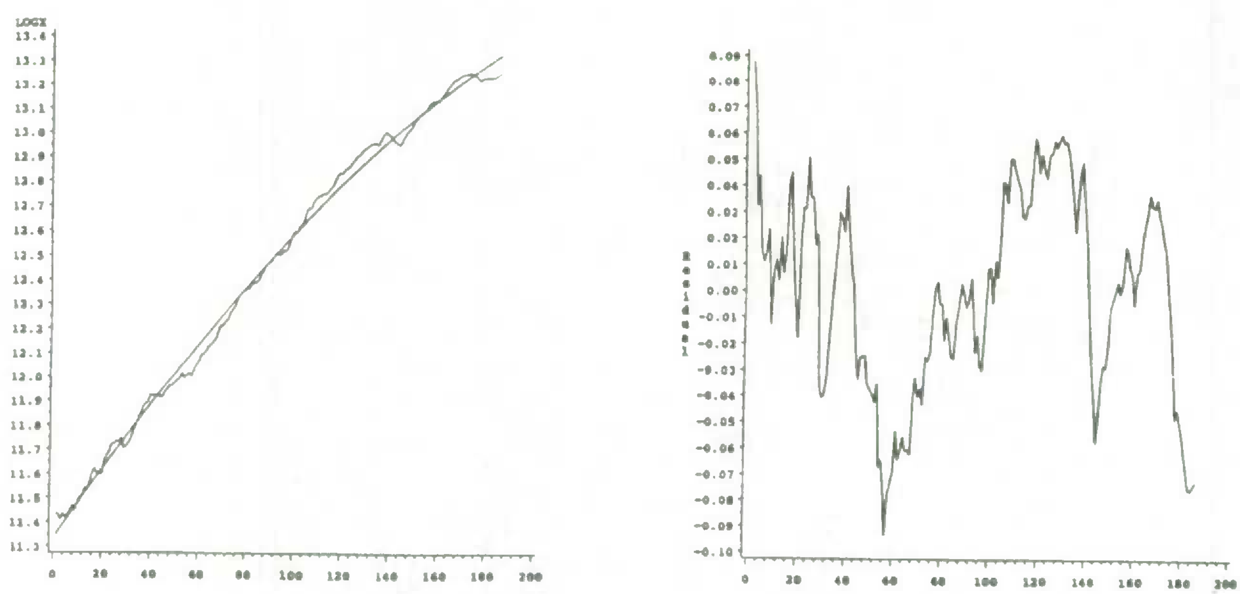


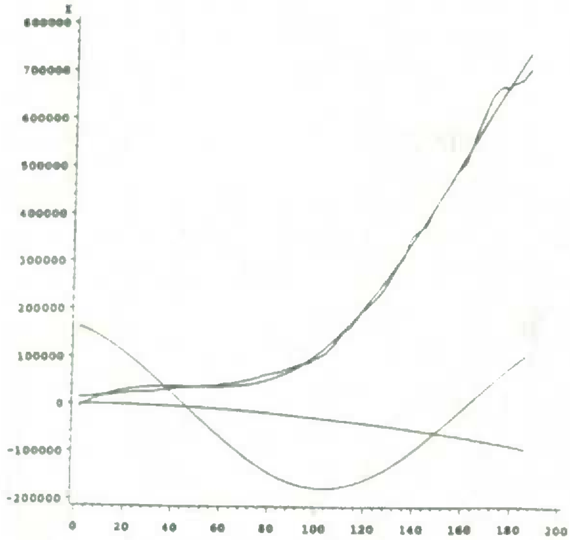
Fig.3.2 Periodogram and $\hat{Z}_j$-statistics



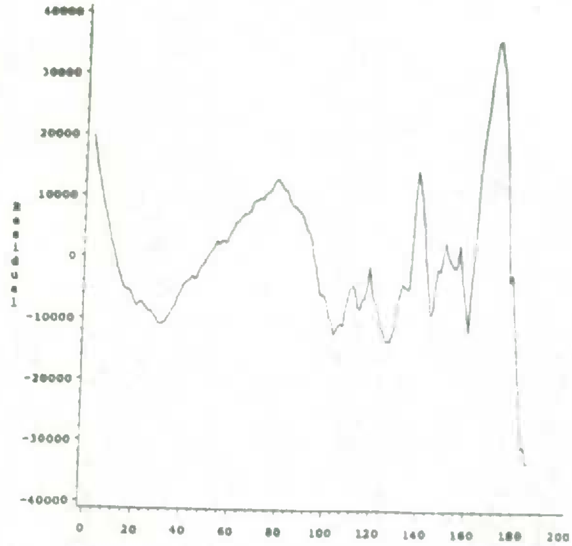
(a)

(b)

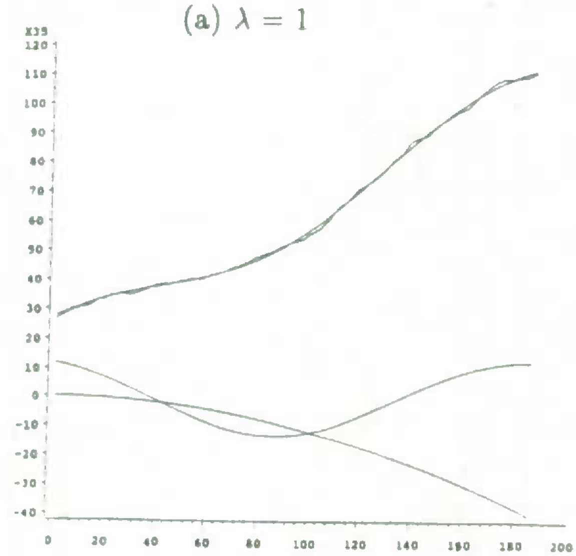
Fig.5.1 Quadratic regression fitting and residuals of real GDP



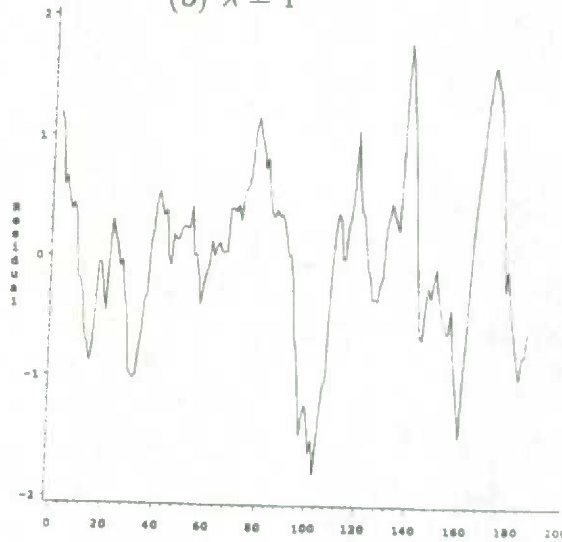
(a) $\lambda = 1$



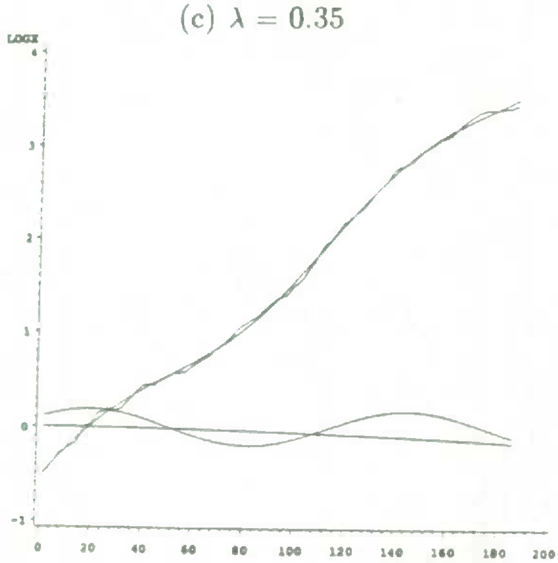
(b) $\lambda = 1$



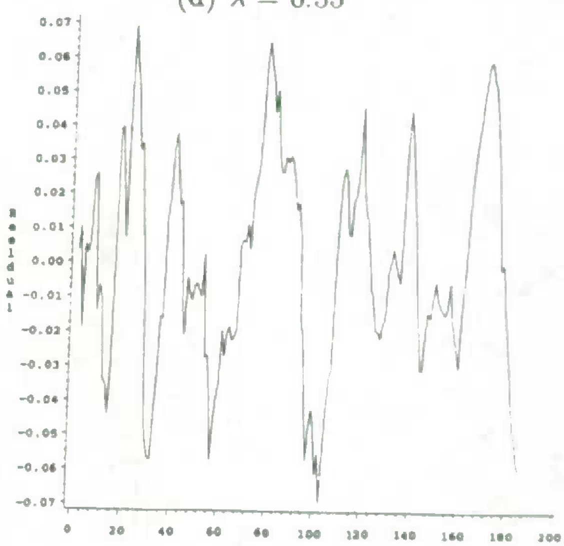
(c) $\lambda = 0.35$



(d) $\lambda = 0.35$

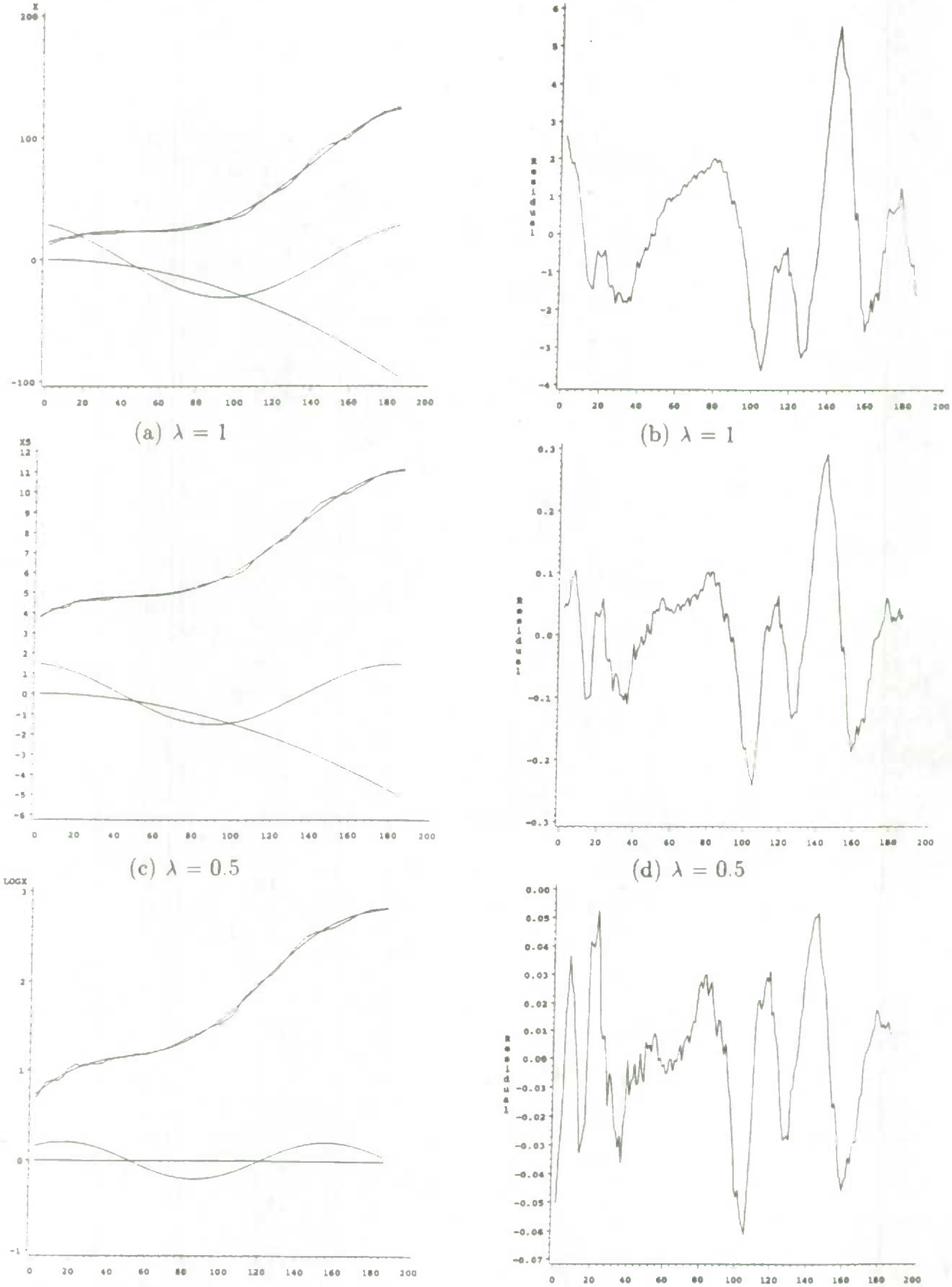


(e) $\lambda = 0$



(f) $\lambda = 0$

Fig.5.2 Regression fitting and residuals of GDPC

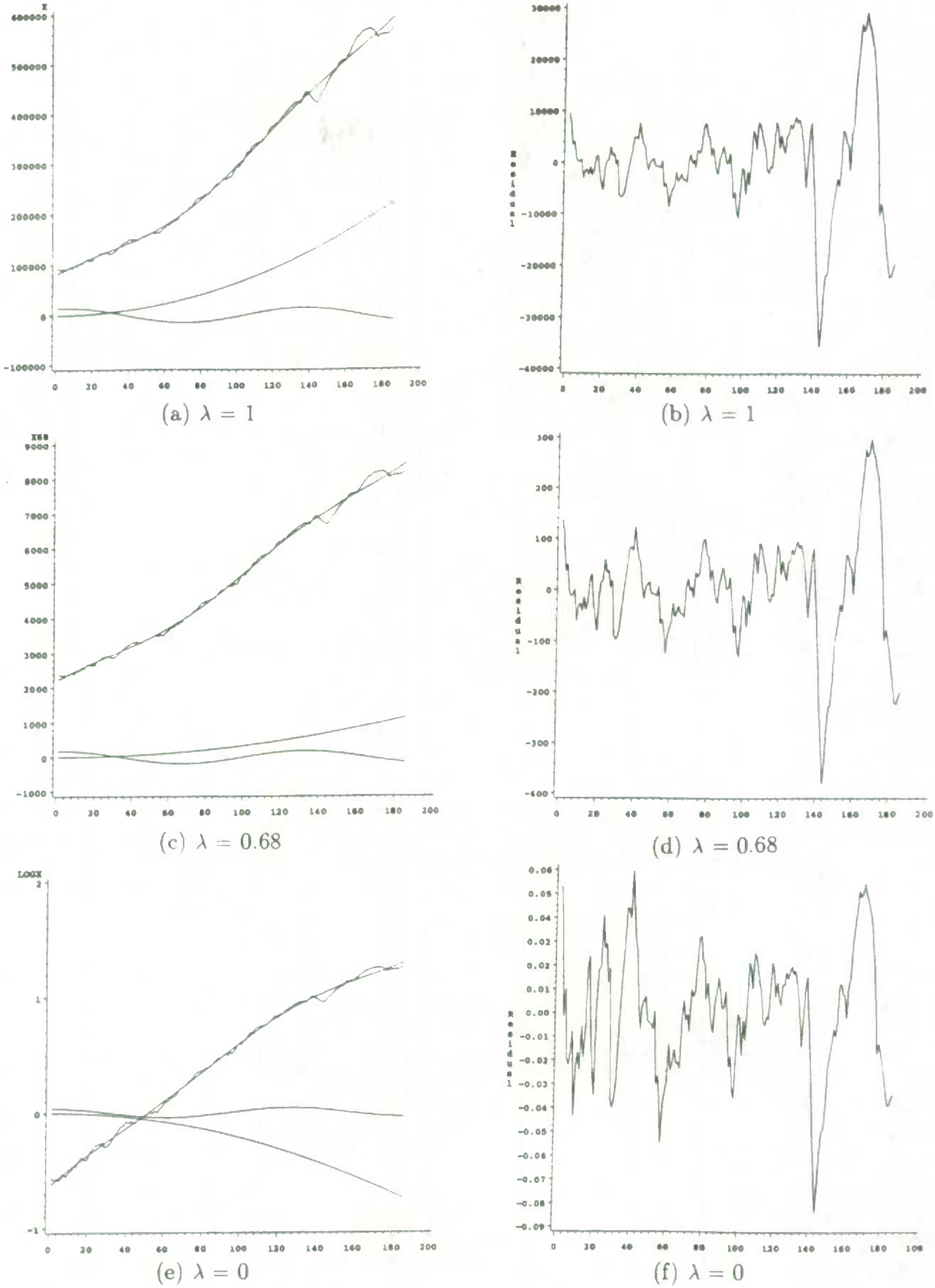


(a) $\lambda = 1$ (b) $\lambda = 1$

(c) $\lambda = 0.5$ (d) $\lambda = 0.5$

(e) $\lambda = 0$ (f) $\lambda = 0$

Fig.5.3 Regression fitting and residuals of the IPI

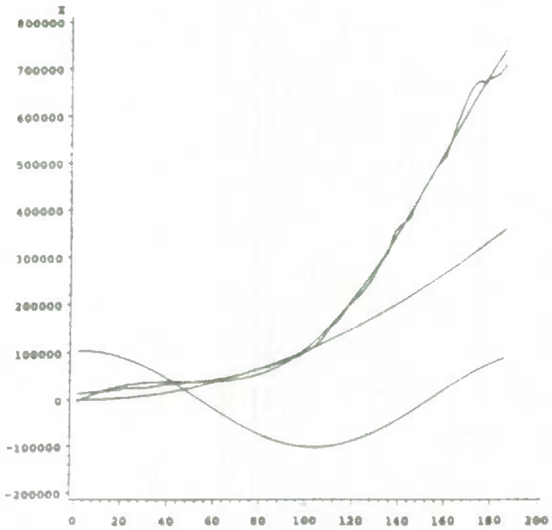


(a) $\lambda = 1$ (b) $\lambda = 1$

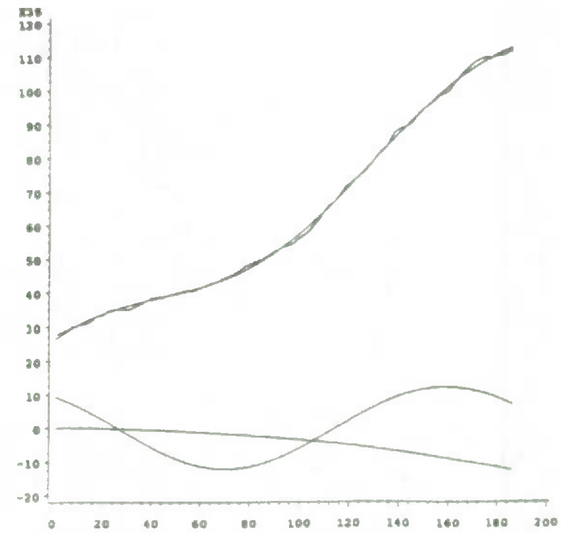
(c) $\lambda = 0.68$ (d) $\lambda = 0.68$

(e) $\lambda = 0$ (f) $\lambda = 0$

Fig.5.4 Regression fitting and residuals of GDP



(a) $\lambda = 1$



(b) $\lambda = 0.35$

Fig.5.5 Regression fitting of GDPC ($M = 2$, for comparision)

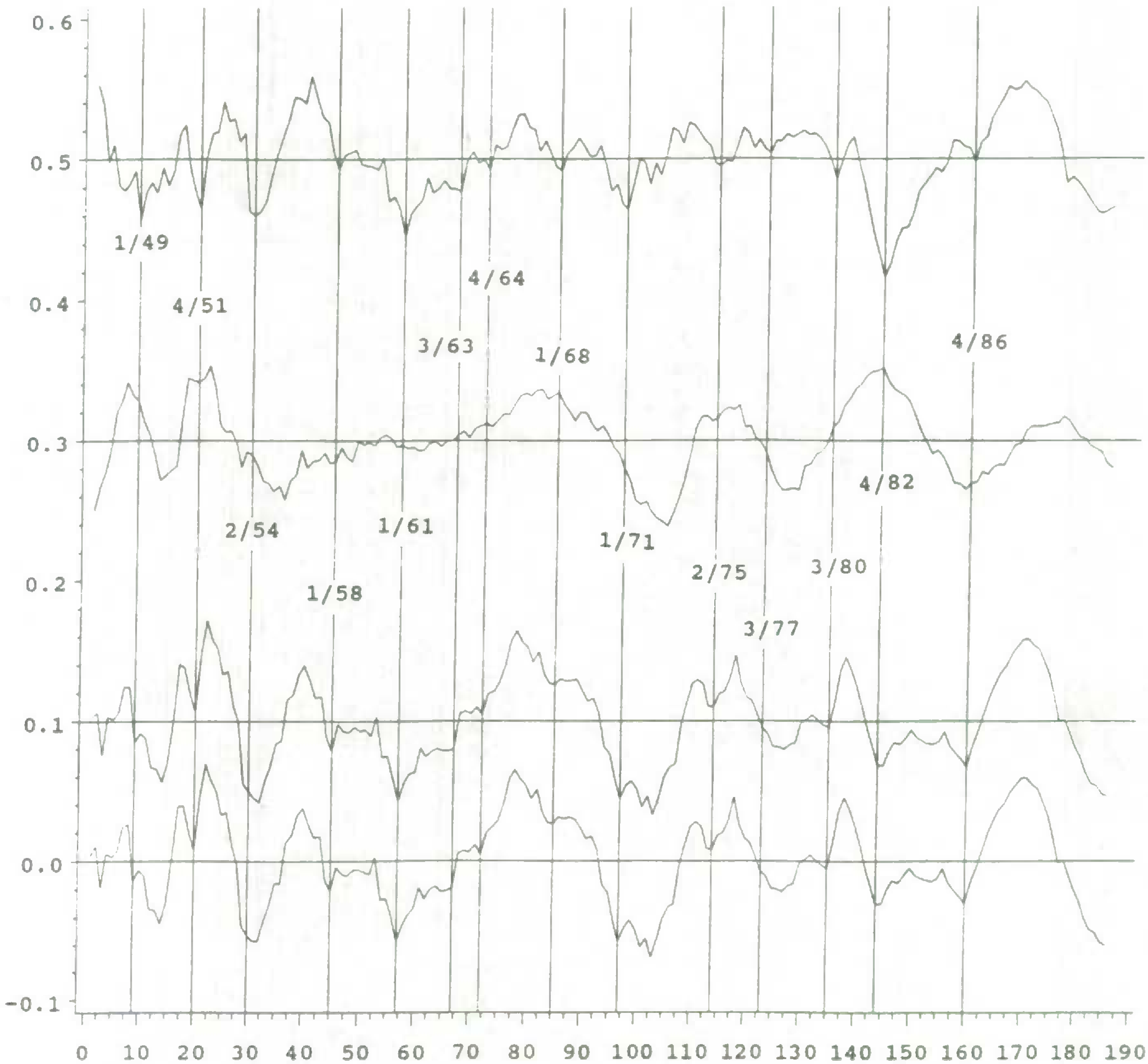
Table 5.1 Final estimates of ω_0 for GDPC and the IPI

	λ	$\hat{Z}_j$	$M = 3$		$M = 2$		MP	
			ω_0	p	ω_0	p	ω_0	p
GDPC	1	4.35	0.0269	58.39y	0.0317	49.55y	0.0258	60.85y
	0.35	5.79	0.0320	49.09y	0.0354	44.37y	0.0328	47.93y
	0	2.45	0.0504	31.17y	0.0509	30.86y	0.0515	30.53y
IPI	1	6.94	0.0294	53.26y	0.0342	45.92y	0.0292	53.78y
	0.5	4.70	0.0335	46.89y	0.0368	42.68y	0.0363	43.22y
	0	4.27	0.0452	34.75y	0.0499	31.48y	0.0487	32.23y

Table 5.2 Estimated parameters

	GDPC ($M = 3$)			GDPC ($M = 2$)		
λ	1	0.35	0	1	0.35	0
ω_0	0.0269	0.0320	0.0504	0.0317	0.0354	0.0509
p	58.39y	49.09y	31.17y	49.55y	44.37y	30.86y
ρ	175450	12.9885	0.1874	103462	9.6042	0.1849
ϕ	0.1172 π	0.1175 π	-0.3119 π	-0.0500 π	0.0272 π	-0.3608 π
a_0	-173347	13.9732	9.3597	-109398	16.5205	9.3663
a_1	4841.42	0.67309	0.023762	2168.91	0.53522	0.023522
a_2	-2.6724	-0.001154	-4.372×10^{-6}	10.3870	-0.000361	-2.927×10^{-6}
q	11556	4.9905	0.0189	44930	1.5611	0.0127
σ	11638	0.7097	0.0318	11358	0.7189	0.0321
ρ/σ	15.08	18.30	5.89	9.11	13.36	5.76

	IPI ($M = 3$)			GDP ($M = 3$)		
λ	1	0.5	0	1	0.68	0
ω_0	0.0294	0.0335	0.0452	0.0479	0.0479	0.0494
p	53.26y	46.89y	34.75y	32.79y	32.79y	31.80y
ρ	31.4526	1.5005	0.2039	14646	179.40	0.03938
ϕ	0.1014 π	0.0521 π	-0.2341 π	-0.0886 π	-0.0280π	0.0154 π
a_0	-18.1550	2.2233	2.5554	66686	2033.6	11.3253
a_1	1.13945	0.067039	0.011958	1690.60	29.025	0.014786
a_2	-0.002795	-0.000147	-1.84×10^{-7}	6.6027	0.033136	-2.153×10^{-5}
q	12.0870	0.6357	0.0008	27688	143.30	0.09311
σ	1.8863	0.1025	0.0250	10107	108.20	0.02675
ρ/σ	16.68	14.64	8.16	1.45	1.38	1.48



From top to bottom:
GDP at constant prices, moved up 0.5
implicit price index, moved up 0.3
the sum of above two, moved up 0.1
GDP at current prices

Fig.6.1 Business cycles, Canada 1/1947 -2/1993

```
4:51           3:53           11:56          10:59          3:62
1/51           2/53           4/56           4/59           3/62
  \_            _/ \_          _/ \_          _/ \_          _/ \_
    \_        _/     \_      _/     \_      _/     \_      _/     \_      _/
      \     /          \   /          \   /          \   /          \   /
      12:51          10:54          8:58           3:61           5:63
      4/51           2/54           1/58           1/61           3/63
```

```
   3:66           2:69           2:74           5:76           2:79
   2/66           4/68           4/73           2/76           1/79
    _/ \_          _/ \_          _/ \_          _/ \_          _/ \_
  /      \_      _/     \_      _/     \_      _/     \_      _/     \
           \   /          \   /          \   /          \   /
           2:68           12:70          10:75          7:77
           1/68           1/71           2/75           3/77
```

Fig.6.2 Peaks and troughs of business and growth cycles

Table 6.1 Troughs of business cycles

	GDP		IPI	GDPC	
m	1/49	9	down fast, 2/50 up	2/50	14
f	4/51	20	on peak area	4/51	20
m	2/54	30	down	4/54	32
	1/58	45	fluctuating	1/58	45
	1/61	57	fluctuating	1/61	57
*	(3/63)	67	fluctuating	(1/63)	65
	(4/64)	72	fluctuating	(4/64)	72
f	1/68	85	peak → down fast	(1/68)	85
dm	1/71	97	down fast	3/72	102
f	2/75	114	on peak area	1/75	113
dm	(3/77)	123	down fast	3/78	127
f	3/80	135	up fast	(3/80)	135
f	4/82	144	on peak area	4/82	144
d	(4/86)	160	on trough area	4/84	160

* The values of GDPC at 1/61 and 3/63 are very close.

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Appended figures from Zarnowitz (1992)

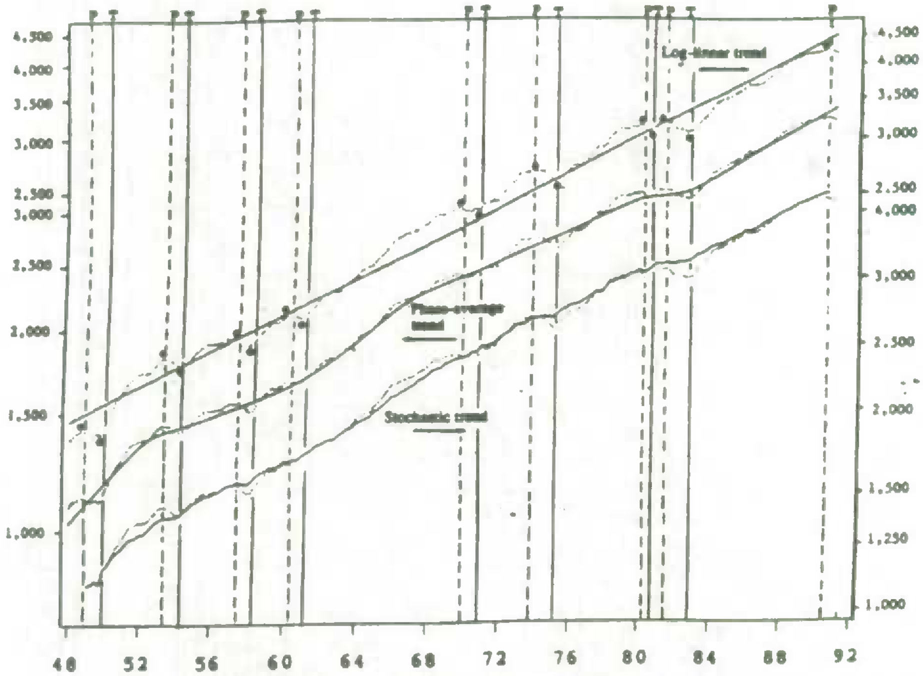

Fig. 6.1 Seasonally adjusted levels of real GNP: Specific cycles and two trend estimates, 1948–91
Note: Broken vertical lines represent business cycle peaks (P); solid vertical lines represent business cycle troughs (T). Dotted curves represent RGNP; solid curves represent trends in RGNP. Dots identify peaks and troughs of specific cycles in RGNP (shown for the top curve only).

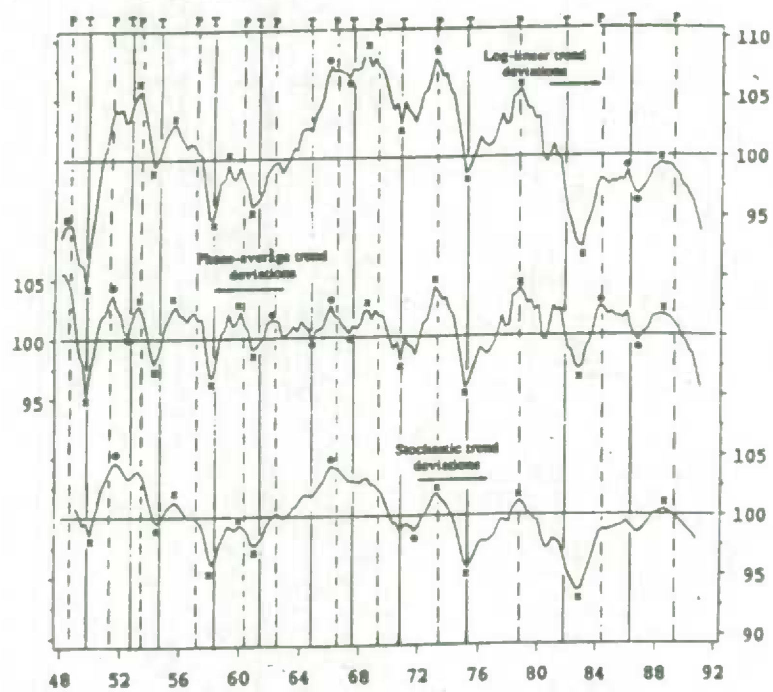

Fig. 6.2 Real GNP: Deviations from log-linear trend (A) and from phase-average trend (B), 1948–85
Note: Trend level = 100. Broken vertical lines represent growth cycle peaks (P); solid vertical lines represent growth cycle troughs (T). Dots identify peaks and troughs of specific cycles; crosses identify those turning points that match growth cycle peaks or troughs.