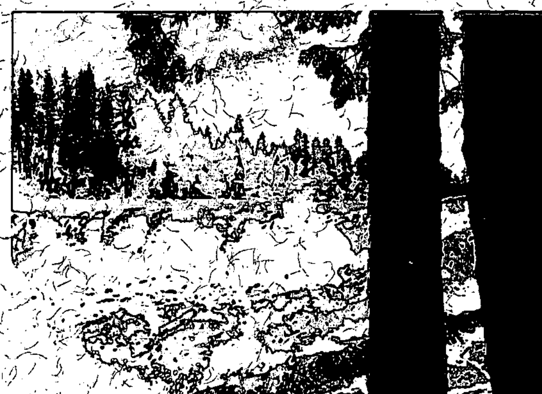
Environment Canada — Environnement Canada

Conservation and Protection — Conservation et Protection

On the Dynamics of Phosphorus in Lake Systems

Bernard C. Kenney



NHRI PAPER NO. 45

IWD SCIENTIFIC SERIES NO. 182

INLAND WATERS DIRECTORATE
NATIONAL HYDROLOGY RESEARCH INSTITUTE
NATIONAL HYDROLOGY RESEARCH CENTRE
SASKATOON, SASKATCHEWAN, 1990

(Disponible en français sur demande)

Canada

Environment Canada
Conservation and Protection

Environnement Canada
Conservation et Protection

On the Dynamics of Phosphorus in Lake Systems

Bernard C. Kenney

NHRI PAPER NO. 45

IWD SCIENTIFIC SERIES NO. 182

INLAND WATERS DIRECTORATE
NATIONAL HYDROLOGY RESEARCH INSTITUTE
NATIONAL HYDROLOGY RESEARCH CENTRE
SASKATOON, SASKATCHEWAN, 1990

(Disponible en français sur demande)

Canada

Printed on paper that contains recovered waste

Published by authority of
the Minister of the Environment

Cat. No. En 36-502/182E

ISBN 0-662-18283-9

Contents

Table

Illustrations

Abstract

The application of first-order linear dynamics to predict eutrophication in lakes is reviewed within the framework of the phosphorus loading concept. Lack of understanding of the sedimentation term and use of the steady state assumption have impeded predictions in the past.

A model of the dynamics of total phosphorus in lakes is derived which shows that lakes respond as a forced system to changes in inflow phosphorus concentration. Although the concentration of phosphorus in a lake is shown to scale with the inflow concentration, the time dependence of flushing and sedimentation is important to successful modelling.

Résumé

L'application d'une dynamique linéaire du premier ordre servant à la prévision de l'eutrophisation des lacs est étudiée en fonction de la charge en phosphore. Une mauvaise compréhension du terme de la sédimentation et l'utilisation de l'hypothèse du régime permanent ont gêné les prévisions par le passé.

Un modèle de la dynamique du phosphore total d'un lac est établi. Ce modèle montre que les lacs réagissent comme un système forcé aux variations de la concentration en phosphore du débit entrant. Bien que la concentration en phosphore d'un lac corresponde à celle du débit entrant, il est montré qu'il importe de cerner les caractéristiques de la vidange et de la sédimentation afin de bien modéliser la dynamique.

Management Perspective

Many lakes in Canada experience large blooms of algae during the summer months that degrade water quality and limit potential uses. The algae thrive on high concentrations of phosphorus that often originate from municipal sewage effluents. Some understanding of how phosphorus behaves in lakes is necessary in order to evaluate the potential impact of improvements in sewage treatment.

In this report, the theory of phosphorus dynamics is derived for lakes controlled by flushing and sedimentation. The phosphorus loading concept is then reviewed. It is shown that the use of static models and a lack of understanding of phosphorus sedimentation have impeded the prediction of lake phosphorus.

On the Dynamics of Phosphorus in Lake Systems

Bernard C. Kenney

INTRODUCTION

In several classic papers, Vollenweider (1968, 1975, 1976) presented methods to predict lake eutrophication using the phosphorus loading concept. This concept was based upon an empirical observation that chlorophyll production increases with the supply of phosphorus (or loading) to lakes. The phosphorus loading concept was quantified in the form of nomographs called phosphorus loading diagrams, which have been used to predict changes in the trophic state of lakes from changes in loading. In general, the concept is called the nutrient loading concept and can be applied to any plant nutrient or fertilizer in addition to phosphorus.

In the present paper, the phosphorus loading concept is examined within the framework of equilibrium input-output models. Some theoretical and practical shortcomings of several phosphorus models are then discussed. Finally, the dynamics of total phosphorus in lakes is explored using first-order linear theory with time-dependent coefficients.

INPUT-OUTPUT MODELS

Sawyer (1947) reported that when the phosphorus concentration in a lake was greater than 10 to 20 mg/m³, algal blooms frequently occurred. The lower limit of Sawyer's observation was used by Vollenweider (1975, 1976) as the criterion to quantify eutrophication. The problem of predicting the trophic state of a lake is then reduced to one of predicting when the concentration of phosphorus in a lake, P_L, exceeds 10 mg/m³ from estimates of phosphorus input or loading, L, to the lake. The phosphorus loading is defined as the mass of total phosphorus entering a lake per unit time divided by the surface area of the lake. The required result may be expressed mathematically as a relationship between P_L and L (plus other as yet unspecified variables). The relationship may be expressed in terms of an unknown function, f, as

$$f(P_L, L,,,,) = 0. \qquad (1)$$

The usual empirical approach applied to such problems is to measure P_L and L for a number of cases and determine the unknown function, f. The empirical approach is basically a curve-fitting exercise, and the resultant curve (the unknown function, f) is often assumed to be universal. Once f is known, one may determine (or "predict") P_L from estimates of L for other lakes. Conversely, a critical loading can be established for a lake by setting P_L to 10 mg/m³ in equation (1). How well this model predicts the unknown case is strongly dependent upon the validity of the assumption of similarity among lakes.

In engineering applications, the empirical approach produces good results when the data used in determining the unknown function are judiciously chosen to encompass the complete range of situations for which the resultant curve may be used. In the context of lake eutrophication, use of the empirical approach requires that data from a wide spectrum of lakes be included when determining f. It is also implicitly assumed that other factors such as geological and geochemical differences between lake basins are unimportant.

In order to include important parameters such as the flushing rate of a lake, Vollenweider (1975) proposed that more sophisticated phosphorus loading models be tied to the principle of conservation of mass. To this end, he derived several input-output models. Because the understanding of input-output models is important to the discussion of the phosphorus loading concept to follow, a simple derivation of these models is presented in this paper.

Two input-output models are considered. The first has a single inlet and a single outlet from a well-mixed lake. Removal of phosphorus from the lake is achieved solely through the outlet. The second model is the same as the first, but with the addition of a second term for the removal of phosphorus in the form of sedimentation. This second model is equivalent to the nonconservative model presented by Vollenweider (1975, p. 58).

A Well-Mixed Model with a Single Time Scale

Consider a well-mixed "model lake" with a single inflow and a single outflow. Let Q be the volume flow of water into the lake and assume the lake level is constant with a mean depth, z. The surface area of the lake is A and the lake concentration of total phosphorus is P_L. The concentration of phosphorus in the inflowing water is P_i and in the outflowing water is P_L (because the lake is assumed well mixed). The usual fundamental variables in the problem are, therefore, P_i, P_L, Q, A, z, and t, where t is time.

Phosphorus loading, L, was used by Vollenweider (1975) as a fundamental quantity in his derivation of input-output models. Phosphorus loading, however, is actually a derived quantity or parameter. When all the phosphorus enters the lake with the inflowing water, $L = P_i Q/A$. The above fundamental variables are used in the present section to simplify the derivation of the models. To facilitate comparison with Vollenweider (1975), certain equations are also expressed in terms of phosphorus loading.

The problem is to determine P_L at any time t after the continuous introduction of phosphorus at the inflow, P_i, beginning suddenly at time t = 0. The assumption can be made, without loss of generality, that the initial lake concentration was zero at time t = 0. If the problem were to be approached from experimental point of view, the first task would be to perform a dimensional analysis of the independent variables (Taylor, 1974). The pertinent dimensions in this problem are mass $\underline{M}$, length $\underline{L}$, and time $\underline{T}$. Taylor's convention will be used to list the dimensions of the fundamental variables. For [ ] read "the dimensions of." For = read "are."

$$[P_i] = \underline{ML^{-3}}$$
$$[P_L] = \underline{ML^{-3}}$$
$$[Q] = \underline{L^3T^{-1}}$$
$$[A] = \underline{L^2}$$
$$[z] = \underline{L}$$
$$[t] = \underline{T}$$

For this simple problem, two dimensionless groups may be found, either by inspection or by using the Buchingham Π theorem (Taylor, 1974). The result of the dimensional analysis is

$$\frac{P_L}{P_i} = \text{Func}\left(\frac{Qt}{Az}\right). \qquad (2)$$

Equation (2) says that the dimensionless concentration P_L/P_i is some unknown function of the dimensionless variable Qt/Az. The actual functional relationship depends upon equations governing the process. Using the empirical approach, the unknown function would be determined by the experiment--by correlating P_L/P_i against Qt/Az, for example.

In this case, it is not necessary to conduct a series of experiments to determine the unknown function. The governing equation for the model lake can be written down directly using the principle of conservation of mass as

$$\frac{dP_L}{dt} = \frac{P_i Q}{Az} - \frac{P_L Q}{Az}. \qquad (3)$$

In this differential equation, dP_L/dt is the rate of change of concentration of phosphorus in the lake expressed as the difference between the rate at which the concentration is increasing in the lake due to the inflow and the rate at which the concentration is decreasing because of losses at the outflow. Because the assumption was made that phosphorus enters the lake only with the inflowing water, equation (3) may also be written in terms of the phosphorus loading as

$$\frac{dP_L}{dt} = \frac{L}{z} - \frac{P_L Q}{Az}. \qquad (4)$$

Equation (4) is equivalent to equation (2.5) in Vollenweider (1975).

Equation (3) is a first-order ordinary differential equation that can be readily solved by separation of variables. The solution, subject to the initial conditions, $P_L = 0$ at $t = 0$, is

$$\frac{P_L}{P_i} = 1 - \exp(-Qt/Az). \qquad (5)$$

Thus, the unknown function in equation (2) has been found for this simple model lake. If the governing equation were different because of a different set of initial assumptions (say, about mixing), then this unknown function would also be different. This solution to equation (3) is called the step function response of the lake with respect to phosphorus. It describes the increase in lake concentration of phosphorus with time after the sudden start of phosphorus addition at the inflow. Using similar techniques, a sudden decrease or even continuous changes in the inflow phosphorus concentration can also be calculated.

The step function response is shown in Figure 1 in terms of the time scale $\tau_w = Az/Q$ of the system. By definition, when $t = \tau_w$, then $P_L = - 1/e = 63\%$ of its final value. The time scale, Az/Q, in this simple problem is the same as that often used for real lakes. It is known as the water residence time or water renewal time.

Two characteristics of Figure 1 are important for discussions to be presented later in this paper. First, at steady state (i.e., as $t \rightarrow \infty$), the lake concentration is equal to the inflow concentration for this model (i.e., $P_L = P_i$). Second, a reasonable estimate of the final steady state value (say 99%) requires a time of $t > 5\tau_w$. For all values of $t < 5\tau_w$, $P_L < P_i$.

The analysis that leads to equation (5) is exact. If a physical model of the model lake were constructed in a laboratory such that all the initial assumptions were met, then equation (5) could also be determined experimentally. How well equation (5) applies to a real lake is solely dependent upon how well the model assumptions apply to the real lake. The validity of the assumptions to real lakes will be discussed later.

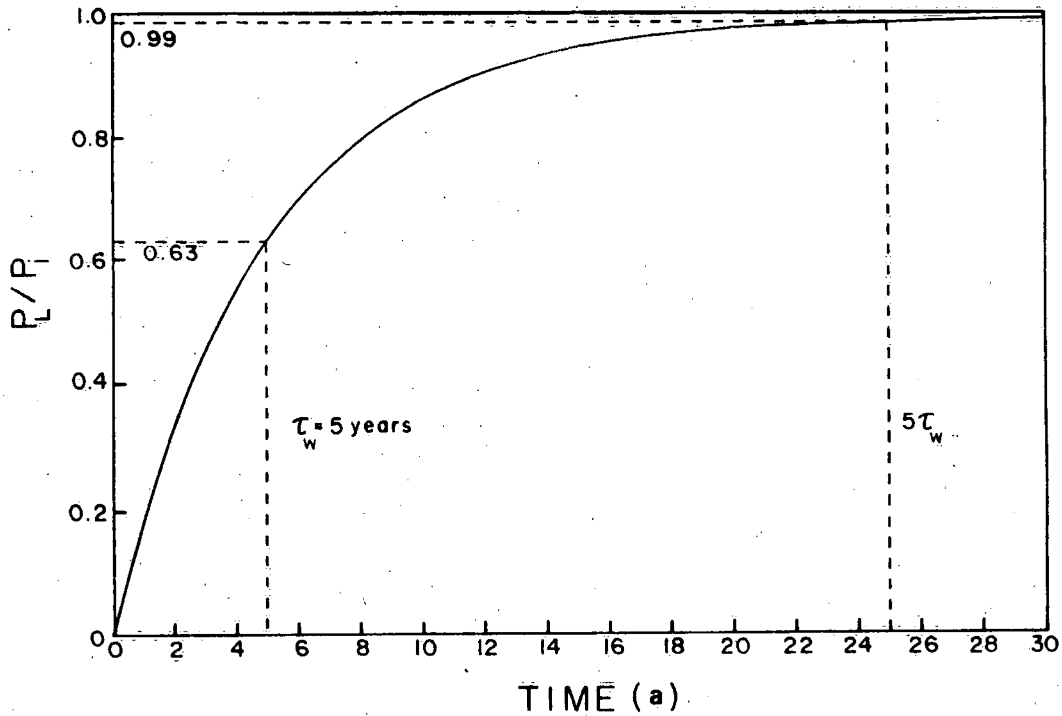


Figure 1. Time response of input-output model with a single scale: water renewal time.

A Model with Two Time Scales

As an extension of the previous model, consider that the element being introduced into the lake (phosphorus) may be removed from the lake by some other means than outflow, i.e., sedimentation. The assumption is made that the rate of removal by sedimentation is proportional to the concentration of phosphorus in the lake, P_L, and the proportionality constant will be called σ. The fundamental variables are now P_i, P_L, Q, A, z, and σ where $[\sigma] = T^{-1}$. The result of a dimensional analysis for this new model is

$$\frac{P_L}{P_i} = \text{Func}\left(\frac{Qt}{Az}, \sigma t\right). \qquad (6)$$

In this case, the nondimensional lake concentration is an unknown function of two independent nondimensional variables. If the unknown function were to be determined by experiment, the task would be more formidable, particularly if the unknown function were nonlinear.

In this example, however, the governing equation may be written down directly, as before, with the addition of another term for the sedimentation process, as

$$\frac{dP_L}{dt} = \frac{P_i Q}{Az} - \frac{Q\,P_L}{Az} - \sigma P_L. \qquad (7)$$

(source) (sink) (sink)

The equation governing the new model lake now has one source term and two sink terms. This equation may also be solved by separation of variables. The solution, again subject to the initial conditions $P_L = 0$ at $t = 0$, is

$$\frac{P_L}{P_i} = \frac{1}{1 + \frac{Az\sigma}{Q}} [1 - \exp(-Qt/Az)\exp(-\sigma t)]. \qquad (8)$$

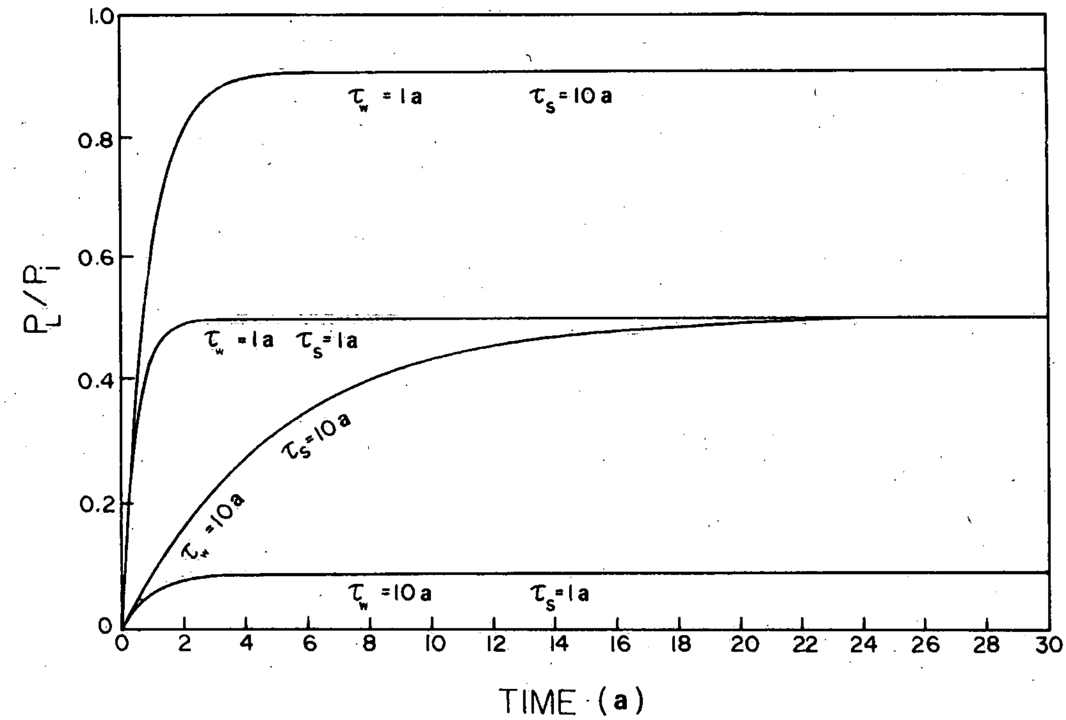


Figure 2. Time response of input-output model with two time scales: water renewal and sedimentation.

Once again the unknown function, Func (Qt/Az, σt), has been found. It is seen that there are two time scales in this problem. The water renewal time, $\tau_w = Az/Q$, as before, and a second time scale, $\tau_s = 1/\sigma$, which will be called the sedimentation time scale.

The form of the solution is shown in Figure 2. In the steady state,

$$\frac{P_L}{P_i} = \frac{1}{1 + \frac{Az\sigma}{Q}} = \frac{1}{1 + \frac{\tau_W}{\tau_S}} \cdot (t \rightarrow \infty) \qquad (9)$$

The steady state value of the lake concentration is no longer simply equal to the inflow concentration, but $P_L < Pi$ always. The actual steady state value of P_L is dependent not only upon the sedimentation time scale, but also on the water renewal time. Therefore, two vastly different lakes--one with a high flushing rate and a high sedimentation rate and the second with low flushing and low sedimentation--can have the same steady state value of lake concentration (Fig. 2). However, the time required to reach steady state is much longer in the latter case. In general, the higher the flushing rate (i.e., the smaller τ_w) for a given sedimentation rate, the closer the steady state value of P_L will be to P_i. Conversely, the higher the sedimentation (i.e., the smaller τ_s) for a given flushing rate, the smaller the final value of P_L.

Because of the form of the solution for this model, an overall time constant, τ_o, can also be defined by

$$\tau_o = \frac{\tau_W}{1 + \frac{\tau_W}{\tau_S}}. \qquad (10)$$

The overall time scale of a system is always less than the water renewal time for any finite value of sedimentation. As before, 99% of the steady-state value is reached in $5\tau_o$.

THE PHOSPHORUS LOADING MODELS

The Hydraulic Loading Model

The hydraulic loading model presented in Vollenweider (1976) was based on the input-output model with one source (inflow) and two sinks (outflow and sedimentation) described in the previous section. The results of the model were presented as a plot of phosphorus loading versus hydraulic loading, where hydraulic loading was defined as the mean depth divided by the water renewal time of a lake. The derivation of the hydraulic loading model, however, contained an error in the treatment of the sedimentation term. The nature and consequences of this error are discussed in the present section.

The requisite input-output model was expressed in equation (7) using the fundamental variables of the problem. Equation (7) can be rewritten then in terms of the phosphorus loading as

$$\frac{dP_L}{dt} = \frac{L}{z} - \frac{P_L}{\tau_w} - \sigma P_L. \quad (11)$$

The solution to equation (11) is the required result, that is, a relationship between L and P_L. It is unnecessary to integrate equation (11) to obtain the solution, however, because the assumption was made in Vollenweider (1976) that all lakes are in steady state with respect to phosphorus. Therefore, setting $dP_L/dt = 0$ in equation (11) produces the required result of the hydraulic loading model,

$$L = P_L \left(\frac{z}{\tau_w} + z\sigma\right). \quad (12)$$

When z, τ_w, and σ are known, equation (12) describes the relationship between L and P_L (at least within the assumptions made in deriving the model).

The application of equation (12) to real lakes is hampered by the fact that the sedimentation time scale, σ^{-1}, is neither generally known nor readily measured. In Vollenweider (1975), it was stated that σ could not be measured, but must be determined from the model itself. This was done by rewriting equation (12) in the form

$$\sigma = \frac{1}{z}\left(\frac{L}{P_L} - \frac{z}{\tau_w}\right). \quad (13)$$

Data for P_L, L, z, and τ_w were then used to determine σ for 21 different lakes. Although the sedimentation constant can be determined for individual lakes that are in steady state using this method, equation (13) is not an independent equation. If equation (13) is substituted back into equation (12), an identity results. That is,

$$L = P_L\left(\frac{z}{\tau_w} + z\sigma\right)$$

$$= P_L\left[\frac{z}{\tau_w} + \frac{z}{z}\left(\frac{L}{P_L} - \frac{z}{\tau_w}\right)\right] = L. \quad (14)$$

This identity (14) means that equation (13) cannot be used as a functional relationship for σ in equation (12). A general relationship for σ that is valid for all lakes must be derived independently of either equation (12) or equation (13).

In Vollenweider (1976), the estimates of σ for the 21 lakes were plotted against the mean depth in an attempt to obtain an independent equation for σ. In this case, mean depth was an unfortunate choice of parameters because the correlation that was found between σ and z was a spurious self-correlation that resulted from plotting a ratio against its own denominator. This ratio correlation was examined by Kenney (1982) as part of an analysis of the forms of spurious self-correlation that result from the use of a common term. The result of the correlation was

$$\sigma = \frac{10}{z}. \qquad (15)$$

Therefore, instead of applying equation (13) directly to equation (12), which would have resulted in the identity (14), the spurious relation described by equation (15) was substituted into equation (12) to obtain the final result of the hydraulic loading model, that is,

$$L = P_L \left(\frac{z}{\tau_w} + 10\right). \qquad (16)$$

When the Sawyer criterion was substituted, the critical phosphorus loading, L_c, was found for a lake:

$$L_c = 10 \frac{z}{\tau_w} + 100. \ (mg/m^2a) \qquad (17)$$

Alternatively, it may be shown that the hydraulic loading model may be obtained directly from the spurious correlation and equation (13):

$$\sigma = \frac{10}{z} = \frac{1}{z} \left(\frac{L}{P_L} - \frac{z}{\tau_w}\right). \qquad (18)$$

Equation (18) shows that the acceptance of the spurious correlation is equivalent to assuming

$$10 = \frac{L}{P_L} - \frac{z}{\tau_w}. \qquad (19)$$

Equation (19) is not a good approximation for the data, however, because (L/P_L - z/τ_w) varies from 3 to 60 (with a mean of 15 and a standard deviation of 14). Nevertheless, equation (16) may be obtained directly from equation (19) alone by simply rearranging the terms.

The Advanced Model

The next phosphorus loading model to be considered was also derived in Vollenweider (1976), and it contains both mean depth and the water renewal time as parameters. It is referred to as the advanced model in the present paper.

The advanced model result relating L and P_L is

$$L = P_L \frac{z}{\tau_w} (1 + \sqrt{\tau_w}). \qquad (20)$$

Again the final result may be expressed as a critical phosphorus loading that is not to be exceeded by substitution of Sawyer's (1947) criterion that $P_L \leq 10$ mg/m³.

In the derivation presented in Vollenweider (1976), the advanced model results from the application of the steady state assumption to equation (11). It is not necessary, however, to invoke equation (11) in order to derive the advanced model. The advanced model may be derived directly from the regression between τ_p/τ_w and τ_w, where τ_p is the residence time for phosphorus in the lake (Fig. 3).

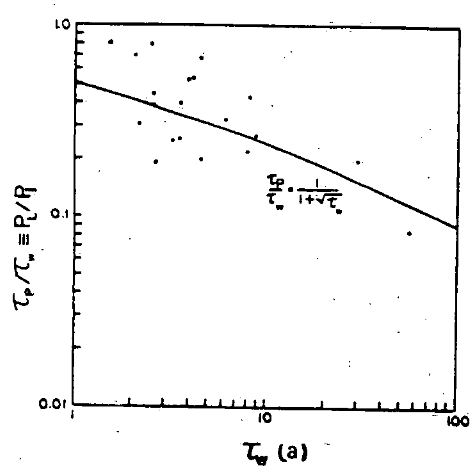


Figure 3. Relative phosphorus residence time as a function of water residence time (after Vollenweider, 1976).

Rather than use the actual power law regression equation for the data in Figure 3, Vollenweider used the following analytical representation instead:

$$\frac{\tau_p}{\tau_w} = \frac{1}{1 + \sqrt{\tau_w}}. \qquad (21)$$

Equation (21) was chosen over the power law equation because it more closely represented the expected asymptotic behaviour of highly flushed lakes (lakes with short water-renewal times). Although equation (21) represents the 21 data points in Figure 3 almost as well as the power law equation, it must be noted that the success of the regression (i.e., the high correlation coefficients obtained) is strongly dependent upon the two data points with large water-renewal times. The validity of equation (21) is discussed later, but first, the advanced model result is derived directly from equation (21).

The phosphorus residence time, τ_p, is defined by Vollenweider (1976) in analogy to the water residence time as "the hypothetical time necessary to bring the phosphorus concentration of a lake to its present level starting from concentration equal to 0." The residence time of phosphorus relative to that of water, τ_p/τ_w, was shown in Vollenweider (1976, p. 62, equation [6]) to equal the ratio of the average lake concentration of phosphorous, P_L, and the average inflow concentration, P_i. Therefore, equation (21) may be rewritten as

$$\frac{\tau_p}{\tau_w} = \frac{P_L}{P_i} = \frac{1}{1 + \sqrt{\tau_w}}. \qquad (22)$$

The assumption that the entire phosphorus loading to each lake was contained in the inflowing water was implicit in Vollenweider (1976). Using this assumption, one may write equation (22) as

$$\frac{P_L}{L}\frac{Q}{A} = \frac{1}{1 + \sqrt{\tau_w}}. \qquad (23)$$

After rearranging the terms in equation (23),

$$L = P_L \frac{Q}{A} \frac{z}{z} (1 + \sqrt{\tau_w}), \qquad (24)$$

and recalling that $\tau_w = Az/Q$, one arrives at the advanced model equation. Thus, the advanced model follows directly from an empirical regression analysis.

It is instructive, however, to further examine equation (11) to see how it is related to the advanced model. By comparing equation (22) to equation (9), one can see that the advanced model result can be derived from the input-output model with two time scales if the assumption is made that $\sigma = 1/\sqrt{\tau_w}$. When this assumption is made, however, a very different input-output model results. The differential equation governing phosphorus for the advanced model lake is not equation (7) or equation (11), but rather,

$$\frac{dP_L}{dt} = \frac{L}{z} - \frac{P_L}{\tau_w} - \frac{P_L}{\sqrt{\tau_w}}. \qquad (25)$$

(source) (sink) (sink)

Note that, although equation (25) contains one source term and two sink terms, the two sink terms are not independent. Rather than a model based on two independent time scales (τ_w, σ^{-1}), the differential equation governing phosphorus for the advanced model depends solely on the water renewal time, τ_w, Rewriting equation (25) as

$$\frac{dP_L}{dt} = \frac{L}{z} - P_L \left(\frac{1 + \sqrt{\tau_w}}{\tau_w}\right) \qquad (26)$$

and substituting equation (21) produces the simplification

$$\frac{dP_L}{dt} = \frac{L}{z} - \frac{P_L}{\tau_p}. \qquad (27)$$

The solution of equation (27) (by separation of variables) shows that the phosphorus residence time, τ_p, is the effective time scale of the advanced model. There is an important difference between the effective time scale of the advanced model described here and the overall time scale (equation [10]) discussed previously. The overall time scale contains two independent variables that describe two independent processes --flushing and sedimentation; the effective time scale describes only one process--flushing.

When the input-output models were first derived, the assumption was made that flushing and sedimentation were independent processes. This assumption appears to be reasonable for most lakes. Flushing is controlled by hydrological parameters such as precipitation, size of drainage basin, lake size, and morphometry. From a physical perspective, sedimentation is a nonstationary, random process that is largely controlled by lake turbulence and, therefore, ultimately by the wind. From a biological perspective, sedimentation is controlled by the rate at which biological processes convert dissolved phosphorus into particulate form through the growth of algae. In addition to being a nonstationary, random process, algal growth is highly nonlinear and directly coupled to many physical processes occurring in lakes. Sedimentation is likely to be quantitatively different in shallow lakes and lakes that are deep enough to stratify because the sediment flux through the thermocline is trapped, at least until fall overturn. Significant coupling between flushing and sedimentation rates is to be expected only when the hydraulic flow through the lake dominates lake turbulence generation (i.e., for lakes that are only slight widenings of rivers with high flows). Although such lakes do exist, they are not common. Therefore, the assumption that flushing and sedimentation are independent processes appears to be reasonable for most lakes.

From an alternative viewpoint, the advanced model lake may be seen as one with a special type of sedimentation, that is, sedimentation with a time scale proportional to the square root of the water renewal time. Table 1 compares typical values of the effective time scale of the advanced model lake with the water renewal time, and this special sedimentation time scale. According to Table 1, the phosphorus in lakes with long water-renewal times is dominated by sedimentation. For lakes with short water-renewal times, the effects of

Table 1. Some typical values of the effective time scale, τ_p, the water renewal time, τ_w, and the special sedimentation time scale, τ_s, according to the advanced model.

τ_p (years)	τ_w (years)	τ_s (years)
0.5	1	1.0
0.83	2	1.4
1.25	4	2.0
2.4	10	3.16
9.1	100	10.0
25.5	700	26.5
30.7	1000	31.6

flushing and sedimentation are similar in magnitude. Thus, for a lake with a long water-renewal time, such as Lake Tahoe (τ_w = 700 a), the effective time scale of the advanced model lake with respect to phosphorus is much shorter (25 a) than τ_w. As was shown earlier, it takes 5τ to approach a steady state condition. Therefore, Lake Tahoe would reach steady state with respect to phosphorus after 150 years according to the advanced model rather than 3500 years if there were no sedimentation. With such a large difference in time scales, it is very important for lake management to know which time scale is correct. The validity of the advanced model and some important consequences of the steady state assumption are discussed in the next section.

THE STEADY STATE ASSUMPTION

All of the phosphorus loading models described in Vollenweider (1976) make use of the assumption that lakes are in steady state with respect to phosphorus. Unfortunately, substantial error may be introduced by ignoring the time-dependent term in the differential equations. An assessment of the impact of the steady state assumption on the advanced model is made in this section.

It was shown previously that the advanced model is actually based on the regression between P_L/P_i and τ_w (Fig. 3). Here, the steady state assumption is crucial to the success of the model. Figures 1 and 2 show the time-dependent behaviour of P_L/P_i for two input-output models prior to steady state. For the input-output model with a single time scale (Fig. 1), the steady state lake phosphorus concentration was always equal to the inflow phosphorus concentration. For practical purposes, steady state is achieved after a time equal to $5\tau_w$. The important point here is that $P_L < P_i$ for all time less than $5\tau_w$. Furthermore, P_L may be very much less than P_i, as illustrated in Figure 1.

It is further shown in Figure 2 that (even at steady state) P_L is always less than P_i for an input-output model with two time scales (flushing and sedimentation). Prior to reaching a steady state condition, P_L may be very much less than P_i for all practical cases. The question is, how much less? And, why is it less? Is P_L less than P_i because of high sedimentation in a particular lake or is that lake simply far from its steady state value? The answers to these questions have a profound effect on the validity of the advanced model.

If all the lakes used in the regression (Fig. 3) were actually at steady state with respect to phosphorus, then $\sigma = 1/\sqrt{\tau_w}$ might be a reasonable estimate of the effect of sedimentation of phosphorus (or any other removal process for phosphorus not dealt with explicitly). However, if P_L/P_i is smaller than its steady state value for any of the lakes in Figure 3, then the effect of this too small value is simply absorbed in the sedimentation term. In other words, if P_L/P_i is small because the lake is far from steady state, the advanced model automatically assigns this lake a high sedimentation--because high sedimentation (small τ_s) is required for small P_L/P_i at steady state (see Fig. 2). Such a trend is visible in Table 1. Lakes with long water-renewal times (those lakes least likely to be at steady state) are dominated by sedimentation--according to the advanced model. But, is the advanced model correct? To properly assess the impact of the steady state assumption on the advanced model, one must return to the origin of the model (Fig. 3). As was noted previously, the success of the regression is very dependent upon the two data points in Figure 3 with large τ_w (i.e.,

τ_w = 30 years and τ_w = 56 years). When these two points are removed, the correlation coefficient changes from one that is significant at the 1% level (-0.674) to one that is not significant at the 5% level (-0.412). In other words, when these two data points are removed, there is no significant correlation between P_L/P_i and τ_w (at the 5% level) and the basis for the advanced model disappears. The validity of the advanced model is, therefore, strongly dependent upon data from two lakes--both of which have large water-renewal times. In addition, these two lakes are the least likely to be in equilibrium because they require 150 and 280 years respectively to reach steady state (ignoring sedimentation). The heavy weight given to sedimentation for lakes with long water-renewal times may be a direct result of using two lakes that are far from equilibrium as the basis for the advanced model. The accuracy of any predictions made with the advanced model must be suspect until such time as the impact of sedimentation processes is fully assessed.

Another potential concern is that the advanced model may be verified by using data from other lakes equally far from equilibrium. Since the large cultural input of phosphorus to lakes is of relatively recent origin, most lakes with long water-renewal times are included in this category. For now, the advanced model may appear to be valid. Overestimation of phosphorus sedimentation by the advanced model, however, together with concomitant lake management (i.e., increasing the phosphorus load), could have serious effects for lakes with τ_w as steady state is eventually reached.

Larsen and Mercier (1976) presented a similar analysis to Vollenweider (1976) using data from a different series of lakes. The basic input-output model and the use of the steady state assumption were identical, but the analysis was formulated in terms of the retention coefficient, R = 1 - P_L/P_i. The Larsen and Mercier result,

$$R = \frac{1}{1 + \tau_W^{-1/2}}, \qquad (28)$$

is equivalent to Vollenweider (1976) equation (22), which may be readily shown by substitution of the definitions of R into equation (28). Figure 4 shows the data from the 20 lakes used by Larsen and Mercier to obtain equation (28) plotted as P_L/P_i versus τ_w. These lakes fit equation (22) quite well and might be used as additional evidence in support of the advanced model. Unfortunately, the same criticism (i.e., the blanket application of the steady state assumption of all lakes) applies equally well to Larsen and Mercier's analysis. Furthermore, Larsen and Mercier selected the 20 lakes (out of a possible 36 lakes for which data were presented) that gave the best empirical relationship between R and lake characteristics. The process of subjectively selecting the lakes that produced the largest regression coefficient is questionable. When all the lakes from Larsen and Mercier (1976) and Vollenweider (1976) are plotted (Fig. 4), more scatter is evident. This scatter may be attributed to differences in sedimentation, the degree of unsteadiness of the various lakes, or to other factors not included in the model. Certainly, the three lakes in Figure 4 with the longest water renewal times do have the smallest values of P_L/P_i--but is this because of high sedimentation or simply because these lakes are far from steady state?

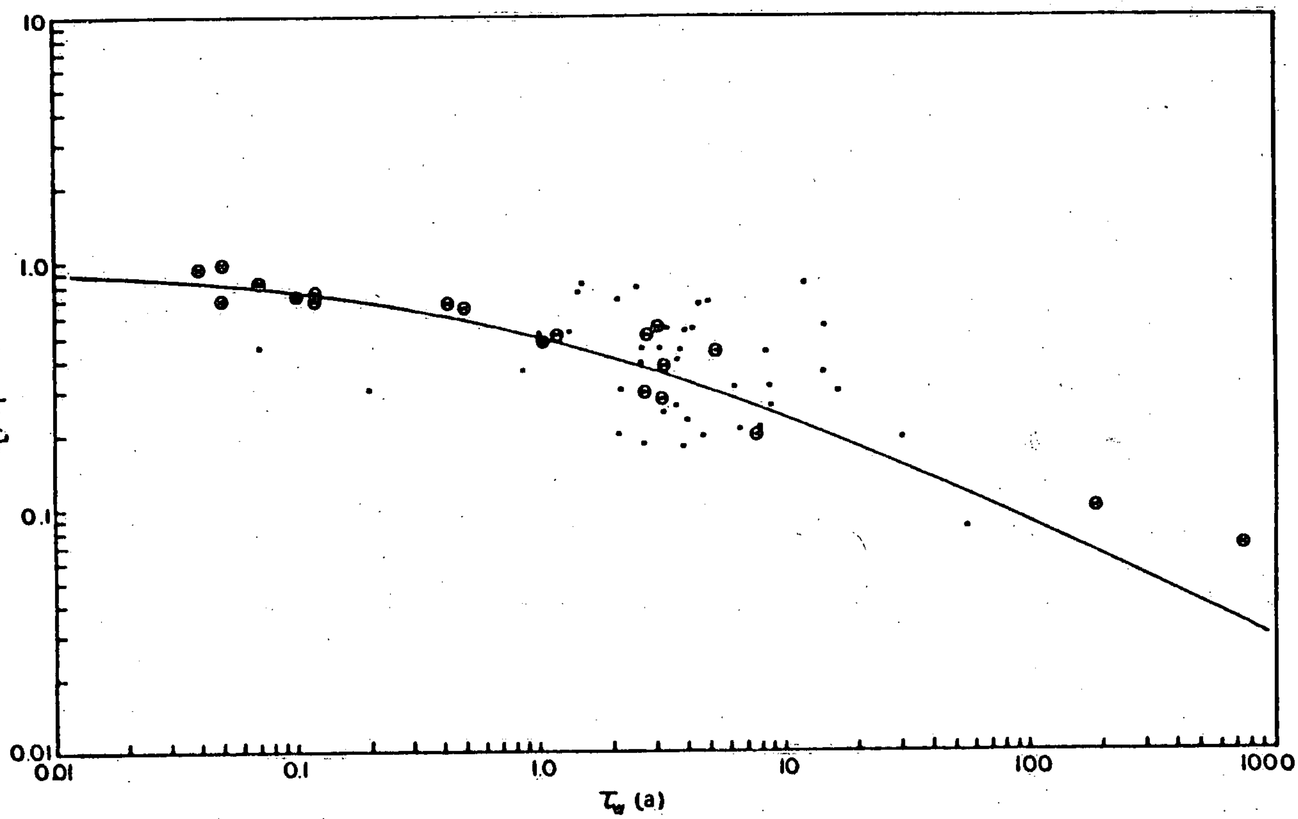


Figure 4. Nondimensional lake phosphorus, P_L/P_i, plotted against water residence time, τ_w, for 54 lakes from Larsen and Mercier (1976) and Vollenweider (1976). The 20 lakes used by Larsen and Mercier are circled.

Equations (22) and (28) and Figures 3 and 4 all assume either implicitly or explicitly that P_L/P_i is a function of τ_w alone. It was shown, however, in the derivation of the basic model underlying both the Vollenweider and the Larsen and Mercier analyses that P_L/P_i is a function of τ_w, σ, and t (cf. equation [6]). These conflicting requirements can be resolved if σ and/or t are constants; or if P_L/P_i is constant with variations in σ and/or t; or if σ and/or t are functions of τ_w; or some combination of the above. The conclusion of Vollenweider's advanced model is that P_L/P_i is constant with variations in t (i.e., the steady state assumption) and σ is a function of τ_w. There appears to be no compelling mathematical or physical evidence to support such a conclusion.

THE EDMONDSON AND LEHMAN MODEL

Edmondson and Lehman (1981) (hereafter referred to as EL) presented a lake eutrophication model that they verified with data from Lake Washington. The starting point of the EL model is the equation for the well-mixed model with a single time scale (equation [13] in this paper, equation [10] in EL). The EL model is of interest because it introduces a new concept in the treatment of sedimentation of lake phosphorus. The conventional wisdom of phosphorus sedimentation is based on a heuristic equation in which phosphorus sedimentation is proportional to the concentration of phosphorus in the lake; the higher the phosphorus concentration, the greater the sedimentation. The EL paper suggests that sedimentation of lake phosphorus may be more accurately modelled by the assumption that sedimentation is proportional to phosphorus inflow, I (where $I = P_iQ$, using the symbols of the present paper). It will be shown below that the EL suggestion is incorrect.

The sedimentation of lake phosphorus was accounted for in the EL model simply by reducing the phosphorus inflow using the proportionality constant, $\underline{f}$. $\underline{f}$ was found in EL as the slop of the linear regression line between phosphorus sedimentation, P_{sed}, and I. P_{sed} was itself calculated from the other terms in the phosphorus budget by difference. That is, the phosphorus sedimentation was equal to the annual phosphorus inflow minus the annual phosphorus outflow, P_o, minus the annual change in the mass of phosphorus in the lake, dP, as measured on 1 January of successive years. In effect,

$$\underline{f} = \text{Func}\ (I,\ P_o,\ dP). \qquad (29)$$

It will be shown below, however, that the EL model equation may be written as

$$dP = \text{Func}\ (I,\ P_o,\ \underline{f}) \qquad (30)$$

or, alternatively, as

$$P_o = \text{Func}\ (I,\ dP,\ \underline{f}). \qquad (31)$$

Since there was no new information added between equations (29) and (31), the essence of the EL model is contained in $\underline{f}$ and the linear regression line shown in Figure 5. Figures 6 and 7, which are presented to verify the EL model, are only restatements of Figure 5. The scatter from the line of 100% agreement in Figures 6 and 7 is a re-expression of the scatter from the linear regression line in Figure 5. If all the data points in Figure 5 had fallen on the regression line, then the EL model "results" in Figures 6 and 7 would have shown perfect agreement. In actual fact, the EL model is a simple tautology.

To show the tautology more clearly requires a detailed examination of Figure 5. This figure is a plot of the annual flux of P to the sediments, P_{sed}, versus P-loading, I. The least squares regression line for the data in Figure 5 is

$$\langle P_{sed} \rangle = \underline{f}\ I + \text{constant}. \qquad (32)$$

If the annual phosphorus inflow to the lake is known for any year, say the jth year, then the least squares estimate of the annual sedimentation of phosphorus for the jth year, $\langle P_{sed} \rangle_j$; can be found from equation (32).

The EL model was given in discrete analog form as (EL equation [12]),

$$P_{j+1} = I_j(1 - \underline{f}) + P_j(1 - Q_j/V) \qquad (33)$$

where P_j is the phosphorus content of the lake on 1 January of successive years. However, EL compared the model prediction of annual outflow with the measured values (Fig. 6) so that the terms of the model equation must be rearranged to

$$P_jQ_j/V = I_j(1 - \underline{f}) - (P_{j+1} - P_j)$$

$$= \text{PREDICTED ANNUAL OUTFLOW}. \qquad (34)$$

Equation (34) is the EL model equation for predicting phosphorus outflow. Values of I_j, P_j, and P_{j+1} were

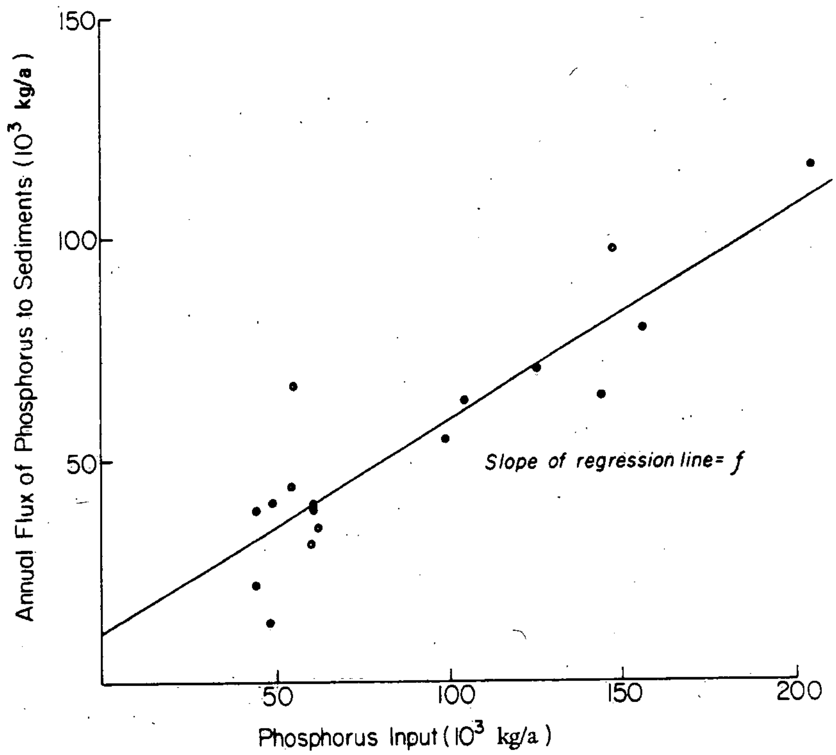


Figure 5. Annual flux of phosphorus to the sediments as calculated from the phosphorus budget of Lake Washington versus phosphorus input (after Edmondson and Lehman, 1981).

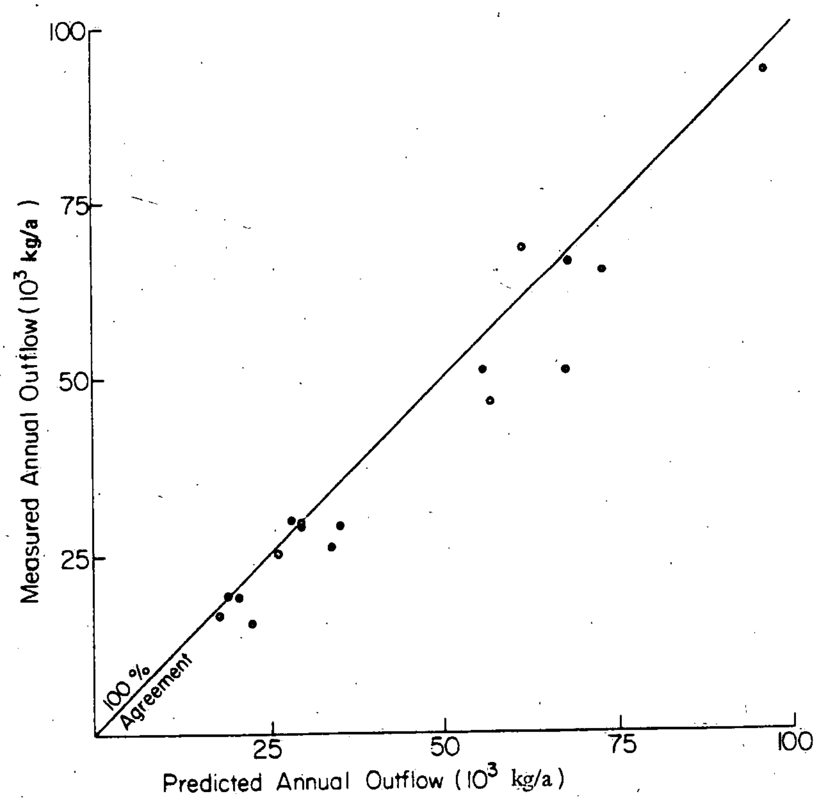


Figure 6. Predicted phosphorus outflow from the Edmondson and Lehman model versus the measured outflow (after Edmondson and Lehman, 1981).

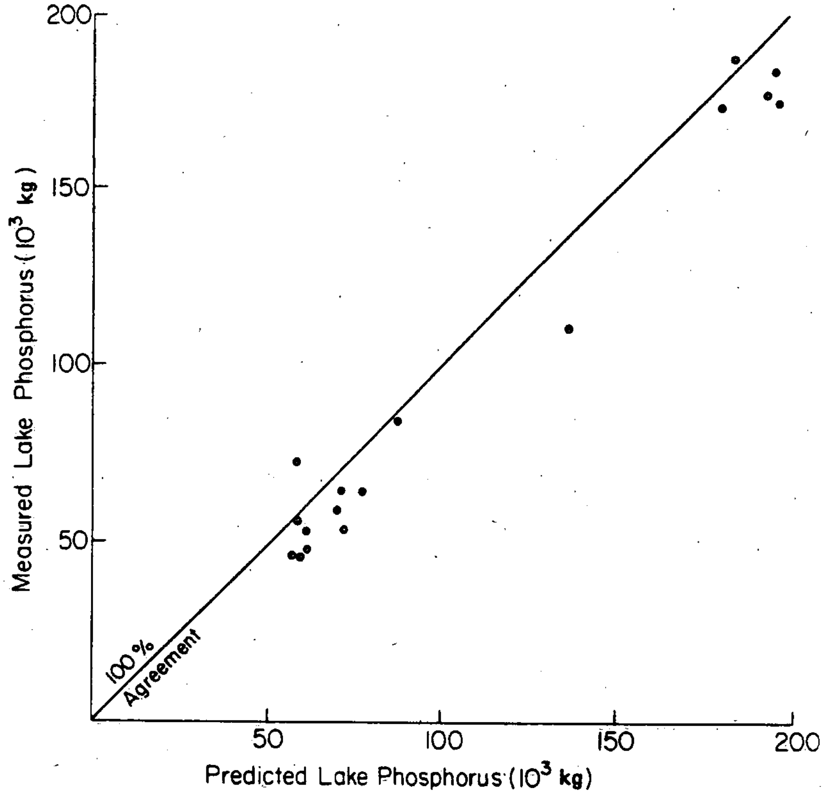


Figure 7. Lakes phosphorus predicted by the Edmondson and Lehman model versus the measured lake phosphorus (after Edmondson and Lehman, 1981).

substituted in EL to find values of the predicted annual phosphorus outflow for the jth year. The value of $\underline{f}$ was constant and equal to the 0.49 slope found in Figure 5. These predicted outflows were compared with the measured outflows in Figure 6. A similar procedure was followed for the lake P results given in Figure 7.

Equation (34) can be expanded as

$$P_j Q_j / V = I_j - \underline{f} I_j - (P_{j+1} - P_j) + P_{o_j} - P_{o_j} = I_j - P_{o_j} - dP_j - \underline{f} I_j + P_{o_j} . \quad (35)$$

But,

$$P_{sed_j} = (I - P_o - dP)_j \quad (36)$$

because P_{sed} was found by difference using the budget method. Furthermore, from equation (32), $\underline{f}I$ is the least squares estimate of P_{sed} minus a constant, that is,

$$\underline{f} I_j = \langle P_{sed} \rangle_j - \text{constant}. \quad (37)$$

Substituting equations (36) and (37) into equation (35) gives the annual phosphorus outflow predicted by the EL model as

$$P_j Q_j / V = P_{sed_j} - \langle P_{sed} \rangle_j + \text{constant} + P_{o_j} . \quad (38)$$

For those data points that fall on the regression line, P_{sed} is identically equal to $\langle P_{sed} \rangle$. For all other data points, P_{sed} is approximately equal to $\langle P_{sed} \rangle$ except for scatter about the least squares regression line. Therefore, equation (38) reduces to

$$P_j Q_j / V = P_{o_j} + \text{constant} \pm \text{scatter}. \quad (39)$$

Thus, we have returned to our starting point. The predicted outflow phosphorus is identically equal to the measured outflow phosphorus plus a constant plus or minus some scatter. The constant, which arose because of the non-zero intercept of the regression line in Figure 5, explains why most of the predicted values lie to the right of the theoretical line of 100% agreement between prediction and observation in Figure 6. As stated earlier, the scatter in Figure 6 is simply a reflection of the scatter in Figure 5.

The same arguments apply for Figure 7, where the predicted lake phosphorus generally lies to the right of line of 100% agreement with the observed lake phosphorus.

LAKE PHOSPHORUS DYNAMICS

In this section, the nature of the input-output models derived previously is further explored, and several factors affecting the application of these models to real lakes are considered. It will be assumed that phosphorus is the correct element to model and that Sawyer's criterion is also correct. The prediction of eutrophication, therefore, may be reduced to one of predicting when $P_L > 10\ mg/m^3$, as before. Because one is interested in the long-term effects of phosphorus loading to a lake, it is the steady state value of P_L that one would like to predict.

The steady state result relating phosphorus loading and lake phosphorus concentration was given in equation (12) for the input-output model with two time scales, flushing and sedimentation. The critical phosphorus loading, P_{L_c}, for a specific lake may be found by substituting Sawyer's criterion in equation (12). That is,

$$L_c = 10z \left(\frac{1}{\tau_w} + \sigma\right) (mg/m^2/a). \quad (40)$$

Within the assumption made, equation (40) is the "predictive" equation for lake eutrophication. For lakes where z, τ_w, and σ are known, the critical phosphorus loading may be calculated directly from equation (40). Of course, σ is not generally known, and it is this deficiency that led to the problems previously described.

The Sedimentation Constant, σ

There are several indirect methods to determine σ for a specific lake depending upon the type of phosphorus loading to that lake. Each method depends critically on the validity of the input-output model with two time scales. All experimental error in the measurement of the data (as well as any error from other sources or sinks not explicitly accounted for by the model) is simply absorbed in the sedimentation constant.

The Steady State Method

The first method was discussed briefly when considering the hydraulic loading model. It may be applied to a specific lake if that lake is known to be in steady state with respect to phosphorus. To assume that a lake is in steady state without corroborative data is futile. The existence of steady state conditions may be determined by measuring L and P_L for a period of time. If L and P_L are constant for a time equal to about τ_o or longer, then it is safe to assume that the lake is in steady state. If data are also available for z (and τ_w), then σ may be determined from equation (13).

If one is only interested in determining the critical phosphorus loading for a specific lake in steady state, it is not necessary to calculate the sedimentation constant, σ. For a specific lake in steady state, the ratio, L/P_L, is a constant. Therefore, a single measurement of L and P_L will produce the constant for that specific lake--within the requisite assumptions, of course. The critical phosphorus loading may then be found from

$$\frac{L_c}{10} = \text{CONSTANT} = \left(\frac{L}{P_L}\right)_{\text{steady state}} \qquad (41)$$

Although equation (41) applies to each lake separately, the value of the constant will vary greatly from lake to lake.

The Step Function Response Method

The second method of determining the sedimentation constant may be applied if there has been a sudden change in the phosphorus loading (either increase or decrease), and if the lake phosphorus concentrations, P_L, are measured following the change. The overall time constant of the lake, τ_o, may be determined by fitting an exponential curve to the P_L versus time data. Once τ_o is known, σ may be found from equation (10) and the critical phosphorus loading for that specific lake may again be calculated from equation (40).

The Forced Response Method

For many lakes, the phosphorus loading is neither constant nor subject to sudden changes to a new level, but rather is continuously changing. One must examine the forced response of the lake to determine σ in these cases.

In order to simplify the discussion of forced systems, an analysis of the forced response of the input-output model with a single time scale is presented first. The first-order linear equation governing the input-output model with one time scale (equation 3) may be rewritten as

$$\tau_w \frac{dP_L}{dt} + P_L = P_i\ (t) \qquad (42)$$

where the phosphorus concentration in the inflowing water, P_i, is now the time-dependent forcing function for the system. The simplest forcing is given by

$$P_i(t) = P_o + P_1 \sin \omega t \qquad (43)$$

where P_1 is the amplitude of a simple harmonic oscillation about some constant level of phosphorus concentration, P_o.

Equations (42) and (43) may be readily solved in terms of the relative amplitude of the output oscillations and the phase shift between the input and the output, but for brevity, the derivation of the solution is omitted. After initial transients have decayed, the output of the forced system (P_L', the oscillating part of P_L in this example) occurs at the same frequency as the forcing function (i.e., at ω). The amplitude of the output oscillations and the phase shift between the input and the output are determined by the characteristics of the system. In equation (42) there is only one parameter that characterizes the entire system--the water renewal time, τ_w. The relative amplitude of the response, G (i.e., the magnitude of the output oscillation relative to the input oscillations or forcing), is given by

$$G = \frac{P_L'}{P_1} = \frac{1}{(1 + \omega^2 \tau_w^2)^{1/2}}. \quad (44)$$

The output oscillations lag the input by a phase shift, ϕ, given by

$$\phi = \tan^{-1} \omega\tau_w. \quad (45)$$

The forced system response is totally described by G and ϕ as a function of nondimensional frequency $\omega\tau_w$. For low frequency forcing (i.e., small $\omega\tau_w$), the output (P_L') essentially follows the input (P_1) with little attenuation or phase shift. On the other hand, high frequency forcing ($\omega\tau_w >> 1$) is very strongly attenuated. The lake phosphorus lags the input oscillations by approximately 90 degrees at high frequency.

As an example, consider a hypothetical lake with a water renewal time of one year subjected to forcing with an annual cycle. From equations (44) and (45) the relative response is only 0.16 with a phase shift of 81 degrees. This means that oscillations in phosphorus concentration of the inflow of 100 mg/m^3, for example, would result in a 16-mg/m^3 oscillation in the phosphorus concentration in the lake. Thus annual oscillations in inflow phosphorus are high frequency oscillations for a lake with a one-year water renewal time. Faster oscillations in inflow (e.g., monthly, daily) are more highly attenuated in this hypothetical lake and require extremely large amplitude forcing even to be detectable in the outflow.

Except that the response characteristics are determined by two parameters (τ_w and σ), the forced response of the input-output model with two time scales is similar to that described above. The governing equation (equation [7]) may be rewritten to show the role of forcing as

$$\tau_w \frac{dP_L}{dt} + (1 + \sigma\tau_w)P_L$$

$$= P_o + P_1 \sin \omega t. \quad (46)$$

The solution of equation (46) for the relative amplitude is

$$G = \frac{1}{[1 + (\frac{\omega\tau_w}{1 + \sigma\tau_w})^2]^{1/2}}. \quad (47)$$

The phase shift is now

$$\phi = \tan^{-1}(\frac{\omega\tau_w}{1 + \sigma\tau_w}). \quad (48)$$

For cases where a lake is subjected to simple harmonic forcing, that is, a sinusoidal oscillation of a single frequency, either equation (47) or equation (48) can be used to determine the sedimentation constant within the range of nondimensional frequencies, $0.1 \leq \omega\tau_w/(1 + \sigma\tau_w) \leq 10$. Outside of this frequency range, the accuracy to which σ can be determined is very poor. The use of either equation to determine σ requires a measurement of the water renewal time plus a time history of inflow and lake phosphorus concentrations of sufficient length to determine ω from the measurements. To use equation (47), the relative response, G, is calculated from the time histories of the inflow and outflow leaving σ as the only unknown in equation (47). Similarly, the phase shift, ϕ, is measured directly from the time histories thereby permitting the use of equation (48) to determine σ. Reasonable accuracy requires data over at least one complete cycle of period $T = 2\pi/\omega$.

For cases where the forcing is broadband (i.e., composed of many different frequencies), the phase shift between the input oscillations and the output may still be determined using cross-spectral analysis. However, spectral techniques require data over many complete cycles of oscillation so that sufficient data may not be available for the spectral approach if the forcing is predominantly low frequency. It must be emphasized that all the methods for the determination of σ described herein require time histories of the inflow and lake phosphorus concentrations over several τ_w. Consequently, the spectral approach to the determination of σ is likely to be applied (in the near future) only to lakes with small τ_w.

Time-Dependent Coefficients

As mentioned before, the successful application of input-output models for predicting eutrophication depends solely upon the validity of the assumptions made in the derivation of the models--the mathematics is exact. The validity of one assumption that greatly affects the application of these models is now considered.

It was assumed throughout this paper that the coefficients of the first-order differential equations (i.e., τ_w, σ) were constants. Most real lakes, however, are subjected to large variations in flow (Q) throughout the year, with peak flows usually occurring during spring runoff for many temperate lakes. Changes in lake level (z) and volume (Az) may also be large, particularly for lakes used for hydroelectric purposes. These variables directly affect the water renewal time ($\tau_w = Az/Q$), making τ_w time dependent.

The factors that affect the sedimentation constant, σ, are not as well understood. It is known that sedimentation is a nonstationary, random process that can be positive or negative at various times. A negative sedimentation flux occurs to some extent in the water column of all lakes and is very common in shallow lakes. Because physical sedimentation is controlled principally by the level of turbulence in a lake, large seasonal variations in σ are expected to reflect the differences between turbulence under winter ice and during open water in temperate lakes.

More realistic models require time-dependent coefficients, therefore, for both τ_w and σ. For example,

$$\tau_W(t)\,\frac{dP_L(t)}{dt} + [1 + \sigma(t)\tau_W(t)]\,P_L(t) = P_i(t) \qquad (49)$$

now replaces equation (46) where the time dependence is given explicitly by (t). Before equation (49) can be solved, the temporal variation in τ_w and σ must be known (or independently predicted) either in the form of time histories for random variations or in the form of analytical functions for periodic variations. The model forcing, $P_i(t)$, must also be input to the model. The forcing may be in the form of time histories of measured phosphorus inflow to a lake for a hindcast of the lake response. Anticipated changes in phosphorus inflow could also be input to the model either as time histories or as analytical functions in order to forecast future lake changes.

Temporal variations in the coefficients may be more important than variations in the phosphorus concentration of the inflow (forcing) for many lakes. In complicated lakes, the forcing and the time-dependent coefficients may even be coupled, say, by the flow through the lake, Q. It may be stated, in general, that a large spring runoff tends to make the spring lake phosphorus concentration equal to the spring inflow concentration by greatly reducing the instantaneous value of τ_w. Ice cover, on the other hand, tends to enhance the sedimentation sink.

A quasi-stationary approximation may allow analytical solutions to equation (49) if the variations in the coefficients are slow compared to the forcing frequency. Since seasonal variations in τ_w and σ are often predominant, the quasi-stationary approximation will seldom apply. In general, equations like equation (49) require numerical solutions.

Several examples of the application of the theory developed here to real lakes may be found in Kenney (1990a, 1990b).

ACKNOWLEDGMENTS

I thank K. Patalas, F.M. Boyce, D.W. Schindler, C. Gray, M. Waiser, and R.A. Vollenweider for comments on earlier drafts of the manuscript. J. Mollison is gratefully acknowledged for preparation of the figures.

REFERENCES

Edmondson, W.T., and J.T. Lehman. 1981. The effect of changes in the nutrient income on the condition of Lake Washington. Limnol, Oceanogr., 26: 1-29.

Kenney, B.C. 1982. Beware of spurious self-correlations! Water Resour. Res., 18: 1041-1048.

Kenney, B.C. 1990a. Dynamics of phosphorus in a chain of lakes: The Fishing Lakes. NHRI Pap. No. 44, Sci. Ser. No. 176, National Hydrology Research Institute, National Hydrology Research Centre, Inland Waters Directorate, Environment Canada, Saskatoon, Sask.

Kenney, B.C. 1990b. Lake dynamics and the effects of flooding on total phosphorus. Can. J. Fish. Aquat. Sci., 47: 480-485.

Larsen, D.P., and H.T. Mercier. 1976. Phosphorus retention capacity of lakes. J. Fish. Res. Board Can., 33: 1742-1750.

Sawyer, C.N. 1947. Fertilization of lakes by agricultural and urban drainage. J. N. Engl. Water Works Assoc., 61: 109-127.

Taylor, E.S. 1974. Dimensional analysis for engineers. London: Oxford University Press.

Vollenweider, R.A. 1968. Scientific fundamentals of the eutrophication of lakes and flowing water, with particular reference to phosphorus and nitrogen as factors in eutrophication. OECD Tech. Rep. DAS/CS1/68.27. 159 p.

Vollenweider, R.A. 1975. Input-output models with special reference to the phosphorus loading concept in limnology. Schweiz. Z. Hydrol., 37: 53-84.

Vollenweider, R.A. 1976. Advances in defining critical loading levels for phosphorus in lake eutrophication, Mem. Ist. Ital. Idrobiol., 33: 53-83.