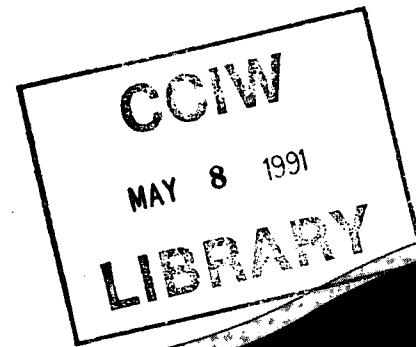
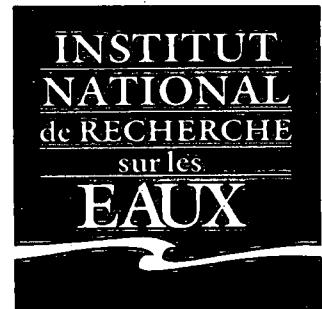
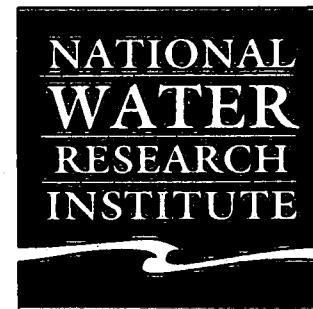


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AN APPROXIMATION OF LIKELIHOOD FUNCTION  
WITH APPLICATION TO ENVIRONMENTAL DATA

A.H. El-Shaarawi and A. Naderi

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**AN APPROXIMATION OF LIKELIHOOD FUNCTION  
WITH APPLICATION TO ENVIRONMENTAL DATA**

**by**

**A.H. El-Shaarawi and A. Naderi**

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**NOVEMBER 1990  
NWRI Contribution #90-162**

## MANAGEMENT PERSPECTIVE

Statistical methods for the analysis of environmental data are frequently restricted to the use of the normal or lognormal distributions as models for describing the data. This limits the scope of applications and can result in misleading interpretation especially when the true probability model deviates substantially from the one used in the analysis. The basic reasons for the use of these distributions are the simplicity of producing summary statistics and conducting tests of significance. In this paper, the class of normal and lognormal models is extended to the family of the location and scale parameter models. This extension leads to a more appropriate analysis of environmental data. Furthermore, a simplified closed form estimates for the characteristics of this class is given which can be easily computed. As an illustration of these methodologies several examples using data on the concentrations of contaminants from the Niagara River is presented.

## PERSPECTIVES DE GESTION

Les méthodes statistiques pour l'analyse des données environnementales sont souvent limitées à l'utilisation des distributions normale ou log-normale, comme modèles pour la description des données. Cela restreint la portée des applications et peut entraîner une mauvaise interprétation, surtout lorsque le modèle de probabilité réel s'écarte sensiblement de celui qui est utilisé dans l'analyse. Raisons fondamentales de l'emploi de ces distributions : il est très simple d'obtenir des statistiques sommaires et d'effectuer les tests de signification. Dans la présente communication, on étend la classe des modèles normaux et log-normaux à la famille des modèles de paramètres de moyenne et de variance. Cette extension permet une analyse plus adéquate des données environnementales. De plus, on donne une estimation complète et simplifiée pour les caractéristiques de cette classe, qui peut être facilement informatisée. Pour illustrer ces méthodes, on présente plusieurs exemples avec les données relatives aux concentrations de contaminants de la rivière Niagara.

AN APPROXIMATION OF LIKELIHOOD FUNCTION  
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**ABSTRACT**

Likelihood function for the location and scale family of distributions is approximated so that closed form estimates for the location and scale parameters can be obtained. The results are then extended to the particular case of t-distributions with unknown degrees of freedom. The data is transformed so that the transformed data nearly follows a location and scale family of t-distribution. Simulation studies are done to examine the performance of the estimators and an application using data on the concentrations of  $\alpha$ -BHC in Niagara River is presented.

**Key words:** Likelihood function, approximate likelihood function, log relative likelihood function, t-distribution, censored samples.

## RÉSUMÉ

On procède à l'approximation de la fonction de vraisemblance pour la moyenne et la variance de distribution de façon à pouvoir obtenir des estimations complètes pour les paramètres de moyenne et de variance. Les résultats sont ensuite élargis aux cas particuliers de la loi de t à degrés de liberté inconnus. Les données sont transformées de telle façon que les données transformées suivent presque une moyenne et une variance selon la loi de t. Des études de simulation sont effectuées pour évaluer la performance des estimateurs et on présente une application utilisant les données relatives aux concentrations de  $\alpha$ -BHC dans la rivière Niagara.

Mots clés : fonction de vraisemblance, fonction de vraisemblance approchée, fonction de vraisemblance log-relative, loi de t, échantillons censurés

## INTRODUCTION

In many situations it is not possible to obtain closed form solutions to the maximum likelihood equations. This paper presents an approximation to the likelihood function of the location and scale family of distributions. This approximation results in closed form estimates for the location and scale parameters for both complete and censored samples. The closed form estimates for the particular case of t-distribution with unknown degrees of freedom are also presented. Tiku and Suresh (1991) have modified likelihood equations rather than likelihood function to obtain closed form estimates for the location and scale parameters of the location and scale family of t-distributions with known degrees of freedom. Simulation studies are conducted to examine the performance of the estimators presented in this paper. Also, an application using data on the concentrations of  $\alpha$ -BHC in water samples from the Niagara River is presented. The data is transformed so that the transformed data nearly follows a location and scale family of t-distribution. This provides a wider class of distributions than that obtained under the Box and Cox (1964) transformation.

### ESTIMATING THE PARAMETERS OF THE LOCATION AND SCALE FAMILY OF DISTRIBUTIONS

Let  $\Phi(x)$  and  $\phi(x)$  respectively denote the probability distribution function and the probability density function of  $x_i$  ( $i=1,\dots,n$ ). Then the likelihood function for the location and scale family of distributions is proportional to

$$L = \bar{\sigma}^n \prod_{i=1}^n \phi\left(\frac{x_i - \mu}{\sigma}\right),$$

where  $x_1, \dots, x_n$  are the ordered observations. Let  $y_i = \frac{x_i - \mu}{\sigma}$  and  $E(y_i) = t_i$ . Then  $t_i$  may be estimated by  $\tilde{t}_i = \Phi^{-1}(P_i)$ , where  $P_i = i/n+1$ . If

$$s_i^{(j)} = \frac{d \ln \phi(t_i)}{dt_i^j} \mid t_i = \tilde{t}_i, j = 1, 2, \text{ and } k = \ln \phi(\tilde{t}_i) - \frac{1}{2} (s_i^{(1)})^2 / s_i^{(2)}.$$

Then, using Taylor expansion, we have

$$\ln \phi(y_i) = k + \frac{1}{2} s_i^{(2)} \left\{ y_i + \frac{s_i^{(1)}}{s_i^{(2)}} - \tilde{t}_i \right\}^2 + o(\frac{1}{n}).$$

Hence,

$$\ln L = n k - n \ln \sigma + \frac{1}{2} \sum_{i=1}^n s_i^{(2)} \left\{ y_i + \frac{s_i^{(1)}}{s_i^{(2)}} - \tilde{t}_i \right\}^2,$$

and thus

$$L \approx L^* = \exp(n k) \bar{\sigma}^n \prod_{i=1}^n \exp\left(\frac{s_i^{(2)}}{2} \left\{ y_i + \frac{s_i^{(1)}}{s_i^{(2)}} - \tilde{t}_i \right\}^2\right).$$

we have

$$l^* = \ln L^* = n k - n \ln \sigma + \sum_{i=1}^n \frac{s_i^{(2)}}{2} \left\{ \frac{x_i - \mu}{\sigma} + \frac{s_i^{(1)}}{\frac{s_i^{(2)}}{2} - \tilde{t}_i} \right\}^2.$$

Equating  $\frac{\partial l^*}{\partial \mu}$  and  $\frac{\partial l^*}{\partial \sigma}$  to zero we obtain

$$\mu = \left( \sum_{i=1}^n s_i^{(2)} x_i + \sigma \sum_{i=1}^n (s_i^{(1)} - \tilde{t}_i) \frac{s_i^{(2)}}{s_i^{(1)}} \right) / \sum_{i=1}^n s_i^{(2)} = A + B\sigma,$$

and

$$\begin{aligned} & [n+B \left\{ \sum_{i=1}^n s_i^{(2)} (B + \tilde{t}_i) - \sum_{i=1}^n s_i^{(1)} \right\}] \sigma^2 + \left[ \sum_{i=1}^n (x_i - A) \{ s_i^{(1)} - s_i^{(2)} (\tilde{t}_i + 2B) \} \right] \sigma \\ & + \sum_{i=1}^n s_i^{(2)} (x_i - A)^2 = D\sigma^2 + E\sigma + F = 0. \end{aligned}$$

Hence,

$$\sigma = \frac{(-E + \sqrt{E^2 - 4FD})}{2D} \quad (1)$$

and

$$\mu = A + B\sigma. \quad (2)$$

The above equations provide closed form estimates for the location and scale parameters. For the t distribution with  $v$  (fixed) degrees of freedom we have

$$\begin{aligned} s_i^{(1)} &= -\frac{v+1}{v} \frac{\tilde{t}_i}{1 + \tilde{t}_i/v}, \text{ and } s_i^{(2)} = -\frac{v+1}{v} \frac{\tilde{t}_i^2/v}{(1 + \tilde{t}_i/v)^2}. \end{aligned}$$

For large  $v$ ,  $s_i^{(1)} \rightarrow -\tilde{t}_i$  and  $s_i^{(2)} \rightarrow -1$ . Hence, we obtain the usual maximum likelihood estimates in the normal case. The above equations along with Equations (1) and (2) provide closed form estimates for the location and scale parameters in the case of t distribution when the degrees of freedom is known. When the degrees of freedom is not known, for each  $v$  in  $\{1, 2, \dots, n_1\}$ , we first obtain the estimate  $\hat{\mu}_v$ , and  $\hat{\sigma}_v$ , and then evaluate the function

$$g(v) = n(\ln k_v - \ln \hat{\sigma}_v) - \left(\frac{v+1}{2}\right) \sum_{i=1}^n \ln\left(1 + \frac{(x_i - \hat{\mu}_v)^2}{v\hat{\sigma}_v^2}\right).$$

The estimate  $\hat{v}$  of  $v$  is then chosen so that  $\hat{g}(\hat{v}) \geq g(v)$  for all values of  $v$  in  $\{1, 2, \dots, n_1\}$ .

The asymptotic variance - covariance matrix for  $\hat{\mu}$  and  $\hat{\sigma}$  can be obtained by noting that

$$\frac{\partial^2 \ln L^*}{\partial \mu^2} = \frac{1}{\sigma^2} \sum_{i=1}^n s_i^{(2)},$$

$$\frac{\partial^2 \ln L^*}{\partial \mu \partial \sigma} = \frac{1}{\sigma^2} \sum_{i=1}^n s_i^{(2)} \left\{ \frac{x_i - \mu}{\sigma} + \frac{s_i^{(1)}}{s_i^{(2)}} - \tilde{t}_i \right\} + \frac{1}{\sigma^3} \sum_{i=1}^n s_i^{(2)} (x_i - \mu),$$

$$\frac{\partial^2 \ln L^*}{\partial \sigma^2} = \frac{n}{\sigma^2} + \frac{2}{\sigma^3} \sum_{i=1}^n s_i^{(2)} \left\{ \frac{x_i - \mu}{\sigma} + \frac{s_i^{(1)}}{s_i^{(2)}} - \tilde{t}_i \right\} (x_i - \mu) + \frac{1}{\sigma^4} \sum_{i=1}^n s_i^{(2)} (x_i - \mu)^2.$$

The results of the previous section can be extended to the censored samples. Consider Type I censoring and let  $D$  be the censoring level. Furthermore, let  $n_0$  and  $n$  be the number of censored and uncensored observations, respectively. The likelihood function is

$$L_1 = C_1 \sigma^{-n} (\Phi(\zeta))^{n_0} \prod_{i=1}^n \phi(\zeta_i) ,$$

where  $\zeta = \frac{D-\mu}{\sigma}$ ,  $\zeta_i = \frac{x_i-\mu}{\sigma}$  ( $i = 1, \dots, n$ ) and  $x_1 \leq x_2 \leq \dots \leq x_n$ . Define

$$\hat{\zeta} = \Phi^{-1}\left(\frac{n_0}{n_0+n}\right), \quad \hat{\zeta}_i = \Phi^{-1}\left(\frac{n_0+i}{n_0+n+1}\right).$$

By noting that

$$\ln \Phi(\zeta) \approx K_1 + \frac{1}{2} S^{(2)} \left\{ \zeta + \frac{S^{(1)}}{S^{(2)}} - \hat{\zeta} \right\}^2 ,$$

$$\text{where } S^{(j)} = \frac{d^j \ln \Phi(\zeta)}{d\zeta^j} \mid \zeta = \hat{\zeta}, \quad K_1 = \ln \Phi(\tilde{t}_1) - \frac{1}{2} \frac{(S^{(1)})^2}{S^{(2)}} ,$$

and that

$$\ln \phi(\zeta_i) \approx K_2 + \frac{1}{2} S_i^{(2)} \left\{ \zeta_i + \frac{S_i^{(1)}}{S_i^{(2)}} - \hat{\zeta}_i \right\}^2 ,$$

$$\text{where } S_i^{(j)} = \frac{d^j \ln \phi(\zeta_i)}{d\zeta_i^j} \mid \zeta_i = \hat{\zeta}_i, \quad K_2 = \ln \phi(\tilde{t}_i) - \frac{1}{2} \frac{(S_i^{(1)})^2}{S_i^{(2)}} ;$$

the log likelihood function can be written as

$$\ln L_1 \approx \ln L_1^* = \ln C_2 - n \ln \sigma + \frac{n_0}{2} S^{(2)} \left\{ \zeta + \frac{S^{(1)}}{S^{(2)}} - \hat{\zeta} \right\}^2 + \frac{1}{2} \sum_{i=1}^n S_i^{(2)} \left\{ \zeta_i + \frac{S_i^{(1)}}{S_i^{(2)}} - \hat{\zeta}_i \right\}^2$$

$\frac{S_i^{(1)}}{S_i^{(2)}} - \hat{\zeta}_i \right\}^2$ , where  $C_2$  is a constant. Hence,

$$\frac{\partial \ln L_1^*}{\partial \mu} = n_0 S^{(2)} \left( -\frac{1}{\sigma} \right) \left\{ \zeta + \frac{S^{(1)}}{S^{(2)}} - \hat{\zeta} \right\} - \frac{1}{\sigma} \sum_{i=1}^n S_i^{(2)} \left\{ \zeta_i + \frac{S_i^{(1)}}{S_i^{(2)}} - \hat{\zeta}_i \right\},$$

$$\frac{\partial \ln L_1^*}{\partial \sigma} = -\frac{n}{\sigma} - n_0 S^{(2)} \left( \frac{\zeta}{\sigma} \right) \left\{ \zeta + \frac{S^{(1)}}{S^{(2)}} - \hat{\zeta} \right\} -$$

$$\sum_{i=1}^n S_i^{(2)} \left( \frac{\zeta_i}{\sigma} \right) \left\{ \zeta_i + \frac{S_i^{(1)}}{S_i^{(2)}} - \hat{\zeta}_i \right\}.$$

By solving the equations  $\frac{\partial \ln L^*}{\partial \mu} = 0$  and  $\frac{\partial \ln L^*}{\partial \sigma} = 0$  we obtain

closed form solutions for  $\mu$  and  $\sigma$  as in the complete sample case.

Similar results for Type II censoring can be obtained by replacing

$\zeta = \frac{D-\mu}{\sigma}$  by  $\eta = \frac{x_1-\mu}{\sigma}$  and noticing that  $\hat{\eta} = \Phi^{-1}\left(\frac{n_0}{n+1}\right)$ .

The estimation results provided in this paper may be used to fit a distribution to the data. The idea is to transform the data

$x_i$  ( $i=1, \dots, n$ ) to  $y_i$  ( $i=1, \dots, n$ ) where

$$y_i = \begin{cases} \frac{\lambda}{x_i - 1} & \text{if } \lambda \neq 0, \\ \frac{\lambda}{\ln x_i} & \text{if } \lambda = 0, \end{cases}$$

and where transformation parameter  $\lambda$  is chosen so that the transformed data nearly follows a location and scale family of t-distribution.

In practice, for each  $\lambda$  in  $\{0, \lambda_1, \dots, \lambda_{n_0}\}$ , the estimates  $\hat{\mu}_\lambda$ ,  $\hat{\sigma}_\lambda$  and

$\hat{\nu}_\lambda$  for  $\mu$ ,  $\sigma$  and  $\nu$ , respectively, are obtained and then the estimate

$\hat{\lambda}$  of  $\lambda$  is chosen as the value that maximizes  $h(\hat{\mu}_\lambda, \hat{\sigma}_\lambda, \hat{\nu}_\lambda) =$

$$n(\ln k_\nu - \ln \hat{\sigma}) + (1 - \frac{1}{\lambda}) \sum_{i=1}^n \ln(1+\lambda y_i) - (\frac{\nu+1}{2}) \sum_{i=1}^n \ln(1 + \frac{(y_i - \hat{\mu}_\lambda)^2}{\nu \hat{\sigma}_\lambda^2}) .$$

The estimates  $\hat{\lambda}$ ,  $\hat{\mu}_\lambda$  and  $\hat{\sigma}_\lambda$  then completely determine the distributional form of the original data.

## SIMULATION RESULTS AND APPLICATION

Simulation experiments were conducted to evaluate the performance of the estimates described in this paper. For given sample size  $M$  and degrees of freedom  $\nu$ , samples from t-distribution with location and scale parameters 0 and 1, respectively, were generated using the International Mathematical and Statistical Libraries (IMSL, 1987). Tables 1a and 1b give the estimation results using the exact and modified likelihood functions, respectively. These results indicate

that the estimates resulting from the use of the modified likelihood function provide good approximations to the exact maximum likelihood estimates. Simulation experiments were also performed to examine the dependency of the approximate estimates of the mean and standard deviation on the true values of the location and scale parameters. The results are shown in Figures 1a, 1b, 2a and 2b. These Figures compare the approximate and exact estimates of the mean and standard deviation for high (40) and low (5) degrees of freedom. The results suggest that approximate estimates are close to the exact estimates except for one case when estimating the standard deviation with low degrees of freedom. In this case, the differences between the approximate and exact estimate of the standard deviation increases with  $\sigma$ . Table 2 presents the simulation results for the estimation of degrees of freedom of t-distribution from a sample of size 100. These results indicate that the estimates are close to the true value when estimating a relatively high degrees of freedom. The simulation results for the case of censored samples are shown in Tables 3a and 3b. The methods of this paper were also applied to data representing the concentrations (nanograms per liter) of  $\alpha$ -BHC in water samples from Niagara River collected by Environment Canada at Niagara-on-the-Lake stations for 1987-88 study period. Information regarding the data can be found in the report prepared by Data Interpretation Group (1989). Tables 4a gives the estimate of the transformation parameter as well as the estimates of the location and scale parameters in addition to the estimate of the degrees of freedom for the transformed data. Table 4b compares the approximate estimates of the mean and standard deviation of the data to the sample mean and standard deviation. The results indicate that the approximate

estimates are close to the estimates of the sample mean and standard deviation. The log relative likelihood function for the transformation parameter  $\lambda$  is shown in Figure 3a. A 95% confidence interval for  $\lambda$  consists of those values of  $\lambda$  for them the values of the log relative like function lie above the solid line. Figure 3b shows the QQ-plot for the transformed data. This plot indicates that there is a good agreement between the observed and fitted model.

#### CONCLUDING REMARKS

This paper provides closed form estimates for the location and scale family of distributions in general and the location and scale family of t-distribution in particular. The simulation results indicate good agreement between the exact and approximate estimates of t-distribution except when estimating the standard deviation in the case of low degrees of freedom. Hence, the approximate estimates presented here can be used when the transformed data follows nearly a location and scale family of t-distribution with a relatively high degrees of freedom. This, of course, generates a wide class of distributions for the original data that includes the class of distributions generated by the Box-Cox Power transformation. Furthermore, although todays fast computers can provide solutions to the exact likelihood equations; they require initial solutions which in many cases are difficult to guess. The approximate estimates presented here may always be considered as good initial solutions.

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## FIGURES

- Figure 1a. Approximate - Exact estimates of the mean and standard deviation as a function of  $\mu$  (5 degrees of freedom).
- Figure 1b. Approximate - Exact estimates of the mean and standard deviation as a function of  $\mu$  (40 degrees of freedom).
- Figure 2a. Approximate - Exact estimates of the mean and standard deviation as a function of  $\sigma$  (5 degrees of freedom).
- Figure 2b. Approximate - Exact estimates of the mean and standard deviation as a function of  $\sigma$  (40 degrees of freedom).
- Figure 3a. Log relative likelihood function for  $\lambda$  from a-BHC data.  
The solid line determines a 95% confidence interval for  $\lambda$ .
- Figure 3b. QQ-plot for the transformed data.

Sample Size	Degrees of Freedom	$\hat{\mu}$	$\hat{\sigma}$	$v_{11}$	$v_{12}$	$v_{22}$
20	5	-.00591	.98132	.06401	.00002	.02115
	10	-.00087	.95971	.05474	.00005	.02256
	20	-.00033	.95936	.05118	.00004	.02308
	30	-.00122	.96964	.05061	-.00001	.02371
	40	-.00642	.96651	.04989	-.00002	.02375
	50	.01332	.96350	.04942	.00002	.02374
60	5	-.00471	.99215	.02189	.00003	.00678
	10	.00503	.98739	.01919	-.00002	.00767
	20	-.00599	.99012	.01788	.00001	.00803
	30	-.00422	.98930	.01740	-.000009	.00812
	40	-.00415	.98465	.01709	.000008	.00813
	50	.00424	.98351	.01694	-.00001	.00812
100	5	.00153	.99566	.01324	-.000007	.00389
	10	-.00409	.99665	.01168	-.000001	.00459
	20	-.00296	.99139	.01079	.000007	.00484
	30	-.00448	.98952	.01048	4.60580E-7	.00488
	40	-.00269	.99642	.01040	.000003	.00493
	50	-.00239	.99183	.01027	2.41323E-7	.00493

Table 1a. Exact estimates of the location and scale parameters of t-distribution along with the elements of the observed information matrix.

Sample Size	Degrees of Freedom	$\hat{\mu}$	$\hat{\sigma}$	$v_{11}$	$v_{12}$	$v_{22}$
20	5	-.00741	1.09533	.09986	.00667	.13134
	10	-.00219	.99299	.06381	-.00038	.03896
	20	-.00141	.97483	.05595	-.00011	.03046
	30	-.00110	.97808	.05421	-.00004	.02836
	40	-.00603	.97287	.05265	.00018	.02735
	50	.01276	.96864	.05167	-.00034	.02675
60	5	-.00875	1.10831	.02962	.00031	.03787
	10	.00609	1.01775	.02102	-.00002	.01248
	20	-.00681	1.00082	.01870	.00005	.01006
	30	-.00373	.99519	.01791	.00004	.00937
	40	-.00464	.98789	.01737	.00002	.00898
	50	.00498	.98701	.01716	-.00002	.00887
100	5	.00275	1.10875	.01714	-.00008	.01621
	10	-.00369	1.03003	.01280	.00004	.00761
	20	-.00376	1.00059	.01111	.000007	.00595
	30	-.00456	.99459	.01065	.00002	.00558
	40	-.00299	1.00019	.01061	.00001	.00547
	50	-.00242	.99411	.01039	.00001	.00533

Table 1b. The estimation results using the modified likelihood function.

Degrees of Freedom	Estimate
5	7.7
10	13.8
20	22.3
30	31.6
40	40.7
50	51.4

**Table 2.** Estimates of degrees of freedom of t-distribution from a sample of size 100.

No. of data	No. of Uncensored data	Degrees of Freedom	Detection Limit	$\hat{\mu}$	$\hat{\sigma}$
20	16	5	-1.	.00106	1.00888
20	16	30	-1.	-.00630	.98167
100	81	5	-1.	-.00233	.99954
100	83	30	-1.	.00315	.99515

Table 3a. The simulation results for low-level censoring. The actual values of  $\mu$  and  $\sigma$  are 0 and 1 respectively.

No. of data	No. of Uncensored data	Degrees of Freedom	Detection Limit	$\hat{\mu}$	$\hat{\sigma}$
20	10	5	0.	.00295	1.01385
20	9	30	0.	-.01806	.99401
100	50	5	0.	-.00269	1.00431
100	50	30	0.	.00552	.99948

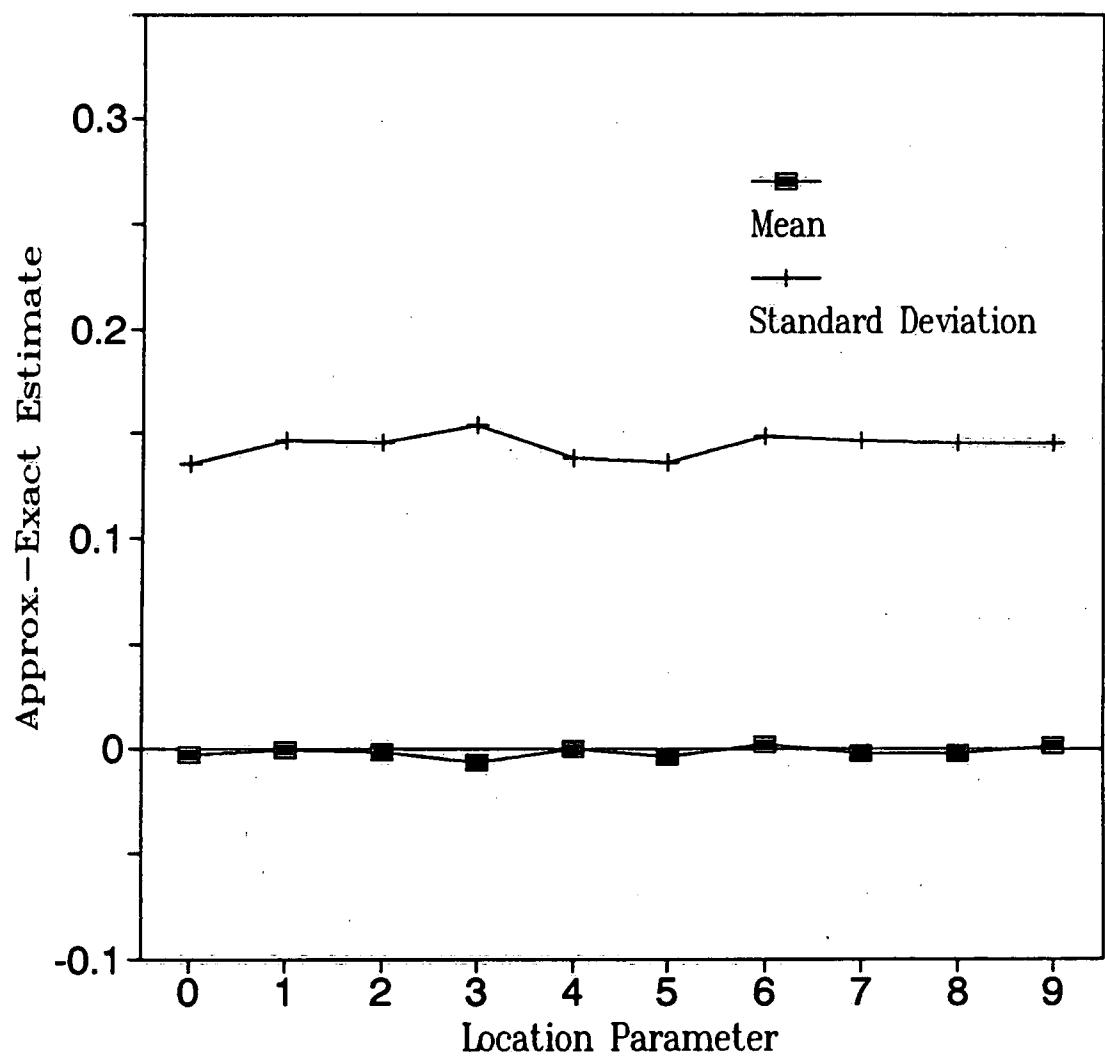
Table 3b. The simulation results for high-level censoring. The true values of  $\mu$  and  $\sigma$  are 0 and 1 respectively.

Data	Sample Size	$\hat{\lambda}$	$\hat{\nu}$	$\hat{\mu}$	$\hat{\sigma}$
a-BHC	44	.85	$\infty$	1.49115	.66334

**Table 4a.** Estimates of the transformation parameter for A-BHC in Niagara on the lake water as well as degree of freedom, and location and scale parameters for the transformed data.

Data	Sample Size	Estimate of the mean	Estimate of the Standard deviation	Sample Mean	Sample Standard Deviation
a-BHC	44	2.63863	.77737	2.63727	.76571

**Table 4b.** Comparison between the estimates of the mean and standard deviation of A-BHC data with sample mean and standard deviation.



**Figure 1a.** Approximate - Exact estimates of the mean and standard deviation as a function of the location parameter ( 5 degrees of freedom ).

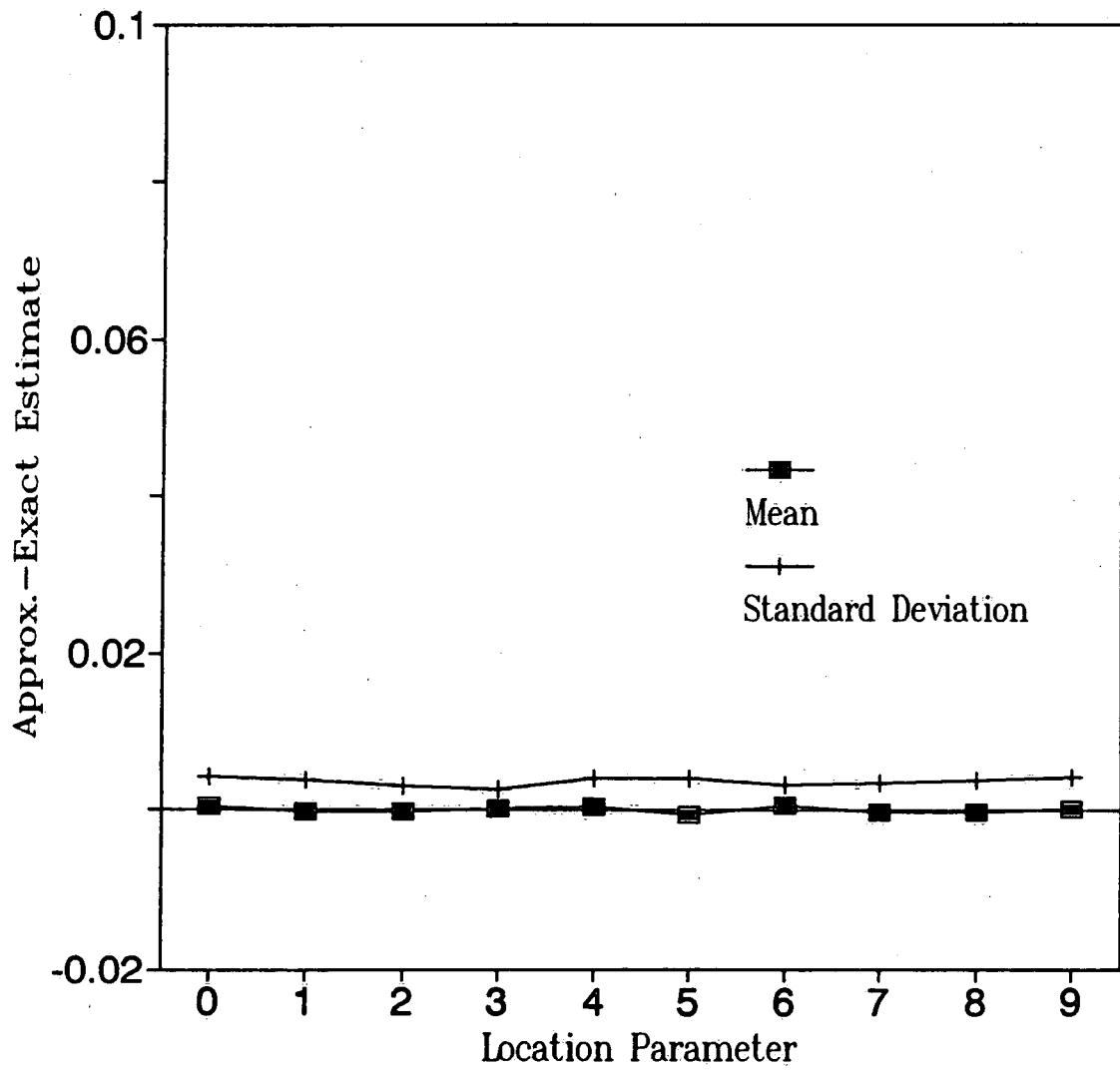


Figure 1b. Approximate - Exact estimates of the mean and standard deviation as a function of the location parameter ( 40 degrees of freedom ).

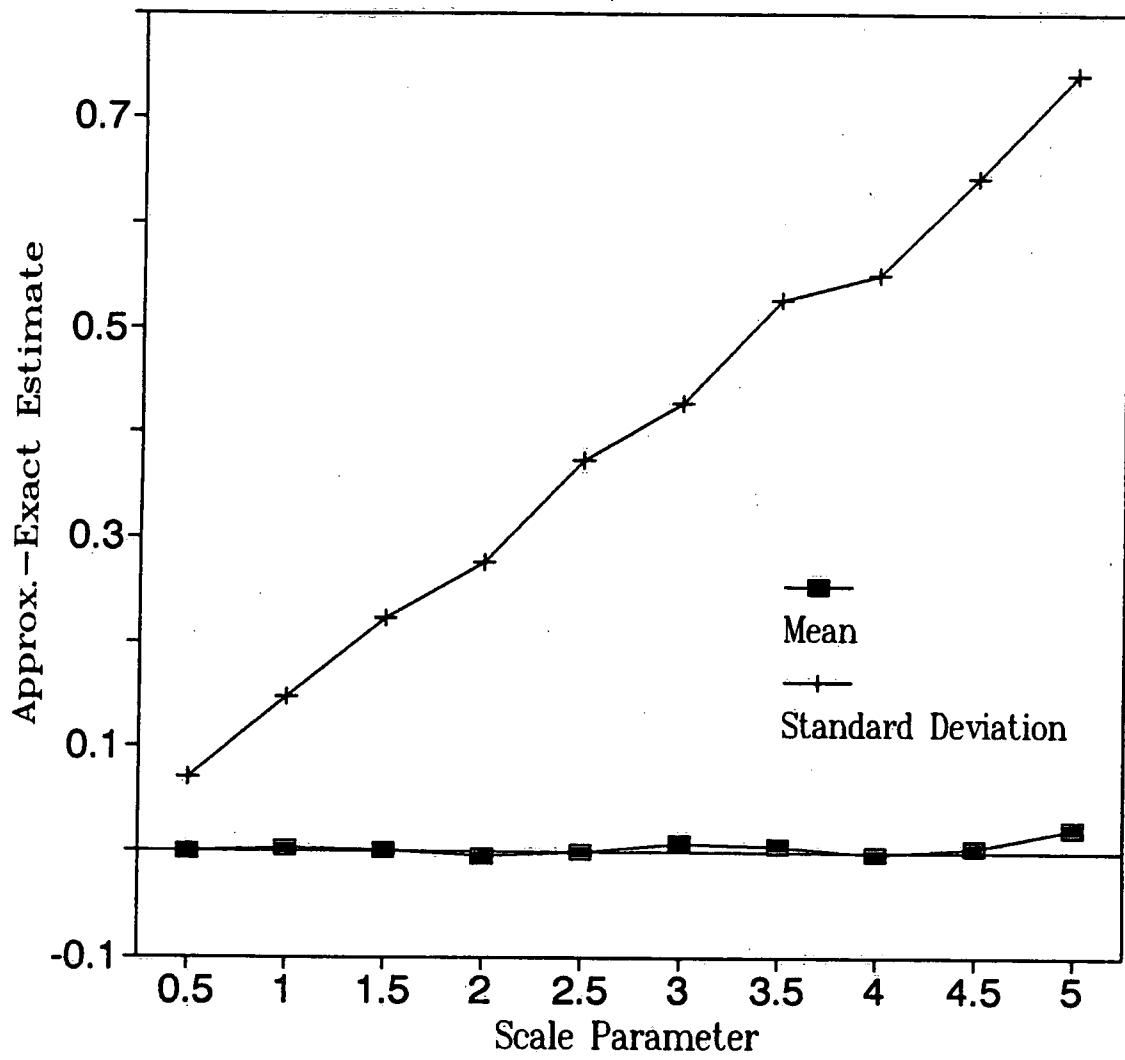


Figure 2a. Approximate - Exact estimates of the mean and standard deviation as a function of the scale parameter ( 5 degrees of freedom ).

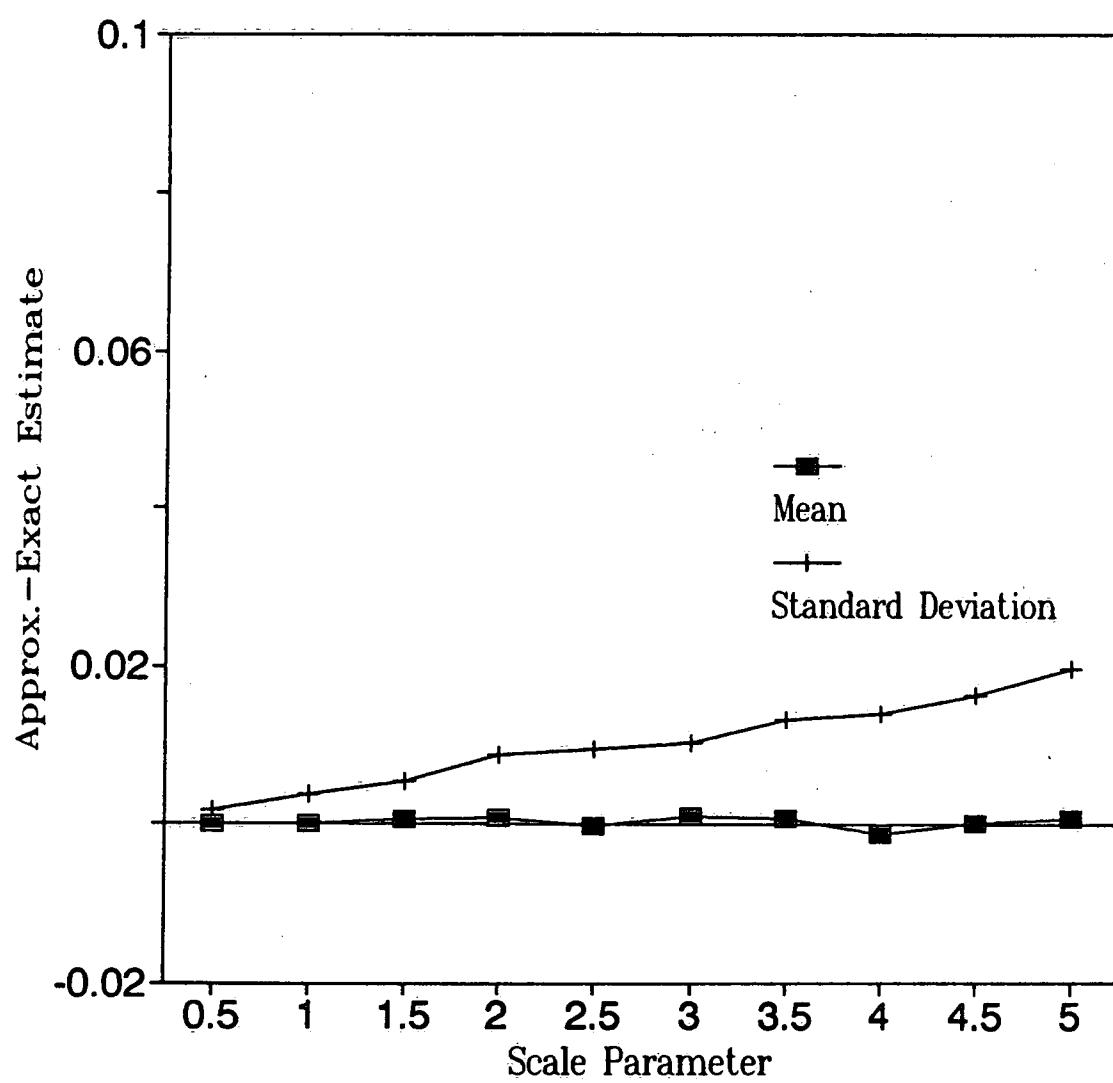


Figure 2b. Approximate - Exact estimates of the mean and standard deviation as a function of the scale parameter ( 40 degrees of freedom ).

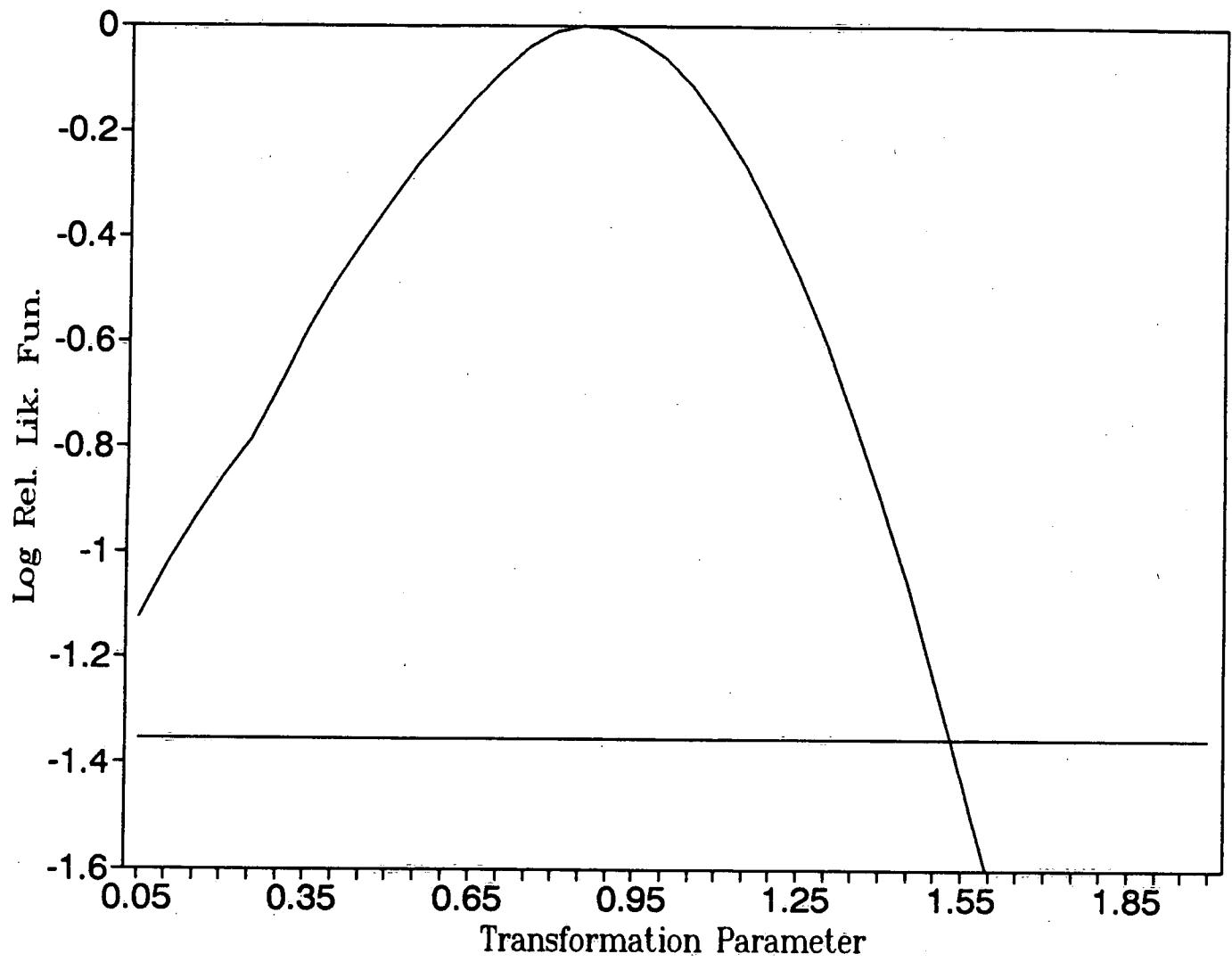


Figure 3a. Log relative likelihood function for the transformation parameter. The solid line determines a 95% confidence interval for the transformation parameter.

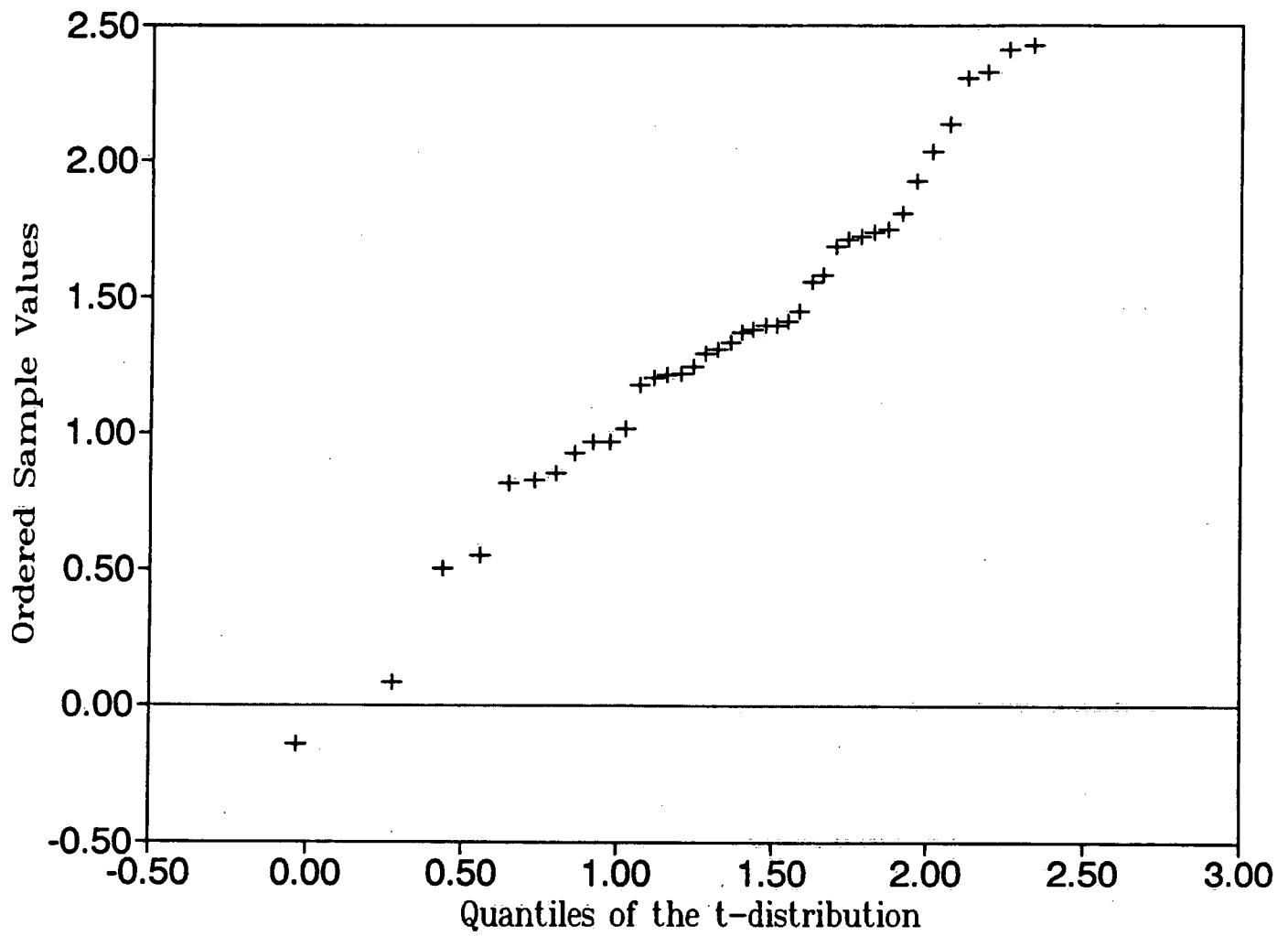
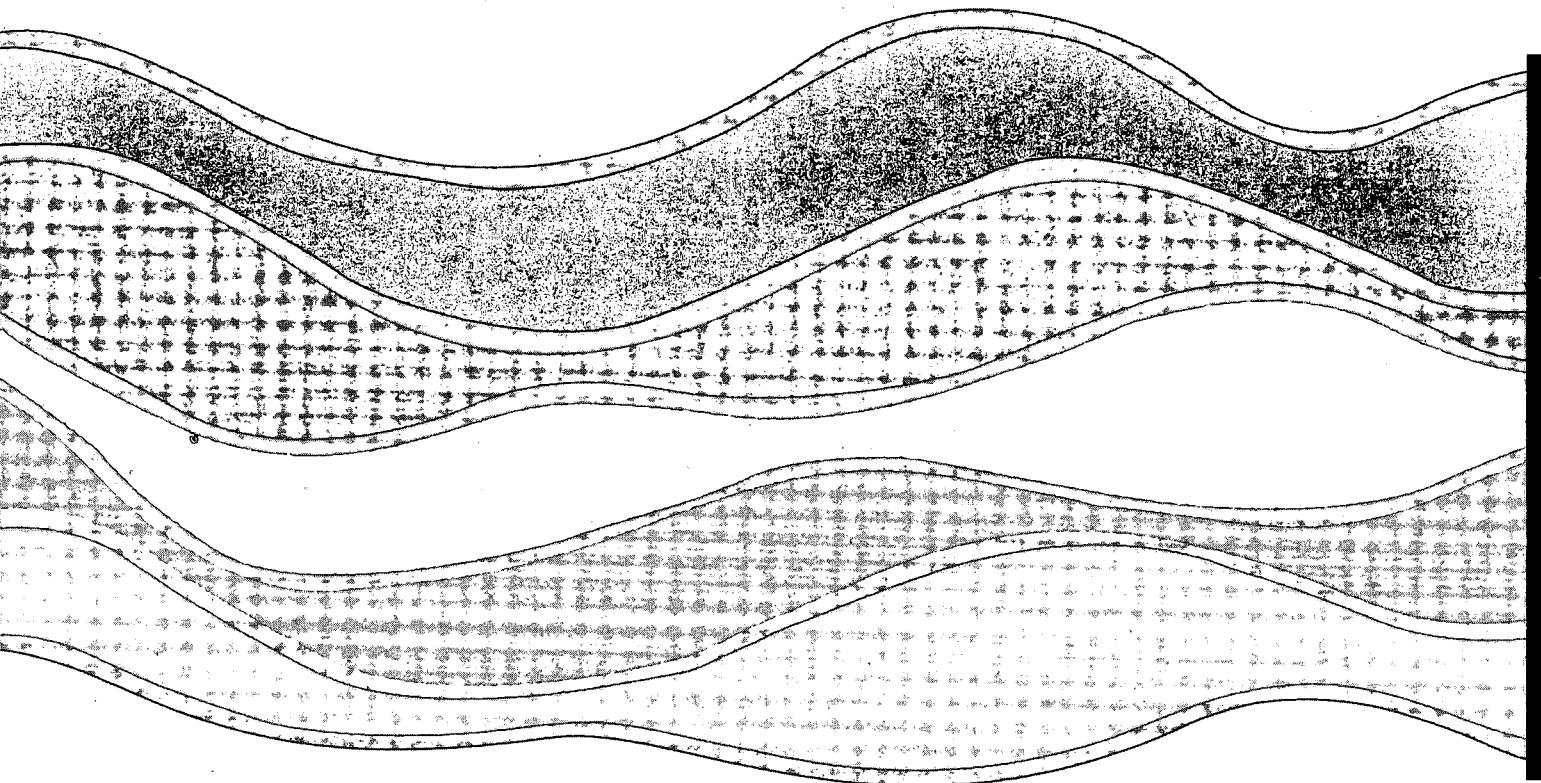


Figure 3b. QQ - plot for the transformed data .

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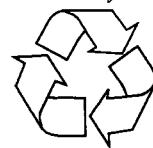


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