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EVOLVING VELOCITY FIELDS UNDER COALESCING WAVE GROUPS

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MANAGEMENT PERSPECTIVE

The transfer of toxics at the air-water interface, the mixing of the surfave waters of lakes and oceans and the design of offshore structures are all dependent on, amongst other things, complex nonlinear interactions of the surface waves. This paper describes some of the nonlinear processes for wave groups and as such is a valuable resource document for work in these programme areas.

PERSPECTIVE DE GESTION

Le transfert de substances toxiques à l'interface eau-air, le mélange des eaux de surface des lacs et des océans et le dessin des structures marins dépendent, entre autres facteurs, des interactions complexes non-linéaires entre les vagues de surface. Cette étude décrit certains processus non-linéaires pour les groupes des vagues et constitue comme telle un précieux document de référence pour ces domaines.

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ABSTRACT

Much of ocean wave theory is built upon the assumption that the ocean surface is made up of a field of random waves, each propagating independently according to the laws of linear theory. Increasingly, however, as more accurate prediction models are required, attention has been focussed on the understanding of phenomena - such as wave group propagation - which are inherently nonlinear. For instance, it is now known that nonlinear interactions among components in a wave group can have important consequences on the growth and evolution of individual waves.

A fully nonlinear method, based on Fenton's Fourier wave theory, was employed to study the evolution of wave groups in a wave tank. With our numerical model, we looked at the coalescence of a prototype wave group, the Gaussian wave packet. In particular, the evolution of the velocity field associated with the coalescing waves was investigated. The theoretical results have been compared with measurements made using a surface follower and an acoustic current metre in the 100 metre laboratory tank at the National Water Research Institute.

RÉSUMÉ

La théorie moderne des vagues est fondée en grande partie sur la premisse que la surface de la mer est composée d'un champ d'ondes, chacune d'elles se propageant suivant les lois de la théorie linéaire. Toutefois, face au besoin de modèles de prédiction plus précis, la compréhension des phénomènes non-linéares, tel que la propagation des groupes de vagues, est devenu nécessaire. Par exemple, c'est déjà connu que les interactions non-linéaires entre les composantes d'un groupe de vagues peuvent avoir des conséquences importantes sur la croissance et l'évolution des vagues individuelles.

Notre étude de l'évolution des groupes de vagues dans un canal à houle se fonde sur la usage d'une méthode non-linéaire qui est basée sur la théorie des ondes Fourier d'aprés Fenton. Avec notre algorithme numérique, nous étudions la coalescence d'un ensemble Gaussien des vagues utilisé comme prototype. Spécifiquement, l'évolution du champ de velocité du groupe de vagues sera étudiée. Les résultats théoriques seront comparés aux donnés expérimentales faites dans la canal à houle de 100 mètres à l'INRE avec un courantomètre acoustique couplé à un dispositif de mésure du déplacement de la surface.

1. INTRODUCTION AND FORMULATION

We consider the problem of the two dimensional, irrotational evolution of nonlinear wave groups travelling through an incompressible, inviscid fluid in wave tank of length L. We employ a Cartesian coordinate system (x, y) centred at a bottom corner of the tank with x in the horizontal direction and y pointing upwards. The water surface is denoted $y = \eta(x,t)$. For convenience, we use L and the gravitational constant g to render all variables dimensionless. Under the above assumptions, the fluid motion in the wave tank is described by Laplace's equation

$$\nabla^2\phi = 0, \tag{1}$$

with boundary conditions

$$\phi_y = 0 \text{ on } y = 0 \tag{2}$$

$$\eta_t + \phi_x\eta_x - \phi_y = 0 \text{ on } y = \eta(x,t) \tag{3}$$

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + \eta = 0 \text{ on } y = \eta(x,t) \tag{4}$$

$$\phi \text{ and } \eta \;\; 2\pi\text{-periodic in } x \tag{5}$$

and initial conditions

$$\eta(x,0) = f(x) \tag{6}$$

$$\eta_t(x,0) = g(x), \tag{7}$$

where the subscripts denote partial differentiation. Note that boundary condition (5) effectively allows the group to pass though one tank wall and reappear out of the other. Although not physically realistic, this artifice permits us to follow the evolution of the wave group over distances beyond one tank length. The reader is referred to Drennan, Fenton and Donelan (1990) where reflections off the end walls are taken into account.

The complexity of the above system, which is nonlinear and involves a free boundary, has focussed most attention on numerical solutions. Longuet-Higgins and Cokelet (1976) and Dold and Peregrine (1984) have employed boundary integral techniques to follow the evolution of a plunging wave almost to the point of breaking. Such techniques, however, are long and complicated to implement. For flows in which the wave surface, $\eta(x,t)$ remains single valued, there exists an alternative approach which is both accurate and relatively simple.

2. THE FOURIER METHOD

The Fourier or pseudospectral method employs truncated Fourier series to approximate horizontal (spatial) variation with a finite difference scheme for temporal variation. The method was proposed by Orszag (1971) and has subsequently been refined by Fenton and Rienecker (1982) in their investigations of solitary wave interaction. A useful summary of recent work applying the Fourier method to the steady wave problem can be found in Sobey (1989).

In Fourier wave theory, it is assumed that ϕ and η are 2π -periodic in the x-direction. We can then write

$$\phi(x, y, t) = \sum_{j=0}^{N-1} A_j(t) \frac{\cosh jy}{\cosh jd} e^{-ijx} \tag{8}$$

where N is the order of truncation, $i = \sqrt{-1}$ and d is the average depth. (The deep water case is handled by shifting the coordinate axis to the mean water level and replacing the hyperbolic function with exponentials.) This form satisfies Laplace's equation (1) as well as the bottom (2) and periodicity (5) conditions. The N coefficients $A_j(t)$ are found at each point in time by satisfying the kinematic (3) and dynamic (4) boundary conditions at N discrete points ($x_m = \frac{m\pi}{N-1}, m = 0, \dots, N-1$) of the free surface; the N free surface values $\eta(x_m, t)$ are found at the same time. With the Fourier method, the spatial derivatives are easily calculated from the original Fourier series. From the series for η,

$$\eta(x_m, t) = \eta_m = \sum_{j=0}^{N-1} B_j(t) e^{-ijx_m}, \tag{9}$$

the discrete Fourier transform yields

$$B_j(t) = \mathcal{F}(\eta) = \frac{2}{N-1} \sum_{n=0}^{N-1} \eta_n e^{(i\frac{jn\pi}{N-1})}. \tag{10}$$

The spatial derivative of η is then given by

$$\frac{\partial \eta}{\partial x}(x_m, t) = -i \sum_{j=0}^{N-1} j B_j(t) e^{-ijx_m}, \tag{11}$$

— i.e. the inverse Fourier transform of (jB_j). We note here that the accuracy of this step, and indeed of the method itself, relies on the capability to approximate the unknown functions ϕ and η by *truncated* Fourier series. If the functions are sufficiently smooth (nonsharp), the Fourier coefficients will decay with increasing frequency at an almost exponential rate, and the approximation will be an excellent one. If, however, there is a slope discontinuity, as with a sharp crest, the coefficients will decay slowly and a large number of them will be required to adequately represent the function — in this case truncation may result in errors that will rapidly render the solution useless. In practice, the magnitude of N, and therefore the steepness of waves that can be approximated using Fourier theory,is limited by the available computer memory.

In order to advance the solution in time, a second order finite difference scheme is employed. In particular, we use

$$\eta(x_m, t + \Delta) = \eta(x_m, t - \Delta) + 2\Delta \frac{\partial \eta}{\partial t}(x_m, t) + O(\Delta^3) \tag{12}$$

where Δ denotes the time step. Note that in order to advance the solution in time, we require initial values at times 0 and Δ. Given η and η_t at time 0 (by (6) and (7)), we calculate the A_j's from (3) by solving a linear system of equations. We then solve for η and the A_j's at time Δ using a first order finite difference scheme with ten steps of $\Delta/10$. Then, given solution vectors A and η at two times t and $t-\Delta$, the calculation of $A_j(t+\Delta)$ and $\eta(x_m, t+\Delta)$ proceeds as follows: at time t,

$$\begin{aligned}
\text{calculate} \quad \frac{\partial \phi}{\partial x} &= -i \sum_{j=0}^{N-1} j A_j \frac{\cosh j\eta_m}{\cosh jd} e^{-ijx_m} \\
\frac{\partial \phi}{\partial y} &= i \sum_{j=0}^{N-1} j A_j \frac{\sinh j\eta_m}{\cosh jd} e^{-ijx_m} \\
\frac{\partial \phi}{\partial t} &= -\eta - \frac{1}{2}\left(\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right),
\end{aligned}$$

$$\text{solve } N \text{ x } N \text{ system} \quad \frac{\partial \phi}{\partial t} = \sum_{j=0}^{N-1} \frac{dA_j}{dt} \frac{\cosh j\eta_m}{\cosh jd} e^{-ijx_m} \quad \text{for} \quad \frac{dA_j}{dt},$$

$$\begin{aligned}
\text{calculate} \quad \frac{\partial \eta}{\partial x} &= i\mathcal{F}^{-1}(j\mathcal{F}(\eta)) \\
\frac{\partial \eta}{\partial t} &= \frac{\partial \phi}{\partial y} - \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x},
\end{aligned}$$

$$\begin{aligned}
\text{and finally} \quad A_j(t + \Delta) &= A_j(t - \Delta) + 2\Delta \frac{dA_j}{dt}(t) \\
\eta(x_m, t + \Delta) &= \eta(x_m, t - \Delta) + 2\Delta \frac{\partial \eta}{\partial t}(x_m, t).
\end{aligned}$$

In summary, we note that the Fourier method requires two independent approximations. The first is that the Fourier series be truncated — this was discussed above. The second is the finite difference approximation to the time derivative. This latter error can be controlled to some degree through the choice of Δ, at the expense of increased computational time. A linear stability analysis for the scheme has been carried out by Fenton and Rienecker (1982) with the result that $\Delta \leq (N-1)^{-1/2}$. This yields an upper bound on the largest time step which can be employed at a given truncation order. As a check on the accuracy of the solution, the conservation of total energy requirement was strictly enforced. A sudden change in energy at some time step was a sign that stability was lost in the solution — this was typically corrected by decreasing the time step.

3. NUMERICAL RESULTS

We choose as our initial conditions the Gaussian wave packet given by

$$\eta(x) = -\frac{A}{D^{1/4}} e^{-\frac{16\pi^2 B^2}{T^2 g^2 D}(x-gTt/4\pi)^2}$$
$$\sin(\frac{4\pi^2 x}{DgT^2} - \frac{2\pi t}{DT} - \frac{4B^4 t^2 x}{Dg} + \frac{1}{2}\arctan 4B^2 x/g) \qquad (13)$$

where $D = 1 + 16B^4x^2/g^2$. In a recent set of experiments conducted in the National Water Research Institute's 100 metre wave tank by Doering (1991), low amplitude Gaussian wave packets were produced at the wave board and allowed to propagate linearly down the tank past a wave gauge where their passage was recorded. The dispersed wave trains were then reversed, amplified and fed as input signals to the wave board. The passage of the coalescing wave group was then recorded using a wave gauge and a Minilab SD-12 acoustic current meter. The current meter was connected to a surface follower which kept it at a fixed horizontal position and a constant distance below the surface. Details of the measurements may be found in Doering (1991).

By reversing the Gaussian wave packet and setting the parameters (A, B, T and t) in (13) to coincide with the experimental values we are able to simulate the group coalescence. The algorithm described above was coded in Matlab and implemented on a 386-computer. The following results are obtained using 256 Fourier components in the series approximations (i.e. $N = 256$) and a time step of size $\Delta = 0.005$. Figures 1 and 2 illustrate the comparison between laboratory measurements and numerical predictions. In Figure 1, we show time series of the horizontal velocity component, u, at a distance of 26.7 metres down the tank and a constant 1.6 cm. below the surface. The agreement between the surface follower measurements (a) and numerical solution (b) as the wave group passes is very good, although there are some discrepancies at larger times (greater than 12 seconds) where the numerical solution is losing accuracy. This is related to the formation of a sharp crest associated the maximum coalescence of the group and could be corrected by increasing the number of Fourier components. Figure 2 shows measured (a) and calculated (b) time series of the wave height, at a location coincident with the velocity measurements and, again, the agreement between the two is very good.

Finally, in Figure 3, we exploit the advantages of a numerical solution to look at space series of the horizontal surface velocity (equivalent to 256 wave staffs). The curves in the figure show the space series at three times, 0 (initial condition), 7.1 and 14.2 seconds, and illustrate the evolution of the horizontal velocity field as the wave group coalesces. Note that during the short time period involved, the maximum velocity occuring in the group has increased by close to 50 % ! Although much of this is, of course, due to linear superposition, nonlinear interactions result in about ten per cent of the gain (and this increases quickly with increasing amplitude).

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Figure 1a - Time series of u, 1.6 cm below surface

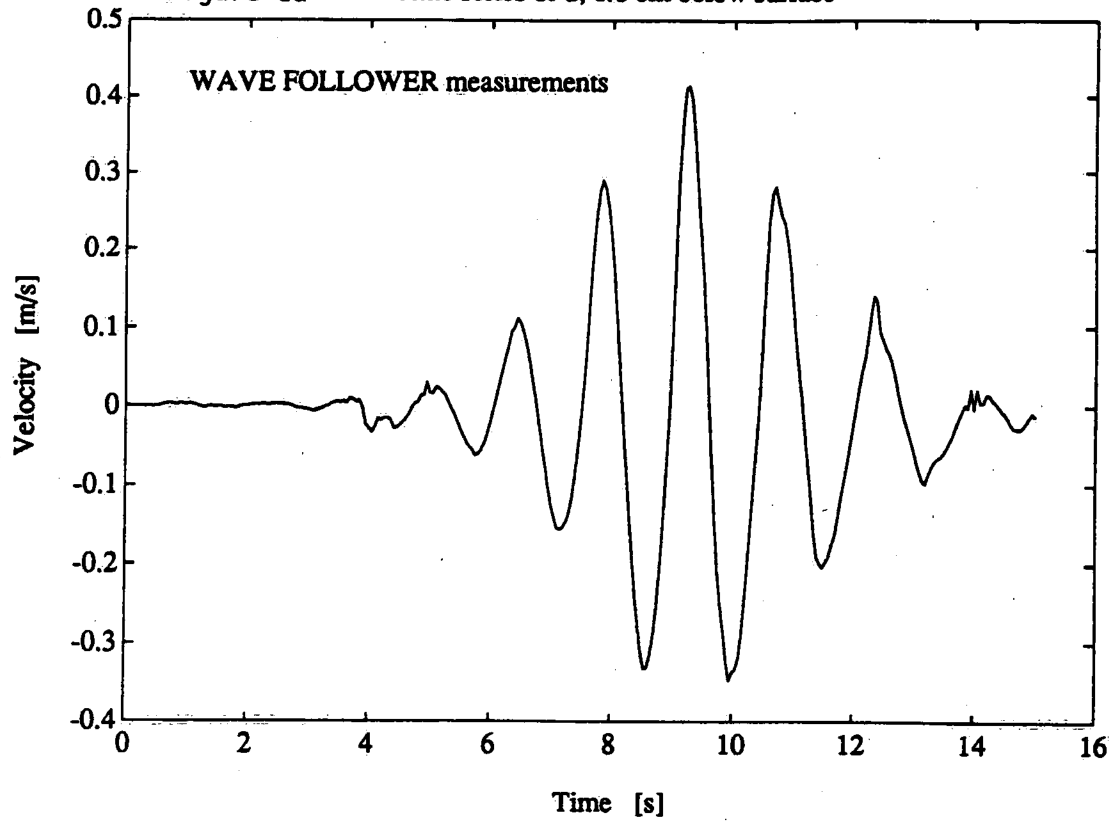


Figure 1b - Time series of u, 1.6 cm below surface

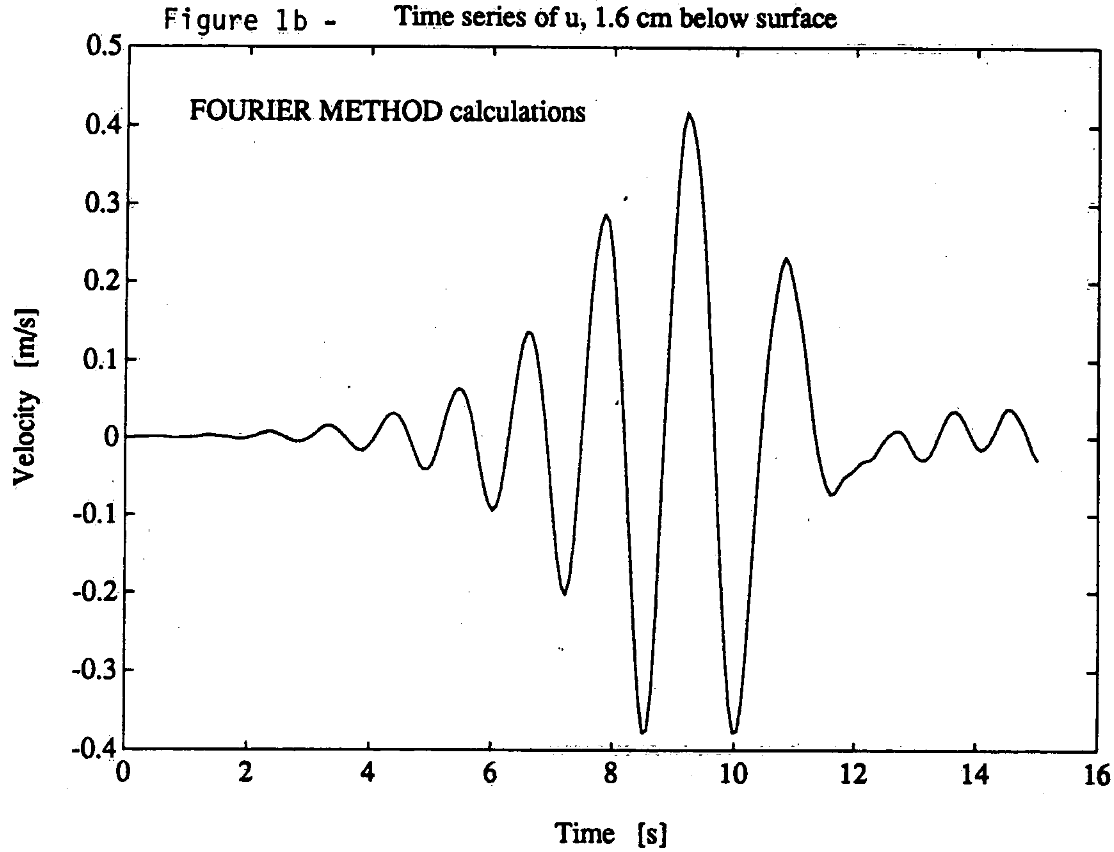


Figure 2a - Time series of h, 26.7 m from wave board

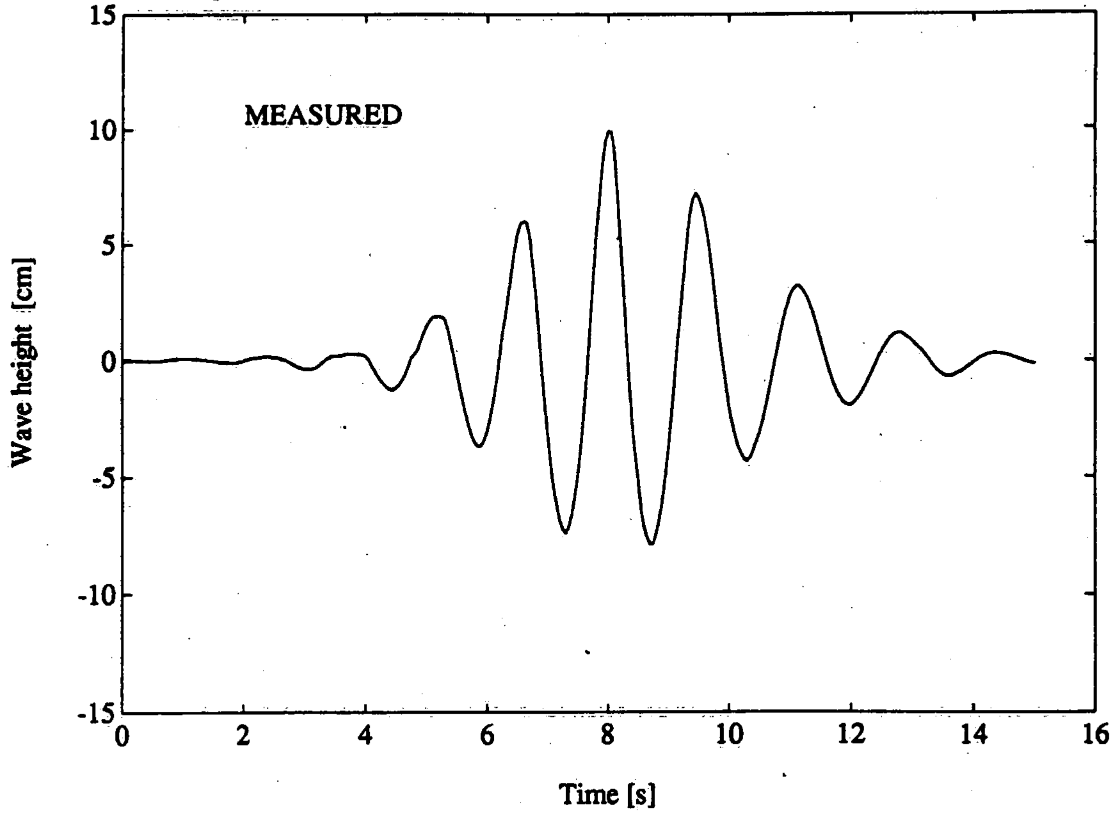


Figure 2b - Time series of h

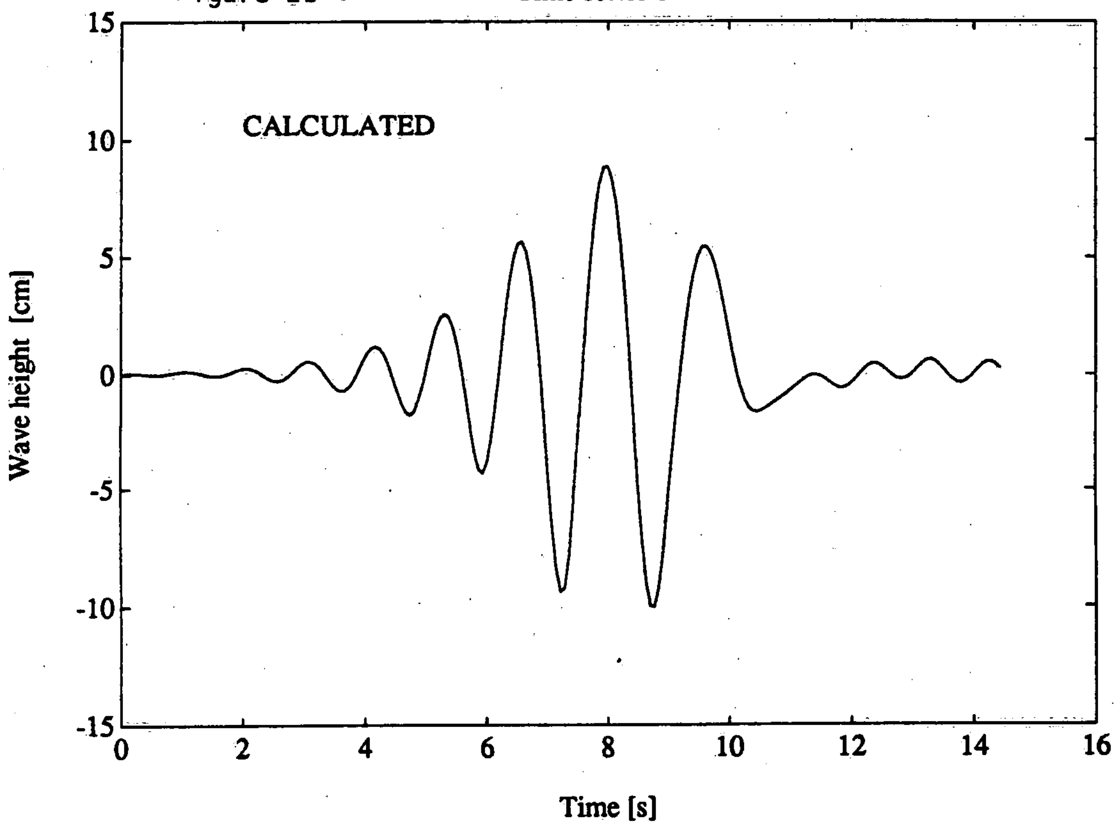
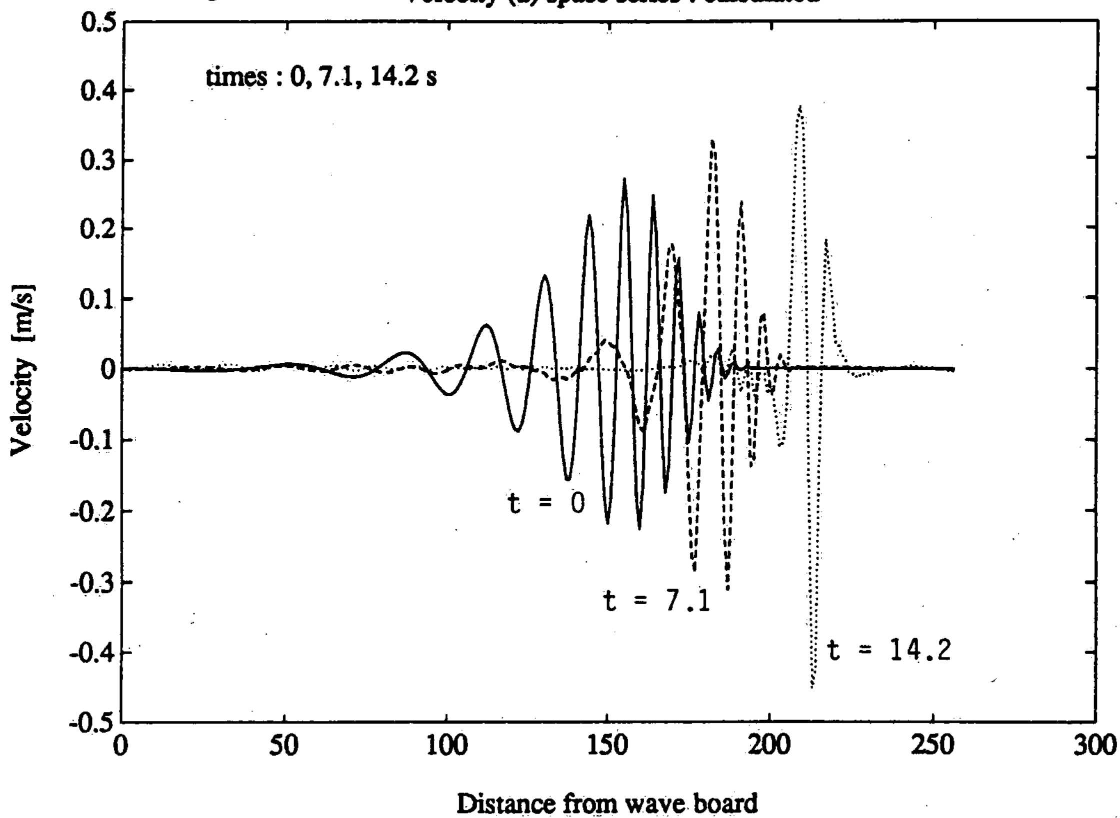


Figure 3 - Velocity (u) space series : calculated



(Full length = 256)

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