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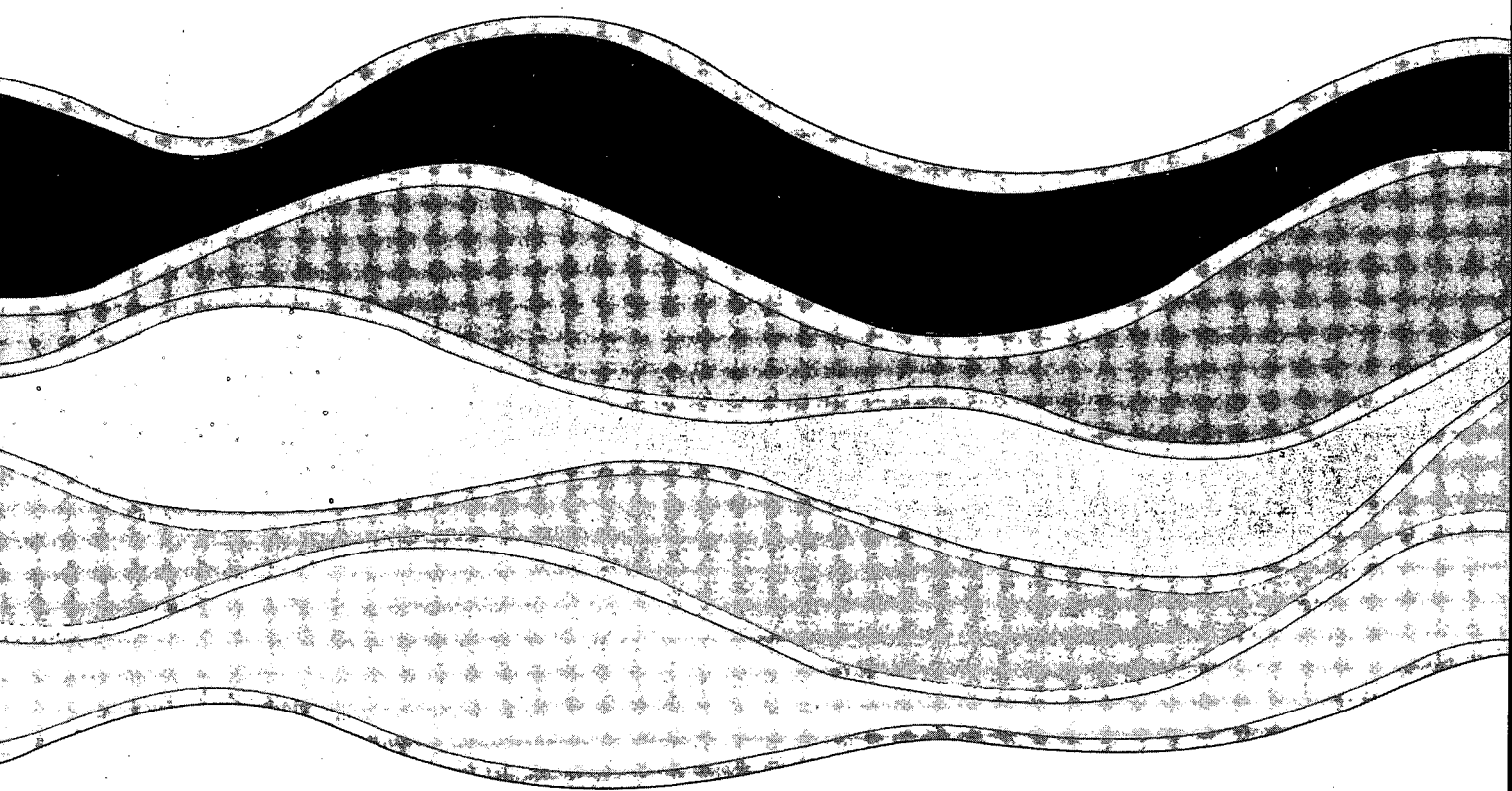
THE EVOLUTION OF THE JOINT
PROBABILITY DENSITY FUNCTION OF WAVE
HEIGHT AND PERIOD DURING SHOALING

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MANAGEMENT PERSPECTIVE

Coastal engineering design practice depends heavily on theoretical and empirical models of the distribution of wave properties. Most emphasis has been placed on the distribution of heights of waves since there is a considerable body of theory and experience in this area. However, many design criteria hinge on the forces produced by the water flowing by coastal structures and structures moored in coastal waters. Estimates of these forces depend not only on the heights of waves but also on their periods. This work extends the body of knowledge of the joint distribution of heights and periods of waves from deep water right up to the breaker zone. The results and conclusions in this report give marine and coastal design engineers the material they need to improve both the economy and safety of their designs.

SOMMAIRE À L'INTENTION DE LA DIRECTION

La méthode de conception des travaux maritimes est fortement tributaire de modèles théoriques et empiriques de distribution des propriétés des vagues. On a surtout insisté sur la distribution des hauteurs des vagues puisqu'il existe une masse de connaissances théoriques et beaucoup d'expérience dans ce domaine. Toutefois, de nombreux critères de conception dépendent des forces exercées par l'écoulement de l'eau auquel sont soumises des structures côtières et des structures mises en place dans la zone côtière. L'estimation de ces forces est fonction non seulement de la hauteur des vagues mais également de leur période. Ces travaux élargissent les connaissances de la distribution mixte de la hauteur et de la période des vagues depuis les eaux profondes jusqu'à la zone de déferlement. Les résultats et les conclusions du présent rapport fournissent aux ingénieurs responsables d'ouvrages de mécanique navale et d'ouvrages côtiers le matériel nécessaire pour rendre leurs conceptions plus économiques et plus sûres.

ABSTRACT

Laboratory data were conducted on a 1:40 beach slope to investigate the evolution of the joint distribution of wave heights and periods during shoaling. The data are compared to the joint distribution proposed by Longuet-Higgins (1983). For $d/L > 0.1$ the shapes of the observed and predicted distributions agree reasonably well but the modal values of H and T are not particularly well predicted especially as d/L approaches 0.1.

RÉSUMÉ

Des expériences de laboratoire ont été effectuées sur une pente de plage de 1:40 afin d'étudier l'évolution de la répartition conjointe de la hauteur des vagues et de leur période pendant le déprofondissement. Les données sont comparées à la distribution conjointe proposée par Longuet-Higgins (1983). Lorsque le rapport d/L est supérieur à 0,1, la forme des distributions observées et prévues concordent assez bien, mais les valeurs modales de H et de T ne sont pas spécialement bien prévues, notamment lorsque d/L s'approche de 0,1.

1. INTRODUCTION

The joint probability density function (jpdf) of wave height and period is of both practical and theoretical importance. The design of breakwaters is often based on formulae which determine the weight of the armour stone required based solely on a design wave height; that is, without consideration of the associated wave period. Yet studies have shown that the stability of breakwater armour units is related not only to the design wave height but also to wave period (Losada and Giménez-Curto, 1979); wave height sequencing or wave grouping is also of considerable importance here. This is, of course, not particularly surprising as it is known that the wave-induced forces on a structure arise from pressures, velocities, and accelerations, all of which have a wave period dependence. Improved design methods for structures subjected to wave-induced loads must therefore consider both wave height and period. The jpdf of height and period is thus central to the design process. While some attention has focussed on the jpdf of height and period for deep water waves (i.e., a relatively narrow-banded Gaussian process), very little can be found regarding the jpdf of height and period for shoaling or shallow water waves (a regime that is typically neither narrow-banded nor Gaussian). Yet this is precisely the location of coastal structures.

Furthermore, the jpdf of wave height and period can be used to derive other marginal and joint probability densities of interest, such as elevation versus slope. Wave slope or steepness is, of course, related to wave breaking and the occurrence of whitecapping; these processes are linked to the physical exchange of gases (e.g., CO_2) and environmental contaminants across the air-sea interface.

The purpose herein is to examine the evolution of the jpdf of wave height and period that occurs as a result of shoaling. A brief review of existing models for the jpdf of height and period is outlined in the following section; particular emphasis is placed

on the model of Longuet-Higgins (1983). The experiments that were undertaken for the present study are described in §3, followed by a discussion in §4; again, the emphasis is on the ability of Longuet-Higgins (1983) to predict the evolution of the jpdf into intermediate water depths. A summary of the results follows in §5.

2.   A REVIEW OF EXISTING THEORIES

An approximate jpdf for wave amplitude and period in random noise processes was first proposed by Wooding (1955) who extended Rice's (1944, 1945) work on the distribution of intervals between successive zeros for a narrow spectrum of random noise. Longuet-Higgins (1957) independently proposed a similar formulation for the jpdf of amplitude and period, which he later distilled to a simplified form (Longuet-Higgins, 1975) and applied to ocean waves. Although several comparisons have been made between this jpdf and data, it will not be considered any further as it has been superseded by Longuet-Higgins (1983).

The modified jpdf proposed by Longuet-Higgins (1983) is

$$p(R,\tau) = \frac{2}{\nu\sqrt{\pi}}\frac{R^2}{\tau^2}L(\nu)\exp\left\{-R^2\left[1+\frac{(1-\frac{1}{\tau})^2}{\nu^2}\right]\right\},$$

where

$$L(\nu) = \frac{1}{\frac{1}{2}\left[1+\sqrt{1+\nu^2}\right]}$$

and

$$R = \frac{H}{2\sqrt{2m_0}}, \quad \tau = \frac{T}{\overline{T}}.$$

R and τ are the normalized wave height period, respectively, H is the wave height, T the period, and $\overline{T} = 2\pi\frac{m_0}{m_1}$, where the $^-$ denotes a mean value. ν is a spectral width parameter, which is defined by the lowest three spectral moments, viz.,

$$\nu = \sqrt{\frac{m_0 m_2}{m_1^2} - 1}.$$

This jpdf is formally valid for $\nu^2 \ll 1$. The probability density of a specific height-period combination is uniquely defined by the single parameter ν.

Several other models have also been proposed for the deep water jpdf of wave height and period. Cavanié et al. (1976) developed a jpdf starting from the work of Cartwright and Longuet-Higgins (1956). Unlike the jpdf of Longuet-Higgins (1983), Cavanié et al. (1976) define wave heights and periods using a crest-to-crest definition, such that the period is given by the time between successive crests while the height is given by the vertical difference between a crest and the succeeding trough. This jpdf is formally valid for narrow-banded spectra, and a given height-period combination is uniquely defined by the spectral width parameter ε, which was defined by Cartwright and Longuet-Higgins (1956) as a function of the spectral moments m_0, m_2, and m_4.

Lindgren (1972) developed a model, known as "WAMP", which is based on properties of a normal process near a local maximum. The WAMP model differs from those of Cavanié et al. (1976) or Longuet-Higgins (1983), in that the results depend on the full covariance function and not only on a few spectral moments, such as those used to compute ε or ν. In particular, the WAMP model uses the covariance function and its first four derivatives to compute the jpdf of wave height and period. Wave height and period in WAMP are both defined by a crest to succeeding trough approach; this definition for wave height is the same as that used by Cavanié et al. (1976), however, this definition for wave period is known as $T_{1/2}$, as it is the time between a maximum and the succeeding minimum, not the time between two successive maxima (or minima). The approximations used in this model are considered to be accurate for most narrow and moderate banded processes provided the spectra of these processes have a distinct cut-off point. This last point was examined by Srokosz and Challenor (1987) who showed that the jpdf of heights and periods predicted by WAMP for a JONSWAP shaped spectrum was extremely sensitive to the ratio of spectral cut-off

to peak frequency. The model predictions were not deemed to be sensitive to low frequency cut-offs.

3.   THE LABORATORY EXPERIMENTS

To examine the evolution of the jpdf of wave height and period that occurs during shoaling, laboratory experiments were conducted in the wind-wave flume at the Canada Centre for Inland Waters. The flume is 103 m long, 4.5 m wide, and has a maximum water depth of 1.5 m. "Random" waves were generated using the GEDAP software package developed by the National Research Council of Canada (Funke and Mansard, 1984). The GEDAP package makes second-order corrections to the wave-board drive signal to (help) suppress spurious waves that arise through the mechanical generation of waves. All of the "random" wavetrains were created from DHH target spectra, after Donelan et al. (1985). The parameters of a DHH spectrum are similar to those of a JONSWAP spectrum, however, the DHH spectrum has an f^{-4} tail rather than the f^{-5} tail that is characteristic of a JONSWAP spectrum. Moreover, a DHH spectrum exhibits stronger peak enhancement at short fetches and greater directional spreading at high frequencies than a JONSWAP spectrum. Five peak periods (T_p=1.11, 1.25, 1.42, 1.67, 2.0, and 2.5 s) and two peak enhancement or wave age values (U/c_p=0.83 (fully developed) and 5.0 (strongly forced)) were used. Each realization contained approximately 500 waves; four realizations of each spectral peak and wave age parameter were created, giving a total of forty realizations.

The experiments were conducted on an impervious (plywood) beach of slope 1:40. Wave heights and periods were measured using surface-piercing capacitance-type wave wires of 1.1 mm diameter. The electronics packages for these wave probes were designed and built by the technical support team at the National Water Research Institute. Wave probe calibration data shows that these instruments are very linear

$(r^2 > 0.999)$ and have excellent long term gain stability. Ten wave probes were installed on the 1:40 beach slope. The first wave probe was located at the toe of the beach, which was 27.7 m from the mean position of the waveboard. The remaining nine wave probes were installed on 4 m centers.

Wave reflection from the beach was measured using a wave-wire array. This beach yielded a reflection coefficient of approximately 4% for the longest peak period of 2.5 s waves. Shorter waves were, of course, reflected less, while longer period waves arising from radiation stress effects associated with wave groupiness were more strongly reflected.

Analog outputs were lowpass filtered at 10 Hz then sampled digitally (with 12 bit resolution) at 20 Hz.

4.   DISCUSSION

The model of Longuet-Higgins (1983) was chosen for comparison to the data because it is less sensitive to the cutoff frequency than the models of Lingren (1972) or Cavanié et al. (1976). Even so, the Longuet-Higgins choice of spectral width is also sensitive to cut-off frequency. Figure 1 shows the variation of m_0, m_1, and m_2 as a function of f/f_p for a fully-developed DHH spectrum with $f_p = 0.6$ Hz. f is the upper limit of integration used in the computation of the spectral moments and f_p is the peak spectral frequency. m_0 and m_1 rapidly converge by $f/f_p \approx 3$, whereas m_2 is slower to converge. To compute the spectral width parameter ε used by Cavanié et al. (1976) requires m_4, yet this moment increases monotonically because the tail of a DHH spectrum has an f^{-4} slope. This makes their model extremely sensitive to the cut-off frequency. Figure 2 shows the variation in ν as function of f/f_p for a fully developed and strongly forced DHH spectrum for which $f_p = 0.6$ Hz; these two particular cases

were selected because their jpdf of height and period are discussed in detail later. Although ν is not as sensitive to the cut-off frequency as in the other models a cut-off frequency must nonetheless be chosen; moreover, it should be consistently defined. As such ν was computed using frequencies from $\frac{1}{2}f_p \to 3.0f_p$, which is the band containing approximately 99% of the integrated spectral variance. The lower limit was imposed to avoid the small contribution from longwaves, which have a variable spatial contribution and thus impose a small "jitter" in the value of ν.

The jpdf of height and period for a fully developed train of waves with peak period $T_p = 1.67$ s in 0.833 m of water, is shown in figure 3. The dotted contours indicate the jpdf of Longuet-Higgins (1983) for $\nu = 0.31$. The general shape of these two jpdfs is clearly similar. However, the mode of the theoretical jpdf lies approximately 10% to the left of the data; this observation was found to be true in general of the other realizations outlined in §3. Therefore, the normalizing period was increased by 10% in the model; this is shown in figure 4.

The evolution of the jpdf for a fully developed train of waves with $T_p = 1.67$ s is shown in figures 5(a–c). The dashed line indicates the maximum theoretical wave height (Miche, 1944), i.e., $(H/L)_{max} = 0.142\tanh(2\pi d/L)$, where d and L are the local depth and wavelength, respectively. It is interesting to note that this curve nicely envelopes both the data and the jpdf proposed by Longuet-Higgins (1983). It is the latter observation that is particularly intriguing as the jpdf is simply a mathematical model for random noise and embodies no wave physics to limit the wave height. Figure 5c indicates that the modal values of H and T are not well predicted by the model at this depth. A short time series of surface elevation for these three water depths is shown in figure 5d. The pronounced skewness of the waves at the shallowest depth $(d = 0.231$ m) indicates the existence of strong nonlinearities associated with shoaling.

Figures 6(a–d) shows the corresponding (to figures 5a–d) plots for a strongly forced train of waves. Comparison of figures 6a and 6b indicates a slight reduction in wave height due to breaking, whereas figure 6c shows a jpdf that has been significantly altered by shoaling and breaking, and is not well modeled by Longuet-Higgins' jpdf, which, in this case, exceeds the breaking limit. Again the data in figure 6c is nicely enveloped by the maximum theoretical (H/L) given above. Figure 6d clearly shows the reduction in wave height between the depths of 0.535 and 0.231 m, and the strong nonlinear nature of the waves in these water depths.

A measure of the error between the data and the theoretical jpdf is given by computing the time average mean square surface velocity given by the data and by the theoretical jpdf. Figure 7 shows the relative error between the theoretical jpdf and the data for this quantity as a function of d/L. For this particular parameter, related to the horizontal force on coastal structures, the model of Longuet-Higgins provides a good fit to the data for $d/L > 0.1$.

5.   CONCLUSIONS

The evolution of the jpdf of height and period was investigated using laboratory data collected on a 1:40 beach slope. These data were compared to the jpdf for wave height and period proposed by Longuet-Higgins (1983). This jpdf was selected for comparison because it is not as sensitive to the cut-off frequency as other models that depend on higher-order moments of the spectrum (i.e., Cavanié et al., 1976) or of the covariance function (Lingren, 1972). The comparison between the laboratory data and Longuet-Higgins' jpdf indicates that his model, with a minor adjustment to the normalizing period, provides a good fit to the data over a wide range of depths to wavelengths. Once the waves approach the breaker zone the increase in nonlinearities degrades the fit considerably. The use of Longuet-Higgins (1983) model for the jpdf

of heights and periods is recommended for depth to wavelength ratios in the range of $\infty > d/L > 0.1$.

ACKNOWLEDGMENTS

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Figure 2. Variation of the spectral width parameter ν as a function of f/f_p for a fully-developed ($U/c_p = 0.83$, — line) and strongly forced ($U/c_p = 5.00$, - - - line) DHH spectrum with $f_p = 0.6$ Hz.

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Figure 4. As for figure 3 except the modal location of the normalized period for Longuet-Higgins' distribution has been increased by 10%.

Figure 5. (a–c) Contours of the jpdf of wave height and period, in decreasing water depths, for a fully-developed (i.e., $U/c_p = 0.83$) DHH spectrum of waves with $T_p = 1.67$ s. The dotted contours show the jpdf given by Longuet-Higgins (1983). (a) $d = 0.833$ m, $d/L = 0.26$, $\nu = 0.31$. (b) $d = 0.535$ m, $d/L = 0.19$, $\nu = 0.31$. (c) $d = 0.231$ m, $d/L = 0.11$, $\nu = 0.39$. (d) Short segments of surface displacement corresponding to the jpdfs shown in (a), (b), and (c), being the bottom, middle, and upper traces, respectively. Note the segments have been separated for plotting purposes.

Figure 6. (a–c) Contours of the jpdf of wave height and period, in decreasing water depths, for a strongly forced (i.e., $U/c_p = 5.00$) DHH spectrum of waves with $T_p = 1.67$ s. The dotted contours show the jpdf given by Longuet-Higgins (1983). (a) $d = 0.833$ m, $d/L = 0.26$, $\nu = 0.26$. (b) $d = 0.535$ m, $d/L = 0.19$, $\nu = 0.30$. (c) $d = 0.231$ m, $d/L = 0.11$, $\nu = 0.42$. (d) Short segments of surface displacement corresponding to the jpdfs shown in (a), (b), and (c), being the bottom, middle, and upper traces, respectively. Note the segments have been separated for plotting purposes.

Figure 7. Relative error in the mean square time average velocity integrated over all H and T space, between the predicted jpdf and that given by the data as a function of d/L. + – fully-developed, * – strongly forced DHH spectrum of waves, respectively.
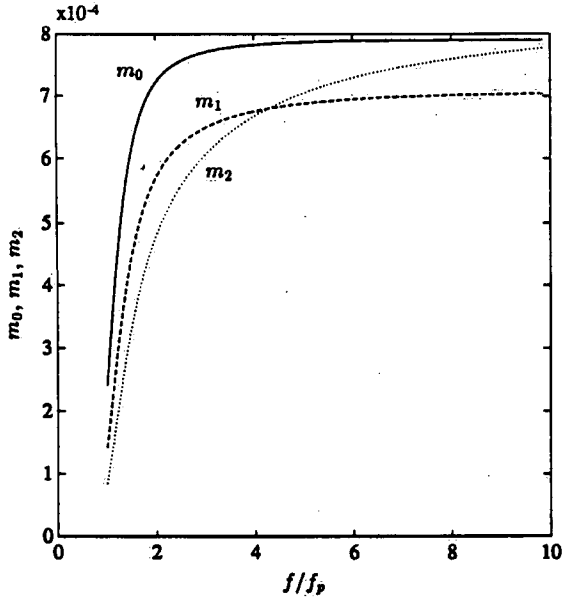
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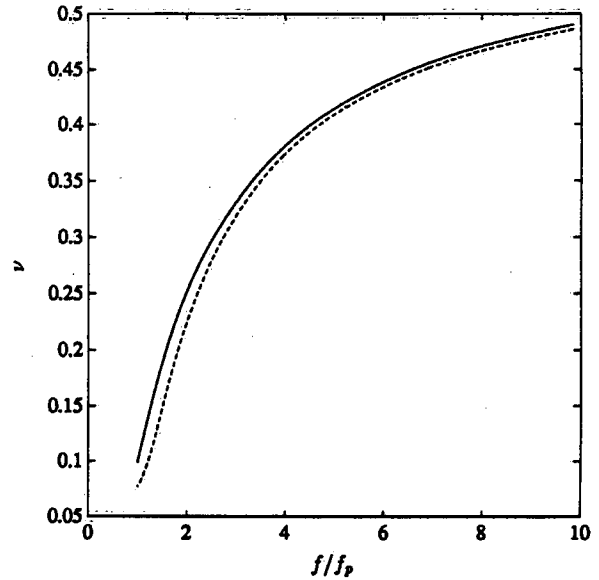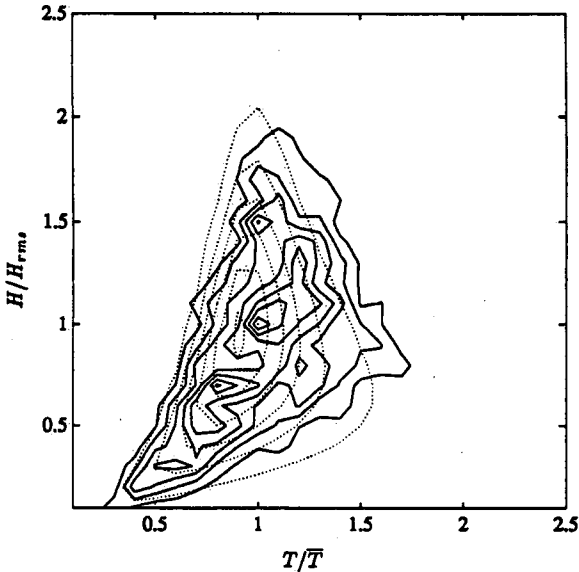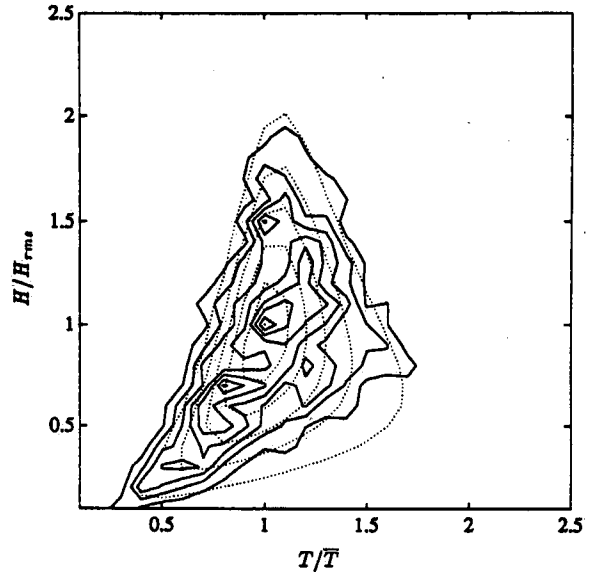
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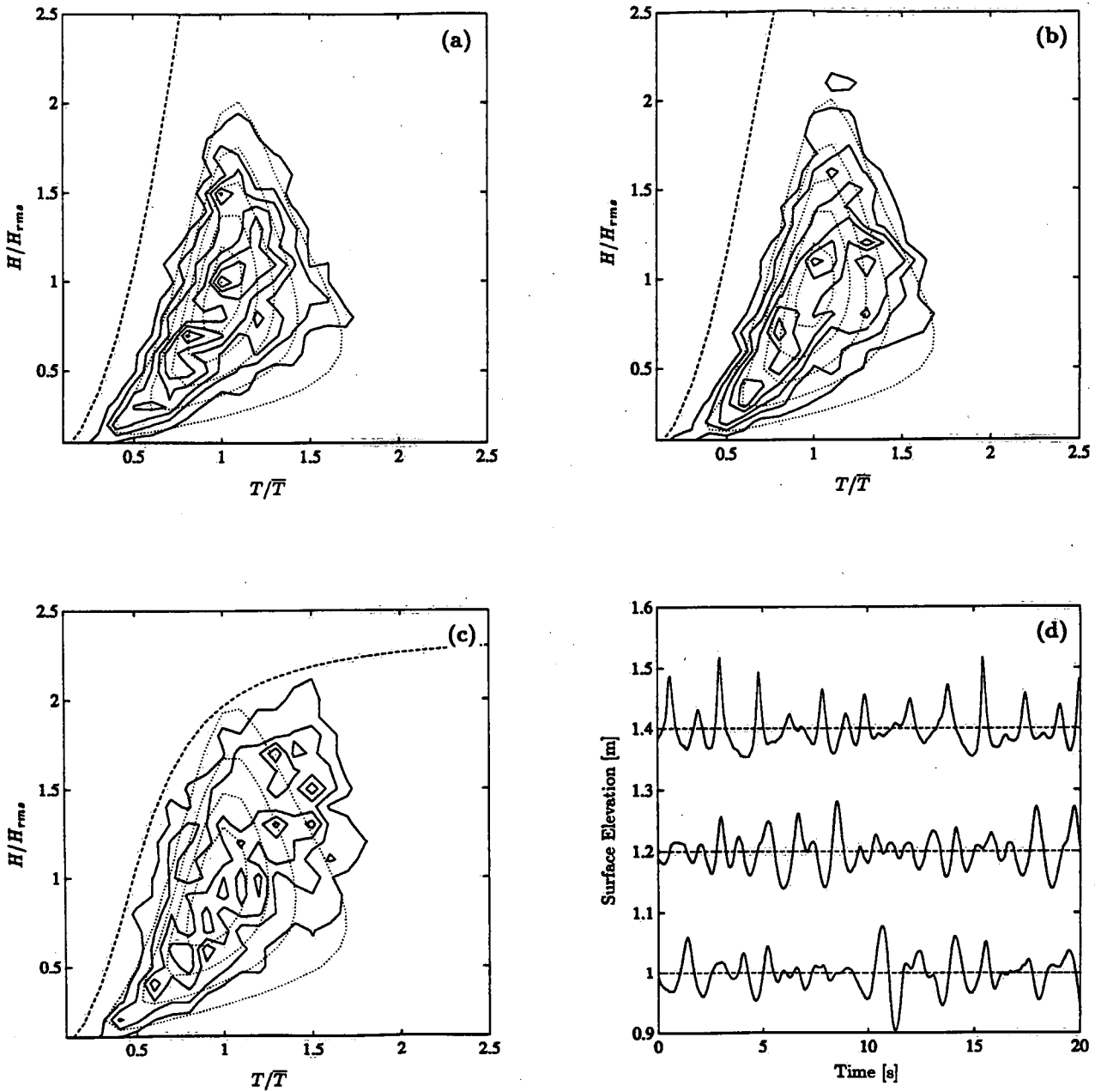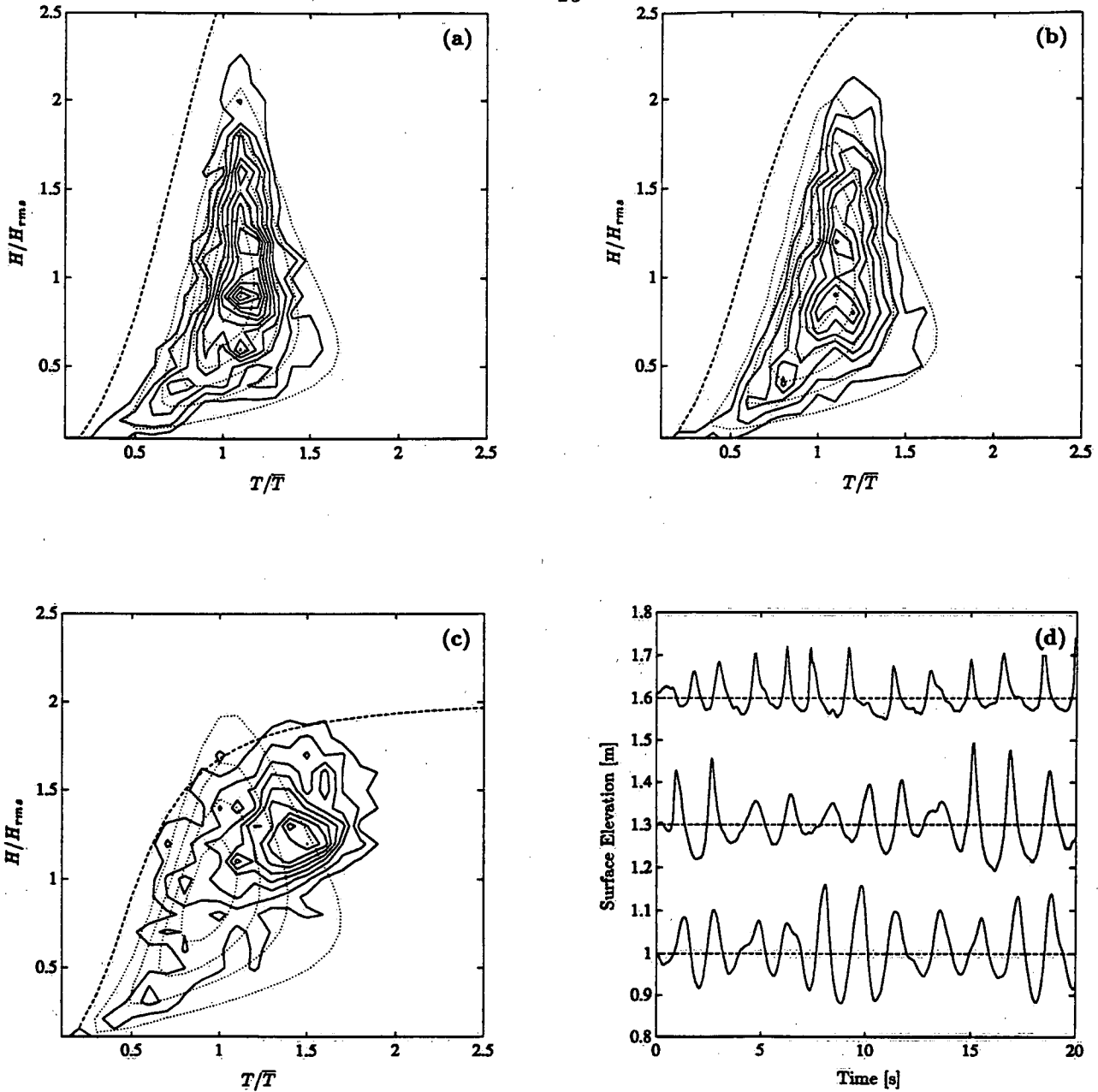
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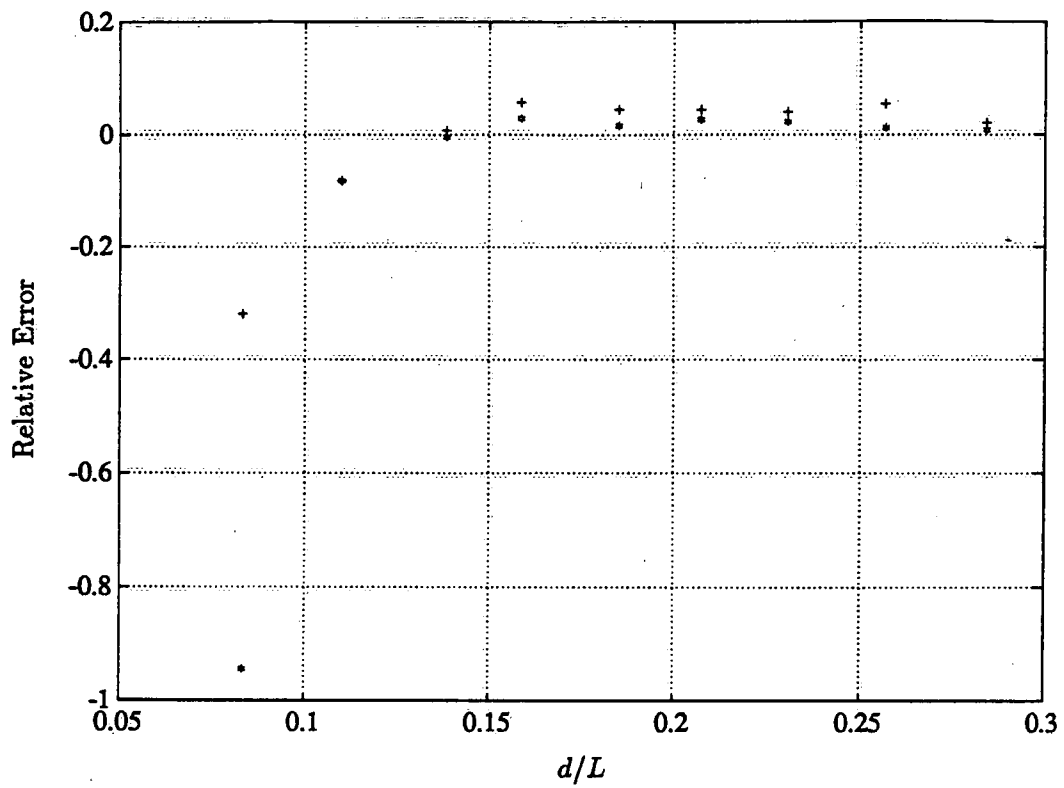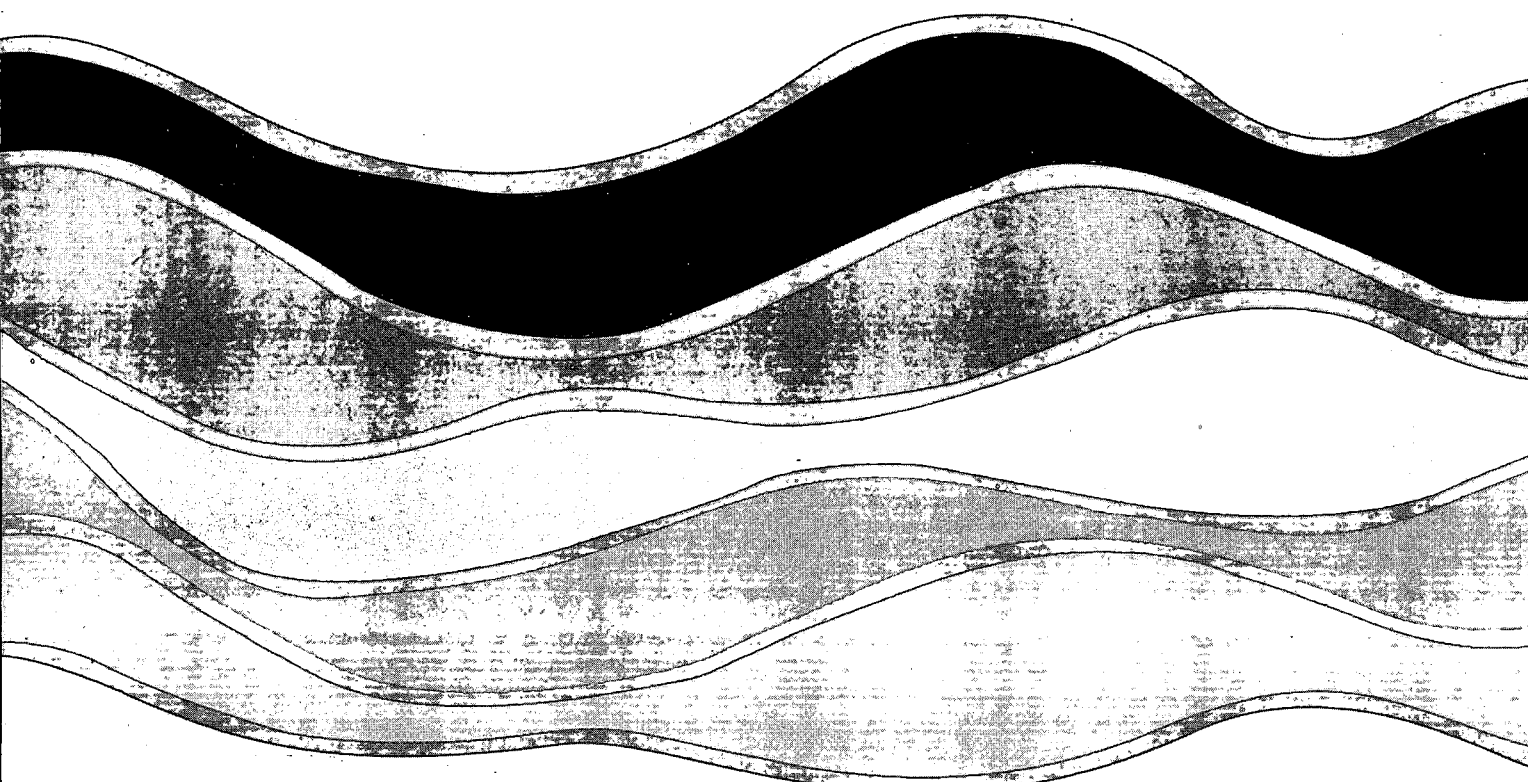
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