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# SENSITIVITY ANALYSIS OF PARTIALLY ORDERED SETS AND RANKING OBJECTS OF ENVIRONMENTAL INTEREST 

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## MANAGEMENT PERSPECTIVE

The environmental hazard of chemicals is a problem of international concern. Ranking of chemcials is a procees through which decision makers decide which chemicals need to be regulated. In this paper we have analyzed in detail two issues. One is the process of ranking itself. Our opinion is that indices are not suitable to identify hazardous contaminants since, indices hide information. The second issue is that the criteria used to assess a chemical might not be the best possible. Therefore, in this paper we have used a different method, based on graph theory, to rank toxic contaminants. We have also developed a method to assess which criteria are critical in ranking toxic contaminants, which could be excluded or not even measured.

## SOMMAIRE À L'INTENTION DE LA DIRECTION

Les produits chimiques présentent des dangers pour l'environnement partout dans le monde. Par un processus de classement, les décideurs établissent quels produits doivent être réglementés. Dans le présent document, nous analysons en détail deux questions relatives à ce problème. La première est le processus de classement lui-même. Nous pensons que les indices ne sont pas des outils valables pour le repérage des contaminants dangereux parce qu'ils cachent de l'information. En deuxième lieu, nous examinons le fait que les critères utilisés dans l'évaluation des substances pourraient ne pas être les meilleurs possibles. C'est pourquoi nous préconisons ici une autre méthode de classement des substances toxiques, fondée sur la théorie des graphes. Nous avons aussi élaboré une méthode permettant de déterminer les critères essentiels dans le classement de ces substances et ceux qui peuvent être exclus ou pas même mesurés.


#### Abstract

A set of objects can be analyzed by several tools. When ranking is a concern, then lattice theory and its graphical representation (Hasse diagrams) are useful. This paper introduces a new approach to analyze Hasse diagrams with respect to ranking in the environmental field. Set theoretical and lattice theoretical concepts such as cardinality, successor sets and the intersection of these sets have been used to compute a new intersection matrix $\mathbf{D}$ that identifies the relation between objects. As an example we analyze a published data set with the conclusion that the waste disposal site E-5 (located at Windsor Malden) is a site worthy of further investigation because it is a site representative of many others and L-26 (located at) Ed Johnston Construction is interesting because of its special geological features. We have also studied the importance that each attribute has for ranking. For this purpose we have introduced a new matrix, $\mathbf{W}$, which quantifies the dissimilarity of different Hasse diagrams and a sensitivity measure $\sigma(i)$ that analyzes the importance of criteria, by which objects are characterized. A crude estimation of the upper bound of $\sigma(\mathrm{i})$ is given.


## RÉSUMÉ

Plusieurs outils peuvent être utilisés pour l'analyse d'un ensemble d'objets. Quand il s'agit de classer des objets, la théorie des treillis et sa représentation graphique (diagrammes de Hasse) peuvent être mis à profit. Dans le présent document, nous proposons une nouvelle façon d'analyser les diagrammes de Hasse pour des problèmes de classement qui se posent dans le domaine de l'environnement. Nous avons utilisé des concepts des théories des ensembles et des treillis, comme la cardinalité, les ensembles successeurs et l'intersection de ces ensembles, pour calculer une nouvelle matrice d'intersection $\mathbf{D}$ qui établit la relation qui existe entre les objets. À titre d'exemple, nous avons analysé un ensemble de données publiées pour arriver à la conclusion que le site d'élimination de déchets E-5 (Windsor Malden) mérite d'être étudié plus avant parce qu'il est représentatif de bon nombre d'autres sites, et que le site L-26 (Ed Johnstone Construction) est intéressant en raison de ses caractéristiques géologiques particulières. Nous avons aussi examiné l'importance de chacun des attributs pour le classement. Ä cette fin, nous avons établi une nouvelle matrice, $\mathbf{W}$, qui quantifie la dissimilarité de différents diagrammes de Hasse, ainsi qu'une mesure de sensibilité, $\sigma(i)$, qui analyse l'importance des critères de caractérisation des objets. Nous donnons une estimation brute de la limite supérieure de $\sigma(i)$.

## INTRODUCTION

Hasse diagrams (Davey and Priestley, 1990) have been used to rank chemicals according to environmental hazard (Halfon and Reggiani, 1986; Brüggemann and Halfon, 1989), to compare waste disposal sites (Halfon, 1989), to compare mathematical models (Reggiani and Marchetti, 1975; Halfon, 1983a,b), in QSAR studies (Brüggemann et al., 1991; Randic, 1991) in problems of regional pollution (Brüggemann et al., 1994; Münzer et al., 1994 ), and in the evaluation of data sources (Voigt and Brüggemann, 1993). The basis of this method is the assumption that we can perform a ranking while avoiding the use of an ordering index (Halfon and Reggiani, 1986). In our application, Hasse diagrams present information not only on the ranking but, most importantly, they show whether the criteria, characterizing the objects, lead to ambiguities in the ranking: For example, an object might be ranked higher according to one criterion but lower according to another. These two objects are not ordered because their data are "contradictory" to each other. This ambiguity is not evident when we use an index for ranking, but it is immediately evident by the presence or absence of lines in a Hasse diagram. In this paper we investigate a method to extract information from Hasse diagrams since the casual user of the ranking method of Halfon and Reggiani (1986) might become confused by the large number of lines and/or lack of lines in a Hasse diagram and might not be able to use the large amount of information present in this graph. Furthermore, we study the influence of the choice of criteria to rank a set of objects (precise definitions follow later). The ranking of a set of objects depends not only on the numerical values, but even more on the choice of criteria.

The results of this analysis are two matrices, $\mathbf{D}$ and $\mathbf{W}$, that identify the main features of the structure of Hasse diagrams and quantify the influence of criteria on ranking. The textbooks of Harary (1969), Preparate and Yeh (1973) and Davey and Priestley (1990) present useful background information on graphs, sets, partially ordered sets (posets) and Hasse diagrams.

## Hasse diagrams

Hasse diagrams visualize the order relations of posets. They are oriented graphs (digraphs). A digraph consists of a set $E$ of objects drawn as circles in Hasse diagrams. In our applications the circles near the top of the page (of the Hasse diagram) indicate objects that are
most hazardous according to the criteria used to rank them: These objects have no predecessors ${ }^{1}$, are called maximal elements, abbreviated as "maximals." A line in the Hasse diagram indicates that the two objects connected by that line are "comparable" with each other, lack of sequences of connecting lines indicates that there are contradictions (a complete explanation with examples may be found in Halfon and Reggiani, 1986).

## DEFINITIONS

Criteria include both quantitative and qualitative properties.
An attribute is a quantitative, measurable, criterion. We denote these attributes as \#1, \#2, ..., \#n. It is convenient to denote the full attribute set as A. A family of $p:=2^{n}-1$ attribute sets will be considered in our analysis, namely the power set of $\underline{A}$ without the empty set. Each subset of attributes is denoted by $A_{i}$, with $A_{i} \subseteq$ $A$ and will be used to perform a sensitivity analysis (see later).

Data are the numerical values corresponding to each criterion by which a given object is characterized.

An object is the item of interest. Each object is characterized by a tuple of data. Objects are ranked graphically by Hasse diagrams, applying as order relation the usual $\leq$ relation of the components of the tuple. The set of $m$ objects is called $E$.

Equivalent objects in Hasse diagrams: Different objects that have the same data with respect to a given set of attributes. They are elements of an equivalence class with the equivalence relation "equality of the characterizing tuples."

A case is a shorthand notation for an analysis by Hasse diagrams of $m$ objects and with a defined attribute set $\mathbf{A}_{i}$. Thus, a given set of attributes induces a Hasse diagram.
${ }^{1}$ They are not "covered" by other objects (Davey and Priestley, 1990)

## Definition of key elements and successor sets

Substructures within a Hasse diagram, i.e., relations among objects as well as the importance of criteria in ranking are investigated with the help of key elements and successor sets. Any object of the poset can be chosen as a starting point to begin the analysis. We call this object, a "key element". For convenience, all chosen key elements are considered to be elements of a set $K$, a subset of $E$.

In a Hasse diagram objects are connected by lines. Analysis of the successors of a key element implies a search of all objects located lower than, or equivalent to, that of the key element and connected to it by a path, being a sequence of connecting lines. The set of all successors of key element k is denoted as $G(\mathrm{k})^{2}$. The properties of the successor set $G(\mathrm{k})$ and its relation with successor sets of other key elements are first used to analyze the structure of a Hasse diagram, later they will be used to perform the sensitivity analysis. We write the cardinality of the successor set $\underline{G}(\mathbf{k})$ as card $\underline{G}(\mathbf{k})$. The cardinality of a set is the number of objects in each set.

## RELATIONS BETWEEN ELEMENTS OF POSETS

To investigate the global structure of the relations between any two elements of the posets, we perform this analysis mathematically. This structural relation might be hidden within the geometrical representation of the Hasse diagram. Thus, we introduce the symmetrical matrix D, whose entries are calculated from the cardinalities of all intersections of pairs of successor sets:

$$
\begin{equation*}
D_{i j}:=\operatorname{card}[G(i) \cap G(j)] . \tag{1}
\end{equation*}
$$

The relation between two objects $i, j$ can be examined with the help of the triple ( $D_{i i}, D_{i j}, D_{i j}$ ) and three possible outcomes are possible:

1) $D_{i j} \not D_{i j}>D_{i j}$, the $\varepsilon$-relation.

[^0]2) $D_{i i} \approx D_{i j} \approx D_{i j}$, the $\varphi$-relation.
3) $D_{i i} \rightarrow D_{i j} ; D_{i j} \geq D_{i j}$, the $\pi$-relation.

This process of searching for relations between objects is called "unfolding" the structure of the Hasse diagram. In the $\varphi$-relation the two key elements have many successor elements in common: they are comparable with respect to all attributes to almost the same set of objects. In the $\varepsilon$-relation the two key elements may be taken as representatives for two quite different sets of objects. In the $\pi$-relation one key element has few successors. The key element may either be geometrically located near the bottom of the diagram or connected directly with elements at the bottom level.

## APPLICATION OF THE METHOD TO A PUBLISHED DATA SET

Halfon (1989) has discussed the ranking of 38 waste disposal sites according to geological and pollution characteristics. The interpretation of the Hasse diagrams in his paper might be difficult because of the many circles and lines. We apply the concept of matrix $\mathbf{D}$ to his Hasse diagram both to explain its use and to improve on the interpretation of his Fig. 3:

## Choice of the key elements

Let us focus on three maximals, namely the sites, E-5, E-7 and L-26, which differ with respect to their geologic characteristics: these sites (objects) are now the key elements. Table 1 gives the needed background information, by listing the elements of the successor sets. The cardinalities of the three successor sets are:

$$
\begin{aligned}
& \operatorname{card} G(\mathrm{E}-5)=24 \\
& \operatorname{card} G(\mathrm{E}-7)=19 \\
& \operatorname{card} G(\mathrm{~L}-26)=7
\end{aligned}
$$

The part of the intersection matrix $\mathbf{D}$, referring only to the three objects of interest here, is:

|  |  |  | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | 1 | E-5 | 26 | 17 | 7 |
| $\mathrm{D}=$ | 2 | E-7 | 17 | 19 | 5 |
|  | 3 | L-26 | 7 | 5 | 7 |

Note that the columns are numbered one to three.

## Comparison of key elements E-5 and E-7

The difference of the cardinalities of the two successor sets is seven ( $D_{11}-D_{22}$ ). E-5 has more successors (26) than E-7 (19). Therefore we conclude that even if E-5 and E-7 are ranked as maximals, E-5 is a site worthy of further investigation, because many other sites can be compared with E-5.

The cardinality of the intersection ( $D_{12}=D_{21}$ ) of these two successor sets is 17 , only slightly smaller than the cardinality $\left(\mathrm{D}_{22}\right)$ of $G(\mathrm{E}-7)$ which is 19 . E-7 and $\mathrm{E}-5$ are two key elements in a " $\varphi$-relation". Even if the attributes of E-5 and E-7 make these two sites incomparable to one another, they are nevertheless comparable to many other objects at once.

## Comparison of key elements E-5 and L-26

Analysis of the intersection matrix, $\mathbf{D}$, shows that

$$
\mathrm{D}_{11} \gg \mathrm{D}_{33} \quad \mathrm{D}_{13}=\mathrm{D}_{33}
$$

therefore, although L-26 is on the highest level of the Hasse diagram, L-26 has few successors. L-26 and E-5 are in " $\pi$-relation" which is hidden by the geometrical representation in the Hasse diagram of Halfon (1989). Additionally, since $\mathrm{D}_{13}=\mathrm{D}_{33}$, L-26 has a successor set that is a subset of E-5. That is, all the successors of L-26, i.e., the elements of $G(\mathrm{~L}-26)$, have geological and pollution attributes that are all smaller than those of E-5. L-26 itself is the only exception (by definition a key element does not belong to its own successor set). Therefore, there is at least one attribute, whose value makes L-26 incomparable with the rest of the data set of objects in $G(E-5)$. This is important if further geological measurements are performed with respect to hazard assessment. Analogously it can be found by the matrix D that also L-26 and E-7 are in a " $\pi$-relation".

## SENSITIVITY ANALYSIS OF THE RANKING IN RESPECT TO ITS ATTRIBUTES

The ranking of the objects is sensitive to the set of attributes. To quantify the importance of an attribute on ranking the basic idea is to compare Hasse diagrams induced by different attribute sets with each other. In order to do this again the concept of successor sets generated by a key element is the starting point. The notation of successor set must be expanded to include all the actual combinations of attributes. Within the generalization of having a family of attribute
sets the successor set depends not only on the key element but on which attributes are used. Therefore the following notation $G(\mathrm{k}, A)$ or $G\left(\mathrm{k}, A_{i}\right)$ is used, where $G$ is the successor set, k denotes some arbitrary chosen key element, $A$ is the full set of attributes and $A_{i}$ is a subset of attributes, $A_{i} \subseteq A$.

The influence of each attribute can be quantified by counting the elements of the symmetrized difference between two successor sets as follows:
$\mathrm{W}\left(\mathrm{k}, A_{\mathrm{i}}, A_{\mathrm{j}}\right):=\operatorname{card}\left\{\left[G\left(\mathrm{k}, A_{\mathrm{i}}\right) \backslash G\left(\mathrm{k}, A_{\mathrm{j}}\right)\right] \cup\left[G\left(\mathrm{k}, A_{\mathrm{j}}\right) \backslash G\left(\mathrm{k}, A_{\mathrm{i}}\right)\right]\right\}$
For a given key-element, two Hasse diagrams (given by two arbitrary sets of attributes) are more dissimilar, the more the successor sets $G\left(k, A_{i}\right)$ and $G\left(k, A_{j}\right)$ differ. The equivalent form of the symmetrized difference is

$$
\begin{equation*}
\left[G\left(k, A_{i}\right) \cup G\left(k, A_{\mathrm{j}}\right)\right] \backslash\left[G\left(\mathrm{k}, A_{\mathrm{i}}\right) \cap G\left(\mathrm{k}, A_{\mathrm{j}}\right)\right] \tag{3}
\end{equation*}
$$

which is easier to evaluate than the right hand side of Eq. 2. The cardinality of the symmetrized set difference is also called the Hamming-distance (Bollobás, 1986) between sets.

To simplify notation, we write $\mathrm{W}(\mathrm{k}, \mathrm{i}, \mathrm{j})$ for $\mathrm{W}\left(\mathrm{k}, A_{\mathrm{i}}, A_{\mathrm{j}}\right)$. We also note that each

$$
\begin{equation*}
W(k, i, j) \geq 0 \text { and } W(k, i, j)=W(k, j, i) . \tag{4}
\end{equation*}
$$

The matrix itself is denoted by $W(k)$. Thus we have:

$$
\begin{align*}
& W(k, 1,1), W(k, 1,2), \ldots ., W(k, 1, p) \\
& W(k, 2,1), W(k, 2,2), \ldots ., W(k, 2, p) \\
W(k)= & \ldots . . . . . . . . . . . . . . . . . . . . . . . . . . . . ~ \tag{5}
\end{align*}
$$

$$
\mathbf{W}(k, p, 1), \mathbf{W}(k, p, 2), \ldots ., \mathbf{W}(k, p, p)
$$

with $p=2^{n}-1$ (see definition of attribute). This matrix $\mathbf{W}$ is the key for the sensitivity analysis of ranking, each entry of $\mathbf{W}$ is the cardinality of the symmetrized difference (Eq. 2) of two successor sets which are constructed from a given key element $k$ and the two Hasse diagrams induced by two attribute subsets. Thus the rows and columns of this matrix are indicated by those two given subsets $\underline{\boldsymbol{A}}_{\boldsymbol{i}}$ and $\underline{\boldsymbol{A}}_{\boldsymbol{j}}$. This large but symmetrical matrix needs not be analyzed in its entirety all the times because we are only interested in some few attribute sets. The sensitivity analysis can be performed with the following steps:

1) Since we are interested only in comparisons of the full attribute set $\boldsymbol{A}$ with subsets $\underline{A}$, only one row of the matrix $\mathbf{W}$ is of interest. Since this is an example, we can choose the first one without loss of generalization, thus we are left with $\mathbf{W}(k, 1,1), \mathbf{W}(k, 1,2), \ldots$, $\mathbf{W}(\mathbf{k}, 1, \mathrm{p})$, where the index 1 denotes the full attribute set $\underline{\mathbf{A}}$.
2) To see the influence of attributes on a Hasse diagram we compare the Hasse diagrams induced by $\underline{\mathbf{A}}$ with those induced by the attribute sets with only $\mathrm{n}-1$ attributes. Therefore the effect of dropping exactly one attribute is given by the remaining $n$ entries of the first row, $\mathrm{W}(\mathrm{k}, 1,2), \ldots ., \mathrm{W}(\mathrm{k}, 1, \mathrm{n}+1)$.
3). The remaining $n$ matrix elements of the first row $W\left(k, A_{1}, \mathbf{A}_{1}\right), \ldots ., W\left(k, A, A_{n}\right)$ are put together to form a "sensitivity tuple" of the key element $k, s(k):=\left[W\left(k, A, \mathbf{A}_{1}\right), \ldots\right.$ $\left.\mathbf{W}\left(\mathbf{k}, \mathbf{A}_{1}, \mathbf{A}_{\mathrm{a}}\right)\right]$. Note that the enumerations of the subset $\mathbf{A}_{i}$ are as follows:

$$
\begin{align*}
& \underline{\mathbf{A}}_{i}=\{\# 1, \ldots, \# 1-1, \# \mathrm{i}+1, \ldots, \# \mathrm{n}\} \\
& \underline{\mathbf{A}}_{1}=\{\# 2, \ldots, \# \mathrm{n}\}  \tag{6}\\
& \underline{\mathbf{A}}_{\mathbf{a}}=\{\# 1, \ldots, \# \mathrm{n}-1\}
\end{align*}
$$

4) $\quad s(k)$ can also be written as $\left[s_{1}, \ldots, s_{n}\right]$. The larger $s_{i}$ the larger is the symmetrized difference between $\underline{G}(k, \underline{A})$ and $\underline{G}\left(k, \mathbf{A}_{i}\right)$ and correspondingly the larger the influence of attribute \#i on the position of key element $k$ within the Hasse diagram under $\underline{A}$ compared with that ünder $\underline{\mathbf{A}}_{\mathbf{i}}$.
5) The matrix $W(k)$ depends on the selection of the key element $k$. If however, more objects are to be analyzed we generalize as follows:

$$
\begin{align*}
\mathrm{W}(K, \mathrm{i}, \mathrm{j})=\Sigma & \mathrm{W}(\mathrm{k}, \mathrm{i}, \mathrm{j})  \tag{7}\\
& \mathrm{k} \in K \subseteq E
\end{align*}
$$

where $K$ is any set of key elements, in a shorter notation $\mathbf{W}(K)=\sum \mathbf{W}(\mathbf{k})$.
6) All objects are selected as key elements. Therefore instead of $W(k), W(E)$ is to be investigated. $\mathbf{W}(E)$ is the total matrix of the set $E$. We note that a crude upper limit of
$W(E, i, j)$ can be found simply by comparing a poset of $m$ solely non comparable elements with a poset where all $m$ elements are equivalent to each other. Together with Eq. 4:

$$
\begin{equation*}
0 \leq W(E, i, j) \leq m(m-1) \tag{8}
\end{equation*}
$$

7) $W(E)$ will be used as a measure of sensitivity. Accordingly we suggest to quantify the sensitivity by:

$$
\begin{equation*}
\sigma(i)=W\left(E, A, A_{i}\right) \quad 1 \leq i \leq n \tag{9}
\end{equation*}
$$

with the enumeration scheme of (6). According to (8), $\sigma(\mathrm{i})$ has values between 0 and m (m-1).

## SUMMARY

Hasse diagrams are graphical tools that visualize posets. This visualization, however, must be interpreted by a user and this interpretation may be subjective and / or even incomplete. The method presented in this paper remedies this deficiency by introduction of the matrix $\mathbf{D}$ and identifies mathematically how attributes used for ranking influence the creation of Hasse diagrams by the introduction of the matrix $\mathbf{W}$. We have presented an analysis of Hasse diagrams applied to environmental problems. Set theoretical and lattice theoretical concepts such as cardinality, successor sets and the intersection of these sets have been used to compute the intersection matrix $\mathbf{D}$. This matrix quantifies the relation among key elements and leads to a " $\varphi$ relation", " $\varepsilon$-relation" or " $\pi$-relation" that help us understand the hidden structures of Hasse diagrams. We have also studied the importance that each attribute has for ranking. For this purpose we have introduced a matrix, $\mathbf{W}$, for which the concept of successor sets again is the basis and which quantifies the similarity of different Hasse diagrams with respect to a given key element and then the sensitivity measure $\sigma(\mathrm{i})$ is suggested.

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Table 1: Successor set of the sites E-5, E-7 and L-26

| E-5 | E-7 | L-26 |
| :---: | :---: | :---: |
| $\mathrm{K}-3$ | $\mathrm{~K}-3$ |  |
| $\mathrm{~K}-4$ | $\mathrm{~K}-4$ |  |
| $\mathrm{~K}-5$ | $\mathrm{~K}-5$ |  |
| $\mathrm{~A}-2$ |  |  |
| $\mathrm{E}-2$ |  |  |
| $\mathrm{E}-3$ | $\mathrm{E}-3$ | $\mathrm{E}-4$ |
| $\mathrm{E}-4$ | $\mathrm{E}-4$ |  |
| $\mathrm{~L}-1$ | $\mathrm{~L}-1$ |  |
| $\mathrm{~L}-2$ |  |  |
| $\mathrm{~L}-3$ |  |  |
|  | $\mathrm{~L}-5$ |  |
| $\mathrm{~L}-6$ | $\mathrm{~L}-6$ |  |
|  | $\mathrm{~L}-7$ |  |
| $\mathrm{~L}-8$ | $\mathrm{~L}-8$ | L |
| $\mathrm{~L}-9$ | $\mathrm{~L}-9$ | $\mathrm{~L}-10$ |
| $\mathrm{~L}-10$ | $\mathrm{~L}-10$ | L |
| $\mathrm{~L}-12$ | $\mathrm{~L}-12$ | $\mathrm{~L}-12$ |
| $\mathrm{~L}-13$ | $\mathrm{~L}-13$ |  |
| $\mathrm{~L}-14$ | $\mathrm{~L}-14$ |  |
| $\mathrm{~L}-16$ | $\mathrm{~L}-16$ | $\mathrm{~L}-17$ |
| $\mathrm{~L}-17$ | $\mathrm{~L}-17$ |  |
| $\mathrm{~L}-19$ | $\mathrm{~L}-19$ | $\mathrm{~L}-20$ |
| $\mathrm{~L}-20$ | $\mathrm{~L}-25$ |  |
| $\mathrm{~L}-25$ | $\mathrm{~L}-29$ |  |


| card | 24 | 19 | 7 |
| :--- | :--- | :--- | :--- |




[^1]Think Recycling!


Pensez à reç̀cler.


[^0]:    ${ }^{2}$ Note the similar concept of "down-sets" and order ideals generated by some elements in Davey and Priestley (1990). We also write " $\underline{G}(k)$ is generated by object k."

[^1]:    

