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By:
A.R. Piggott

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Author(s): **Piggott, A.R.**

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EC Priority/Issue: This work was performed as a contribution to the GLAP II program focusing on the remediation of areas of concern and to the national priorities including sustainable development, groundwater, and toxic substances. It also supports the Nuclear Fuel Waste Management Program and the Atomic Energy Control Act. These and forthcoming results are applicable to the assessment of the proposed low-level radioactive materials disposal facility at the Chalk River Laboratory of Atomic Energy of Canada Limited. This work began in 1995 and is expected to continue through at least 1997.

Current Status: Rock masses are fractured over a wide range of scales. Groundwater flow in the low-permeability rock masses that are candidates for the disposal of nuclear fuel waste is often dominated by conduction in fractures. This paper introduces a method of constructing a three-dimensional model of a fractured rock mass from measurable data. Mathematical procedures can then be applied to the model to predict the rate of contaminant transport in the fractures.

Next Steps: A scoping analysis of fracture trace length data from the Chalk River Laboratory of AECL will be completed by mid 1997. External sources of funding for this research are being developed in collaboration with scientists from AECL.

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Fractal relations for the diameter and trace length of disc-shaped fractures

Andrew R. Piggott

National Water Research Institute, Burlington, Ontario, Canada

Abstract. Analytic relations are developed between the fractal parameters of a random, isotropic population of disc-shaped fractures and the parameters of the corresponding population of fracture traces expressed in outcrop. These relations indicate that a fractal distribution of fracture diameters translates to a fractal distribution of trace lengths and that the parameters for diameter may be uniquely determined from the parameters for trace length. Probabilistic results demonstrate the accuracy of the analytic relations and identify nonideal behaviors at the limits of the computed trace length data. It is expected that these relations will be useful in the three-dimensional characterization of fracture systems in rock from in situ trace length data.

Introduction

Rock masses are ubiquitously fractured over a wide range of spatial scales, from microcracks visible in thin section to features apparent in remote sensing data. Knowledge of fracturing as a function of scale is critical to many geological pursuits, including the estimation of groundwater flow in the low-permeability rock masses that are candidates for the disposal of nuclear fuel waste. Much of the research that has been conducted relative to discretely fractured rock masses is based on the assumption that the constituent fractures are finite, linear features distributed within a two-dimensional domain. These idealized fracture systems are reminiscent of the patterns of fracture traces that are observed in outcrop. While this assumption is sufficient to acknowledge the influence of fracture system geometry on rock mass behavior, it does not allow the three-dimensional aspects of geometry to be addressed. For example, the percolation behaviors for two- and three-dimensional groundwater flow in discontinuously conductive rock masses are significantly different [*Piggott and Elsworth*, 1992], and therefore the validity of estimating three-dimensional groundwater flow using a two-dimensional model is uncertain.

In geological settings such as granitic plutons, it is appropriate to represent a three-dimensional fracture system as a population of finite, two-dimensional features. Many researchers have adopted a conceptual model of this geometry, wherein each feature is assumed to be planar and circular, or disc-shaped. The geometry of the system is then described in terms of the positions of the fractures relative to x, y, and z coordinate axes, X, the orientations of the fractures relative to these axes, V, and the diameters of the fractures, D. This rather simple model facilitates a range of calculations such as determining fracture system connectivity [e.g., *Billaux et al.*, 1989; *Charlaix et al.*, 1984; *Gueguen and Dienes*, 1989] and simulating groundwater flow within the fractures [e.g., *Cacas et al.*, 1990; *Long et al.*, 1985; *Piggott and Elsworth*, 1989].

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The concept of fractal geometry [*Mandelbrot*, 1983] is an appealing approach to the characterization of fractured rock masses, the principal benefit being that detailed and realistic geometries may be stated using a minimal set of parameters. Examples of the expression of fracture geometry and density as a function of scale using fractal methods are reported by *Barton and Larsen* [1985], *La Pointe* [1988], and *Kulatilake et al.* [1996]. This paper addresses the relation between the fractal parameters of a population of disc-shaped fractures and the parameters of the population of traces lengths that results from the intersection of the fracture system with a planar outcrop surface. These relations provide a method of characterizing a three-dimensional fracture system using data that is measurable in situ.

In addition to the standard conceptual model of disc-shaped fractures, this study assumes that fractures are uniformly and randomly distributed within a rock mass and have a uniform and random distribution of orientation defined by the dip direction, α, and dip angle, β, of the fractures [*Priest*, 1985]. The orientation of the outcrop surface does not influence the results, and therefore the relations developed herein may be applied to an outcrop surface with any orientation (e.g., a horizontal or sloping ground surface exposure or a mined excavation at depth). Each fracture that intersects the outcrop surface results in a trace length of L, such that $0 \leq L \leq D$ where the specific value of L is defined by the position and orientation of the fracture relative to the outcrop surface. This sampling forms a convolution between the distributions of fracture diameter and trace length. Inversion of this convolution allows the fractal parameters for diameter to be determined from the parameters for trace length.

Mathematical Development

In this application, a population of disc-shaped fractures is deemed fractal if the number of fractures per unit volume with a diameter greater than or equal to D may be approximated using

$$N_D = \frac{a_D}{D^{F_D}}, \tag{1}$$

where F_D is the fractal dimension of the distribution of diameters and a_D is a constant of proportionality. This form of number-size relation has been applied to a range of fragmented geologic media [*Turcotte*, 1986]; applications to fracture trace length data are reported by *Scholz et al.* [1993] and *Watanabe and Takahashi* [1995]. It is also possible to describe this distribution in terms of a volumetric density of fractures of diameter D, ρ_D, where this density is related to (1) through

$$N_D = \int_D^{\infty} \rho_D dD , \tag{2}$$

and therefore

$$\rho_D = \frac{a_D F_D}{D^{F_D+1}} . \tag{3}$$

Figure 1 depicts a disc-shaped fracture intersecting a planar outcrop surface. View A shows the fracture and outcrop surface in three-dimensional perspective and indicates the linear trace of the intersection of the fracture with the outcrop surface. View B shows the dip of the fracture relative to the outcrop surface, β, and the distance between the fracture center and outcrop surface, z. View C shows the diameter of the fracture, D, and the length of the intersection of the fracture with the outcrop surface, L. The trace length, diameter, dip angle, and position of the fracture are related via

$$\beta = \sin^{-1}\frac{2z}{\sqrt{D^2-L^2}} \tag{4}$$

where the permissible range of values of z is

$$0 \le z \le \frac{1}{2}\sqrt{D^2-L^2} . \tag{5}$$

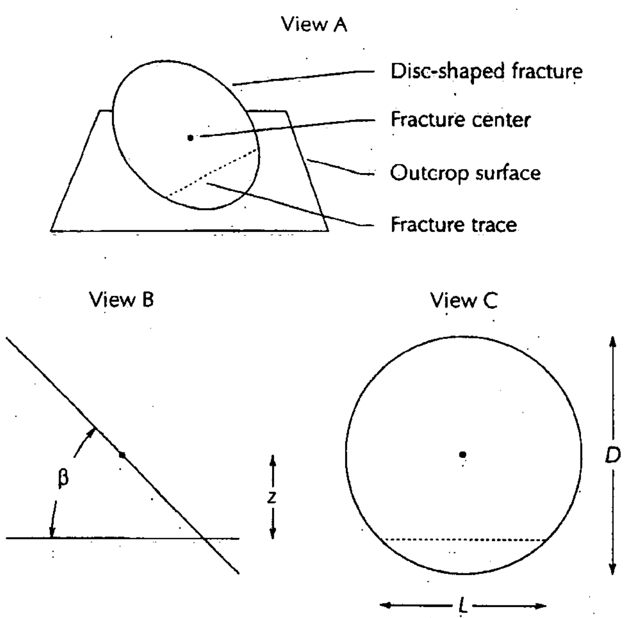

Figure 1. Geometry of a disc-shaped fracture intersecting a planar outcrop surface.

The probability that the scenario shown in Figure 1 results in a trace length of L may be determined from the probability density function for the dip angle, p_β, as [*Guttman et al.*, 1982]

$$p_L(D,z) = p_\beta \frac{d\beta}{dL} , \tag{6}$$

where

$$p_\beta = \frac{2}{\pi} \tag{7}$$

for a uniform, random distribution of dip angles. Differentiating (4) with respect to trace length yields

$$\frac{d\beta}{dL} = \frac{2zL}{(D^2-L^2)\sqrt{D^2-L^2-4z^2}} , \tag{8}$$

and substituting (7) and (8) into (6) yields

$$p_L(D,z) = \frac{4L}{\pi(D^2-L^2)}\frac{z}{\sqrt{D^2-L^2-4z^2}} . \tag{9}$$

The number of fracture traces of length L per unit area of an outcrop surface resulting from the intersection of fractures of diameter D with the surface is

$$\rho_L(D,z) = 2\rho_D p_L(D,z)\Delta z . \tag{10}$$

Here fractures on either side of the outcrop surface are considered, and Δz is an increment of the distance from the surface. Integrating (10) over (5) yields the areal density of traces of length L due to fractures of diameter D

$$\rho_L(D) = \frac{2}{\pi}\frac{L}{\sqrt{D^2-L^2}}\rho_D . \tag{11}$$

All fractures with a diameter of $D \ge L$ have a finite probability of producing a trace length of L upon intersection with an outcrop surface. Thus the areal density of traces of length L resulting from fractures of any diameter may be determined by integrating (11) with respect to fracture diameter

$$\rho_L = \frac{2}{\pi}\int_L^{\infty}\frac{L}{\sqrt{D^2-L^2}}\rho_D dD . \tag{12}$$

Substituting (3) into (12) and integrating the result yields

$$\rho_L = \frac{a_D F_D}{\sqrt{\pi}}\frac{\Gamma(\frac{1+F_D}{2})}{\Gamma(\frac{2+F_D}{2})}\frac{1}{L^{F_D}} , \tag{13}$$

where Γ denotes the gamma function.

The number of fracture traces per unit area of an outcrop surface with a length greater than or equal to L is the areal equivalent of (1) and may be calculated as

$$N_L = \int_L^{\infty}\rho_L dL . \tag{14}$$

Substituting (13) into (14) and integrating the result yields

$$N_L = \frac{a_D}{\sqrt{\pi}} \frac{F_D}{F_D - 1} \frac{\Gamma(\frac{1+F_D}{2})}{\Gamma(\frac{2+F_D}{2})} \frac{1}{L^{F_D - 1}} . \quad (15)$$

Equation (15) has a fractal form which is analogous to that of (1); that is,

$$N_L = \frac{a_L}{L^{F_L}} . \quad (16)$$

This indicates that a population of disc-shaped fractures exhibiting a fractal distribution of diameters results in a fractal distribution of trace lengths when expressed in outcrop. Further, it is possible to determine the fractal dimension and coefficient for diameter from the parameters for trace length via

$$F_D = F_L + 1 \quad (17)$$

and

$$a_D = a_L \sqrt{\pi} \frac{F_L}{F_L + 1} \frac{\Gamma(\frac{3+F_L}{2})}{\Gamma(\frac{2+F_L}{2})} . \quad (18)$$

From (15), intersecting a fracture system with a planar outcrop surface reduces the fractal dimension for trace length by a unit value relative to the fractal dimension for fracture diameter. A similar relation may be used to determine the fractal dimension of a fracture surface with rough, fractal topography from sections cut through and across the surface [e.g., *Piggott and Elsworth*, 1995] and of a fractal population of spheres from sections cut through the population [*Sammis et al.*, 1987].

Verification Exercises

Equation (15) may be verified using a probabilistic approach that forms a population of fractures with the parameters a_D and F_D. The first step in this approach is to specify the smallest fracture diameter that is required in the results, D_{min}. The total number of fractures is then determined from (1) as

$$n_T = \frac{a_D}{D_{min}^{F_D}} \Delta x \Delta y \Delta z , \quad (19)$$

where Δx, Δy, and Δz are prescribed dimensions of the fracture system. Fracture centers are generated from uniformly distributed, random values with $0 \le x \le \Delta x$, $0 \le y \le \Delta y$, and $0 \le z \le \Delta z$. The orientations of the fractures are computed from uniformly distributed, random dip directions and angles using

$$V = \begin{bmatrix} \cos\alpha \ \sin\beta \\ \sin\alpha \ \sin\beta \\ \cos\beta \end{bmatrix} \quad (20)$$

with $0 \le \alpha \le 2\pi$ and $0 \le \beta \le \pi/2$. Finally, the required distribution of diameters is generated using

$$D = \frac{D_{min}}{(1 - \xi)^{1/F_D}} , \quad (21)$$

where ξ is a uniformly distributed, random value with $0 \le \xi \le 1$. The intersection of each fracture with the outcrop surface is calculated using linear algebra [*Anton*, 1981], constraining the intersection segment to match the circular shape of the fracture. In this case, the outcrop surface is positioned at the center of the fracture system and oriented perpendicular to the z axis. Those fractures which intersect the outcrop surface yield trace lengths, L_i, which may be sorted into descending order and plotted relative to

$$N_L = \frac{i}{\Delta x \Delta y} \quad (22)$$

for comparison with the analytic result computed using (15).

This probabilistic approach has been implemented as a FORTRAN algorithm where the required random values are generated using the RAN2 function described by *Press et al.* [1992]. The following examples assume a fracture system with dimensions of $\Delta x = \Delta y = \Delta z = 100$ m, where the smallest fracture represented in the results is $D_{min} = 1$ m. A value of $a_D = 1$ is assumed for fractal dimensions of $F_D = 2$, 3, and 4 and results in a total of $n_T = 10^6$ fractures for all three realizations. Figure 2 shows the distribution of the 9542 trace lengths rendered using $F_D = 3$. The region containing the fracture centers is obvious as the highly populated portion of Figure 2. Traces that extent beyond this region are the result of fractures that are centered within the populated volume but which intersect the outcrop surface beyond the boundaries of the volume.

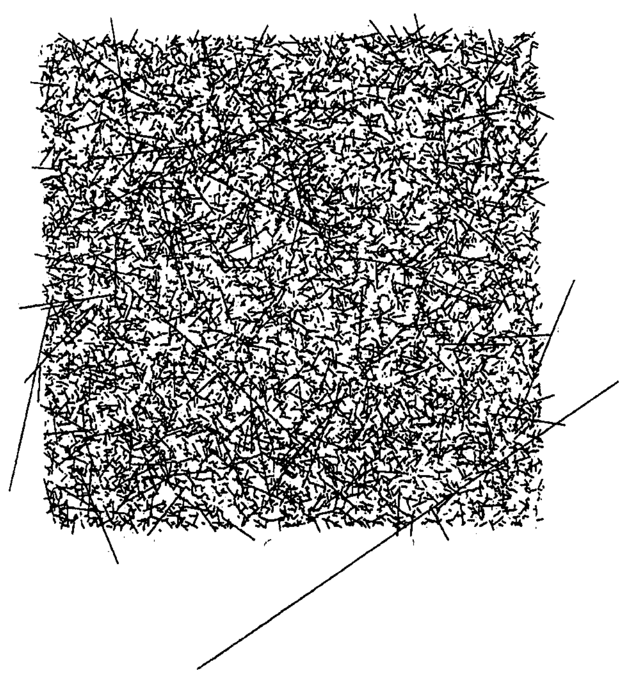

Figure 2. Fracture traces for a fracture system with the fractal parameters $a_D = 1$ and $F_D = 3$. The dimension of the highly populated portion of the figure is 100 m.

Figure 3 compares the analytic results determined using (15) to the probabilistic results determined using (22). A linear relation between N_L and L is indicative of a fractal distribution when plotted in the logarithmic form of Figure 3. The two sets of results compare closely for trace lengths between 1 and 10 m, confirming the accuracy of the analytic relations. There is a substantial and consistent discrepancy between the sets of results for trace lengths of less than 1 m. This range of values is the product of fractures with diameters greater than 1 m, as the smallest diameter represented in the results is $D_{min} = 1$ m. Thus the discrepancy at the lower end of the range of data is indicative of a lack of fractures of corresponding dimension. In situ, this can result from selective sampling practices; for example, mapping fracture traces using geophysical techniques which are most responsive to larger scale features. The results also diverge for trace lengths greater than 10 m. Here the departure is less substantial and consistent than at the lower end of the range of data. This discrepancy is the result of the sparse sampling of larger fracture diameters. In situ, a scarcity of larger fractures and the truncation of fracture traces at the limits of an outcrop region may cause the characterization results to deviate from the fractal form. Similar departures are apparent in the trace length data reported by *Scholz et al.* [1993] and are attributed to similar sampling limitations.

Conclusions

Fractal geometry is a useful method of representing the geometry and scale dependence of fracturing in a range of geologic settings. A number-size relation for the frequency of fractures of a prescribed dimension is one method of characterizing a fracture system in fractal form. The analytic relations developed in this paper indicate that for a population of disc-shaped fractures, a fractal distribution of fracture diameters results in a fractal distribution of trace lengths when the three-dimensional fracture system is intersected by a planar outcrop surface. The fractal parameters of the distributions are uniquely related, and therefore a population of fracture diameters, which cannot be directly measured in situ, may be characterized in fractal form through the interpretation of trace length data, which can be readily measured in outcrop. The probabilistic results reported in this paper suggest that the departure of trace length data from a fractal form at the upper and lower limits of a range of data may be indicative of sampling limitations. Thus it may be reasonable to interpret trace length data in terms of a fractal model if an intermediate range of data displaying fractal behavior is apparent.

The relations developed in this paper apply to homogeneous and isotropic fracture systems. The principal implication of this limitation is likely to be that the relations are not applicable to fracture systems with statistically disparate groupings of orientation. While similar analytic relations and probabilistic results may be feasible for more complex models of fracture system geometry, the acquisition of data to support these models and the robust determination of the fractal parameters of the models may prove to be daunting tasks. An effort to collect and interpret fracture trace length data for a granitic rock mass over varying scales of approximately 20 m to 10 km is currently in progress. The results that have been achieved to date indicate a reasonable match relative to the analytic relations developed in this paper, with fractal parameters that are roughly consistent with those used in the verification exercises. As a result, the analytic relations reported in this paper may have considerable value in practical application.

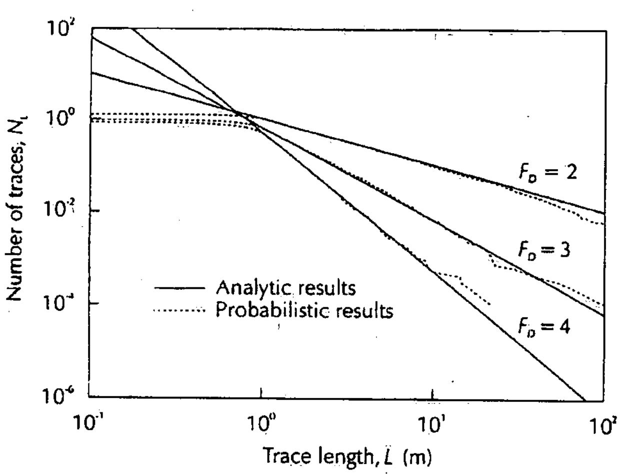

Figure 3. Analytic and probabilistic distributions of fracture trace length for fracture systems with the fractal parameters $a_D = 1$ and $F_D = 2$, 3, and 4.

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A. R. Piggott, National Water Research Institute, 867 Lakeshore Road, Burlington, Ontario, L7R 4A6, Canada. (e-mail: Andrew.Piggott@CCIW.ca)

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Environment Canada Environnement Canada

Canada

Canada Centre for Inland Waters
P.O. Box 5050
867 Lakeshore Road
Burlington, Ontario
L7R 4A6 Canada

National Hydrology Research Centre
11 Innovation Boulevard
Saskatoon, Saskatchewan
S7N 3H5 Canada

St. Lawrence Centre
105 McGill Street
Montreal, Quebec
H2Y 2E7 Canada

Place Vincent Massey
351 St. Joseph Boulevard
Gatineau, Quebec
K1A 0H3 Canada

Centre canadien des eaux intérieures
Case postale 5050
867, chemin Lakeshore
Burlington (Ontario)
L7R 4A6 Canada

Centre national de recherche en hydrologie
11, boul. Innovation
Saskatoon (Saskatchewan)
S7N 3H5 Canada

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105, rue McGill
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H2Y 2E7 Canada

Place Vincent-Massey
351 boul. St-Joseph
Gatineau (Québec)
K1A 0H3 Canada