97-142

Environment Canada

Water Science and Technology Directorate

Direction générale des sciences et de la technologie, eau

Environnement Canada

A Grid Generating Algorithm for Simulating a Fluctuating Water Table Boundary in Heterogeneous Unconfined Aquifers

By:
A. Crowe, S. Shikaze, F. Schwartz

97-142

A Grid Generating Algorithm for Simulating a Fluctuating Water Table Boundary in Heterogeneous Unconfined Aquifers

by

A.S. Crowe[1], S.G. Shikaze[1] and F.W. Schwartz[2]

[1]National Water Research Institute,
Canada Centre for Inland Waters,
P.O. Box 5050,
Burlington, Ontario, L7R 4A6,
Canada.

[2]Department of Geological Sciences,
The Ohio State University,
125 South Oval Mall,
Columbus, Ohio 43210,
U.S.A.

for submission to
Advances in Water Resources

December 1997

NWRI Cont # 97-142

Management Perspective

Title: A Grid Generating Algorithm for Simulating a Fluctuating Water Table Boundary in Heterogeneous Unconfined Aquifers.

Authors and affiliations:

Allan Crowe[1], Steven G. Shikaze[1] and Frank W. Schwartz[2]

[1] Groundwater Remediation Project, National Water Research Institute, Environment Canada, Canada Centre for Inland Waters, Burlington, Ontario.

[2] Department of Geological Sciences, The Ohio State University, 125 South Oval Mall, Columbus, Ohio 43210, U.S.A.

NWRI Contribution Number: 97-142

Citation: Advances in Water Resources.

EC Priority/Issue:

Environmental stress caused by human activities have contributed to a deterioration in the heath and biodiversity of the marsh at Point Pelee National Park. As part of the GL 2000 program, NWRI is undertaking an assessment of the hydrogeological environment at Point Pelee. Ultimately, this project will have wider implications for the control of nutrient loading to coastal wetlands of the Great Lakes, and the conservation of these fragile ecosystems.

Current status:

It is suspected that nutrients originating from the Park's septic systems may be contributing to the high nutrient levels in the marsh. In order to assess this possibility, and evaluate new septic system designs and proceed with remedial measures, a numerical model is being developed to both simulate the present hydrogeological process currently occurring, and assess remedial options. Because there are not any numerical models that can accurately simulate groundwater - surface water interactions and conditions occurring in the coastal areas of the great lakes, a model must be developed. This paper presents the numerical technique for generating the grid used by the model.

Next steps:

The numerical methods incorporated into the model have been validated, the model will be used to simulate the hydrogeological environment at Point Pelee and assess the transport of contaminants from the septic systems to the marsh.

ABSTRACT

A method for generating finite element grids that calculates the position of a fluctuating water table and the formation of seepage faces within a heterogeneous unconfined aquifer is described. Our approach overcomes limitations with existing techniques, with respect to numerical accuracy and heterogeneities, by allowing the water table to rise or decline through hydrostratigraphic boundaries, yet maintain numerical and conceptual accuracy with respect to hydrostratigraphic geometry. The algorithm involves (1) a limited stretching of elements along the water table if the change in the position of the water table is small with respect to the vertical grid spacing, and (2) the addition or removal of nodes and elements to the finite element mesh along the water table as the change becomes large with respect to the vertical grid spacing. This technique is applicable to any 2-D or 3-D finite element code that contains an automatic finite-element grid generator.

KEY WORDS

finite element method, free surface, grid generation, groundwater, hydrogeology, modelling, unconfined aquifer, water table

INTRODUCTION

The objective of this paper is to present a method for generating finite element grids that calculate the position of a fluctuating water table and the formation of seepage faces within a heterogeneous unconfined aquifer. Specifically, our algorithm allows the water table to rise or decline through hydrostratigraphic boundaries, yet maintain numerical and conceptual accuracy with respect to hydrostratigraphic geometry. Although we make use of a two-dimensional domain that is discretized with triangular finite elements, the method proposed here may be incorporated into any 2-D or 3-D finite element code that contains an automatic finite element grid generator.

This work provides an important step forward over other approaches that suffer from a variety of limitations. For example, in some schemes the elevation of the nodes along the water table is fixed (i.e., the grid does not deform) throughout the simulation but the calculated value of heads along the water table can change[10,9,7]. With the geometry of the cells remaining the same throughout the simulation, errors can result because the hydraulic head along the water table is not equal to the elevation of the water table. Also, these methods cannot account for layering of hydrostratigraphic units through which the water table may rise or fall.

Other schemes match water-table elevations and hydraulic heads by allowing the mesh to deform through time[6,2,1,8,5,3,4]. The elements that are deformed might include only the top row of elements or all in the domain. Although this approach is more accurate, problems can arise when the mesh expands or contracts through layer boundaries. In this case, the initial hydrostratigraphy is not preserved as elements stretch past layer boundaries. These techniques are inappropriate for problems where the water table moves through more than one hydrostratigraphic unit. Finally, with elements able to deform in an unknown manner, numerical inaccuracies may creep in due problems of aspect ratio.

METHOD

In a free-surface problem, both the hydraulic head distribution and the water table configuration are unknown. The main criterion for accurately simulating the position of the water table is that the elevation of a node *i* along the water table, $\delta(x,t)$, is equal to the hydraulic head at the water table node, $h(x,t)$, at all times. Also, if the water table is at ground surface, a seepage face will form and the value of hydraulic head will be equal to the elevation of the ground surface.

Our approach considers only the saturated part of the flow system (below the water table). Positive and negative fluxes are used as boundary conditions along the top of the domain, causing the water table to rise or fall. Figure 1 shows a typical cross section and boundary conditions that are described below. The governing equation for transient groundwater flow in the saturated zone (S in Fig. 1) is:

$$\frac{\partial}{\partial x_i}\left[K_{ij}\frac{\partial h}{\partial x_j}\right] = S_s\frac{\partial h}{\partial t} \qquad (1)$$

where K_{ij} is the hydraulic conductivity tensor [L/T], h is hydraulic head [L], S_s is the specific storage coefficient [L^{-1}], t is time [T], x_i is the coordinate vector and i, j =1,2 [L]. The initial conditions are:

$$h(x,z,0) = h_o(x,z) \tag{2}$$

$$\delta(x,z,0) = \delta_o(x,z) \tag{3}$$

where δ is the elevation of the free surface (F in Fig. 1) above a datum [L], δ_o is the initial elevation of the free surface [L], h_o is the initial hydraulic head [L]. The boundary conditions for equation (1) are:

$$h(x,z,t) = H(x,z,t) \qquad \text{on b-c (Fig. 1)} \tag{4}$$

$$K_{ij}\frac{\partial h}{\partial x_i}n_i = -Q(x,z,t) \tag{5}$$

$$\delta(x,z,t) = h(x,z,\delta,t) \qquad \text{on F (Fig. 1)} \tag{6}$$

$$K_{ij}\frac{\partial h}{\partial x_i}n_i = \left[R - S_y\frac{\partial \delta}{\partial t}\right]n_3 \qquad \text{on F (Fig. 1)} \tag{7}$$

$$h(x,z,t) = z \qquad \text{on a-b (Fig. 1)} \tag{8}$$

where H is the hydraulic head on a constant head boundary [L], R is the rate of vertical recharge along the free surface [L/T], n_i is the unit outward normal vector, S_y is the specific yield and Q is the flux along a specified-flux boundary [L/T].

Equations (2) and (3) are initial conditions which state that the hydraulic head values and the elevations of the water table must be specified at the start of the simulation. Equation (4) represents a first-type or Dirichlet boundary condition where specified values of hydraulic head are assigned along the boundary. The value of the specified head at these boundaries can change in time, and constant head nodes can be turned off and on during a simulation. Equation (5) represents a second-type, or a Neumann boundary condition, where a specified flux across a boundary is assigned. Equations (6) and (7) represent the boundary conditions along the free surface, depending on whether or not recharge fluxes are present. Equation (8) represents a free-surface boundary, where the hydraulic head is equal to elevation of the ground surface. The boundary value problem defined by equations (1) through (8) is described in further detail by Neuman and Witherspoon[6].

The two-dimensional form of equation (1), subject to initial and boundary conditions, is solved in a vertical, two-dimensional cross section using a standard finite-element technique. Although, we use a triangular finite-element mesh, the procedure would apply with most element types (e.g., quadrilateral finite elements, 3-D domains). The finite-element equations are formulated using the Galerkin method[8]. Our algorithm for generating the finite-element grid satisfies the following conditions:

- the position of the water table can rise or fall over time as a result of boundary conditions that can change in time,
- all nodes along the water table are located at $\delta(x,t) = h(x,t)$,

- the interfaces between hydrostratigraphic units within the saturated zone are always located at nodes,
- a single element does not cross over the interface between two hydrostratigraphic units,
- seepage faces form where nodes are located at $z(x,t) = h(x,t) = z_g(x,t)$ (where z_g = the elevation of the ground surface),

Our method involves a combination of a limited stretching of elements along the water table and/or the addition or removal of nodes and elements along the water table. If the change in the position of the water table is small with respect to the vertical grid spacing, the elements along the water table are stretched or compressed. If the change in position is large with respect to the vertical grid spacing, new elements and nodes are added or removed.

The first step is to discretize the computational domain into triangular finite elements using an automatic mesh generator. This initial mesh depends on the geometry of the domain, the boundary conditions and the initial elevation of the water table. The grid spacing (Δx, Δz) is small relative to the scale of the problem in order to represent the hydrostratigraphic units and to position nodes along the interface between units. The grid generator assigns an elevation to each node along the uppermost row of the mesh that is equal to the assigned value of hydraulic head of the water table at that node.

With both the elevation of the water table and hydraulic heads as unknowns, an iterative solution is required. The adjustment of the finite-element mesh is illustrated in

Figure 2. At the beginning of a time step, the elevation of the nodes along the water table is fixed and the hydraulic heads within the flow domain are calculated (Fig. 2a, 2g). The difference between the elevation and the calculated head for each node along the water table is compared. If any nodal difference is greater than a specified convergence tolerance, the nodes along the water table are repositioned vertically to a location corresponding to the calculated value of hydraulic head (the x position remains constant). Because only the nodes along the water table are allowed to move, only the top row of elements are stretched or compressed. Changing the vertical dimension of an element produces a new vertical spacing of $\Delta\zeta$. All remaining elements below the uppermost row of elements remain at a constant vertical spacing of Δz (Fig. 2b, 2h). At the end of each iteration, numerical convergence is tested by calculating a residual based on the difference between the head and the elevation of the nodes along the water table. The solution has converged when the residual is less than a user-defined tolerance. Moving to the next time step, the process is repeated with the opportunity to change the mesh again (Fig. 2c, 2e, 2i, 2k).

The procedure outlined above is used with most finite-element codes that allow the grid to deform as the shape of the flow domain changes. However, in our method, at the beginning of each new time step, if an element is stretched more than $^1/_4\Delta z$ beyond a regular grid spacing ($\Delta\zeta > {}^5/_4\Delta z$) we form a new node and a new element. The new node is inserted at the regular Δz spacing, and the new element is inserted along the water table with a vertical element spacing of $\Delta\zeta_{new} = \Delta\zeta_{old} - \Delta z$ (Fig. 2e). If an element stretches less than $^1/_4\Delta z$ beyond the regular grid spacing ($\Delta\zeta < {}^5/_4\Delta z$), only the top two elements are stretched, and a new node is not inserted (Fig. 2b). These two stretched elements are formed from the regular Δz spacing to the present position of the water table where $\Delta\zeta_{new} = \Delta z + \Delta\zeta_{old}$ (Fig. 2f). Similarly, if a node at the water table declines by more than $^3/_4\Delta z$ of the regular grid spacing ($\Delta\zeta < {}^1/_4\Delta z$), the node immediately below this water table node is removed (Fig. 2k). If the decline of a water table node is less than $^3/_4\Delta z$, the z position of this node is simply lowered to the current value, thereby compressing the finite element, with no removal of nodes or elements (Fig. 2h).

Because all elements, except those at the water table, are maintained at the original vertical grid spacing of Δz, unit boundaries remain unchanged (Fig. 3). The only instance where the mesh may not coincide with the unit boundaries occurs when the water table passes into a new geologic unit. Initially, the changes in the water table elevation may result in water-table elements stretching less than $^3/_4\Delta z$ from $\Delta\zeta = \Delta z$. If the stretched node exists at the interface between two units, the stratigraphy will not be preserved because new elements are not formed and the new stratigraphic unit will not exist in the model. With time, the water table will continue to rise to a point where a new element will form, at which time the stratigraphy will be once again be accurately represented. The error resulting from this slight misrepresentation is small. Once the new elements form, element boundaries will be placed at the proper Δz spacing. Hence, the error will be insignificant. However, there may be a case in this situation where the scheme may have difficulty converging to a stable solution. This problem will be addressed in more detail.

Our scheme also allows seepage faces to form when the water table intersects the ground surface. Nodes and elements along the seepage face are inserted as the seepage face expands or removed as the seepage face contracts according to the above criterion. Nodes are not allowed to be repositioned above the ground surface. At the ground surface, nodes are redesignated as constant-head nodes along the seepage face. If the water table falls below the ground surface, the constant-head nodes along the seepage face revert to regular nodes.

Sometimes, when the water table moves through a geologic unit of contrasting hydraulic conductivity, convergence problems can develop. For example, if the water table rises from a low to a high hydraulic conductivity unit, the elements in the lower K unit will stretch until the hydraulic heads increase beyond $^3/_4\Delta z$. After this, new water-table elements will form, but these new elements will have the higher K assigned to them, and the low K elements will shrink to the regular element size. Because these new elements have a higher K, the hydraulic head along the water table may decrease resulting in a drop in the water table. If these high K elements shrink to below $^3/_4\Delta z$, they will be removed and the low K elements will be stretched upwards. This may result in an increase in hydraulic head and result in the formation of new high K elements yet again. This entire sequence may repeat itself in a oscillatory manner and convergence might never be achieved. To rectify this problem, an algorithm is included that identifies these oscillations. In such cases, the criterion for forming a new element is decreased from $^3/_4\Delta z$ to $^1/_{10}\Delta z$, and the criterion for removing an element is increased from $^3/_4\Delta z$ to $^9/_{10}\Delta z$. This fix provides convergence.

APPLICATION

Following is a comparison of simulation trials with our adaptive griding scheme with a scheme that allows various numbers of rows of elements to deform. Specifically, we allow one, four or all of the rows of elements to deform. This latter case is the treatment incorporated in other codes. The comparison involves two different simulation problems. The first is the simple case of a homogeneous, isotropic aquifer. The second case provides a layer with a contrasting hydraulic conductivity through which the water table moves.

Figure 4 presents a schematic of the domain and the boundary conditions for the domain. The initial water table elevation is five metres, which coincides with the top boundary of the domain. With time, the water table will rise due to recharge and the size of the domain in the vertical direction will increase as new elements are formed. A uniform recharge rate of 0.51 m/year is applied across top boundary and a constant head node is specified at the top right corner of the domain with a hydraulic head value equal to five metres. All other boundaries in the domain are no-flow.

Case 1 - Homogeneous Aquifer

The physical and numerical parameters for Case 1 are tabulated in Table 1. Figure 5 shows results from the four different solution methods at a time equal to 100 days and at steady-state. The hydraulic head along the water table is plotted versus the horizontal (x) distance. At steady state, the maximum elevation of the water table is approximately 4.6 m above the starting water-table elevation. For this simple case, all four solution schemes result in essentially identical results. This result is expected because although elements can deform significantly, hydraulic parameters remain the same and vertical stretching of the elements is minimal which would preclude problems related to the aspect ratio

Case 2 - Heterogeneous Aquifer

The second example includes a layer of lower hydraulic conductivity that is present at an elevation of four to six metres (Fig. 4). Because the initial domain is only five metres thick, the simulation techniques that involve only stretching of elements will not preserve the location and thickness of this low conductivity layer. Our scheme, however, preserves the geometry of the low-K layer. Figure 6 shows the hydraulic head at the water table versus horizontal distance for the four different schemes. As the simulations proceed, problems related to the treatment of the low conductivity layer become apparent for schemes that only allow the elements to stretch. If one or four rows of elements are stretched, the steady-state water table exceeds that from our method by approximately 3 and 1 m, respectively. The excess head develops because the low-K unit is unrealistically enlarged through element stretching. If all elements are stretched, the water table is lower by approximately 0.1 m, as compared to our method. The match is more favorable because the thickness of the low conductivity layer remains closer to the actual thickness. However, the position of the low conductivity layer is distorted.

Thus, in some layered cases, the solution methods that simply stretch the elements produce results that can be significantly different from those produced by our method

CONCLUSIONS

Several models are capable of simulating both head distributions and water-table configurations within a unconfined aquifer. Various schemes are used. The simplest involves calculating the position of the water table with a fixed grid or a grid. The calculated heads along the water table can change but the position of the nodes along the water table remain fixed. Other models calculate the position of the water table by having nodes and cells/elements stretch or compress in order to match the elevation of the water table nodes to the respective values of hydraulic head. However, these models are most applicable to problems where the water table fluctuates within a single homogeneous unit. They are not appropriate for systems with changing hydraulic parameters within the zone over which the water table fluctuates.

Our scheme calculates the position of a fluctuating water table and the formation of seepage faces within a heterogeneous unconfined aquifer. More importantly, it maintains the distribution of hydraulic parameters by careful regeneration of the grid as appropriate. This technique is applicable to any 2-D or 3-D finite element code that contains an automatic finite-element grid generator.

REFERENCES

1. Bonnerot, R. and P. Jamet. 1977. Numerical computation of the free boundary for the two-dimensional Stefan problem by space-time finite elements. Journal of Computational Physics, 25(2):163-181.
2. France, P.W., C.J. Parekh, J.C. Peters and C. Taylor. 1971. Numerical analysis of free surface seepage problems. Am. Soc. Civil Eng. Journal of the Irrigation and Drainage Division 97(IR1):165-171.
3. Huyakorn, P.S. and G.F. Pinder. 1983. Computational Methods in Subsurface Flow. Academic Press, Inc., 473 pp.
4. Knupp, P. 1995. A moving mesh algorithm for 3-D regional groundwater flow with water table and seepage face. Advances in Water Resources, 19(2):83-95.
5. Lynch, D.R. and W.G. Gray. 1980. Finite element simulation of flow in deforming regions. Journal of Computational Physics, 36(2):135-153.
6. Neuman, S.P. and P.A. Witherspoon. 1971. Analysis of nonsteady flow with a free surface using the finite element method. Water Resources Research, 7(3):611-623.
7. McDonald, M.G. and A.W. Harbaugh. 1984. A Modular Three-Dimensional Finite-Difference Groundwater Flow Model. U.S. Geological Survey, Open File Report, 528pp.
8. Pinder, G.F. and W.G. Gray. 1977. Finite Element Simulation in Surface and Subsurface Hydrology. Academic Press, Inc., 295 pp.
9. Townley, L.R. and J.L. Wilson. 1980. Description of and User's Manual for a Finite Element Aquifer Flow Model Aquifem-1. R.M. Parsons Laboratory for Water Resources and Hydrodynamics, M.I.T., Report Number 252, 294 pp.
10. Trescott, P.C. 1975. Documentation of Finite-Difference Model for Simulation of Three-Dimensional Ground-Water Flow. U.S. Geological Survey, Open File Report 75-438.

Table 1. Hydrogeological and numerical parameters used in the simulations.

	Case 1	Case 2
Δx	4.0 m	4.0 m
Δz (initial size)	0.5 m	0.5 m
Δt; deltin	0.5 days, 1.02	2 days, 1.02
Δt_{max}	5 days	5 days
K_1	10^{-5} m/s	10^{-5} m/s
K_1	10^{-5} m/s	2×10^{-6} m/s
n	0.3	0.3
S_s	0.0005	0.0005
S_y	0.2	0.2

NOTE: deltin is the factor by which to increase each successive time step.

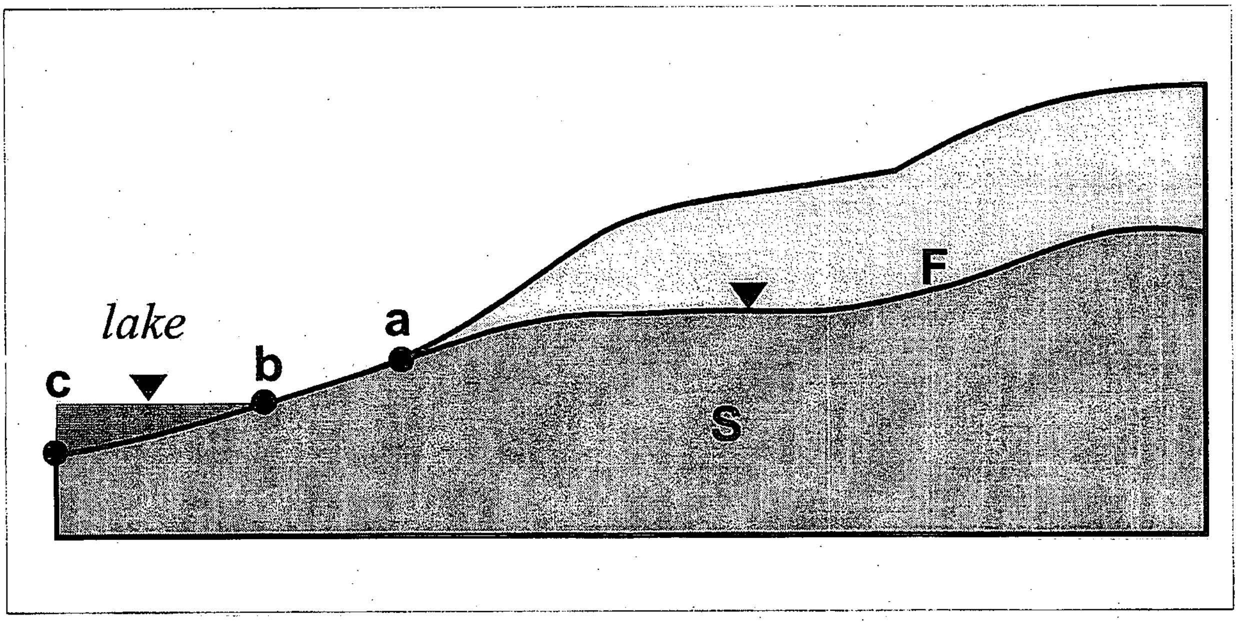

Figure 1. Schematic cross section showing the computation domain (S), the free surface (F), seepage face (a-b), and constant head boundary (b-c).

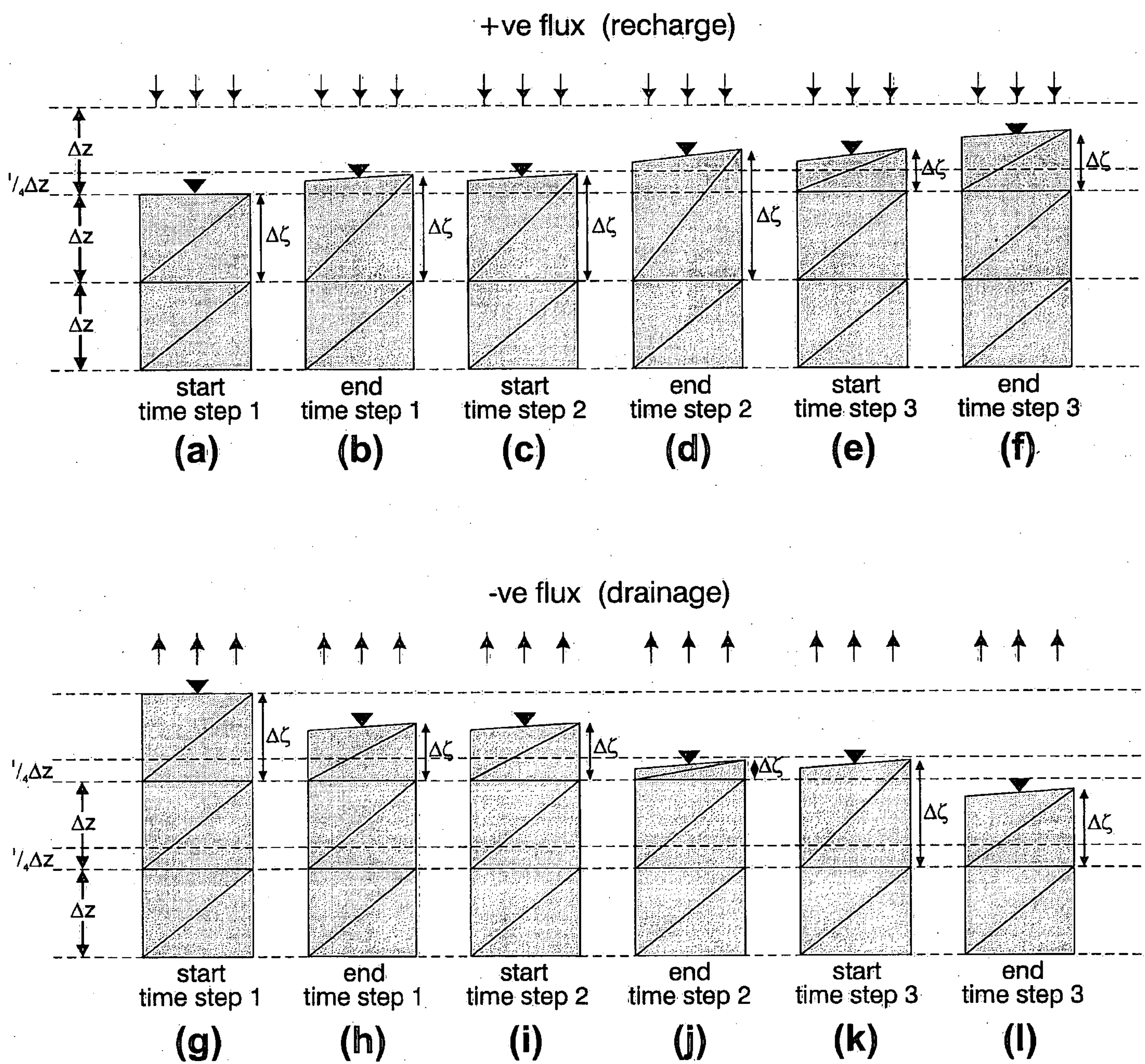

Figure 2. The procedure through which a finite element stretches/compresses and is added/removed as the water table rises or falls, respectively.

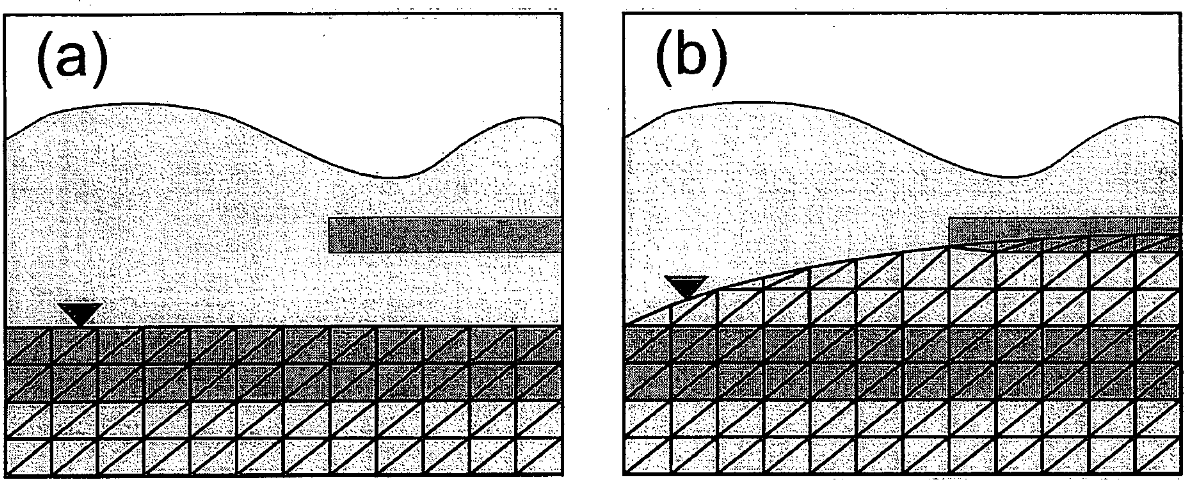

Figure 3. A cross section illustrating the reforming of the finite element grid through stretching and adding nodes and elements as the water table rises; (a) initial grid, (b) grid after several time steps.

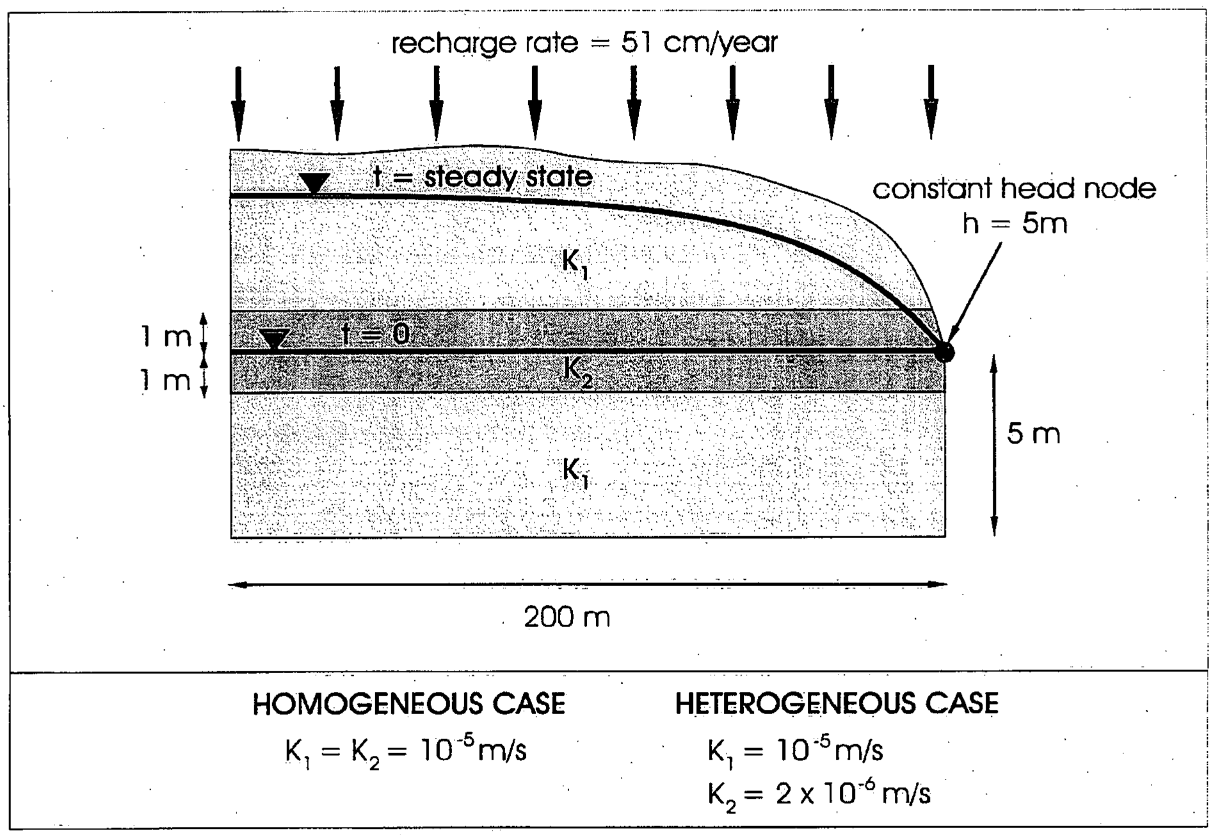

Figure 4. Conceptual model and boundary conditions for the sensitivity analysis for case 1 and case 2

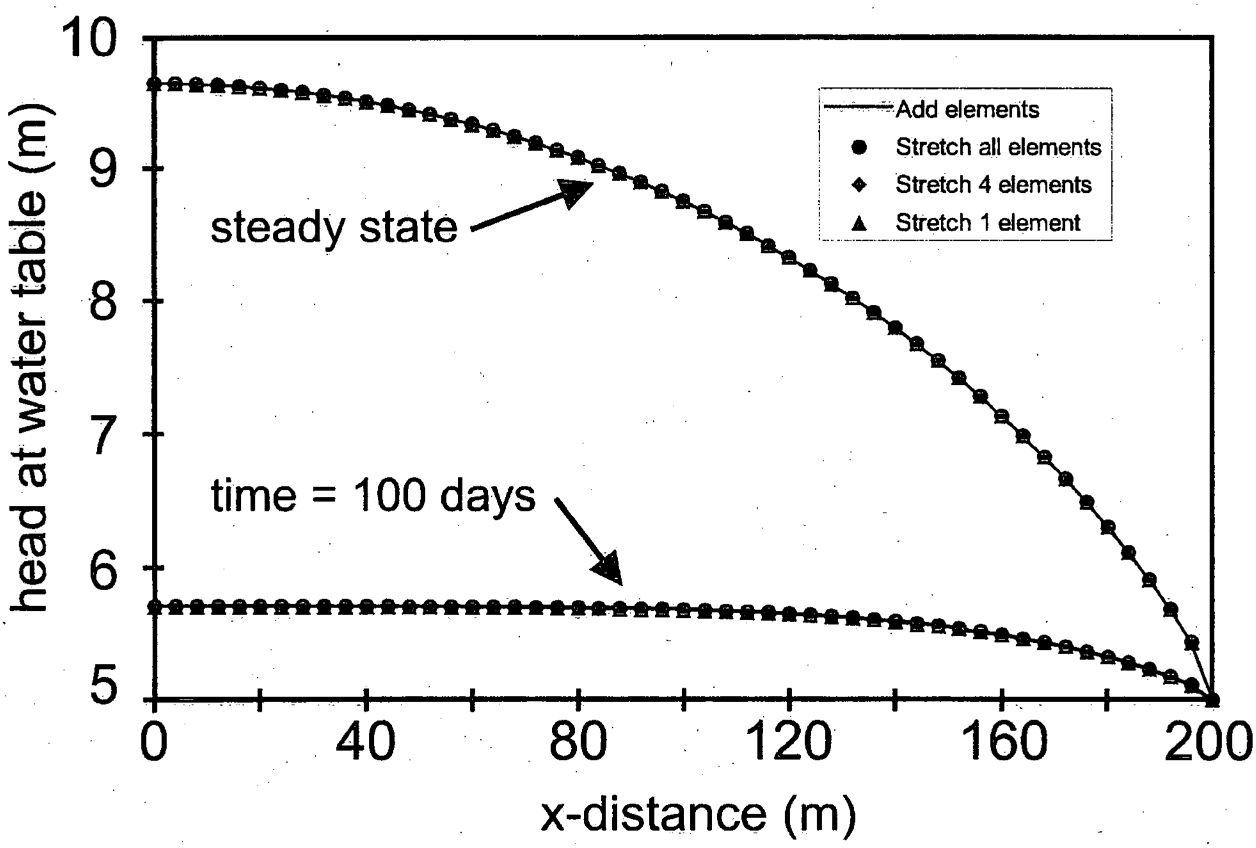

Figure 5. Comparison of current numerical methods that involve only element stretching with the algorithm that stretches and adds new elements; case 1 - homogeneous aquifer.

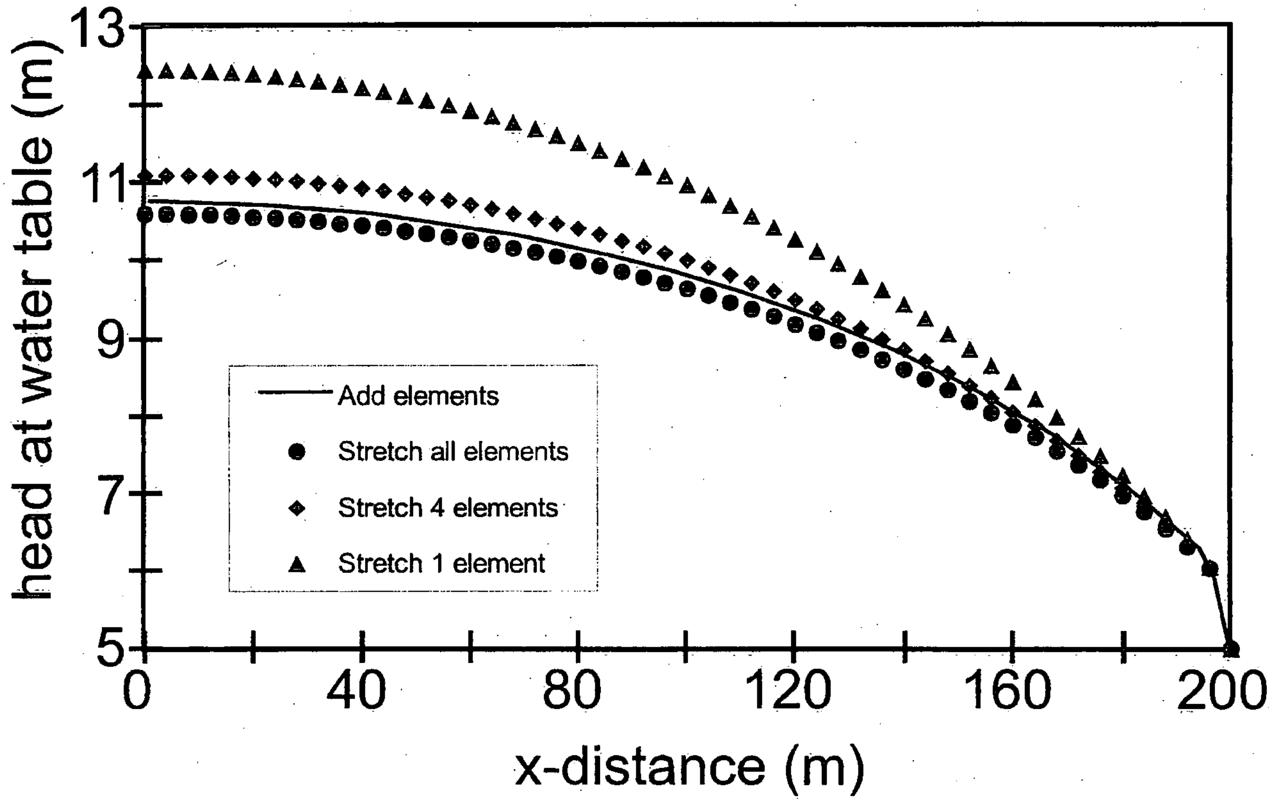

Figure 6. Comparison of current numerical methods that involve only element stretching with the algorithm that stretches and adds new elements; case 2 - heterogeneous aquifer.

Environment Canada Environnement Canada

Canada

Canada Centre for Inland Waters
P.O. Box 5050
867 Lakeshore Road
Burlington, Ontario
L7R 4A6 Canada

Centre canadien des eaux intérieures
Case postale 5050
867, chemin Lakeshore
Burlington (Ontario)
L7R 4A6 Canada

National Hydrology Research Centre
11 Innovation Boulevard
Saskatoon, Saskatchewan
S7N 3H5 Canada

Centre national de recherche en hydrologie
11, boul. Innovation
Saskatoon (Saskatchewan)
S7N 3H5 Canada

St. Lawrence Centre
105 McGill Street
Montreal, Quebec
H2Y 2E7 Canada

Centre Saint-Laurent
105, rue McGill
Montréal (Québec)
H2Y 2E7 Canada

Place Vincent Massey
351 St. Joseph Boulevard
Gatineau, Quebec
K1A 0H3 Canada

Place Vincent-Massey
351 boul. St-Joseph
Gatineau (Québec)
K1A 0H3 Canada