INLAND WATERS BRANCH

DEPARTMENT OF ENERGY, MINES AND RESOURCES

Analysis of wave motion in a rectangular channel using electrical network analogy

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Preface

In this publication an attempt is made to study the wave motion in a rectangular channel using the method applied to electrical network analysis. This method provides a clear and systematic approach to solving problems of wave motion. Also presented are various ways of electrical analogue to the tidal motion in a channel.

The study described in this publication was carried out during the period the Tides and Water Levels Section functioned as a part of the Water Survey of Canada in the Inland Waters Branch.

Acknowledgements

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Linearized Hydrodynamic Equation of One-dimensional Flow

In a river or estuary, the flow of water is predominantly in one direction and, therefore, the motion of water can be considered as one-dimensional motion. In this publication a channel refers to the whole or a portion of a river or an estuary.

Assuming a channel can be divided into sections of uniform width and depth and also constant De Chezy's coefficient, each section can be defined as a rectangular channel. At the boundary between channels, perpendicular to the flow, the following conditions can be assumed to exist:

$$q_{in} = q_{out} \qquad (1\text{-}1)$$

$$h- = h+ \qquad (1\text{-}2)$$

where q_{in} and q_{out} denote the horizontal flow entering and leaving the boundary while h- and h+ represent the vertical movement with respect to mean water level at the left-hand side and right-hand side of the boundary.

The hydrodynamic equation of fluids consists of two equations: the equation of continuity and the equation of motion.

For an incompressible fluid, the equation of continuity in a rectangular channel is

$$\frac{\partial q}{\partial x} + b_o \frac{\partial h}{\partial t} = 0 \qquad (1\text{-}3)$$

where b_o is the width of the channel, i.e., the length of the boundary.

By applying the following assumptions that:

1. the effect of wind force in negligible;
2. the flow is a laminar flow;
3. the Coriolis acceleration is not taken into consideration;
4. the convective derivative of the acceleration is negligibly small; and
5. the bottom of the channel is parallel to the datum;

the equation of motion of the fluid can be expressed in the following equation: [Dronkers (1)]

$$\frac{\partial q}{\partial t} + \lambda q + g a_o \frac{\partial h}{\partial x} = 0 \qquad (1\text{-}4)$$

where a_o is the mean cross-sectional area of the channel, g is the gravitational acceleration and λ is defined as follows:

$$\lambda = \frac{8\rho}{3\pi} \frac{g\, q_m}{C^2 a_o h_o} \qquad (1.5)$$

where q_m is the mean of q in the channel, C is De Chezy's coefficient of the channel, ρ is the density and h_o is the mean water height with respect to the bottom. The variables a_o, b_o and h_o satisfy the following condition:

$$a_o = b_o\, h_o \qquad (1\text{-}6)$$

Equations (1-3) and (1-4) are the basic equations considered in this publication.

The Solution of the Hydrodynamic Equation

2-1 GENERAL SOLUTION

Equation (1-4) can be rewritten:

$$\frac{\partial h}{\partial x} + \frac{\lambda}{ga_o} \cdot q + \frac{1}{ga_o} \cdot \frac{\partial q}{\partial t} = 0 \qquad (2\text{-}1)$$

If we define:

$$C_W = b_o$$
$$R_W = \frac{\lambda}{ga_o} \qquad (2\text{-}2)$$
$$L_W = \frac{1}{ga_o}$$

then (1-3) and (2-1) become:

$$\frac{\partial q}{\partial x} + C_W \frac{\partial h}{\partial t} = 0 \qquad (2\text{-}3)$$

$$\frac{\partial h}{\partial x} + R_W\, q + L_W \frac{\partial q}{\partial t} = 0 \qquad (2\text{-}4)$$

Assuming that the wave is a periodic function of time t:

$$h = \frac{1}{2}\left(H\varepsilon^{jwt} + H^*\varepsilon^{-jwt}\right) \qquad (2\text{-}5)$$

$$q = \frac{1}{2}\left(Q\varepsilon^{jwt} + Q^*\varepsilon^{-jwt}\right) \qquad (2\text{-}6)$$

where the superscript denotes the conjugate of the variable, and by substituting (2-5) and (2-6) into (2-3) and (2-4) we obtain

$$\frac{\partial Q}{\partial x} + j\omega C_W H = 0 \qquad (2\text{-}7)$$

$$\frac{\partial H}{\partial x} + \left(R_W + j\omega L_W\right) Q = 0 \qquad (2\text{-}8)$$

If we define

$$Y = j\omega C_W \qquad (2\text{-}9)$$

$$Z = R_W + j\omega L_W \qquad (2\text{-}10)$$

Equations (2-7) and (2-8) become

$$\frac{\partial Q}{\partial x} + YH = 0 \qquad (2\text{-}11)$$

$$\frac{\partial H}{\partial x} + ZQ = 0 \qquad (2\text{-}12)$$

From (2-11) and (2-12)

$$\frac{\partial^2 H}{\partial x^2} - \gamma^2 H = 0 \qquad (2\text{-}13)$$

where γ is called the complex propagation constant and is defined as

$$\gamma = \sqrt{ZY} \qquad (2\text{-}14)$$

The solution of (2-13) is

$$H = K_1\varepsilon^{\gamma x} + K_2\varepsilon^{-\gamma x} \qquad (2\text{-}15)$$

where K_1 and K_2 are the constants to be determined by the boundary conditions of the channel.

From (2-12) and (2-15) the solution for Q is obtained as

$$Q = \frac{1}{Z_C}\left(-K_1\varepsilon^{\gamma x} + K_2\varepsilon^{-\gamma x}\right) \qquad (2\text{-}16)$$

where Z_C is called the characteristic impedance of the channel and equal to

$$Z_C = \frac{Z}{\gamma}$$
$$= \sqrt{\frac{Z}{Y}} \qquad (2\text{-}17)$$

The instantaneous power associated with the wave is defined by Proudman (2) as

$$P = \rho g h q \qquad (2\text{-}18a)$$
$$= \frac{1}{4}\rho g\left(H\varepsilon^{jwt} + H^*\varepsilon^{-jwt}\right)\left(Q\varepsilon^{jwt} + Q^*\varepsilon^{-jwt}\right)$$
$$= \frac{1}{4}\rho g\left(HQ\varepsilon^{j2wt} + H^*Q^*\varepsilon^{-j2wt} + H^*Q + HQ^*\right) \qquad (2\text{-}18b)$$

Therefore the time average active tidal power can be defined as

$$P = \frac{1}{2}\rho g \mathrm{Re}\left(H.Q^*\right) \qquad (2\text{-}19a)$$

or

$$P = \frac{1}{2}\rho g \mathrm{Re}\left(H^*Q\right) \qquad (2\text{-}19b)$$

The propagation constant, γ, the characteristic impedance, Z_C, and the power, P, are discussed separately in the following sub-sections.

2.2 PROPAGATION CONSTANT γ

From (2-14), the propagation constant is defined as

$$\gamma = \sqrt{ZY} \tag{2-20}$$

By substituting (2-9) and (2-10) into (2-20), we obtain

$$\gamma = j\omega \sqrt{L_W C_W} \sqrt{1 - j\frac{R_W}{\omega L_W}} \tag{2-21}$$

If we define

$$\Phi = \frac{R_W}{\omega L_W} \tag{2-22}$$

then (2-21) becomes

$$\gamma = j\omega \sqrt{L_W C_W} \sqrt{1 - j\Phi} \tag{2-23}$$

By substituting (2-2) into (2-22) and (2-23), γ and Φ can be expressed in terms of the dimension of the channel and its λ.

$$\Phi = \frac{\lambda}{\omega} \tag{2-24}$$

$$\gamma = j\omega \sqrt{\frac{b_o}{ga_o}} \sqrt{1 - j\Phi} \tag{2-25}$$

or

$$\gamma = \frac{j\omega}{\sqrt{gh_o}} \sqrt{1 - j\Phi} \tag{2-26}$$

Equation (2-26) can be written in polar form as:

$$\gamma = \frac{\omega}{\sqrt{gh_o}} (1 + \Phi^2)^{\frac{1}{4}} \varepsilon^{\frac{j}{2}(\pi - \tan^{-1}\Phi)} \tag{2-27}$$

where the absolute value of γ is defined as:

$$|\gamma| = \frac{\omega}{\sqrt{gh_o}} (1 + \Phi^2)^{\frac{1}{4}} \tag{2-28}$$

Since γ is a complex number, we may define that

$$\gamma = \alpha + j\beta \tag{2-29}$$

where

$$\alpha = |\gamma| \cos\left(\frac{\pi}{2} - \frac{1}{2}\tan^{-1}\Phi\right) \tag{2-30}$$

$$\beta = |\gamma| \sin\left(\frac{\pi}{2} - \frac{1}{2}\tan^{-1}\Phi\right) \tag{2-31}$$

α and β are called the attenuation constant and the phase shift respectively and are always positive. By using trigonometrical manipulation equations (2-30) and (2-31) become

$$\alpha = \frac{\omega}{\sqrt{2gh_o}} \sqrt{\sqrt{1 + \Phi^2} - 1} \tag{2-32}$$

$$\beta = \frac{\omega}{\sqrt{2gh_o}} \sqrt{\sqrt{1 + \Phi^2} + 1} \tag{2-33}$$

The variation of α and β with respect to frequency ω may be investigated as follows:

Substituting in (2-32) and (2-33), the value of α in (2-24), we obtain

$$\alpha = \frac{\omega}{\sqrt{2gh_o}} \sqrt{\sqrt{1 + \left(\frac{\lambda}{\omega}\right)^2} - 1} \tag{2-34}$$

$$\beta = \frac{\omega}{\sqrt{2gh_o}} \sqrt{\sqrt{1 + \left(\frac{\lambda}{\omega}\right)^2} + 1} \tag{2-35}$$

If (a) $\omega \ll \lambda$ that is $\frac{\lambda}{\omega} \gg 1$

then
$$\alpha \simeq \beta \simeq \frac{\sqrt{\omega\lambda}}{\sqrt{2gh_o}} \tag{2-36}$$

(b) $\omega = \lambda$ that is $\frac{\lambda}{\omega} = 1$

then
$$\alpha \simeq \frac{0.64\lambda}{\sqrt{2gh_o}} \tag{2-37}$$

$$\beta \simeq \frac{1.55\lambda}{\sqrt{2gh_o}} \tag{2-38}$$

(c) $\omega \gg \lambda$ that is $\frac{\lambda}{\omega} \ll 1$

$$\alpha \simeq \frac{0.705\lambda}{\sqrt{2gh_o}} \tag{2-39}$$

$$\beta \simeq \frac{1.41\omega}{\sqrt{2gh_o}} \tag{2-40}$$

From the above discussion the asymptotic plot of α and β versus frequency ω may be obtained as shown in Figure 2-1.

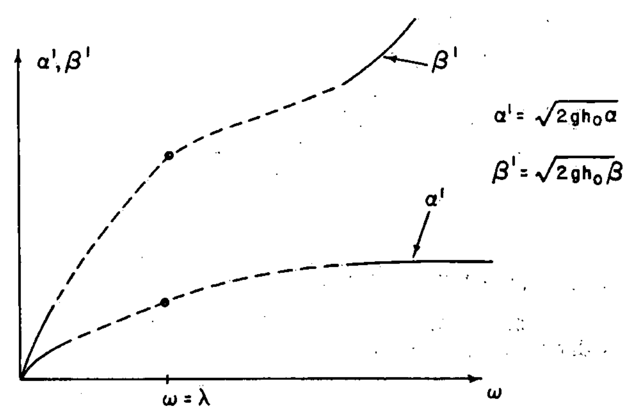

Figure 2-1

Therefore, we might conclude that at low frequency both α and β are functions of $\sqrt{\omega}$, but at high frequency α tends to be constant while β becomes linearly proportional to ω.

Similarly, the asymptotic plot of α and β in terms of λ may be obtained as shown in Figure 2-2.

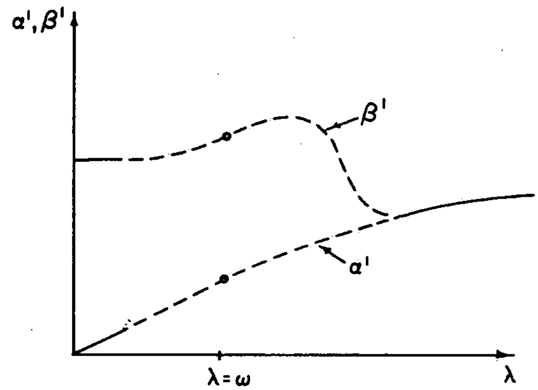

Figure 2-2

It is obvious that for channels with small λ the attenuation constant α is linearly proportional to λ while the phase shift β is almost the same, however, for channels with large λ both α and β are proportional to $\sqrt{\lambda}$.

Moreover, for constant $\frac{\lambda}{\omega}$, the propagation constant is independent of the width of the channel and inversely proportional to the square root of its height as shown in Figure 2-3.

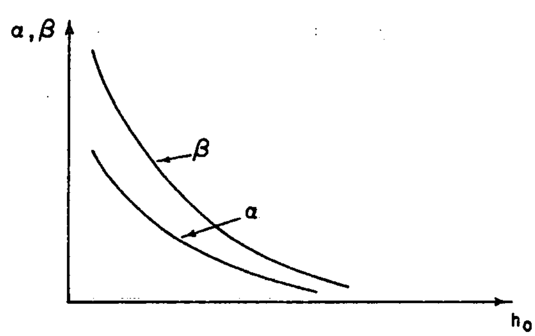

Figure 2-3

2-3 CHARACTERISTIC IMPEDANCE

According to (2-17), the characteristic impedance Z_C is defined as

$$Z_C = \sqrt{\frac{Z}{Y}} \tag{2-41}$$

By substituting in (2-41) the value of Z and Y as shown in (2-9) and (2-10) we obtain

$$Z_C = \sqrt{\frac{L_W}{C_W}} \sqrt{1 - j\frac{R_W}{\omega L_W}} \tag{2-42}$$

The equation of Z_C can be expressed in terms of the parameter of the channel by substituting equation (2-2) into (2-42),

$$Z_C = \frac{1}{\sqrt{g a_o b_o}} \sqrt{1 - j\left(\frac{\lambda}{\omega}\right)} \tag{2-43}$$

or

$$Z_C = \frac{1}{b_o\sqrt{g h_o}} \sqrt{1 - j\left(\frac{\lambda}{\omega}\right)} \tag{2-44}$$

Since Z_C is a complex number, it can be expressed in polar form as

$$Z_C = |Z_C|\, \varepsilon^{-j\theta} \tag{2-45}$$

where

$$|Z_C| = \frac{\left[1 + \left(\frac{\lambda}{\omega}\right)^2\right]^{\frac{1}{4}}}{b_o\sqrt{g h_o}} \tag{2-46}$$

$$\theta = \frac{1}{2}\tan^{-1}\frac{\lambda}{\omega} \tag{2-47}$$

From (2-47) it is obvious that θ is always positive. Therefore, the phase angle of Z_C is always negative and varies between zero and $-\frac{\pi}{4}$, depending on the value of $\frac{\lambda}{\omega}$.

The asymptotic variation of the magnitude of Z_C in terms of the frequency, ω, can be obtained in a way similar to that for α and β described in the previous section.

If (a) $\omega \ll \lambda$ that is $\frac{\lambda}{\omega} \gg 1$

then

$$|Z_C| \simeq \frac{1}{b_o\sqrt{g h_o}} \sqrt{\frac{\lambda}{\omega}} \tag{2-48}$$

If (b) $\omega = \lambda$ that is $\frac{\lambda}{\omega} = 1$

then

$$|Z_C| \simeq \frac{1.18}{b_o\sqrt{g h_o}} \tag{2-49}$$

If (e) $\omega \gg \lambda$ that is $\frac{\lambda}{\omega} \ll 1$

then

$$|Z_C| \simeq \frac{1}{b_o\sqrt{g h_o}} \tag{2-50}$$

Therefore, the asymptotic plot of $|Z_C|$ versus frequency, ω, is as shown in Figure 2-4,

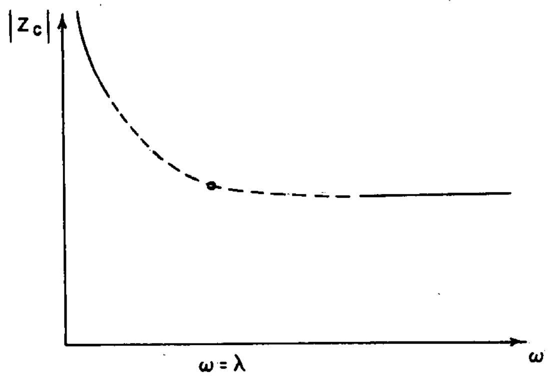

Figure 2-4

which indicates that at low frequency $|Z_C|$ is a function of $\frac{1}{\sqrt{\omega}}$ while at high frequency $|Z_C|$ approaches a constant.

The variation of $|Z_C|$ in channels with different λ can be obtained by the same procedure; the asymptotic plot is shown in Figure 2-5.

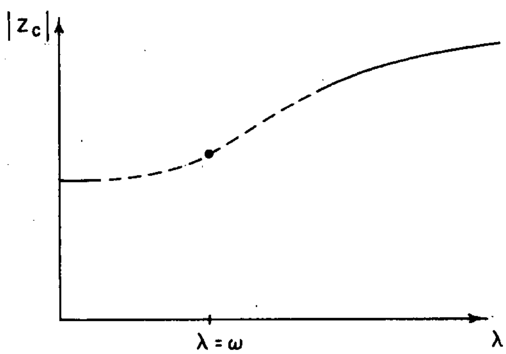

Figure 2-5

Figure 2-5 indicates that for channels with small λ, $|Z_C|$ is almost identical but for channels with large λ the magnitude of Z_C is proportional to $\sqrt{\lambda}$.

Moreover, for constant $\frac{\lambda}{\omega}$ it is obvious that $|Z_C|$ is inversely proportional to the height of the channel h_o.

2-4 THE INTERPRETATION OF THE SOLUTION OF THE HYDRODYNAMIC EQUATION

The solution of the hydrodynamic equation of the rectangular channel was derived in Section 2-1, it is

$$H = K_1 \varepsilon^{\gamma x} + K_2 \varepsilon^{-\gamma x} \qquad (2\text{-}51)$$

$$Q = \frac{1}{Z_C} \left(-K_1 \varepsilon^{\gamma x} + K_2 \varepsilon^{-\gamma x}\right) \qquad (2\text{-}52)$$

By substituting (2-29) and (2-45) into (2-51) and (2-52), we obtain

$$H = K_1 \varepsilon^{(\alpha + j\beta)\ x} + K_2 \varepsilon^{-(\alpha + j\beta)\ x} \qquad (2\text{-}53)$$

$$Q = \frac{1}{|Z_C|} \left[-K_1 \varepsilon^{\alpha x\ +\ j\ (\beta x + \theta)} + K_2 \varepsilon^{-\alpha x\ -\ j\ (\beta x - \theta)}\right] \qquad (2\text{-}54)$$

From these equations we may obtain the instantaneous value of q and h by substituting them into (2-5) and (2-6) which yield:

$$h\ (x_1 t) = K_1 \varepsilon^{\alpha x} \cos\ (\omega t + \beta x) + K_2 \varepsilon^{-\alpha x} \cos\ (\omega t - \beta x) \qquad (2\text{-}55)$$

$$q\ (x_1 t) = \frac{1}{|Z_C|} \left[-K_1 \varepsilon^{\alpha x} \cos\ (\omega t + \beta x + \theta) + K_2 \varepsilon^{-\alpha x} \cos\ (\omega t - \beta x + \theta)\right] \qquad (2\text{-}56)$$

The constant phase velocity associated with each term in (2-55) and (2-56) may be obtained by setting the phase angle as constant and differentiating it with respect to time t.

For the first term it yields

$$V_p = \frac{dx}{dt} = -\frac{\omega}{\beta} \qquad (2\text{-}57)$$

and for the second term it gives

$$V_p = \frac{dx}{dt} = \frac{\omega}{\beta} \qquad (2\text{-}58)$$

which indicates that the first term represents a wave travelling in the opposite direction of x and is called a retrogressive wave while the second term denotes a wave travelling in the same direction of x and is called a progressive wave; the magnitudes of the velocities are identical and equal to $\frac{\omega}{\beta}$.

Both the progressive and retrogressive waves are attenuated by a factor $\varepsilon^{-\alpha|x|}$ along their travelling paths if x is measured from the middle of the channel; from the conclusion in Section 2-2, we may state that

(a) the waves will be attenuated faster in a channel with larger λ;

(b) the waves will be attenuated faster in a shallow channel; and

(c) the high frequency waves will be attenuated faster than the low frequency waves.

From (2-35) we obtain

$$|V_p| = \frac{\omega}{\beta}$$

$$= \sqrt{2gh_o} \left[\sqrt{1 + \left(\frac{\lambda}{\omega}\right)^2} + 1\right]^{-\frac{1}{2}} \qquad (2\text{-}59)$$

The asymptotic plots of $|V_p|$ with respect to λ and ω are shown in Figure 2-6 and Figure 2-7.

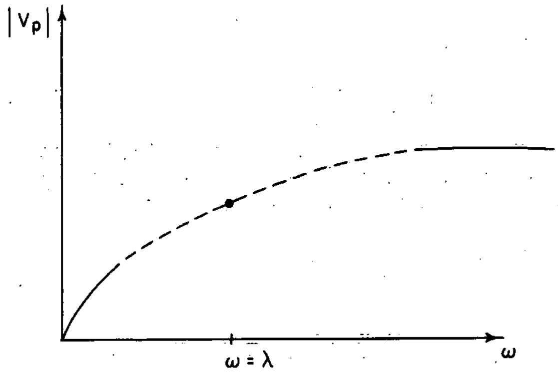

Figure 2-6

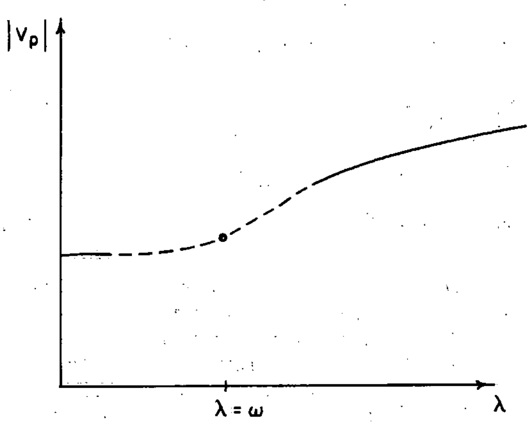

Figure 2-7

From (2-59) it is obvious that the velocity is proportional to the square root of the height of the channel, therefore, the wave travels faster in a deeper channel.

By comparing the phase angle of each term in (2-55) and (2-56), it is obvious that the horizontal flow, q, is always leading the vertical movement, h, by an angle θ. From (2-47) we know that

$$\theta = \frac{1}{2} \tan^{-1} \frac{\lambda}{\omega} \qquad (2\text{-}60)$$

Therefore, the difference between the phase angles of the q and h of each wave is larger in a channel with larger λ, or for the wave with lower frequency.

The ratio between the h and q of each wave is equal to the magnitude of the characteristic impedance, Z_C, of the channel. Therefore, the conclusions in Section 2-3 are applicable to this ratio.

2-5 POWER

According to (2-19) the power associated with the wave is defined as

$$P = \frac{1}{2} \rho g \mathrm{Re}\, (H^*Q) \qquad (2\text{-}61)$$

$$P = \frac{\rho g}{2|Z_C|} \Big[-K_1^2 \varepsilon^{2\alpha x} \cos (2\beta x+\theta) + K_2^2 \varepsilon^{-2\alpha x} \cos (2\beta x-\theta)\Big] \qquad (2\text{-}62)$$

or

$$P = -\frac{\rho g}{2|Z_C|} K_1^2 \varepsilon^{2\alpha x} \cos (2\beta x+\theta) + \frac{\rho g}{2|Z_C|} K_2^2 \varepsilon^{-2\alpha x} \cos (2\beta x-\theta) \qquad (2\text{-}63)$$

The first term represents the power associated with the retrogressive waves while the second term denotes the power associated with the progressive waves.

2-6 DETERMINATION OF Z_C AND $\varepsilon^{\gamma \ell}$ FROM THE BOUNDARY CONDITION

If the horizontal and vertical tides of a rectangular channel are known, and we define that

at $x = o$ $\quad H = H_i$

$$Q = Q_i \qquad (2\text{-}64)$$

$x = \ell$ $\quad H = H_o$

$$Q = Q_o \qquad (2\text{-}65)$$

Substituting these boundary conditions into (2-15) and (2-16) we obtain the following set of equations:

$$H_i = K_1 + K_2 \qquad (2\text{-}66)$$

$$Q_i = \frac{1}{Z_C} (-K_1+K_2) \qquad (2\text{-}67)$$

$$H_o = K_1 \varepsilon^{\gamma\ell} + K_2 \varepsilon^{-\gamma\ell} \qquad (2\text{-}68)$$

$$Q_o = \frac{1}{Z_C} (-K_1 \varepsilon^{\gamma\ell} + K_2 \varepsilon^{-\gamma\ell}) \qquad (2\text{-}69)$$

From (2-66) and (2-67) it is obvious that

$$K_1 = \frac{1}{2} (H_i - Q_i Z_C) \qquad (2\text{-}70)$$

$$K_2 = \frac{1}{2}\left(H_i + Q_i Z_c\right) \qquad (2\text{-}71)$$

From (2-68) and (2-69) we obtain

$$H_o + Z_c Q_o = 2K_2 \varepsilon^{-\gamma \ell} \qquad (2\text{-}72)$$

$$H_o - Z_c Q_o = 2K_1 \varepsilon^{\gamma \ell} \qquad (2\text{-}73)$$

and by multiplying these two equations we get

$$H_o^2 - Z_c^2 Q_o^2 = 4K_1 K_2 \qquad (2\text{-}74)$$

Substituting (2-70) and (2-71) into (2-74) we obtain the following equation

$$H_o^2 - Z_c^2 Q_o^2 = H_i^2 - Z_c^2 Q_i^2 \qquad (2\text{-}75)$$

The equation of Z_c is obtained from (2-75)

$$Z_c^2 = \frac{H_o^2 - H_i^2}{Q_o^2 - Q_i^2} \qquad (2\text{-}76)$$

or

$$Z_c = \sqrt{\frac{H_o^2 - H_i^2}{Q_o^2 - Q_i^2}} \qquad (2\text{-}77)$$

The equation for $\varepsilon^{\gamma \ell}$ can be derived by substituting (2-70) into (2-73) which yields

$$\varepsilon^{\gamma \ell} = \frac{H_o - Z_c Q_o}{H_i - Z_c Q_i} \qquad (2\text{-}78)$$

Substituting equation (2-77) into (2-78) we finally obtain

$$\varepsilon^{\gamma \ell} = \frac{H_o \sqrt{Q_o^2 - Q_i^2} - Q_o \sqrt{H_o^2 - H_i^2}}{H_i \sqrt{Q_o^2 - Q_i^2} - Q_i \sqrt{H_o^2 - H_i^2}} \qquad (2\text{-}79)$$

Equations (2-77) and (2-79) indicate that Z_c and $\varepsilon^{\gamma \ell}$ of the rectangular channel can be obtained in terms of the measurements at both ends of the channel.

Various forms of the Solution of the Hydrodynamic Equation

The constants K_1 and K_2 appearing in the solution of the hydrodynamic equation have been obtained in terms of the boundary condition as shown in (2-70) and (2-71). By substituting these equations into (2-15) and (2-16) we obtain:

$$H_o = \cosh \gamma\ell . H_i - Z_c \sinh \gamma\ell . Q_i \tag{3-1}$$

$$Q_o = -\frac{1}{Z_c} \sinh \gamma\ell . H_i + \cosh \gamma\ell \, Q_i \tag{3-2}$$

These equations can be written in matrix form as

$$\begin{bmatrix} H_o \\ Q_o \end{bmatrix} = \begin{bmatrix} \cosh \gamma\ell & -Z_c \sinh \gamma\ell \\ -\frac{1}{Z_c} \sinh \gamma\ell & \cosh \gamma\ell \end{bmatrix} \begin{bmatrix} H_i \\ Q_i \end{bmatrix} \tag{3-3}$$

By defining a matrix (X) as

$$[X] = \begin{bmatrix} \cosh \gamma\ell & -Z_c \sinh \gamma\ell \\ -\frac{1}{Z_c} \sinh \gamma\ell & \cosh \gamma\ell \end{bmatrix} \tag{3-4}$$

The solution becomes

$$\begin{bmatrix} H_o \\ Q_o \end{bmatrix} = [X] \begin{bmatrix} H_i \\ Q_i \end{bmatrix} \tag{3-5}$$

Equation (3-5) is called the X-form of the solution. It is obvious that the (X) matrix has the following properties:

$$x_{11} = x_{22} = \cosh \gamma\ell \tag{3-6}$$

$$\frac{x_{12}}{x_{21}} = Z_c^2 \tag{3-7}$$

$$x_{11}x_{22} - x_{12}x_{21} = 1 \tag{3-8}$$

where x_{ij} is the element of (X) at row i and column j

By defining

$$(W) = (X)^{-1} \tag{3-9}$$

equation (3-5) may be written as

$$\begin{bmatrix} H_i \\ Q_i \end{bmatrix} = [W] \begin{bmatrix} H_o \\ Q_o \end{bmatrix} \tag{3-10}$$

this is the W-form of the solution where

$$[W] = \begin{bmatrix} \cosh \gamma\ell & Z_c \sinh \gamma\ell \\ \frac{1}{Z_c} \sinh \gamma\ell & \cosh \gamma\ell \end{bmatrix} \tag{3-11}$$

where $$W_{11} = W_{22} = \cosh \gamma\ell \tag{3-12}$$

$$\frac{w_{12}}{w_{21}} = Z_c^2 \tag{3-13}$$

$$w_{11}w_{22} - w_{12}w_{21} = 1 \tag{3-14}$$

If (3-2) is re-arranged as in the following equation:

$$H_i = \frac{Z_c}{\sinh \gamma\ell} \left(-Q_o + \cosh \gamma\ell \, Q_i\right) \tag{3-15}$$

and substituting (3-15) into (3-1), the following equation is obtained:

$$H_o = -Z_c \coth \gamma\ell \, Q_o + \frac{Z_c}{\sinh \gamma\ell} Q_i \tag{3-16}$$

Equation (3-15) and (3-16) may be written in the following matrix form:

$$\begin{bmatrix} H_i \\ H_o \end{bmatrix} = \begin{bmatrix} Z_c \coth \gamma\ell & \frac{-Z_c}{\sinh \gamma\ell} \\ \frac{Z_c}{\sinh \gamma\ell} & -Z_c \coth \gamma\ell \end{bmatrix} \tag{3-17}$$

Again we define a matrix $[Z]$ as

$$[Z] = \begin{bmatrix} Z_c \coth \gamma\ell & \frac{-Z_c}{\sinh \gamma\ell} \\ \frac{Z_c}{\sinh \gamma\ell} & -Z_c \coth \gamma\ell \end{bmatrix} \tag{3-18}$$

the Z- form of the solution becomes

$$\begin{bmatrix} H_i \\ H_o \end{bmatrix} = [Z] \begin{bmatrix} Q_i \\ Q_o \end{bmatrix} \tag{3-19}$$

It is obvious that

$$z_{11} = -z_{22} = Z_c \coth \gamma\ell \tag{3-20}$$

$$z_{12} = -z_{21} = -\frac{Z_c}{\sinh \gamma\ell} \tag{3-21}$$

$$z_{11}z_{22} - z_{12}z_{21} = -Z_c^2 \tag{3-22}$$

If we define a matrix $[Y]$ equal to the inverse of the $[Z]$ matrix

$$[Y] = [Z]^{-1} \tag{3-23}$$

$$[Y] = \begin{bmatrix} \frac{1}{Z_C}\coth\gamma\ell & \frac{-1}{Z_C\sinh\gamma\ell} \\ \frac{1}{Z_C\sinh\gamma\ell} & -\frac{1}{Z_C}\coth\gamma\ell \end{bmatrix} \tag{3-24}$$

Then we may obtain the Y-form of the solution as

$$\begin{bmatrix} Q_i \\ Q_o \end{bmatrix} = [Y] \begin{bmatrix} H_i \\ Q_o \end{bmatrix} \tag{3-25}$$

It is obvious that the $[Y]$ matrix has the following properties:

$$y_{11} = -y_{22} = \frac{1}{Z_C}\coth\gamma\ell \tag{3-26}$$

$$y_{12} = -y_{21} = \frac{-1}{Z_C\sinh\gamma\ell} \tag{3-27}$$

$$y_{11}y_{22} - y_{12}y_{21} = -\frac{1}{Z_C^2} \tag{3-28}$$

Similarly, a G-form solution is obtained as:

$$\begin{bmatrix} Q_i \\ H_o \end{bmatrix} = [G] \begin{bmatrix} H_i \\ Q_o \end{bmatrix} \tag{3-29}$$

where $$[G] = \begin{bmatrix} \frac{-\tanh\gamma\ell}{Z_C} & \frac{1}{\cosh\gamma\ell} \\ \frac{1}{\cosh\gamma\ell} & -Z_C\tanh\gamma\ell \end{bmatrix} \tag{3-30}$$

and the following properties are true:

$$\frac{g_{22}}{g_{11}} = -Z_C^2 \tag{3-31}$$

$$g_{12} = g_{21} = \frac{1}{\cosh\gamma\ell} \tag{3-32}$$

$$g_{11}g_{22} - g_{12}g_{21} = -1 \tag{3-33}$$

Again, if we define

$$[H] = [G]^{-1} \tag{3-34}$$

then we have the H-form solution as

$$\begin{bmatrix} H_i \\ Q_o \end{bmatrix} = [H] \begin{bmatrix} Q_i \\ H_o \end{bmatrix} \tag{3-35}$$

where $$[H] = \begin{bmatrix} +Z_C\tanh\gamma\ell & \frac{1}{\cosh\gamma\ell} \\ \frac{1}{\cosh\gamma\ell} & -\frac{\tanh\gamma\ell}{Z_C} \end{bmatrix} \tag{3-36}$$

$$\frac{h_{11}}{h_{22}} = -Z_C \tag{3-37}$$

$$h_{12} = h_{21} = \frac{1}{\cosh\gamma\ell} \tag{3-38}$$

$$h_{11}h_{22} - h_{12}h_{21} = -1 \tag{3-39}$$

Because the solution of different forms are derived from the same equation, they are correlated to each other. For example, if we write the X-form solution as

$$H_o = x_{11}H_i + x_{12}Q_i \tag{3-40}$$

$$Q_o = x_{21}H_i + x_{22}Q_i \tag{3-41}$$

and re-arrange (3-4) as follows

$$H_i = \frac{1}{x_{21}}\left(Q_o - x_{22}Q_i\right) \tag{3-42}$$

and substitute equation (3-42) into (3-40) we obtain

$$H_o = \frac{x_{11}}{x_{21}} Q_o + \left[\frac{x_{12}x_{21} - x_{11}x_{22}}{x_{21}}\right] Q_i \tag{3-43}$$

By defining

$$\Delta_x = x_{11}x_{22} - x_{12}x_{21} \tag{3-44}$$

we obtain

$$H_o = \frac{x_{11}}{x_{21}} Q_o - \frac{\Delta_x}{x_{21}} Q_i \tag{3-45}$$

Equations (3-42) and (3-45) can be written in matrix form as

$$\begin{bmatrix} H_i \\ H_o \end{bmatrix} = \frac{1}{x_{21}} \begin{bmatrix} -x_{22} & 1 \\ -\Delta_x & x_{11} \end{bmatrix} \begin{bmatrix} Q_i \\ Q_o \end{bmatrix} \tag{3-46}$$

It is obvious that this is the Z-form, where

$$[Z] = \begin{bmatrix} \frac{-x_{22}}{x_{21}} & \frac{1}{x_{21}} \\ \frac{-\Delta_x}{x_{21}} & \frac{x_{11}}{x_{21}} \end{bmatrix} \tag{3-47}$$

or

$$z_{11} = -\frac{x_{22}}{x_{21}}$$

$$z_{12} = \frac{1}{x_{21}} \tag{3-48}$$

$$z_{21} = \frac{-1}{x_{21}}$$

$$z_{22} = \frac{x_{11}}{x_{21}}$$

		[X]	[W]	[Y]	[Z]	[G]	[H]
[X]	$\begin{bmatrix}H_o\\Q_o\end{bmatrix} = [X]\begin{bmatrix}H_i\\Q_i\end{bmatrix}$	$\begin{bmatrix}x_{11} & x_{12}\\x_{21} & x_{22}\end{bmatrix}$	$\frac{1}{\Delta_w}\begin{bmatrix}w_{22} & -w_{12}\\-w_{21} & w_{11}\end{bmatrix}$	$\frac{1}{y_{12}}\begin{bmatrix}-y_{11} & 1\\-\Delta y & y_{22}\end{bmatrix}$	$\frac{1}{z_{12}}\begin{bmatrix}z_{22} & -\Delta z\\1 & -z_{11}\end{bmatrix}$	$\frac{1}{g_{12}}\begin{bmatrix}-\Delta g & g_{22}\\-g_{11} & 1\end{bmatrix}$	$\frac{1}{h_{12}}\begin{bmatrix}1 & -h_{11}\\h_{22} & -\Delta h\end{bmatrix}$
[W]	$\begin{bmatrix}H_i\\Q_i\end{bmatrix} = [W]\begin{bmatrix}H_o\\Q_o\end{bmatrix}$	$\frac{1}{\Delta x}\begin{bmatrix}x_{22} & -x_{12}\\-x_{21} & x_{11}\end{bmatrix}$	$\begin{bmatrix}w_{11} & w_{12}\\w_{21} & w_{22}\end{bmatrix}$	$\frac{1}{y_{21}}\begin{bmatrix}-y_{22} & 1\\-\Delta y & y_{11}\end{bmatrix}$	$\frac{-1}{z_{21}}\begin{bmatrix}-z_{11} & \Delta z\\1 & z_{22}\end{bmatrix}$	$\frac{1}{g_{21}}\begin{bmatrix}-1 & g_{22}\\g_{11} & \Delta g\end{bmatrix}$	$\frac{1}{h_{21}}\begin{bmatrix}-\Delta h & h_{11}\\-h_{22} & 1\end{bmatrix}$
[Y]	$\begin{bmatrix}Q_i\\Q_o\end{bmatrix} = [Y]\begin{bmatrix}H_i\\H_o\end{bmatrix}$	$\frac{1}{x_{12}}\begin{bmatrix}-x_{11} & 1\\-\Delta x & x_{22}\end{bmatrix}$	$\frac{1}{w_{12}}\begin{bmatrix}w_{22} & -\Delta w\\1 & -w_{11}\end{bmatrix}$	$\begin{bmatrix}y_{11} & y_{12}\\y_{21} & y_{22}\end{bmatrix}$	$\frac{1}{\Delta z}\begin{bmatrix}z_{22} & -z_{12}\\-z_{21} & z_{11}\end{bmatrix}$	$\frac{1}{g_{22}}\begin{bmatrix}\Delta g & g_{12}\\-g_{21} & 1\end{bmatrix}$	$\frac{1}{h_{11}}\begin{bmatrix}1 & -h_{12}\\h_{21} & \Delta h\end{bmatrix}$
[Z]	$\begin{bmatrix}H_i\\H_o\end{bmatrix} = [Z]\begin{bmatrix}Q_i\\Q_o\end{bmatrix}$	$\frac{1}{x_{21}}\begin{bmatrix}-x_{22} & 1\\-\Delta x & x_{11}\end{bmatrix}$	$\frac{1}{w_{21}}\begin{bmatrix}w_{11} & -\Delta w\\1 & -w_{22}\end{bmatrix}$	$\frac{1}{\Delta y}\begin{bmatrix}y_{22} & -y_{12}\\-y_{21} & y_{11}\end{bmatrix}$	$\begin{bmatrix}z_{11} & z_{12}\\z_{21} & z_{22}\end{bmatrix}$	$\frac{1}{g_{11}}\begin{bmatrix}1 & -g_{12}\\g_{21} & \Delta g\end{bmatrix}$	$\frac{1}{h_{22}}\begin{bmatrix}\Delta h & h_{12}\\-h_{21} & 1\end{bmatrix}$
[G]	$\begin{bmatrix}Q_i\\H_o\end{bmatrix} = [G]\begin{bmatrix}H_i\\Q_o\end{bmatrix}$	$\frac{1}{x_{22}}\begin{bmatrix}-x_{21} & 1\\\Delta x & x_{12}\end{bmatrix}$	$\frac{1}{w_{11}}\begin{bmatrix}w_{21} & \Delta w\\1 & -w_{12}\end{bmatrix}$	$\frac{1}{y_{11}}\begin{bmatrix}\Delta y & y_{12}\\-y_{21} & 1\end{bmatrix}$	$\frac{1}{z_{11}}\begin{bmatrix}1 & -z_{12}\\z_{21} & \Delta z\end{bmatrix}$	$\begin{bmatrix}g_{11} & g_{12}\\g_{21} & g_{22}\end{bmatrix}$	$\frac{1}{\Delta h}\begin{bmatrix}h_{22} & -h_{12}\\-h_{21} & h_{11}\end{bmatrix}$
[H]	$\begin{bmatrix}H_i\\Q_o\end{bmatrix} = [H]\begin{bmatrix}Q_i\\H_o\end{bmatrix}$	$\frac{1}{x_{11}}\begin{bmatrix}-x_{12} & 1\\\Delta x & x_{21}\end{bmatrix}$	$\frac{1}{w_{22}}\begin{bmatrix}w_{12} & \Delta w\\1 & -w_{21}\end{bmatrix}$	$\frac{1}{y_{22}}\begin{bmatrix}\Delta y & y_{21}\\-y_{12} & 1\end{bmatrix}$	$\frac{1}{z_{22}}\begin{bmatrix}\Delta z & z_{12}\\-z_{21} & 1\end{bmatrix}$	$\frac{1}{\Delta g}\begin{bmatrix}g_{22} & -g_{12}\\-g_{21} & g_{11}\end{bmatrix}$	$\begin{bmatrix}h_{11} & h_{12}\\h_{21} & h_{22}\end{bmatrix}$

TABLE 3-1 CONVERSION TABLE

By using the same procedure we may obtain the relation between different forms of solutions, the results are tabulated in Table 3-1.

In a rectangular channel, the impedance and admittance matrix can be computed if the H and Q at both ends of the channel are known.

Consider the [Z] matrix for example, by applying the conditions in (3-20) and (3-21), the equations become

$$H_i = z_{11}Q_i + z_{12}Q_o \tag{3-49}$$

$$H_o = -z_{12}Q_i - z_{11}Q_o \tag{3-50}$$

From these two equations, the equation for z_{11} and z_{12} can be derived:

$$z_{11} = \frac{H_iQ_i + H_oQ_o}{Q_i^2 - Q_o^2} \tag{3-51}$$

$$z_{12} = -\frac{H_iQ_o + H_oQ_i}{Q_i^2 - Q_o^2} \tag{3-52}$$

The values of z_{22} and z_{21} follow directly from (3-20) and (3-21).

By using the conversion table in Table 3-1, the equation for other matrices can be derived; the same result as (3-51) and (3-52) can be obtained by following the same procedure. The results are tabulated in Table 3-2.

[X]	$x_{11} = x_{22} = \frac{1}{\Delta}(H_iQ_i + H_oQ_o)$ $x_{12} = -\frac{1}{\Delta}(H_i^2 - H_o^2)$ $x_{21} = -\frac{1}{\Delta}(Q_i^2 - Q_o^2)$ $\Delta = H_iQ_o + H_oQ_i$	[Z]	$z_{11} = -z_{22} = \frac{1}{\Delta}(H_iQ_i + H_oQ_o)$ $z_{12} = -z_{21} = -\frac{1}{\Delta}(H_iQ_o + H_oQ_i)$ $\Delta = Q_i^2 - Q_o^2$
[W]	$w_{11} = w_{22} = \frac{1}{\Delta}(H_iQ_i + H_oQ_o)$ $w_{12} = \frac{1}{\Delta}(H_i^2 - H_o^2)$ $w_{21} = \frac{1}{\Delta}(Q_i^2 - Q_o^2)$ $\Delta = H_iQ_o + H_oQ_i$	[G]	$g_{11} = \frac{1}{\Delta}(Q_i^2 - Q_o^2)$ $g_{12} = g_{21} = \frac{1}{\Delta}(H_iQ_o + H_oQ_i)$ $g_{22} = -\frac{1}{\Delta}(H_i^2 - H_o^2)$ $\Delta = H_iQ_i + H_oQ_o$
[Y]	$y_{11} = -y_{22} = \frac{1}{\Delta}(H_iQ_i + H_oQ_o)$ $y_{12} = -y_{21} = -\frac{1}{\Delta}(H_iQ_o + H_oQ_i)$ $\Delta = H_i^2 - H_o^2$	[H]	$h_{11} = \frac{1}{\Delta}(H_i^2 - H_o^2)$ $h_{12} = h_{21} = \frac{1}{\Delta}(H_iQ_o + H_oQ_i)$ $h_{22} = -\frac{1}{\Delta}(Q_i^2 - Q_o^2)$ $\Delta = H_iQ_i + H_oQ_o$

TABLE 3-2

Block Diagram Representation of the Rectangular Channel

From the previous discussion, it is obvious that a rectangular channel can be represented by the block diagram shown in Figure 4-1,

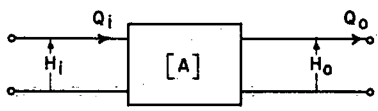

Figure 4-1

where the block represents the channel which is characterized by the [A] matrix, which could be [X], [W], [Y], [X], [G] or [H]. In this diagram H and Q denote the vertical movement and the horizontal flow respectively, the arrow of Q indicates the direction of flow at the end of the channel. The subscripts i and o denote the input and output ends of the channel respectively.

If a channel is divided into two rectangular channels in series, it can be represented by the block diagram shown in Figure 4-2.

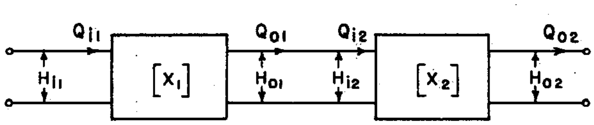

Figure 4-2

where we are using [X] matrix to denote each section of the channel, and their X-form solutions are

$$\begin{bmatrix} H_{o1} \\ Q_{o1} \end{bmatrix} = \begin{bmatrix} X_1 \end{bmatrix} \begin{bmatrix} H_{i1} \\ Q_{i1} \end{bmatrix} \tag{4-1}$$

$$\begin{bmatrix} H_{o2} \\ Q_{o2} \end{bmatrix} = \begin{bmatrix} X_2 \end{bmatrix} \begin{bmatrix} H_{i2} \\ Q_{i2} \end{bmatrix} \tag{4-2}$$

According to the boundary conditions assumed at the beginning of Chapter 1 we know that

$$\begin{aligned} H_{i1} &= H_{i2} \\ Q_{i1} &= Q_{i2} \end{aligned} \tag{4-3}$$

By using these conditions, the following equation can be obtained from (4-1) and (4-2)

$$\begin{bmatrix} H_{o2} \\ Q_{o2} \end{bmatrix} = \begin{bmatrix} X_2 \end{bmatrix} \begin{bmatrix} X_1 \end{bmatrix} \begin{bmatrix} H_{i1} \\ Q_{i1} \end{bmatrix} \tag{4-4}$$

or

$$\begin{bmatrix} H_{o2} \\ Q_{o2} \end{bmatrix} = \begin{bmatrix} X \end{bmatrix} \begin{bmatrix} H_{i1} \\ Q_{i1} \end{bmatrix} \tag{4-5}$$

where

$$\begin{bmatrix} X \end{bmatrix} = \begin{bmatrix} X_2 \end{bmatrix} \begin{bmatrix} X_1 \end{bmatrix} \tag{4-6}$$

From (4-5) it indicates that [X] is the equivalent [X] matrix of two channels in series, the result can be extended to several channels in series and the equivalent [X] matrix is therefore

$$\begin{bmatrix} X \end{bmatrix} = \begin{bmatrix} X_n \end{bmatrix} \begin{bmatrix} X_{n-1} \end{bmatrix} \ldots \begin{bmatrix} X_2 \end{bmatrix} \begin{bmatrix} X_1 \end{bmatrix} \tag{4-7}$$

where n is the total number of channels in series.

If we assume that

$$\begin{bmatrix} X_1 \end{bmatrix} = \begin{bmatrix} \cosh \mu_1 & -Z_{c1} \sinh \mu_1 \\ -\dfrac{1}{Z_{c1}} \sinh \mu_1 & \cosh \mu_1 \end{bmatrix} \tag{4-8}$$

$$\begin{bmatrix} X_2 \end{bmatrix} = \begin{bmatrix} \cosh \mu_2 & -Z_c \sinh \mu_2 \\ -\dfrac{1}{Z_{c2}} \sinh \mu_2 & \cosh \mu_2 \end{bmatrix} \tag{4-9}$$

where

$$\begin{aligned} \mu_1 &= \varepsilon^{\gamma_1 \ell_1} \\ \mu_2 &= \varepsilon^{\gamma_2 \ell_2} \end{aligned} \tag{4-10}$$

by matrix multiplication we obtain

$$\begin{bmatrix} X \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \cosh \mu_1 \cosh \mu_2 \\ + \dfrac{Z_{c1}}{Z_{c2}} \sinh \mu_1 \sinh \mu_2 \end{bmatrix} & - \begin{bmatrix} Z_{c2} \cosh \mu_1 \sinh \mu_2 \\ + Z_{c1} \sinh \mu_1 \cosh \mu_2 \end{bmatrix} \\ - \begin{bmatrix} \dfrac{1}{Z_{c1}} \sinh \mu_1 \cosh \mu_2 \\ + \dfrac{1}{Z_{c2}} \cosh \mu_1 \sinh \mu_2 \end{bmatrix} & \begin{bmatrix} \cosh \mu_1 \cosh \mu_2 \\ + \dfrac{Z_{c2}}{Z_{c1}} \sinh \mu_1 \sinh \mu_2 \end{bmatrix} \end{bmatrix} \tag{4-11}$$

For the special case when

$$Z_{c1} = Z_{c2} = Z_c$$

equation (4-10) can be simplified as

$$[X] = \begin{bmatrix} \cosh\mu & -Z_c\sinh\mu \\ -\dfrac{\sinh\mu}{Z_c} & \cosh\mu \end{bmatrix} \qquad (4\text{-}12)$$

where $\mu = \mu_1 + \mu_2$ (4-13)

In general, if several channels with identical characteristic impedances are in series, the μ of the equivalent channel will be

$$\mu = \sum_{i=1}^{n} \mu_i \qquad (4\text{-}14)$$

The matrix $[X]$ of the equivalent channel will retain the properties in (3-6) to (3-8).

From (2-44) it was shown that

$$Z_c = \frac{1}{b_o\sqrt{gh_o}}\sqrt{1 - j\left(\frac{\lambda}{\omega}\right)} \qquad (4\text{-}15)$$

Therefore, if the λ of channels are the same, then the condition for identical Z_c for these channels is

$$b_o\sqrt{h_o} = \text{constant} \qquad (4\text{-}16)$$

In the case of two channels in parallel, the block diagram representation is as shown in Figure 4-3.

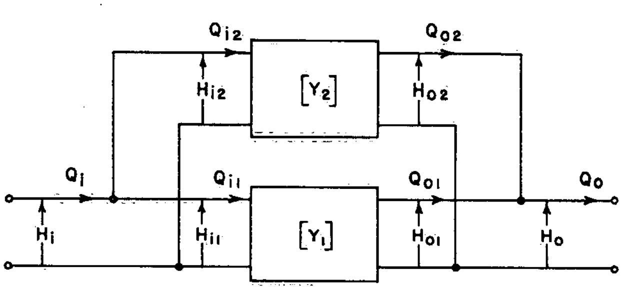

Figure 4-3

The channels are described by Y-form solution as

$$\begin{bmatrix} Q_{i1} \\ Q_{o1} \end{bmatrix} = [Y_1]\begin{bmatrix} H_{i1} \\ H_{o1} \end{bmatrix} \qquad (4\text{-}17)$$

$$\begin{bmatrix} Q_{i2} \\ Q_{o2} \end{bmatrix} = [Y_2]\begin{bmatrix} H_{i2} \\ H_{o2} \end{bmatrix} \qquad (4\text{-}18)$$

with the boundary conditions

$$\begin{aligned} Q_o &= Q_{o1} + Q_{o2} \\ Q_i &= Q_{i1} + Q_{i2} \\ H_o &= H_{o1} = H_{o2} \\ H_i &= H_{i1} = H_{i2} \end{aligned} \qquad (4\text{-}19)$$

By adding (4-17) and (4-18) and applying the boundary condition we obtain

$$\begin{bmatrix} Q_i \\ Q_o \end{bmatrix} = [Y]\begin{bmatrix} H_i \\ H_o \end{bmatrix} \qquad (4\text{-}20)$$

where $[Y] = [Y_1] + [Y_2]$ (4-21)

It is obvious that the result can be generalized as

$$[Y] = \sum_{i=1}^{n} [Y_i] \qquad (4\text{-}22)$$

where n is the number of channels in parallel.

From (3-24)

$$[Y_1] = \begin{bmatrix} \dfrac{\coth\mu_1}{Z_{c1}} & \dfrac{-1}{Z_{c1}\sinh\mu_1} \\ \dfrac{1}{Z_{c1}\sinh\mu_1} & \dfrac{-\coth\mu_1}{Z_{c1}} \end{bmatrix} \qquad (4\text{-}23)$$

$$[Y_2] = \begin{bmatrix} \dfrac{\coth\mu_2}{Z_{c2}} & \dfrac{-1}{Z_{c2}\sinh\mu_2} \\ \dfrac{1}{Z_{c2}\sinh\mu_2} & \dfrac{-\coth\mu_2}{Z_{c2}} \end{bmatrix} \qquad (4\text{-}24)$$

where
$$\begin{aligned} \mu_1 &= \varepsilon^{\gamma_1 \ell_1} \\ \mu_2 &= \varepsilon^{\gamma_2 \ell_2} \end{aligned} \qquad (4\text{-}25)$$

the equivalent $[Y]$ matrix is obtained as

$$[Y] = \begin{bmatrix} \dfrac{\coth\mu_1}{Z_{c1}} + \dfrac{\coth\mu_2}{Z_{c2}} & \dfrac{-1}{Z_{c1}\sinh\mu_1} - \dfrac{1}{Z_{c2}\sinh\mu_2} \\ \dfrac{1}{Z_{c1}\sinh\mu_1} + \dfrac{1}{Z_{c2}\sinh\mu_2} & \dfrac{-\coth\mu_1}{Z_{c1}} - \dfrac{\coth\mu_2}{Z_{c2}} \end{bmatrix} \qquad (4\text{-}26)$$

For the special case when

$$\mu_1 = \mu_2 = \mu \qquad (4\text{-}27)$$

equation (4-26) is reduced to

$$[Y] = \begin{bmatrix} \dfrac{\coth\mu}{Z_c} & \dfrac{-1}{Z_c\sinh\mu} \\ \dfrac{1}{Z_c\sinh\mu} & \dfrac{-\coth\mu}{Z_c} \end{bmatrix} \qquad (4\text{-}28)$$

where $\dfrac{1}{Z_c} = \dfrac{1}{Z_{c1}} + \dfrac{1}{Z_{c2}}$ (4-29)

In general if n channels with identical μ are in parallel then the Z_c of the equivalent channel becomes

$$\frac{1}{Z_c} = \sum_{i=1}^{n} \frac{1}{Z_{ci}} \tag{4-30}$$

Since

$$\mu = \varepsilon^{\gamma \ell} \tag{4-31}$$

substituting (2-26) into (4-31) we obtain

$$\mu = \varepsilon^{\frac{\ell j \omega}{\sqrt{g h_o}} \sqrt{1 - j \left(\frac{\lambda}{\omega}\right)}} \tag{4-32}$$

Therefore, the condition for identical μ for channels with the same λ is

$$\frac{\ell}{\sqrt{h_o}} = \text{constant} \tag{4-33}$$

Analysis of the Wave Motion in a Rectangular Channel

5-1 THE GENERATED SOURCES OF WAVES IN A CHANNEL

Waves in a channel mainly originate from the tidal waves generated in the ocean, and are affected by the runoff discharge of water into the channel. Therefore, we can consider there is an equivalent source at the end of the channel as shown in Figure 5-1.

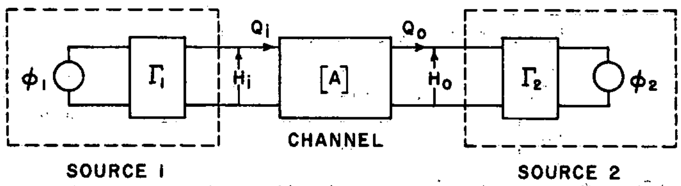

Figure 5-1

Each source consists of a generator ϕ and an internal immittance, Γ, the generator can be a vertical movement generator or horizontal flow generator; the former is associated with an internal impedance as shown in Figure 5-2a while the latter is connected with an internal admittance as shown in Figure 5-2b.

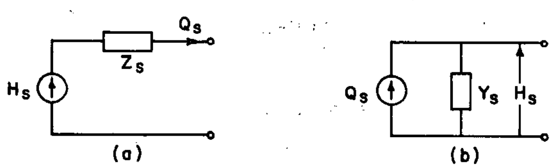

Figure 5-2

The source is considered to be independent of the channel, so that any change in the channel will not affect the source. Therefore, once the values of H_S and Z_S or Q_S and Y_S are determined, they will be considered as constants regardless of any change to the channel.

The matrix [A] of the channel could be [Y], [Z], [G] or [H] depending on the type of sources connected at both ends of the channel as described in Chapter 3 and tabulated in Table 5-1.

ϕ_1	ϕ_2	[A]
H_S	H_S	[Z]
H_S	Q_S	[H]
Q_S	H_S	[G]
Q_S	Q_S	[Y]

Table 5-1

Consider the case where both ends of the channel are connected to the vertical movement generator source as shown in Figure 5-3.

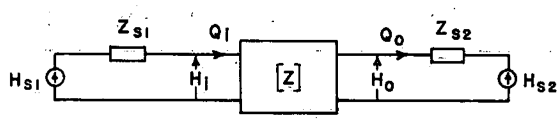

Figure 5-3

The Z-form solution of the channel is

$$\begin{bmatrix} H_i \\ H_o \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} Q_i \\ Q_o \end{bmatrix} \tag{5-1}$$

If we define that

$$H = ZQ \tag{5-2}$$

where H is the vertical movement drop across the impedance and Q is the horizontal flow passing through Z; H is considered positive in the direction of Q.

Then we obtain

$$H_i = H_{S_1} - Z_{S_1}Q_i \tag{5-3}$$

$$H_o = H_{S_2} + Z_{S_2}Q_o \tag{5-4}$$

Substituting (5-3) and (5-4) into (5-1) we obtain

$$\begin{bmatrix} H_{S1} - Z_{S1}Q_i \\ H_{S2} + Z_{S2}Q_o \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} Q_i \\ Q_o \end{bmatrix} \tag{5-5}$$

which can be written as

$$\begin{bmatrix} H_{S1} \\ H_{S2} \end{bmatrix} = \begin{bmatrix} z_{11} + Z_{S1} & z_{12} \\ z_{21} & z_{22} - Z_{S2} \end{bmatrix} \begin{bmatrix} Q_i \\ Q_o \end{bmatrix} \tag{5-6}$$

Using the same procedure, the equations for various combinations of sources at the ends of the channel can be generalized as

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} a_{11} + \Gamma_1 & a_{12} \\ a_{21} & a_{22} + \Gamma_2 \end{bmatrix} \begin{bmatrix} \theta_i \\ \theta_o \end{bmatrix} \tag{5-7}$$

where the relations between ϕ_1, ϕ_2, Γ_1, Γ_2, θ_i, θ_o and $[A]$ matrix are shown in Table 5-2.

ϕ_1	ϕ_2	Γ_1	Γ_2	θ_i	θ_o	$[A]$
H_{S1}	H_{S2}	Z_{S1}	$-Z_{S2}$	Q_i	Q_o	$[Z]$
H_{S1}	$-Q_{S2}$	Z_{S1}	$-Y_{S2}$	Q_i	H_o	$[H]$
Q_{S1}	H_{S2}	Y_{S1}	$-Z_{S2}$	H_i	Q_o	$[G]$
Q_{S1}	$-Q_{S2}$	Y_{S1}	$-Y_{S2}$	H_i	H_o	$[Y]$

Table 5-2

From (5-7) it is obvious that since there are four dependent variables, ϕ_1, ϕ_2, Γ_1 and Γ_2 in two linear equations, we cannot solve for the boundary conditions. However, we might assume that

(a) If the source represents an ocean then the internal impedance is negligible.

(b) If the source is a river then the internal admittance is negligible.

By these assumptions, a river is considered as a horizontal flow source while an ocean is equivalent to a vertical movement source. The overall system is approximated as shown in Figure 5-4.

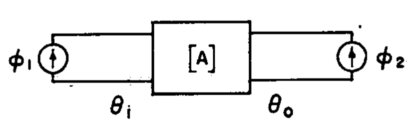

Figure 5-4

$$\text{with} \quad \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = [A] \begin{bmatrix} \theta_i \\ \theta_o \end{bmatrix} \tag{5-8}$$

where the value of ϕ_1 and ϕ_2 are equal to the corresponding boundary conditions.

5-2 WAVE CALCULATION

To calculate the wave in a channel we should determine the sources at both ends of the channel, and from these determine the type of matrix to be used in the calculation according to Table 5-2. Usually, because a channel is divided into sections in series the overall matrix of the channel is obtained in $[X]$ matrix, and we must convert the overall $[X]$ matrix into the type that is the opposite to the type of sources by the conversion table in Table 3-1. By this procedure, it is assumed that (5-8) is determined.

From which we obtain:

$$\begin{bmatrix} \theta_i \\ \theta_o \end{bmatrix} = [A]^{-1} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \tag{5-9}$$

After the value of θ_i and θ_o are determined, the wave at any section can be calculated, proceeding from either end of the channel.

For example, if the schematic diagram of a channel is as shown in Figure 5-5

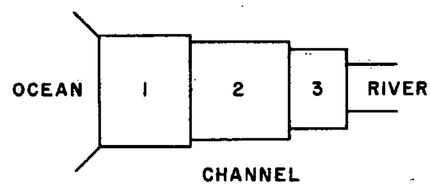

Figure 5-5

which is represented by the following block diagram

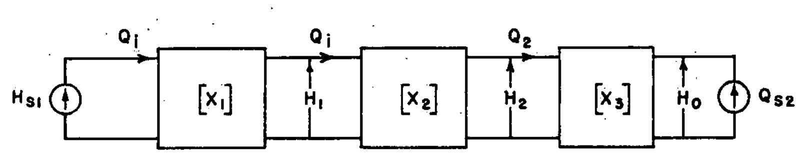

Figure 5-6

The overall $[X]$ matrix is obtained as

$$[X] = [X_3]\,[X_2]\,[X_1] \tag{5-10}$$

From Table 5-2 it follows that we must convert the $[X]$ matrix into $[H]^{-1}$ matrix, that is, $[G]$ matrix, and obtain

$$\begin{bmatrix} Q_i \\ H_o \end{bmatrix} = [G] \begin{bmatrix} H_{s1} \\ -Q_{s2} \end{bmatrix} \tag{5-11}$$

The values of Q_i and H_o are therefore determined.

The values of H_1, Q_1 and H_2, Q_2 can be calculated from

$$\begin{bmatrix} H_1 \\ Q_1 \end{bmatrix} = [X_1] \begin{bmatrix} H_{s1} \\ Q_i \end{bmatrix} \tag{5-12}$$

and

$$\begin{bmatrix} H_2 \\ Q_2 \end{bmatrix} = [W] \begin{bmatrix} H_o \\ Q_{s2} \end{bmatrix} \tag{5-13}$$

5-3 MULTI-TERMINAL CHANNEL

If a channel has more than two terminals then the procedure of calculation should be modified as explained in the following paragraphs.

Suppose a channel has three terminals and is represented by the following block diagram.

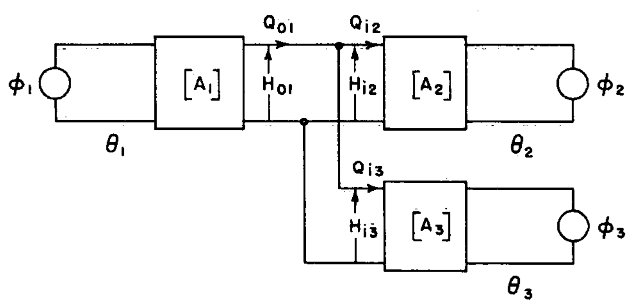

Figure 5-7

Then we have

$$\begin{bmatrix} \phi_1 \\ Q_{01} \end{bmatrix} = [A_1] \begin{bmatrix} Q_1 \\ H_{01} \end{bmatrix} \tag{5-14}$$

$$\begin{bmatrix} Q_{i2} \\ \phi_2 \end{bmatrix} = [A_2] \begin{bmatrix} H_{i2} \\ \theta_2 \end{bmatrix} \tag{5-15}$$

$$\begin{bmatrix} Q_{i3} \\ \phi_3 \end{bmatrix} = [A_3] \begin{bmatrix} H_{i3} \\ \theta_3 \end{bmatrix} \tag{5-16}$$

where the matrix $[A]$ can be any matrix shown in Table 5-2, depending on the source associated with the channel.

By expanding the above three equations we can obtain six linear equations.

$$\phi_1 = a_{11}{}^1\theta_1 + a_{12}{}^1 H_{01} \tag{5-17}$$

$$Q_{01} = a_{21}{}^1\theta_1 + a_{22}{}^1 H_{01} \tag{5-18}$$

$$Q_{i2} = a_{11}{}^2 H_{i2} + a_{12}{}^2\theta_2 \tag{5-19}$$

$$\phi_2 = a_{21}{}^2 H_{i2} + a_{22}{}^2\theta_2 \tag{5-20}$$

$$Q_{i3} = a_{11}{}^3 H_{i3} + a_{12}{}^3\theta_3 \tag{5-21}$$

$$\phi_3 = a_{21}{}^3 H_{i3} + a_{22}{}^3\theta_3 \tag{5-22}$$

where the superscripts of the matrix elements denote the channel numbers.

By applying the boundary conditions that

$$Q_{01} = Q_{i2} + Q_{i3} \tag{5-23}$$

$$H_{01} = H_{i2} = H_{i3} \tag{5-24}$$

we may obtain the following equation from (5-18), (5-19) and (5-21).

$$a_{21}{}^1\theta_1 + a_{22}{}^1 H_{01} = a_{11}{}^2 H_{01} + a_{12}{}^2\theta_2 + a_{11}{}^3 H_{01} + a_{12}{}^3\theta_3 \tag{5-25}$$

which yields

$$H_{01} = \frac{-a_{21}{}^1\theta_1 + a_{12}{}^2\theta_2 + a_{12}{}^3\theta_3}{a_{22}{}^1 - a_{11}{}^2 - a_{11}{}^3} \tag{5-26}$$

By substituting (5-26) into (5-17), (5-20) and (5-22) we obtain

$$\phi_1 = \left[a_{11}{}^1 - \frac{a_{12}{}^1\, a_{21}{}^1}{\Delta}\right]\theta_1 + \frac{a_{12}{}^1\, a_{12}{}^2}{\Delta}\theta_2 + \frac{a_{12}{}^1\, a_{12}{}^3}{\Delta}\theta_3 \tag{5-27}$$

$$\phi_2 = -\frac{a_{21}{}^2\, a_{21}{}^1}{\Delta}\theta_1 + \left[a_{22}{}^2 + \frac{a_{12}{}^2\, a_{21}{}^2}{\Delta}\right]\theta_2 + \frac{a_{21}{}^2\, a_{12}{}^3}{\Delta}\theta_3 \tag{5-28}$$

$$\phi_3 = -\frac{a_{21}{}^3\, a_{21}{}^1}{\Delta}\theta_1 + \frac{a_{31}{}^3\, a_{12}{}^2}{\Delta}\theta_2 + \left[a_{22}{}^3 + \frac{a_{31}{}^3\, a_{12}{}^3}{\Delta}\right]\theta_3 \tag{5-29}$$

where $\quad \Delta = a_{22}{}^{1} - a_{11}{}^{2} - a_{11}{}^{3}$ (5-30)

and can be written in matrix form as

$$\begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\end{bmatrix} = \left[\bar{A}\right] \begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix} \tag{5-31}$$

where $\left[\bar{A}\right]$

$$= \frac{1}{\Delta}\begin{bmatrix} a_{11}{}^{1}\Delta - a_{12}{}^{1}a_{21}{}^{1} & a_{12}{}^{1}a_{12}{}^{2} & a_{12}{}^{1}a_{12}{}^{3} \\ -a_{21}{}^{2}a_{21}{}^{1} & a_{22}{}^{2}\Delta + a_{12}{}^{2}a_{21}{}^{2} & a_{21}{}^{2}a_{12}{}^{3} \\ a_{31}{}^{3}a_{21}{}^{1} & a_{21}{}^{3}a_{12}{}^{2} & a_{22}{}^{3}\Delta + a_{21}{}^{3}a_{12}{}^{3} \end{bmatrix} \tag{5-32}$$

It is obvious that the dimension of the matrix $(\bar{A})$ is equal to the number of terminals of the channel.

From (5-31) the value of θ_1, θ_2, and θ_3 is obtained from

$$\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix} = \left[\bar{A}\right]^{-1} \begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\end{bmatrix} \tag{5-33}$$

By using the result together with the values of ϕ_1, ϕ_2 and ϕ_3 the tides at any section can easily be calculated.

5-4 POWER TRANSFER

According to the definition in (2-19), the average active power is defined as

$$P = \frac{1}{2}\rho g R_e\left(H^*Q\right) \tag{5-34}$$

By defining P_i as the power input to the channel and P_x as the power at x distance from the input end of the channel, we obtain

$$P_i = \frac{1}{2}\rho g \mathrm{Re}\left(H_i{}^*Q_i\right) \tag{5-35}$$

$$P_x = \frac{1}{2}\rho g \mathrm{Re}\left(H_x{}^*Q_x\right) \tag{5-36}$$

Since $\quad H_x = x_{11}H_i + x_{12}Q_i$ (5-37)

$$Q_x = x_{21}H_i + x_{22}Q_i \tag{5-38}$$

we obtain

$$H_x{}^* = x_{11}{}^*H_i{}^* + x_{12}{}^*Q_i{}^* \tag{5-39}$$

therefore

$$H_x{}^*Q_x = x_{11}{}^*x_{21}H_i{}^*H_i + x_{11}{}^*x_{22}H_i{}^*Q_i + x_{12}{}^*x_{21}Q_i{}^*H_i + x_{12}{}^*x_{22}Q_i{}^*Q_i \tag{5-40}$$

Because

$$H_i{}^*H_i = |H_i|^2 \tag{5-41}$$

$$Q_i{}^*Q_i = |Q_i|^2 \tag{5-42}$$

Therefore, the real part of (5-40) is obtained as

$$\mathrm{Re}\left(H_x{}^*Q_x\right) = |H_i|^2\,\mathrm{Re}\left(x_{11}{}^*x_{21}\right) + \mathrm{Re}\left(x_{11}{}^*x_{22}H_i{}^*Q_i\right) + \mathrm{Re}\left(x_{12}{}^*x_{21}Q_i{}^*H_i\right) + |Q_i|^2\,\mathrm{Re}\left(x_{12}{}^*x_{22}\right) \tag{5-43}$$

Since

$$x_{11}{}^*x_{22} = \cosh\gamma x \cosh\gamma^* x = \frac{1}{2}\left(\cosh 2\alpha x + \cos 2\beta x\right) \tag{5-44}$$

$$x_{12}{}^*x_{21} = -Z_c{}^*\sinh\gamma^* x\left[\frac{-1}{Z_c}\right]\sinh\gamma x = \frac{\varepsilon^{j2\theta}}{2}\left(\cosh 2\alpha x - \cos 2\beta x\right) \tag{5-45}$$

$$x_{12}{}^*x_{22} = -Z_c{}^*\sinh\gamma^* x\cosh\gamma x = -\frac{Z_c{}^*}{2}\left(\sinh 2\alpha x - j\sin 2\beta x\right) \tag{5-46}$$

$$x_{11}{}^*x_{21} = \cosh\gamma^* x\left[\frac{-1}{Z_c}\right]\sinh\gamma x = -\frac{1}{2Z_c}\left(\sinh 2\alpha x + j\sin 2\beta x\right) \tag{5-47}$$

where $\quad Z_c = |Z_c|\,\varepsilon^{-j\theta}$ (5-48)

$$= R_e - jX_c \tag{5-49}$$

By defining

$$Y_c = \frac{1}{Z_c} = G_c + jW_c \tag{5-50}$$

and substituting these results into (5-43) we obtain

$$\mathrm{Re}\left(H_x{}^*Q_x\right) = -\frac{|H_i|^2}{2}\left(G_c\sinh 2\alpha x - W_c\sin 2\beta x\right) + \frac{1}{2}\left(\cosh 2\alpha x + \cos 2\beta x\right)R_e\left(H^*Q\right) + \frac{1}{2}\left(\cosh 2\alpha x - \cos 2\beta x\right)R_e\left(\varepsilon^{j2\theta}Q_i{}^*H_i\right) - \frac{|Q_i|^2}{2}\left(R_c\sinh 2\alpha x - X_c\sin 2\beta x\right) \tag{5-51}$$

Since

$$\mathrm{Re}\left(\varepsilon^{j2\theta}Q_i{}^*H_i\right) = \mathrm{Re}\left[\varepsilon^{-j2\theta}\left(H_i{}^*Q_i\right)\right]^* = \cos 2\theta\, R_e\left(H_i{}^*Q_i\right) + \sin 2\theta\,\mathrm{Im}\left(H_i{}^*Q_i\right) \tag{5-52}$$

which shows that the imaginary part of $H_i^*Q_i$ occurs in (5-52), this term is called the reactive power and defined as

$$(P_r)_i = \frac{\rho g}{2} \operatorname{Im}(H_i^*Q_i) \qquad (5\text{-}53)$$

By defining

$$P_{hr} = \frac{\rho g}{2} |H_i|^2 G_c \quad P_{hi} = \frac{\rho g}{2} |H_i|^2 W_c \qquad (5\text{-}54)$$

$$P_{qr} = \frac{\rho g}{2} |Q_i|^2 R_c \quad P_{qi} = \frac{\rho g}{2} |Q_i|^2 X_c \qquad (5\text{-}55)$$

The equation for average power P_x becomes

$$\begin{aligned} P_x = & -P_{hr} \sinh 2\alpha x + P_{hi} \sin 2\beta x \\ & + (\cosh 2\alpha x + \cos 2\beta x) P_i \\ & + (\cosh 2\alpha x - \cos 2\beta x) (\cos 2\theta\, P_i \\ & \qquad + \sin 2\theta\, (P_r)_i) \\ & -P_{qr} \sinh 2\alpha x + P_{qi} \sin 2\beta x \end{aligned} \qquad (5\text{-}56)$$

By collecting terms

$$\begin{aligned} P_x = & -P_{hr} \sinh 2\alpha x + P_{hi} \sin 2\beta x \\ & -P_{qr} \sinh 2\alpha x + P_{qi} \sin 2\beta x \\ & +P_i \left[\cosh 2\alpha x\, (1 + \cos 2\theta) \right. \\ & \qquad \left. + \cos 2\beta x\, (1 - \cos 2\theta)\right] \\ & +(P_r)_i\, (\cosh 2\alpha x - \cos 2\beta x) \sin 2\theta \end{aligned} \qquad (5\text{-}57)$$

equation (5-57) can be written as

$$\begin{aligned} P_x = & -(P_{hr} + P_{qr}) \sin 2\alpha x \\ & +(P_{hi} + P_{qi}) \sin 2\beta x \\ & +2P_i\, (\cos^2\theta \cosh 2\alpha x + \sin^2\theta \cos 2\beta x) \\ & +(P_r)_i\, (\cosh 2\alpha x - \cos 2\beta x) \sin 2\theta \end{aligned} \qquad (5\text{-}58)$$

The power at the output end of the channel is obtained simply by replacing x by ℓ.

$$\begin{aligned} P_o = & -(P_{hr} + P_{qr}) \sin 2\alpha\ell \\ & +(P_{hi} + P_{qi}) \sin 2\beta\ell \\ & +2P_i\, (\cos^2\theta \cosh 2\alpha\ell + \sin^2\theta \cos 2\beta\ell) \\ & +(P_r)_i\, (\cosh 2\alpha\ell - \cos 2\beta\ell) \sin 2\theta \end{aligned} \qquad (5\text{-}59)$$

Electrical Analogue for the Wave Motion in a Channel

6-1 INTRODUCTION

It has been shown that by applying a certain degree of approximation wave motion in a channel can be represented by linear equations. Because similar equations for voltage and current are found in some types of electrical networks, these networks can be used to analogise the channel, with the voltage and current analogous to the vertical movement and the horizontal flow in the channel. There are two types of analogy between the electrical network and channel, voltage-vertical movement analogy and voltage-horizontal flow analogy. The former simulates the vertical movement and horizontal flow by voltage and current respectively while the latter simulates the vertical movement and horizontal flow by current and voltage respectively.

The type of analogy to be used depends on the nature of the problem. For the constant voltage source is more conveniently used in electrical networks. Therefore, if the waves in the channel are generated by an equivalent vertical movement generator then the voltage-vertical movement analogy is used, otherwise the voltage-horizontal flow analogy is applied. Moreover, the value of the equivalent resistance, R, inductance, L, and capacitance, C, of the network will not be the same for different types of analogy, therefore, only the one which requires practical values of R, L and C is chosen.

In either type of analogy there are some different methods to simulate the channel, depending on what kind of mathematical representation of the wave motion in the channel we are referring to.

For convenience, the partial differential equations of the wave motion are rewritten from (1-3) and (1-4).

$$\frac{\partial q}{\partial x} + b_o \frac{\partial h}{\partial t} = 0 \qquad (6\text{-}1)$$

$$\frac{\partial h}{\partial x} + \frac{\lambda}{ga_o} q + \frac{1}{ga_o} \frac{\partial q}{\partial t} = 0 \qquad (6\text{-}2)$$

These equations are called the partial differential form of the wave motion equation.

If the equations are integrated with respect to x and we assume that the time variation of h and q is constant, and q is linearly distributed along a section of the channel of length Δx, then we obtain

$$q_2 - q_1 + b_o \Delta x \frac{\partial h}{\partial t} = 0 \qquad (6\text{-}3)$$

$$h_2 - h_1 + \frac{\lambda}{2ga_o} \Delta x \left(q_2 + q_1\right) + \frac{\Delta x}{2ga_o} \frac{\partial}{\partial t} \left(q_2 + q_1\right) = 0 \qquad (6\text{-}4)$$

where the numeric subscripts represent the waves at two ends of the section respectively.

These equations are called the finite difference form of the wave motion equation.

The above two sets of equations together with the Z- and Y-form solution of the wave motion in a rectangular channel are the basic equations used in the following analogy technique.

6-2 ELECTRICAL ANALOGUE FOR THE PARTIAL DIFFERENTIAL FORM OF THE WAVE MOTION EQUATION

Figure 6-1 represents a section of 2-wire transmission line,

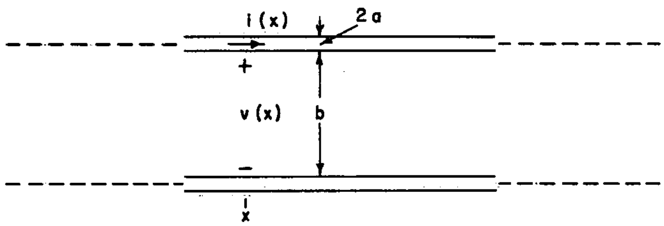

Figure 6-1

where a = radius of the conductor

b = distance between the center of two conductors

i = the current passing the conductor at x

v = the voltage across the two wires at x.

The relationship between voltage and current along the transmission line is represented by the following equations:

$$\frac{\partial i}{\partial x} + Gv + C \frac{\partial v}{\partial t} = 0 \qquad (6\text{-}5)$$

$$\frac{\partial v}{\partial x} + Ri + L\frac{\partial i}{\partial t} = 0 \qquad (6\text{-}6)$$

where R and L represent the series resistance and inductance per unit length while G and C denote the leakage conductance and capacitance per unit length.

The values of R, L, G and C can be approximated by the following equations (King (3)):

$$R = \frac{1}{\pi a}\sqrt{\frac{\mu^1\omega}{2\sigma^1}} \qquad (6\text{-}7)$$

$$L = \frac{1}{\pi a}\sqrt{\frac{\mu^1\omega}{2\sigma^1}} + \frac{\mu}{\pi}\ell n\frac{b}{a} \qquad (6\text{-}8)$$

$$G = \frac{\pi\sigma}{\ell n\frac{b}{a}} \qquad (6\text{-}9)$$

$$C = \frac{\pi\varepsilon}{\ell n\frac{b}{a}} \qquad (6\text{-}10)$$

where μ^1, σ^1 = the permeability and conductivity of the conductor

μ, σ = the permeability and conductivity of the media separating the two wires.

ω = angular frequency of the signal

The above equations are subject to the following restrictions

$$b >> a \qquad (6\text{-}11)$$

$$a\sqrt{\omega\sigma^1\mu^1} >> 10 \qquad (6\text{-}12)$$

(A) VOLTAGE-VERTICAL MOVEMENT ANALOGY

If a medium with very small conductivity is used to separate the wires, then the leakage conductance can be neglected and the equations become

$$\frac{\partial i}{\partial x} + C\frac{\partial v}{\partial t} = 0 \qquad (6\text{-}13)$$

$$\frac{\partial v}{\partial x} + Ri + L\frac{\partial i}{\partial t} = 0 \qquad (6\text{-}14)$$

By comparing these equations with equation (6-1) and (6-2), it is obvious that they are analogous. The analogic parameters are shown in Table 6-1.

Therefore, by defining

$$h = k_h v \qquad (6\text{-}15)$$

$$q = k_q i \qquad (6\text{-}16)$$

$$x_c = k_x x_e \qquad (6\text{-}17)$$

$$t_c = k_t t_e \qquad (6\text{-}18)$$

$$\omega_c = \frac{1}{k_t}\omega_e \qquad (6\text{-}19)$$

where the subscripts c and e denote the channel and electrical network respectively, and substituting (6-15) through (6-19) into (6-2) we obtain

$$\frac{\partial i}{\partial x_e} + \frac{k_h k_x}{k_q k_t} b_o \frac{\partial v}{\partial x_e} = 0 \qquad (6\text{-}20)$$

$$\frac{\partial v}{\partial x_e} + \frac{k_q k_x}{k_h}\frac{\lambda}{ga_o} i + \frac{k_q k_x}{k_t k_h}\frac{1}{ga_o}\frac{\partial i}{\partial t_e} = 0 \qquad (6\text{-}21)$$

CHANNEL	TRANSMISSION LINE
h	v
q	i
b_o	C
$\frac{\lambda}{ga_o}$	R
$\frac{1}{ga_o}$	L

Table 6-1

Therefore, the relationship between the parameters in the transmission line and the channel can be obtained as

$$R = \frac{k_q k_x}{k_h}\frac{\lambda}{ga_o} \qquad (6\text{-}22)$$

$$L = \frac{k_q k_x}{k_t k_h}\frac{1}{ga_o} \qquad (6\text{-}23)$$

$$C = \frac{k_h k_x}{k_q k_t} b_o \qquad (6\text{-}24)$$

The scaling factor must be adjusted to provide a practical value for R, L and C of the transmission line.

(B) VOLTAGE-HORIZONTAL FLOW ANALOGY

From (6-7), if a conductor with very high conductivity is used, then the resistance R can be neglected. The equations of transmission line become

$$\frac{\partial v}{\partial x} + L\frac{\partial i}{\partial t} = 0 \qquad (6\text{-}25)$$

$$\frac{\partial i}{\partial x} + Gv + C\frac{\partial v}{\partial t} = 0 \qquad (6\text{-}26)$$

Obviously, the analogy between the wave motion in a channel and the voltage-current in transmission line is obtained as shown in Table 6-2.

CHANNEL	TRANSMISSION LINE
h	i
q	v
b_o	L
$\frac{\lambda}{ga_o}$	G
$\frac{1}{ga_o}$	C

Table 6-2

The scaling factor in (6-15) and (6-16) are interchanged as

$$h = k_h i \qquad (6\text{-}27)$$

$$q = k_q v \qquad (6\text{-}28)$$

Equations (6-22) through (6-24) are still valid except R is replaced by G, and the expressions for L and C are interchanged.

This kind of analogy has the advantage that the cross-sectional variation of the channel can be accomplished simply by the variation of the size of the wire and the spacing between the wires. Therefore, the same technique can be applied to simulate the non-uniform channel. However, the technique can be used only for one-dimensional flow, and there is some difficulty in the current measurement along the lines; moreover, the mounting of the line will affect the R, G, L and C parameters of the lines, therefore, it is suitable for qualitative study only.

6-3 ELECTRICAL ANALOGY FOR THE FINITE DIFFERENCE FORM OF THE WAVE MOTION EQUATION

(A) VOLTAGE-VERTICAL MOVEMENT ANALOGY

The network in Figure 6-2 is a T-type two-part network. The voltage and current relation for this network are shown in the following equations:

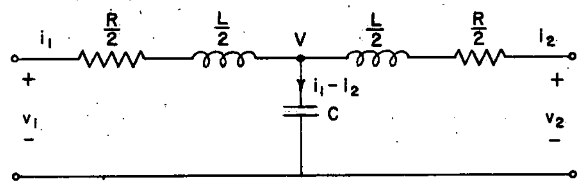

Figure 6-2

$$i_2 - i_1 + C\frac{\partial v}{\partial t} = 0 \qquad (6\text{-}29)$$

$$v_2 - v_1 + \frac{R}{2}(i_1 + i_2) + \frac{L}{2}\frac{\partial}{\partial t}(i_1 + i_2) = 0 \qquad (6\text{-}30)$$

By comparing the above equations with (6-3) and (6-4) it is obvious that they are of the same form. Therefore, the wave motion in a section of a channel can be represented by the voltage and current in the network as shown in Figure 6-2.

Table 6-3 shows the analogy between these two systems

CHANNEL	T-TYPE NETWORK
h	v
q	i
$b_o\Delta x$	C
$\frac{\lambda}{ga_o}\Delta x$	R
$\frac{1}{ga_o}\Delta x$	L

Table 6-3

By using the same proportional relation defined in (6-15) through (6-19), (6-3) and (6-4) become

$$i_2 - i_1 + \frac{k_h}{k_t k_q} b_o\Delta x \frac{\partial v}{\partial t_e} = 0 \qquad (6\text{-}31)$$

$$v_2 - v_1 + \frac{k_q}{k_h}\frac{\lambda}{2ga_o}\Delta x\,(i_2 + i_1)$$

$$+ \frac{k_q}{k_h k_t}\frac{\Delta x}{2ga_o}\frac{\partial}{\partial t}(i_2 + i_1) = 0 \qquad (6\text{-}32)$$

Therefore, the R, L and C components of

the analogical network are obtained as:

$$C = \frac{k_h}{k_t k_q} b_o \Delta x \qquad (6\text{-}33)$$

$$R = \frac{k_q}{k_h} \frac{\lambda}{ga_o} \Delta x \qquad (6\text{-}34)$$

$$L = \frac{k_q}{k_h k_t} \frac{\Delta x}{ga_o} \qquad (6\text{-}35)$$

By adjusting the scaling factors k_h, k_q and k_t a suitable value for R, L and C can be obtained.

(B) VOLTAGE-HORIZONTAL FLOW ANALOGY

The network shown in Figure 6-3 is a π-type two-part network.

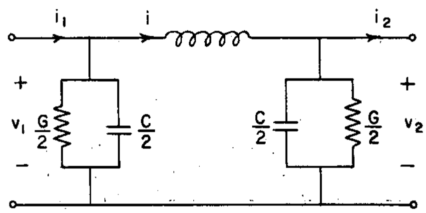

Figure 6-3

The relation between voltage and current in this network is obtained as:

$$v_2 - v_1 + L \frac{\partial i}{\partial t} = 0 \qquad (6\text{-}36)$$

$$i_2 - i_1 + \frac{G}{2}(v_1 + v_2) + \frac{C}{2}\frac{\partial}{\partial t}(v_1 + v_2) = 0 \qquad (6\text{-}37)$$

It is obvious that these equations are the analogues of (6-3) and (6-4) according to the relation shown in Table 6-4.

CHANNEL	π-TYPE NETWORK
h	i
q	v
b_o	L
$\frac{\lambda}{ga_o}$	G
$\frac{i}{ga_o}$	C

Table 6-4

By using the scaling factor defined in (6-27), (6-28) and (6-17) through (6-19) the equations for G, L and C can be derived and they should be the same as (6-33) through (6-35), except that R is replaced by G, and L and C are interchanged.

This type of analogy has the advantage that it has a simple relation between the R, L and C components of the network and the parameters of the channel, and the same technique can be extended to be applied to two-dimensional flow. The only disadvantage of this technique arises from the assumption that $\frac{\partial h}{\partial t}$, $\frac{\partial q}{\partial t}$ are constant throughout the section Δ_x and q is linearly distributed along the section.

The effect can be reduced by decreasing the length of the section which, however, in turn will increase the number of components used in the network.

6-4 ELECTRICAL ANALOGY FOR THE SOLUTION FORM OF THE WAVE MOTION EQUATION

Figures 6-4 and 6-5 show the typical symmetrical T-type, and π-type, two-part networks:

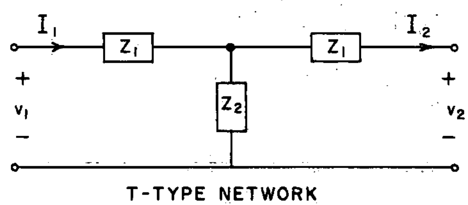

Figure 6-4

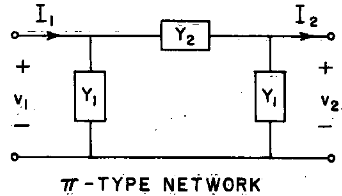

Figure 6-5

The voltage and current in these networks are governed by the following equations:

T-TYPE: $$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_1 + Z_2 & -Z_2 \\ Z_2 & -(Z_1 + Z_2) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{6-38}$$

π-TYPE: $$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_1 + Y_2 & -Y_2 \\ Y_2 & -(Y_1 + Y_2) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{6-39}$$

The solution of wave motion in a rectangular channel can be expressed in many forms as described in Chapter 3. Here, particular interest is directed to the Z-form and Y-form solutions, they are:

$$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = [Z] \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \tag{6-40}$$

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = [Y] \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \tag{6-41}$$

By comparing these equations with (6-38) and (6-39), it is obvious that they are analogous.

(A) VOLTAGE-VERTICAL MOVEMENT ANALOGY

If vertical movement is simulated by voltage, by using the scaling factor defined in (6-15) and (6-16) we may obtain

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \frac{k_q}{k_h} [Z] \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{6-42}$$

or $$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{k_h}{k_q} [Y] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{6-43}$$

By comparing these equations with (6-38) and (6-39), we obtain

T-TYPE: $$Z_2 = \frac{k_q}{k_h} z_{21} \tag{6-44}$$

$$Z_1 + Z_2 = \frac{k_q}{k_h} z_{11} \tag{6-45}$$

π-TYPE: $$Y_2 = \frac{k_h}{k_q} y_{21} \tag{6-46}$$

$$Y_1 + Y_2 = \frac{k_h}{k_q} y_{11} \tag{6-47}$$

From (6-44) and (6-45) we obtain

T-TYPE: $$Z_1 = \frac{k_q}{k_h} (z_{11} - z_{21}) \tag{6-48}$$

Similarly, we obtain

π-TYPE: $$Y_1 = \frac{k_h}{k_q} (y_{11} - y_{21}) \tag{6-49}$$

The equation for the impedance Z and admittance Y of the networks can be expressed in terms of the parameter of the channel by substituting the value for z_{ij} and y_{ij} as shown in (3-18) and (3-24) respectively. They are:

T-TYPE: $$Z_2 = \frac{k_q}{k_h} \frac{z_c}{\sinh \gamma \ell} \tag{6-50}$$

$$Z_1 = Z_2 (\cosh \gamma \ell - 1) \tag{6-51}$$

π-TYPE: $$Y_2 = \frac{k_h}{k_q} \frac{1}{Z_c \sinh \gamma \ell} \tag{6-52}$$

$$Y_1 = Y_2 (\cosh \gamma \ell - 1) \tag{6-53}$$

Therefore, if the wave motion in the channel is represented in Z-form then the T-type network is used, otherwise, we use π-type network to simulate the channel. The components of the network are given by (6-44) and (6-45) or (6-50) and (6-51) for the T-type network and (6-46) and (6-47) or (6-52) and (6-53) for the π-type network.

(B) VOLTAGE-HORIZONTAL FLOW ANALOGY

By using the same procedure, we conclude that if the wave motion in the channel is expressed in Z-form then the channel is simulated by π-type network and the equation for the components of the network are the same as (6-44) and (6-45) or (6-50) and (6-51) except that Z is replaced by Y; otherwise, if the Y-form solution is used, then the channel will be simulated by T-type network and its components are given according to (6-46) and (6-47) or (6-52) and (6-53), with Z replacing Y in the equations. Note that in this kind of analogy, the scaling factors defined in (6-27) and (6-28) are used.

After the impedance Z or admittance Y of the network is calculated, then the corresponding resistance, inductance and conductance can be obtained by the following equations:

T-TYPE: $$R = \mathrm{Re}\,[Z] \tag{6-54}$$

$$L = \frac{1}{\omega_e} \mathrm{Im}\,[Z] \quad \text{if} \quad \mathrm{Im}\,[Z] > o \tag{6-55}$$

$$C = \omega_e \mathrm{Im}\,[Z] \quad \mathrm{Im}\,[Z] < o \tag{6-56}$$

π-TYPE: $$\frac{1}{R} = \mathrm{Re}\,[Y] \tag{6-57}$$

$$L = \omega_e \mathrm{Im}\,[Y] \quad \text{if} \quad \mathrm{Im}\,[Y] < o \tag{6-58}$$

$$C = \frac{1}{\omega_e} \mathrm{Im}\,[Y] \quad \mathrm{Im}\,[Y] > o \tag{6-59}$$

where ω_e is defined by equation (6-19).

This technique will produce more accurate results if a channel can be approximated by sections of uniform rectangular channel. However, some calculations involving complex numbers should be done in order to compute the value of the components in the network. Also, the method can be applied to one-dimensional flow only.

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