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DIFFUSION AND DISPERSION EQUATIONS IN OPEN CHANNEL FLOW

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PREFACE

This manuscript traces the development of the various equations which are used in studying diffusion and dispersion in open-channel flow. Starting from the general convective-diffusion equation, other simpler equations applicable to vertical, transverse, or one-dimensional transport are derived. The meaning of the various transport terms are explained and the equations commonly used for calculating the various dispersion coefficients are derived. This manuscript should be a useful and concise summary for scientists and engineers who are working on diffusion problems in open-channel flows.

PREFACE

Le présent document décrit les étapes de l'élaboration des équations utilisées pour l'étude des phénomènes de diffusion et de dispersion dans un canal à ciel ouvert. A partir de l'équation générale de diffusion-convection, on peut obtenir des équations plus simples, applicables au transport vertical, transversal ou unidimensionnel. Les termes relatifs au transport y sont définis et les équations qui servent généralement au calcul des divers coefficients de dispersion y sont établies. Ce document concis sera sans doute fort utile aux scientifiques et aux ingénieurs qui étudient les problèmes de diffusion dans les canaux à ciel ouvert.

1. <u>Convective Diffusion Equation</u>

Applying the principle of mass conservation to the diffusing substance, one gets the following equation

$$\frac{\partial c}{\partial t} + u_i \frac{\partial c}{\partial x_i} = D_m \frac{\partial^2 c}{\partial x_i \partial x_i} \tag{1-1}$$

where c is the concentration, and u is the velocity -- both are instantaneous values as used in equation (1-1). D_m is the molecular diffusion coefficient.

Separating c and u into their time mean and fluctuating values one can write

$c = C + c'$, $u = U + u'$, $v = V + v'$, $w = W + w'$

Substitute these into equation (1-1) and apply time averaging to the equation:

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} + V\frac{\partial C}{\partial y} + W\frac{\partial C}{\partial z} = D_m\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right) + \frac{\partial\, -\overline{u'c'}}{\partial x} + \frac{\partial\, -\overline{v'c'}}{\partial y} + \frac{\partial\, -\overline{w'c'}}{\partial z} \tag{1-2}$$

The transport due to turbulent fluctuations are usually assumed to be proportional to the gradients of the mean quantities;

$$-\overline{u'c'} = \epsilon_x \frac{\partial C}{\partial x} \quad ; \quad -\overline{v'c'} = \epsilon_y \frac{\partial C}{\partial y} \quad ; \quad -\overline{w'c'} = \epsilon_z \frac{\partial C}{\partial z} \tag{1-3}$$

ε_x, ε_y, ε_z are the turbulent diffusivities of mass in the x,y,z directions respectively. Transport by the turbulent fluctuations are terms turbulent diffusion. The equation now becomes

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} + V\frac{\partial C}{\partial y} + W\frac{\partial C}{\partial z} = D_m\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right) + \frac{\partial}{\partial x}\epsilon_x\frac{\partial C}{\partial x} + \frac{\partial}{\partial y}\epsilon_y\frac{\partial C}{\partial y} + \frac{\partial}{\partial z}\epsilon_z\frac{\partial C}{\partial z} \tag{1-4}$$

Equation (1-4) is the general convective diffusion equation. Very often the physical situation allows some of the terms to be dropped and the equation simplified. This enables some analytical solutions to be obtained. The turbulent diffusion coefficients are generally not known and have to be determined experimentally.

2. Vertical Diffusion

In open-channel flows vertical mixing is usually completed long before the dispersing substance has spread across the channel.

An expression for the vertical diffusion coefficient ε_y can be derived by using Reynolds analogy between diffusivities of mass and momentum. For a wide open channel, assuming logarithmic velocity distribution

$$U(y) = U(h) + \frac{U_*}{K} \ln\left(\frac{y}{h}\right)$$

and a linear shear stress distribution

$$\tau(y) = \tau_o\left(1 - \frac{y}{h}\right)$$

one gets $$\varepsilon_{ym} = \frac{\tau}{\rho \frac{\partial U}{\partial y}} = K U_* y\left(1 - \frac{y}{h}\right) \qquad (2\text{-}1)$$

where ε_{ym} is the vertical diffusivity of momentum, K is the Von Karman constant and h is the flow depth.

The depth averaged value of ε_{ym} is

$$\bar{\varepsilon}_{ym} = \frac{K}{6} U_* h \qquad (2\text{-}2)$$

Assuming $\varepsilon_y = \lambda\varepsilon_{ym}$ one can see that the distribution of ε_y in the vertical direction will be parabolic. Measurements by Al Saffar (1964) showed that ε_{ym} and ε_y had similar variations with depth but ε_y had a higher maximum value. Observations by Vanoni (1946) and Jobson and Sayre (1970) of dye and sediment distributions indicated that $\lambda \cong 1$.

For a steady state 2-dimensional flow with no concentration gradient in the z (transverse) direction and assuming that $\frac{\partial c}{\partial x}$ is small compared with $\frac{\partial c}{\partial y}$ equation (1-4) becomes

$$U\frac{\partial C}{\partial x} = \frac{\partial}{\partial y} \varepsilon_y \frac{\partial C}{\partial y} \qquad (2\text{-}3)$$

If the velocity U and the diffusion coefficient ε_y are considered as constants or depth-averaged values it can be shown, by multiplying both sides of (2-3) by y^2 and integrating, that

$$\varepsilon_y = \frac{U}{2} \frac{d\sigma_y^2}{dx} \qquad (2\text{-}4)$$

where $\sigma_y^2 = \dfrac{\int y^2 C\,dy}{\int C\,dy}$ is the variance of the vertical concentration distribution.

However, if $U = U(y)$ and $\varepsilon_y = \varepsilon_y(y)$ then one cannot use (2-4). ε_y can be calculated from the profiles of the velocity and concentration:

$$\epsilon_y(y) = \frac{\frac{d}{dz}\int U(y)C(y)\,dy}{\frac{\partial C(y)}{\partial y}} \tag{2-5}$$

The time scale for vertical mixing is $T_y = \frac{h^2}{\varepsilon_y}$. From results of concentration distributions calculated from solution of equation (2-3), it can be seen that the time required for vertical mixing is approximately $0.5T_y$. Therefore, the distance L_v required for the concentration to become more or less uniform throughout the depth is

$$L_v = 0.5\,\frac{Uh^2}{\epsilon_y} \tag{2-6}$$

3. Transverse Diffusion and Dispersion

For a steady flow in which the mean vertical velocity is zero and in which molecular diffusion is neglected equation (1-4) can be written as

$$U\frac{\partial C}{\partial x} + W\frac{\partial C}{\partial z} = \frac{\partial}{\partial x}\epsilon_x\frac{\partial C}{\partial x} + \frac{\partial}{\partial y}\epsilon_y\frac{\partial C}{\partial y} + \frac{\partial}{\partial z}\epsilon_z\frac{\partial C}{\partial z} \tag{3-1}$$

In open-channel flow the dispersing substance is often well mixed throughout the depth before significant transverse spreading has occurred. Therefore average values with depth can be used when studying transverse spreading. Let

$$U = \bar{U} + u''; \quad C = \bar{C} + c''; \quad W = \bar{W} + w''; \quad \epsilon_x = \bar{\epsilon}_x + \epsilon_x''; \quad \epsilon_z = \bar{\epsilon}_z + \epsilon_z''$$

The overbar indicates depth-averaged values and the superscript " indicates deviations from the depth averaged values.

Substituting these expressions into equation (3-1) and then averaging the equation over the depth results in the following equation:

$$\bar{U}\frac{\partial \bar{C}}{\partial x} + \bar{W}\frac{\partial \bar{C}}{\partial z} + \frac{\partial \overline{u''c''}}{\partial x} + \frac{\partial \overline{w''c''}}{\partial z} = \frac{\partial}{\partial x}\bar{\epsilon}_x\frac{\partial \bar{C}}{\partial x} + \frac{\partial}{\partial z}\bar{\epsilon}_z\frac{\partial \bar{C}}{\partial z} + \frac{\partial}{\partial x}\overline{\epsilon_x''\frac{\partial c''}{\partial x}} + \frac{\partial}{\partial z}\overline{\epsilon_z''\frac{\partial c''}{\partial z}}$$

When conditions are well mixed vertically the last two terms on the right hand side can be neglected and one can write

$$\bar{U}\frac{\partial \bar{C}}{\partial x} + \bar{W}\frac{\partial \bar{C}}{\partial z} = \frac{\partial\left(\bar{\epsilon}_x\frac{\partial \bar{C}}{\partial x} - \overline{u''c''}\right)}{\partial x} + \frac{\partial\left(\bar{\epsilon}_z\frac{\partial \bar{C}}{\partial z} - \overline{w''c''}\right)}{\partial z} \qquad (3\text{-}2)$$

The terms $\overline{u''c''}$ and $\overline{w''c''}$ represent transport arising out of the deviations of velocities and concentration from the depth average values and are usually called dispersion terms. The diffusion and dispersion terms are generally grouped together and described by the dispersion coefficients e_x and e_z, ie.

$$\left(\bar{\epsilon}_x\frac{\partial \bar{C}}{\partial x} - \overline{u''c''}\right) = \bar{e}_x\frac{\partial \bar{C}}{\partial x} \qquad (3\text{-}3)$$

$$\left(\epsilon_z\frac{\partial \bar{C}}{\partial z} - \overline{w''c''}\right) = \bar{e}_z\frac{\partial \bar{C}}{\partial z} \qquad (3\text{-}4)$$

Equation (3-2) then becomes

$$\bar{U}\frac{\partial \bar{C}}{\partial x} + \bar{W}\frac{\partial \bar{C}}{\partial z} = \frac{\partial}{\partial x}\bar{e}_x\frac{\partial \bar{C}}{\partial x} + \frac{\partial}{\partial z}\bar{e}_z\frac{\partial \bar{C}}{\partial z} \qquad (3\text{-}5)$$

In a straight open channel, where there are no transverse velocities and when the concentration is varying slowly with x (3-5) simplifies to

$$\bar{U}\frac{\partial \bar{C}}{\partial x} = \frac{\partial}{\partial z}\bar{\epsilon}_z\frac{\partial \bar{C}}{\partial z} \qquad (3\text{-}6)$$

There is no theory which allows $\overline{\epsilon_z}$ to be calculated and experimental determination has been used. Multiplying both sides of (3-6) by z^2 and integrating across the channel from y = -B to y = +B

$$\bar{U}\frac{\partial}{\partial x}\int_{-B}^{B} z^2\bar{C}\,dz = \int_{-B}^{B} z^2\frac{\partial}{\partial z}\bar{\epsilon}_z\frac{\partial\bar{C}}{\partial z}\,dz = -2\,\bar{\epsilon}_z\left[z\bar{C}\Big|_{-B}^{B} - \int_{-B}^{B}\bar{C}\,dz\right]$$

$$\therefore\ \bar{\epsilon}_z = \frac{\bar{U}}{2}\,\frac{\frac{\partial}{\partial x}\int_{-B}^{B} z^2\bar{C}\,dz}{\int_{-B}^{B}\bar{C}\,dz - \left[z\bar{C}\right]_{-B}^{B}} \tag{3-7}$$

In the region where the plume has not reached the sides of the channel, c = 0 at z - B and z = -B, then

$$\bar{\epsilon}_z = \frac{\bar{U}}{2}\frac{d}{dx}\frac{\int_{-B}^{B} z^2\bar{C}\,dz}{\int_{-B}^{B}\bar{C}\,dz} = \frac{\bar{U}}{2}\frac{d\sigma_z^2}{dx} \tag{3-8}$$

Equation (3-8) is generally used to determine $\bar{\varepsilon}_z$ from experimental measurements of concentration profiles.

$\bar{\varepsilon}_z$ is usually assumed to be proportional to U_*h or U_*R where R is the hydraulic radius, ie.

$$\bar{\epsilon}_z = \alpha\,U_*R \tag{3-9}$$

Experimentally determined values for α have varied between 0.1 and 0.25. Much higher values have been found for some rivers and curved channels. It can be shown from dimensional analysis that

$$\frac{\bar{\epsilon}_z}{U_*R} = f\left(\frac{U_*}{U}, \frac{W}{R}\right) \tag{3-10}$$

Therefore, the value of α can be expected to vary with the friction factor as well as the width to depth ratio of the channel.

Prych (1970) measured the spreading of salt solutions and showed that with the presence of a transverse density gradient, a secondary current was set up which increased the initial transverse mixing and hence $\bar{\varepsilon}_z$. However, after the initial stages, the value of ε_z returned to that of a flow without density gradient.

For curved or meandering channels transverse velocities exist and the spreading is affected by the advection by transverse velocity as well as by the differential convection $\overline{w''c''}$. Equation (3-5) has to be used, in some instances with a slight modification to allow for the variation of depth across the channel.

4. Longitudinal Diffusion

The turbulent transport by longitudinal diffusion $\overline{\epsilon_x \frac{\partial C}{\partial x}}$ and that by differential convection $\overline{u''c''}$ are additive. However, $\overline{u''c''}$ is usually much larger, making it very difficult to measure $\overline{\epsilon}_x$. It is possible to measure ϵ_x at the surface of a flow by measuring the distribution of floating particles.

One method used is to measure the longitudinal coordinate x_i of particles at a time t after their release. Then the mean distance $\overline{x}(t)$ is

$$\overline{x}(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$$

where N is the number of particles released.

The variance $\sigma_x^2(t)$ is

$$\sigma_x^2(t) = \frac{1}{N-1}\sum_{i=1}^{N} (x_i - \overline{x})^2$$

and

$$\epsilon_x = \frac{1}{2}\frac{d\sigma_x^2(t)}{dt} \tag{4-1}$$

Another method is to measure the time $t_i(x)$ taken by the particles to reach a station x and calculate the mean time and variance of the time distribution.

$$\overline{t}(x) = \frac{1}{N}\sum_{i=1}^{N} t_i(x)$$

$$\sigma_t^2(x) = \frac{1}{N-1}\sum_{i=1}^{N} (t_i - \overline{t})^2$$

Then

$$\epsilon_x = \frac{U^3}{2}\frac{d\sigma_t^2(x)}{dx} \tag{4-2}$$

Measurements by Sayre and Chang (1968) and Engelund (1969) gave values of about $0.5U_*h$ for ϵ_x at the surface.

5. Longitudinal Dispersion

When a slug of tracer has spread over the entire depth of a 2-dimensional channel or the entire cross-section of a natural stream, the governing mass transport equation can be simplified if depth averaged or cross-section averaged values of concentration is required. By averaging the equation, a transport term appears owing to the spatial variation of concentration and velocity. This differential convective transport term which is responsible for an apparent diffusion in the longitudinal direction, is termed longitudinal dispersion. It is often many times larger than the longitudinal diffusion term. The longitudinal dispersion coefficient usually have to be determined experimentally, although for 2-dimensional flows it is possible to derive an analytical expression for it.

5.1 Two-dimensional flow

For a 2-dimensional channel with no concentration or velocity gradient in the lateral direction, equation (1-4) can be rewritten with $C = \bar{C} + c''$, $U = \bar{U} + u''$ and also neglecting the molecular diffusion terms. The overbar indicates depth average and " indicates deviation from the depth average.

$$\frac{\partial(\bar{C}+c'')}{\partial t} + (\bar{U}+u'')\frac{\partial(\bar{C}+c'')}{\partial x} = \frac{\partial}{\partial x}\bar{\epsilon}_x\frac{\partial(\bar{C}+c'')}{\partial x} + \frac{\partial}{\partial y}\epsilon_y\frac{\partial(\bar{C}+c'')}{\partial y} \tag{5.1-1}$$

Averaging the above equation over the depth results in

$$\frac{\partial\bar{C}}{\partial t} + \bar{U}\frac{\partial\bar{C}}{\partial x} + \frac{\partial\overline{u''c''}}{\partial x} = \frac{\partial}{\partial x}\bar{\epsilon}_x\frac{\partial\bar{C}}{\partial x} \tag{5.1-2}$$

$\overline{u''c''}$ is a transport term in the x-direction arising out of the correlation between the spatial deviations. Under certain conditions, one can write

$$\left(\bar{\epsilon}_x\frac{\partial\bar{C}}{\partial x} - \overline{u''c''}\right) = D_L\frac{\partial\bar{C}}{\partial x} \tag{5.1-3}$$

D_L is termed the longitudinal dispersion coefficient. Hence the longitudinal dispersion term includes the differential convective transport $\overline{u''c''}$ in addition to the turbulent diffusion term $\epsilon_x\frac{\partial C}{\partial x}$. Usually $\overline{u''c''}$ is the much larger of the two terms. The equation of mass transport is now

$$\frac{\partial\bar{C}}{\partial t} + \bar{U}\frac{\partial\bar{C}}{\partial x} = D_L\frac{\partial^2\bar{C}}{\partial x^2} \tag{5.1-4}$$

The assumption of a Fickian type diffusion relationship for $\overline{u''c''}$ is not always valid. Fischer (1966) showed that it can be applied only when $c'' \ll \overline{C}$ and the rate of change of C is slow.

For 2-dimensional flow, an analytical expression for D_L can be derived based on the velocity distribution and the value of ε_y. From equation (5.1-1), with longitudinal turbulent diffusion neglected and since $\frac{\partial \overline{C}}{\partial y} = 0$ by definition, one gets

$$\frac{\partial(\overline{C}+c'')}{\partial t} + (\overline{U}+u'')\frac{\partial(\overline{C}+c'')}{\partial x} = \frac{\partial}{\partial y}\varepsilon_y\frac{\partial c''}{\partial y}$$

Putting $\xi = x - \overline{U}t$ (co-ordinate moving with mean velocity) one gets

$$\frac{\partial(\overline{C}+c'')}{\partial t} + u''\frac{\partial(\overline{C}+c'')}{\partial \xi} = \frac{\partial}{\partial y}\varepsilon_y\frac{\partial c''}{\partial y} \qquad (5.1\text{-}5)$$

The assumption that $c'' \ll \overline{C}$ and $\frac{\partial c''}{\partial t} \ll \frac{\partial \overline{C}}{\partial t}$ and that $\overline{C}$ is varying slowly with time allows the equation to be simplified to

$$u''\frac{\partial \overline{C}}{\partial \xi} = \frac{\partial}{\partial y}\varepsilon_y\frac{\partial c''}{\partial y}$$

$$\therefore \varepsilon_y\frac{\partial c''}{\partial y} = \frac{\partial \overline{C}}{\partial \xi}\int_0^y u''dy$$

$$c'' = \frac{\partial \overline{C}}{\partial \xi}\int_0^y \frac{dy}{\varepsilon_y}\int_0^y u''dy$$

$$\therefore u''c'' = \frac{\partial \overline{C}}{\partial \xi}\left(u''\int_0^y \frac{dy}{\varepsilon_y}\int_0^y u''dy\right)$$

One can see now how the assumptions used allow one to write $-\overline{u''c''} = \frac{\partial \overline{C}}{\partial \xi}D_L$

$$\therefore D_L = \frac{-\overline{u''c''}}{\frac{\partial \overline{C}}{\partial \xi}} = -\frac{1}{h}\int_0^h u''dy\int_0^y \frac{dy}{\varepsilon_y}\int_0^y u''dy \qquad (5.1\text{-}6)$$

This is the equation given by Elder (1959) for the dispersion coefficient in a 2-dimensional open channel. For a log velocity distribution, linear shear stress, and assuming $\varepsilon_y = \varepsilon_{ym}$ Elder evaluated $D_L = 5.86\ hU_*$.

For axisymmetric flow in a long straight pipe Taylor (1954) got $D_L = 10.1\ aU_*$ where a is the pipe diameter.

5.2 Longitudinal Dispersion in Natural Streams

The dispersion term in a 2-dimensional channel is caused by variations in velocity with depth. In natural streams where width to depth ratios may be 100 times or more there may be much higher concentration differences between the centre of the stream and the banks than between the surface and the bottom. The tracer would usually have been mixed throughout the depth before it spreads across the whole width of a river. Hence in a natural stream it is the lateral variation in velocity which is important to the dispersion process and values for D_L much greater than Elder's prediction of 5.9 U_*d are found for rivers.

The governing equation for this case can be derived as in the 2-dimensional case but with the lateral variations included. Again, starting from equation (1-4), with v = w = 0 and neglecting molecular diffusion, one can write the variables as the sum of the cross-sectional mean plus deviations from the mean. Then the equation can be averaged over the cross-section. The same assumptions have to be used in order to be able to write

$$-\overline{u''c''} = D_L \frac{\partial \overline{C}}{\partial x}$$

The overbar in this case denotes cross-sectional average and the double primed quantities represent deviations from the average. The governing equation becomes identical to (5.1-4)

$$\frac{\partial \overline{C}}{\partial t} + \overline{U}\frac{\partial \overline{C}}{\partial x} = D_L \frac{\partial^2 \overline{C}}{\partial x^2} \qquad (5.2\text{-}1)$$

An analytical expression can also be derived for D_L. The equivalent of equation (5.1-5) for this case is

$$\frac{\partial(\overline{C}+c'')}{\partial t} + u''\frac{\partial(\overline{C}+c'')}{\partial \xi} = \frac{\partial}{\partial y}\epsilon_y\frac{\partial c''}{\partial y} + \frac{\partial}{\partial z}\epsilon_z\frac{\partial c''}{\partial z} \qquad (5.2\text{-}2)$$

Applying the same assumptions used previously, and also that $\frac{\partial c''}{\partial y}$ is much less than the lateral variation $\frac{\partial c''}{\partial z}$ one gets

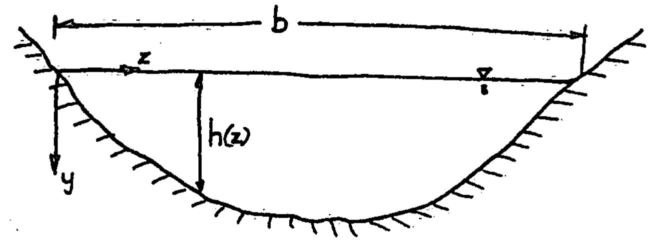
$$u''\frac{\partial \overline{C}}{\partial \xi} = \frac{\partial}{\partial z}\epsilon_z\frac{\partial c''}{\partial z}$$

Integrating over the depth

$$\frac{\partial \bar{C}}{\partial \xi}\underbrace{\int_0^{h(z)} u''\,dy}_{q'(z)} = \int_0^{h(z)} \frac{\partial}{\partial z}\epsilon_z\frac{\partial c''}{\partial z}\,dy$$

Assuming that ε_z and $\frac{\partial c''}{\partial z}$ do not vary with depth, one gets

$$\frac{\partial \bar{C}}{\partial \xi}\,q'(z) = \frac{\partial}{\partial z}\epsilon_z\frac{\partial c''}{\partial z}\,h(z)$$



$$\therefore \frac{\partial \bar{C}}{\partial \xi}\int_0^{z} q'(z)\,dz = h(z)\,\epsilon_z\frac{\partial c''}{\partial z}$$

$$\therefore c'' = \frac{\partial \bar{C}}{\partial \xi}\int_0^{z}\frac{dz}{\epsilon_z h(z)}\int_0^{z} q'(z)\,dz$$

$$\therefore D_L = \frac{-\overline{u''c''}}{\frac{\partial \bar{C}}{\partial \xi}} = -\frac{1}{A}\int_0^{b}\int_0^{h(z)} u''\,dy\,dz\int_0^{z}\frac{dz}{\epsilon_z h(z)}\int_0^{z} q'(z)\,dz$$

$$= -\frac{1}{A}\int_0^{b} q'(z)\,dz\int_0^{z}\frac{dz}{\epsilon_z h(z)}\int_0^{z} q'(z)\,dz \tag{5.2-3}$$

Experiments showed that the initial convective period in which the one-dimensional equation (5.2-1) does not apply was approximately 0.4 times the Eulerian time scale $t_E = \frac{L^2}{\varepsilon_z}$ when L is transverse distance which the tracer has to spread, say the $\frac{1}{2}$ width of a river. Using $\varepsilon_z = 0.23RU_*$,

$$t = 1.8\frac{L^2}{RU_*} \tag{5.2-4}$$

or

$$l = \frac{1.8\,L^2\,\bar{U}}{RU_*} \tag{5.2-5}$$

where ℓ is the downstream distance from injection where the one-dimensional equation starts to apply.

The distance criteria makes it impractical to use the one-dimensional equation for large rivers.

It is interesting to note that for natural streams where transverse gradients dominate the time scale for mixing is proportional to the square of the width and inversely proportional to the depth whereas in 2-dimensional flows, the time scale for mixing is proportional to the depth.

5.3 Experimental Determination of D_L.

The dispersion coefficient can be calculated from the theoretical expressions given above if detailed velocity and depth transverses are made. More often, it is measured from tracer experiments, either by a change of moment method or by a routing procedure relying on the solution of the diffusion equation.

The change of moment method can be derived from equation (5.2-1) which, in a coordinate ξ moving with the mean velocity $\overline{U}$, is

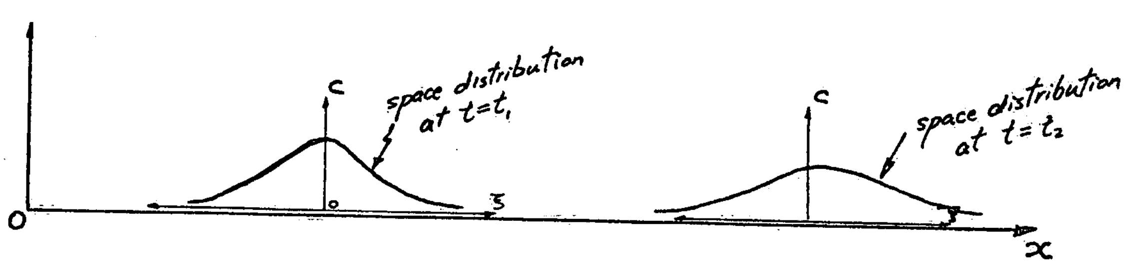
$$\frac{\partial \overline{C}}{\partial t} = D_L \frac{\partial^2 \overline{C}}{\partial \xi^2}$$

$$\therefore \int_{-\infty}^{\infty} \xi^2 \frac{\partial \overline{C}}{\partial t} d\xi = D_L \int_{-\infty}^{\infty} \frac{\partial^2 \overline{C}}{\partial \xi^2} \xi^2 d\xi$$

and
$$D_L = \frac{1}{2} \frac{d\sigma_\xi^2}{dt} \qquad (5.3\text{-}1)$$

where
$$\sigma_\xi^2 = \frac{\int_{-\infty}^{\infty} \xi^2 \overline{C} d\xi}{\int_{-\infty}^{\infty} \overline{C} d\xi} \qquad (5.3\text{-}2)$$

σ_ξ^2 is the variance of the space distribution of concentration about an axis moving with the mean flow velocity.



However, the concentration is usually measured at fixed points at varying time intervals. Fischer showed that the variance of the time concentration is related to σ_ξ^2 by

$$\Delta\sigma_\xi^2 = \bar{U}^2 \Delta\sigma_t^2$$

$$\therefore \quad D_L = \frac{\bar{U}^2}{2} \frac{\sigma_{t_2}^2 - \sigma_{t_1}^2}{\bar{t}_2 - \bar{t}_1} \qquad (5.3\text{-}3)$$

where $\bar{t}_2$ and $\bar{t}_1$ are the mean times of passage of a cloud at stations 1 and 2 respectively. σ_t^2 is the variance of the concentration-time curve at the particular location.

One can also use the following equation if measurement at sufficient stations are taken.

$$D_L = \frac{\bar{U}^3}{2} \frac{d\sigma_t^2}{dx} \qquad (5.3\text{-}4)$$

This derivation applies regardless of the initial concentration distribution and can be used to estimate D_L as long as the stations are within the region where the one-dimensional equation applies.

The routing procedure introduced by Fischer follows from the solution of the one-dimensional equation.

$$\frac{\partial\bar{C}}{\partial t} + \bar{U}\frac{\partial\bar{C}}{\partial x} = D_L \frac{\partial^2\bar{C}}{\partial x^2}$$

With boundary conditions of a delta function type point source at x = 0, t = 0, the solution is

$$\bar{C}(x,t) = \frac{M}{\sqrt{4\pi D_L t}}\, e^{-\frac{(x-\bar{U}t)^2}{4D_L t}} \qquad (5.3\text{-}5)$$

This solution is sometimes matched with field data to obtain D_L. However this is incorrect since this solution does not recognize an initial period in which the equation does not apply. The correct procedure would be to use the concentration distribution existing at the beginning of the diffusive period and solve the equation with that particular distribution as initial condition. From that solution, a concentration distribution for another point downstream can be predicted using an assumed value of D_L. The value of D_L which gives the best fit against the data is then the best value of D_L for that reach.

The governing equation in a co-ordinate moving with the mean flow velocity is

$$\frac{\partial \bar{C}}{\partial t} = D_L \frac{\partial^2 \bar{C}}{\partial \xi^2}$$

Given at time $t = t_o$ a certain spacial concentration distribution $C_o(\xi, t_o)$, the equation can first be transformed using $t_1 = t - t_o$ and then by applying complex Fourier Transform the solution is

$$\bar{C}(\xi, t) = \frac{1}{\sqrt{4\pi D_L (t-t_o)}} \int_{-\infty}^{\infty} C_o(\xi', t_o)\, e^{-\frac{(\xi-\xi')^2}{4D_L(t-t_o)}}\, d\xi' \qquad (5.3\text{-}6)$$

This solution is in terms of space-concentration distribution. To use it for time concentration data, one has to assume that the time concentration distribution measured at x_o is the same as the space-concentration distribution which would have been measured at time t_o where t_o is the time of travel to station x_o, i.e..

$$\bar{C}(\xi, t_o) = \bar{C}(x_o, t)$$

Applying this principle, one can write the time concentration distribution for a point at x_1

$$\bar{C}(x_1, t) = \int_{-\infty}^{\infty} \bar{C}(x_o, \tau)\, \frac{\exp\left\{\frac{-[\bar{U}(t_1 - t_o - t + \tau)]^2}{4D_L(t_1 - t_o)}\right\}}{\sqrt{4\pi D_L (t_1 - t_o)}}\, d\tau \qquad (5.3\text{-}7)$$

Therefore using an upstream time concentration curve, the concentration curve at another point downstream can be calculated from (5.3-7) using some value of D_L. If this does not fit the experimental data, D_L should be adjusted. The value of D_L which enables a good fit is then theoretically the best estimate.

Values of D_L of the order of 230 ft^2/sec. are common. This contrast with values of around 1 ft^2/sec. for the turbulent diffusivities.

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