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THE SYNTHESIS OF NON FLOW REDUNDANT COMMUNICATION NETWORKS

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ABSTRACT

A very important problem in communications is that of the synthesis of a net providing the required channel capacities between the various nodes. More specifically, given the configuration of the system and the expected maximum flow requirements between any two terminals, the synthesis problem deals with determining all the nets providing the given channel (flow) requirements.

Although a general answer to this problem has not yet been given, several related problems have been considered and a satisfactory answer has been given to most of them.[2,3,5,6,7]

In this work, the problem of the synthesis of a net providing the required flows among the various nodes, without capacity redundancies on the channels is considered; the conditions under which such a net is realizable are given and, taking into account the presence of constraints on the channels, a synthesis procedure is suggested and implemented by a computer program, handling the overall problem.

SUMMARY

The present paper is divided into three sections,

In section 1, the first two parts, certain elementary mathematical concepts from set and function theory are presented, while, in the last part, using these concepts, the graph is defined along with certain related concepts, which are further illustrated and discussed in two examples.

In section 2, in the first part, the concepts of a net and a communications net are defined; the weighting and terminal capacity functions are also defined and, using the weighting function of a net, simple operations on nets and semicuts are discussed. In the second part, certain results from the analysis problem are presented.

In section 3, the general synthesis problem is first discussed and the particular synthesis problem being considered is presented. Next, a theorem on the synthesis of a (general) net is given, the presence of constraints on the arcs and their effect on the synthesis procedure is discussed and certain examples, illustrating the synthesis procedure, are given. In the second part, the solution of the synthesis problem for a general net is applied to the case of a communications net.

A short description of the computer program written for the realization of the solution of the synthesis problem, along with the program itself, can be found in the appendix.

It should be noted here, that the theorems in sections 2.2.2., 3.1.1., 3.2.1. and 3.2.2., have been stated slightly differently, by Resh[6].

NOTATION

$\mathbb{R}$	the set of real numbers
$\in$	"belongs to"
$\notin$	"does not belong to"
Z^c	if Z is a subset of some set V, then Z^c will be used to denote all elements of V which do not belong to Z; it is called the "complement" of Z (with respect to V).
X-Y	the set of the elements of X which do not belong to Y.
card X	the number of elements in X (X is assumed finite).

Note: Further details on the elementary set operations used in proving certain theorems, can be found in [4].

Section 1. MATHEMATICAL PRELIMINARIES

In order to make this paper as self-contained as possible, a number of elementary preliminaries are presented in this section.

1.1. Set theory

1.1.1. If X and Y are two sets, then the set

$$\{(x,y)/x\epsilon X,\ y\epsilon Y\}$$

is called the "product" of X and Y and it is denoted by XxY.

1.1.2. Any subset of the set XxY, is called a "relation" from X into Y; if, X = Y = V, then any relation from V into V is called a relation "on" V. In particular, the relation Δ_V, defined as

$$\Delta_V = \{(i,i)/i\epsilon V\}$$

is called the "identity" or "diagonal" relation on V.

1.2 Function theory

1.2.1. A function h, defined on some set X and taking values in a set Y, is denoted by

$$h : X \longrightarrow Y \qquad \text{or} \qquad X \xrightarrow{h} Y$$

1.2.2. Given the function h and a set $D\subset X$, the function

$$h^* : D \longrightarrow Y \text{ such that } h^*(d) = h(d), \text{ for every } d\epsilon D$$

is called the "restriction" of h to D and it is denoted by h/D

1.3. Graph theory

1.3.1. Any pair (V,A) of sets such that V is any set, while, A is a relation on V, is called an "oriented graph" and it is denoted by $\mathcal{G}$.

If V is finite, then $\mathcal{G}$ is called a "finite graph".

If A=VxV, then $\mathcal{G}$ is called a "complete graph".

If $A=V\times V-\Delta_V$, then $\mathcal{G}$ is called a "quasicomplete graph".

If Z is a non-empty proper subset of V, then the set $S_Z = (Z\times Z^c)\cap A$ is called a "semicut" of $\mathcal{G}$, while the corresponding to S_Z set $S_Z \cup S_{Z^c} = [(Z\times Z^c)\cup(Z^c\times Z)]\cap A$ is called a "cut" of $\mathcal{G}$. Observe, that the cuts corresponding to the semicuts S_Z and S_{Z^c} coincide and, therefore, in the case of a finite graph, the number of semicuts is twice the number of cuts in the graph.

<u>1.3.2.</u> Any finite graph $\mathcal{G} = (V,A)$ can be represented geometrically as follows.

The elements of V are represented by points called "vertices" of the graph while the membership relation $(i,j)\in A$ is denoted by joining the vertex i to the vertex j with a line segment bearing an arrowhead which points to the vertex j. Such a (directed) line segment is called a (directed) "arc" of $\mathcal{G}$, if $i\neq j$, and a "loop" of $\mathcal{G}$, if i=j.

Referring to the geometric representation of $\mathcal{G}$, we observe that a semicut S_Z of the graph, consists of all those arcs "emanating" from the set of vertices Z; removal from the graph of the arcs in S_Z, would mean "no connection" <u>from</u> Z to Z^c. Since connections from Z^c to Z might yet exist, the set S_Z was called a semicut as opposed to the set $S_Z \cup S_{Z^c}$ which was called a cut; observe that removal from the graph of the arcs in the cut $S_Z \cup S_{Z^c}$ "separates" the graph into two graphs.

Moreover, if the number of vertices in V is n, then the number of the distinct semicuts in $\mathcal{G}$ is exactly the number of distinct proper non-empty subsets of V, that is,

$$\binom{n}{1} + \ldots + \binom{n}{n-1} = 2^n - 2 = 2(2^{n-1} - 1)$$

and, therefore, the number of distinct cuts in $\mathcal{G}$, is $2^{n-1} - 1$.

1.3.3. The notion of a quasicomplete graph will be of particular interest in this work; hence, the following remarks and definitions.

If $\mathcal{G} = (V,A)$ is a quasicomplete graph, then for every non-empty proper subset Z of V, we have $Z \times Z^c \subset A = V \times V$ and, therefore,

$$S_Z = (Z \times Z^c) \cap A = Z \times Z^c \text{ and } S_Z \cup S_{Z^c} = (Z \times Z^c) \cup (Z^c \times Z)$$

On the other hand, each set of vertices $i_1, \ldots, i_p$, with $i_1 \neq i_p$, in a quasicomplete graph, determines a set of arcs

$$P = \{(i_\lambda, i_{\lambda+1}) \;/\; 1 \leq \lambda \leq p-1\}$$

called a "path" from i_1 to i_p.

To each such path P there corresponds the set of arcs

$$C_P = P \cup \{(i_p, i_1)\}$$

called the "circuit" corresponding to the path P.

Observe that the set

$$P^- = \{(i,j) \in A / (j,i) \in P\} = \{(i_{\lambda+1}, i_\lambda) \;/\; 1 \leq \lambda \leq p-1\}$$

is the ("return") path, from i_p to i_1, corresponding to the path P, while

$$C_{P^-} = P^- \cup \{(i_1, i_p)\} = \{(i,j) \in A / (j,i) \in C_P\}$$

is the circuit corresponding to P^- and that, clearly, $P \cap P^- = \emptyset$ and $C_P \cap C_{P^-} = \emptyset$.

Finally, if the quasicomplete graph $\mathcal{G}$ is also finite with n vertices, it is easy to see that the number of paths from any vertex i to any vertex j, such that $i \neq j$, is

$$\binom{n-2}{0}\,0! + \binom{n-2}{1}1! + \ldots + \binom{n-2}{n-3}(n-3)! + \binom{n-2}{n-2}(n-2)!$$

1.3.4. The following examples illustrate some of the above graph-theoretic concepts.

Example 1 The graph

$$\mathcal{G} = (V,A) = (\{1,2,3,4,5\}, \{(1,2), (1,4), (1,5), (2,1), (2,3), (2,4), (3,4), (4,5)\})$$

is a finite oriented graph, with the following representation

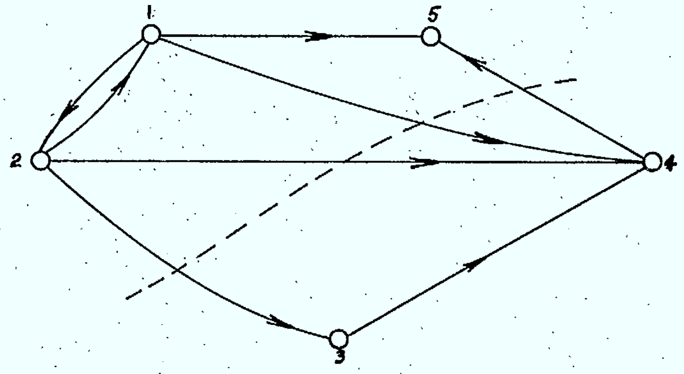

Figure 1

The semicut S_Z corresponding to the non-empty proper subset $Z = \{1,2,5\}$ of V, can be found as follows

$$Z x Z^c = \{1,2,5\} \times \{3,4\} = \{(1,3), (1,4), (2,3), (2,4), (5,3), (5,4)\}$$

and $S_Z = (ZxZ^c) \cap A = \{(1,4), (2,3), (2,4)\}$

Observe that by removing from $\mathcal{G}$ the arcs in S_Z, there still remains the arc (4,5) "connecting" the set Z^c of vertices to the set Z.

Similarly,

$$S_{Z^c} = (Z^c \times Z) \cap A = \{(4,5)\}$$

Observe now that by removing from $\mathcal{G}$ the arcs in S_{Z^c}, there still remain the arcs in S_Z, "connecting" Z to Z^c, while, by removing the arcs in the cut

$$S_Z \cup S_{Z^c} = \{(1,4), (2,3), (2,4), (4,5)\}$$

there is no arc "connecting" neither Z to Z^c nor Z^c to Z and $\mathcal{G}$ is then "separated" into the following two graphs.

$$\mathcal{G}_1 = (\{1,2,5\}, \{(1,2), (1,5), (2,1)\} , \mathcal{G}_2 = (\{3,4\}, \{(3,4)\})$$

Example 2 The finite quasicomplete oriented graph

$$\mathcal{G} = (\{1,2,3,4\}, \{1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4), (4,1), (4,2), (4,3)\})$$

has the following representation

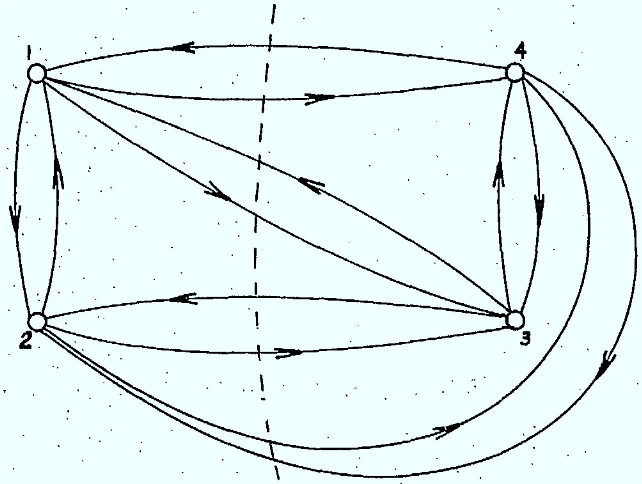

Figure 2

Considering the subset $Z = \{1,2\}$ of V, then

$$S_Z = ZxZ^c = \{1,2\} \times \{3,4\} = \{(1,3), (1,4), (2,3), (2,4)\}$$

$$S_{Z^c} = Z^c xZ = \{3,4\} \times \{1,2\} = \{(3,1), (3,2), (4,1), (4,2)\}$$

$$S_Z \cup S_{Z^c} = \{(1,3), (1,4), (2,3), (2,4), (3,1), (3,2), (4,1), (4,2)\}$$

Again, removal of the arcs in $S_Z \cup S_{Z^c}$, would separate $\mathcal{G}$ into two graphs.

Section 2. NETS AND COMMUNICATIONS NETS

The concepts of a net and a communications net are defined in this section, while, certain results from the analysis problem, which are to be used in the study of the synthesis problem, are also presented.

2.1. Nets and related concepts

2.1.1. Any pair $(\mathcal{G},w)$ such that, $\mathcal{G} = (V,A)$ is a finite, quasicomplete, oriented graph (with n vertices) and w is a real-valued function on A, is called a "net".

The function w is usually referred to as the "weighting function" of the net, while, for each $(i,j) \in A$, $w((i,j))$ is referred to as the "weight" of the arc (i,j).

The requirement, in the above definition, that $\mathcal{G}$ be quasicomplete, is by no means restrictive; that is, if the graph representing the actual net is not quasicomplete, it is a simple matter to adjoin to it the missing arcs, each with zero weight, so as to make it quasicomplete. In other words, it is always possible to represent any net by a quasicomplete graph.

2.1.2. If S_Z is a semicut of $\mathcal{G}$, in the net $(\mathcal{G},w)$, then the real number

$$|S_Z|_w = \Sigma\{w((i,j)) \;/\; (i,j) \in S_Z\}$$

is called the "value" of the semicut S_Z with respect to the weighting function w, and whenever no confusion should arise, $|S_Z|$ will be written instead of $|S_Z|_w$.

2.1.3. The "sum" and the "difference" of two nets $(\mathcal{G}, w_1)$ and $(\mathcal{G}, w_2)$ is defined as

$$(\mathcal{G}, w_1) \pm (\mathcal{G}, w_2) = (\mathcal{G}, w_1 \pm w_2)$$

and, therefore, for any semicut S_Z in $\mathcal{G}$,

$$|S_Z|_{w_1 \pm w_2} = |S_Z|_{w_1} \pm |S_Z|_{w_2}$$

2.1.4. For a net $(\mathcal{G}, w)$, the function

$$t : A \longrightarrow \mathcal{R} \text{ such that } t((i,j)) = \min \{ |S_Z| / (i,j) \in S_Z \}$$

is called the "terminal capacity function" of the net.

Two nets $(\mathcal{G}, w_1)$ and $(\mathcal{G}, w_2)$ are called "equivalent" if they have the same terminal capacity function.

To see that the function t is well defined, it is only necessary to observe that there exists at least one semicut containing (i,j), namely, the semicut $S_{\{i\}}$, and, since $\mathcal{G}$ is finite, so is the number of distinct semicuts and the value of each semicut is finite.

The above definition of the function t was motivated by the fact, that the maximum flow which can be transferred from i to j in a given net, called the "terminal capacity from i to j", was shown ([1], page 11) to be equal to the minimum among the values of the semicuts containing (i,j).

It should be noted here, that what is actually given in the synthesis problem is the expected maximum flow-requirement from any vertex i to any vertex j, that is, the function t is given and it is required to find the arcs and their weights, that is, the function w.

2.1.5. A net $(\mathcal{G},w)$ in which the function w is non-negative, is called a "communications net".

The vertices in a communications net are usually called "terminals" or "nodes" and the arcs, "channels" of the net, while the term "capacity" (of a channel (i,j)) is more often used instead of the term weight (of the arc (i,j)).

2.2. Certain results from the analysis problem

In the analysis problem, the net $(\mathcal{G},w)$ is given and its properties are analyzed. In what follows, after introducing the notation and definitions to be used in the analysis and, later on, in the synthesis problem, certain interesting results from the analysis problem are presented.

2.2.1. The following notation and definitions will be used.

(i) If $t_1, \ldots, t_m$ are, in an increasing order, the distinct values of the terminal capacity function t, then the set of arcs in A on which t has the value t_μ is denoted by A_μ, that is,

$$A_\mu = \{(i,j) \in A/t((i,j)) = t_\mu\} \text{ , } 1 \le \mu \le m.$$

and since the number of arcs in A is n(n-1), obviously $1 \le m \le n(n-1)$.

(ii) The symbol A^*_μ will be used to denote all the arcs in the sets $A_1, \ldots, A_{\mu-1}$, that is

$$A^*_\mu = A \cup \ldots \cup A_{\mu-1} \text{ , } 1 \le \mu \le m+1$$

Clearly, $A^*_1 = \emptyset$ and $A^*_{m+1} = A$.

(iii) Given a circuit C_p in $\mathcal{G}$ and real number α, the net $(\not{\mathcal{G}},b)$ such that

$$b((i,j)) = \begin{cases} \alpha, & \text{if } (i,j) \in C_p \\ -\alpha, & \text{if } (i,j) \in C_{p^-} \\ 0, & \text{otherwise} \end{cases}$$

is called the "bicircuit net" in $\mathcal{G}$, corresponding to the pair (C_p, α).

Moreover, two nets $(\mathcal{G}, w_1)$ and $(\mathcal{G}, w_2)$ are called C_p- equivalent if one of them is the result of the addition to the other of a finite number of bicircuit nets.

2.2.2. Theorem Given a net $(\mathcal{G}, w)$ with terminal capacity function t then for every arc (i,j) in A there exists a semicut S_Z containing (i,j) and such that the restriction t/S_Z, of the function t, attains its maximum at the arc (i,j).

Proof There is always at least one semicut in $\mathcal{G}$, containing the arc (i,j), namely, the semicut $S_{\{i\}} = \{i\} \times \{i\}^c$ and since $\mathcal{G}$ is finite so is the number of distinct semicuts containing (i,j); therefore, for some semicut S_Z among them, according to the definition of the function t, $t((i,j)) = |S_Z|$.

Now, from the definition of t,

$$t((i',j')) \leq |S_Z|, \text{ for every arc } (i',j') \in S_Z$$

Therefore,

$$t((i',j')) \leq |S_Z| = t((i,j)), \text{ for every arc } (i', j') \in S_Z , \quad \text{Q.E.D.}$$

Comments

(i) In this paper, given any net $(\mathcal{G}, w)$, by saying that a semicut S_Z in $\mathcal{G}$ is an " m-restriction" for some arc (i,j) (with respect to the function t) is meant, that the semicut S_Z contains (i,j) and that the restriction t/S_Z, of the function t, attains its maximum at the arc (i,j). Whenever no confusion, concerning the function t, is possible, it will be stated simply that S_Z is an m-restriction for (i,j).

(ii) For any two arcs (i,j) and (i′,j′) in the set A_μ, the following equality is true, namely, $t((i,j)) = t((i', j')) = t_\mu$; if, moreover, this equality implies, that a semicut in $\mathcal{G}$ is an m-restriction for (i,j), if and only if, S_Z is an m-restriction for (i′, j′), then it is called an "essential equality".

(iii) For every semicut S_Z in $\mathcal{G}$, there always exists an arc (i,j) such that S_Z is an m-restriction for (i,j); indeed, since the number of arcs in S_Z is finite, t/S_Z attains its its maximum value at some arc in S_Z.

(iv) The symbol $\mathcal{S}_\mu$ will be used to denote the set of semicuts in $\mathcal{G}$, each of which is an m-restriction for some arc in A_μ.

<u>2.2.3.</u> <u>Theorem</u> Given a net ($\mathcal{G}$,w) whose terminal capacity function t contains only essential equalities, if S_Z is a semicut in $\mathcal{G}$ and $S_Z \in \mathcal{S}_\mu$, then

(i) S_Z is an m-restriction for every arc (i,j) in A_μ and, therefore, $A_\mu \subset S_Z$.

(ii) If $\mu < \mu'$ (which means $t_\mu < t_{\mu'}$), then $S_Z \cap A_{\mu'} = \emptyset$

(iii) $S_Z = A_\mu \cup (S_Z \cap A^*_\mu)$

<u>Proof</u>

(i) S_Z being in $\mathcal{S}_\mu$ implies that there exists an arc (i,j) in A_μ such that S_Z is an m-restriction for (i,j); but, by definition of A_μ, $t((i,j)) = t((i', j'))$ for every arc (i′,j′) in A_μ and since, by assumption each such equality is an essential equality, obviously S_Z is an m-restriction for every arc (i′,j′) in A_μ.

(ii) S_Z being in $\mathcal{S}_\mu$, implies that S_Z is an m-restriction for some arc in A_μ and, therefore, the maximum value of t on S_Z is t_μ. Now, if $S_Z \cap A_{\mu'} \neq \emptyset$, this implies there is some arc (i,j) in $S_Z \cap A_{\mu'}$ and also that $t((i,j)) = t_{\mu'}$ because $(i,j) \in A_{\mu'}$. However, the maximum value of t in S_Z is t_μ and, since $(i,j) \in S_Z$,

$$t((i,j)) \leq t_\mu, \text{ while } t((i,j)) = t_{\mu'}$$

which is a contradiction and, therefore, $S_Z \cap A_{\mu'} = \emptyset$

(iii) By (i) and (ii),

$$A_\mu = S_Z \cap A_\mu \text{ and } (S_Z \cap A_{\mu+1}) \cup \ldots \cup (S_Z \cap A_m) = \emptyset$$

and, using these relations and the distributive law for set operations,

$$\begin{aligned} A_\mu \cup (S_Z \cap A^*_\mu) &= (S_Z \cap A_\mu) \cup (S_Z \cap A^*_\mu) = \\ &= S_Z \cap (A^*_\mu \cup A_\mu) = \\ &= S_Z \cap (A_1 \cup \ldots \cup A_\mu) = \\ &= (S_Z \cap A_1) \cup \ldots \cup (S_Z \cap A_\mu) = \\ &= [(S_Z \cap A_1) \cup \ldots \cup (S_Z \cap A_\mu)] \cup [(S_Z \cap A_{\mu+1}) \cup \ldots \cup (S_Z \cap A_m)] = \\ &= (S_Z \cap A_1) \cup \ldots \cup (S_Z \cap A_m) = \\ &= S_Z \cap (A_1 \cup \ldots \cup A_m) = \\ &= S_Z \cap A = \\ &= S_Z \end{aligned}$$

<u>2.2.4.</u> <u>Theorem</u> Given a net $(\mathcal{G}, w)$ and a function $t : A \longrightarrow \mathcal{R}$, then t is a terminal capacity function for $(\mathcal{G}, w)$, if and only if, for every $(i,j) \in A$,

$$t((i,j)) = \min \{|S_Z| / S_Z \text{ is an m-restriction for } (i,j)\} \quad (1)$$

Proof Assume first that (1) is true; then to establish that t is the terminal capacity function for $(\mathcal{G}, w)$, it is enough to show that if S^*_Z is any semicut containing (i,j) but without being m-restriction for (i,j), then

$$\min \{|S_Z| / S_Z \text{ is an m-restriction for } (i,j)\} \le |S^*_Z| \quad (2)$$

This follows immediately, because if S^*_Z is not an m-restriction for (i,j), there is some $(i',j') \in S^*_Z$ such that $(i',j') \neq (i,j)$ and for which $t((i',j'))$ is the maximum value of t on S^*_Z; therefore,

$$t((i,j)) < t((i',j')) \quad (3)$$

However, by assumption,

$$t((i',j')) = \min \{|S_Z| / S_Z \text{ is an m-restriction for } (i',j')\} \quad (4)$$

and, since S^*_Z is an m-restriction for (i′,j′),

$$t((i',j')) \le |S^*_Z| \quad (5)$$

Thus, (3) and (5) yield (2), which was to be proved.

Assume next that t is the terminal capacity function for $(\mathcal{G}, w)$; in order to show that (1) can serve as a definition of t, it is enough to show that if S^*_Z is any semicut containing (i,j) without being an m-restriction for (i,j) then, $t((i,j)) < |S^*_Z|$.

Indeed, since S^*_Z is not an m-restriction for (i,j), there exists an arc $(i',j') \in S^*_Z$ such that $(i',j') \neq (i,j)$ and for which $t((i',j'))$ is the maximum value of t in S^*_Z. Therefore,

$$t((i,j)) < t((i',j')) = \min \{|S_Z| \; / \; (i',j') \in S_Z\} \leq |S_Z^*|$$

which implies $t((i,j)) < |S_Z^*|$ and that (1) is an equivalent definition for t.

Section 3. A SYNTHESIS PROBLEM

Given a finite quasicomplete oriented graph $\mathcal{G}$ and a function $t : A \longrightarrow \mathcal{R}$, the synthesis problem is,

a). does there exist a function $w : A \longrightarrow \mathcal{R}$ such that the net $(\mathcal{G}, w)$ has as terminal capacity function the given function t?

b). If such function(s) w exist how can they be found?

In communications terminology (a) and (b) can be phrased as follows.

Is it possible, and in which way, to assign capacities to the arcs of the given graph representation so that the maximum flow between any two terminals is the required maximum flow?

It should be noted here, that a solution to this problem, would, at the same time be a synthesis of the cheapest net, meeting the given requirements, under a uniform cost per unit of capacity (because such a net, has no capacity redundancies on its arcs and, therefore, automatically has the minimum total capacity among all possible nets satisfying the given requirements and hence would be the one having the minimum cost).

In what follows, the conditions under which the above synthesis problem has a solution are given and various constraints in synthesizing such a net are considered.

Since the synthesis procedure deals generally with a net, the conditions under which this procedure yields a communications net are presented.

3.1. Synthesizing a net

3.1.1. Theorem Given the finite, quasicomplete, oriented graph

$\mathcal{G} = (V,A)$ and a function $t : A \longrightarrow \mathbb{R}$, containing only essential equalities and such that for every arc (i,j) in A there exists a semicut S_Z in $\mathcal{G}$ which is an m-restriction for (i,j), then a net $(\mathcal{G}, w)$ "realizing" t (that is, having t as its terminal capacity function) is obtained as follows:

Define,

$$w_\mu : A \longrightarrow \mathbb{R} \text{ such that } w_\mu((i,j)) = \begin{cases} x((i,j)) \ , & \text{if } (i,j) \in A_\mu \\ w_{\mu-1}((i,j)) \ , & \text{if } (i,j) \notin A_\mu \end{cases}$$

where, $w_0((i,j)) = 0$, for every $(i,j) \in A$ and the function

$$x : A \ (= A_1 \cup \ldots \cup A_m) \longrightarrow \mathbb{R} \text{ is such that}$$

$$\Sigma\{x((i,j))/(i,j) \in A_\mu\} = t_\mu - \min\{|S_Z|_{w_{\mu-1}} / S_Z \in \mathcal{S}_\mu\}, \text{ for}$$

every $\mu = 1, \ldots, m$. Then, w_m is the required function w and the net is synthesized.

<u>Proof</u> According to theorem in section 2.2.4., it is enough to show that

$$t((i,j)) = \min\{|S_Z|_{w_m} / S_Z \text{ is an m-restriction for } (i,j)\}$$

$$\text{for every } (i,j) \in A \qquad (1)$$

and, since the only values of t on A are the numbers $t_1, \ldots, t_m$, instead of showing (1), it is enough to show that for any $(i,j) \in A$,

$$t_\mu = \min\{|S_Z|_{w_m} / S_Z \text{ is an m-restriction for } (i,j)\},$$

$$\text{for every } \mu = 1, \ldots, m \qquad (2)$$

Now, t_μ is the value of t on the arcs in A_μ and because t contains only essential equalities, $\mathcal{S}_\mu$ is the set of all semicuts which are m-restrictions for (i,j). Then, instead of showing (2) it is only necessary to show that

$$t_\mu = \min \{ |S_Z|_{w_m} / S_Z \in \mathcal{S}_\mu \} \text{ , for every } \mu = 1, \ldots, m \qquad (3)$$

By the definition of the function x it follows:

$$t_\mu = \Sigma \{x((i,j))/(i,j) \in A_\mu\} + \min \{ |S_Z|_{w_{\mu-1}} / S_Z \in \mathcal{S}_\mu \} =$$

$$= \min \{ \Sigma\{x((i,j))/(i,j) \in A_\mu\} + |S_Z|_{w_{\mu-1}} / S_Z \in \mathcal{S}_\mu \} \qquad (4)$$

(because if λ is a constant, then $\lambda + \min \{y_i/i \in I\} = \min \{\lambda + y_i/i \in I\}$

On the other hand, for any semicut $S_Z \in \mathcal{S}_\mu$

$$|S_Z|_{w_m} = \Sigma \{w_m((i,j))/(i,j) \in S_Z\} =$$

$$= \Sigma \{w_m((i,j))/(i,j) \in S_Z \cap A^*_{\mu+1}\}$$

(because $S_Z \subset A = A_1 \cup \ldots \cup A_\mu \cup A_{\mu+1} \cup \ldots \cup A_m$ and, by 2.2.3. (ii), $S_Z \cap A_{\mu+1} = \emptyset$, ... , $S_Z \cap A_m = \emptyset$; therefore, $S_Z \subset A_1 \cup \ldots \cup A_\mu = A^*_{\mu+1}$, which implies $S_Z \cap A^*_{\mu+1} = S_Z$)

$$= \Sigma \{w_m((i,j))/(i,j) \in S_Z \cap A_\mu\} + \Sigma\{w_m((i,j))/(i,j) \in S_Z \cap A^*_\mu\}$$

(because $(S_Z \cap A_\mu) \cup (S_Z \cap A^*_\mu) = S_Z \cap (A_\mu \cup A^*_\mu) = S_Z \cap A^*_{\mu+1}$ and

$$(S_Z \cap A_\mu) \cap (S_Z \cap A^*_\mu) = S_Z \cap A_\mu \cap A^*_\mu = S_Z \cap [A_\mu \cap (A_1 \cup \ldots \cup A_{\mu-1})] =$$

$$= S_Z \cap [(A_\mu \cap A_1) \cup \ldots \cup (A_\mu \cap A_{\mu-1})] = \emptyset)$$

$$= \Sigma \{w_m((i,j))/(i,j) \in A_\mu\} + \Sigma\{w_m((i,j))/(i,j) \in S_Z \cap A^*_\mu\}$$

(because by 2.2.3 (i), $S_Z \cap A_\mu = A_\mu$)

$$= \Sigma \{w_m((i,j))/(i,j) \in A_\mu\} + \Sigma \{w_{\mu-1}((i,j))/(i,j) \in S_Z \cap A^*_\mu\}$$

(because, $(i,j) \in S_Z \cap A^*_\mu$ implies $(i,j) \in A^*_\mu$ and this implies that $(i,j) \notin A_\mu \cup \ldots \cup A_m$, hence, by definition of w_μ, for such an (i,j), $w_m((i,j)) = w_{m-1}((i,j)) = \ldots = w_\mu((i,j)) = w_{\mu-1}((i,j))$)

$$= \Sigma \{x((i,j))/(i,j) \in A_\mu\} + \Sigma \{w_{\mu-1}((i,j))/(i,j) \in S_Z \cap A^*_\mu\}$$

(because, $(i,j) \in A_\mu$ implies $(i,j) \notin A_{\mu+1} \cup \ldots \cup A_m$ which implies, by the definition of w_μ, $w_m((i,j)) = w_{m-1}((i,j)) = \ldots = w_\mu((i,j))$ and for every $(i,j) \in A_\mu$, $w_\mu((i,j)) = x((i,j))$)

$$= \Sigma\{x((i,j))/(i,j) \in A_\mu\} + |S_Z|_{w_{\mu-1}}$$

(because, the obvious relations, $S_Z = (S_Z - A^*_\mu) \cup (S_Z \cap A^*_\mu)$ and $(S_Z - A^*_\mu) \cap (S_Z \cap A^*_\mu) = \emptyset$, imply that,

$$|S_Z|_{w_{\mu-1}} = \Sigma\{w_{\mu-1}((i,j))/(i,j) \in S_Z\} =$$

$$= \Sigma\{w_{\mu-1}((i,j))/(i,j) \in (S_Z - A^*_\mu)\} + \Sigma\{w_{\mu-1}((i,j))/(i,j) \in S_Z \cap A^*_\mu\};$$

but, $(i,j) \in S_Z - A^*_\mu$ implies $(i,j) \notin A^*_\mu = A_1 \cup \ldots \cup A_{\mu-1}$ and, by the definition of w_μ, $w_{\mu-1}((i,j)) = w_{\mu-2}((i,j)) = \ldots = w_0((i,j)) = 0$, for every $(i,j) \in S_Z - A^*_\mu$.

Therefore,

$$\Sigma\{w_{\mu-1}((i,j))/(i,j) \in (S_Z - A^*_\mu)\} = 0 \qquad \text{and, hence,}$$

$$|S_Z|_{w_{\mu-1}} = \Sigma\{w_{\mu-1}((i,j))/(i,j) \in S_Z \cap A^*_\mu\}).$$

Therefore,

$$|S_Z|_{w_m} = \Sigma\{x((i,j))/(i,j) \in A_\mu\} + |S_Z|_{w_{\mu-1}}$$

which, using (4) yields relation (3) Q.E.D.

Comments

(i) The previous theorems required that the given function t contains only essential equalities; this, however, is not actually a restriction.

Indeed, if, while the m-restriction condition is satisfied for every arc in A, the given function t contains non-essential equalities, it can be always perturbed to obtain a new function t_ϵ (which for small enough real positive numbers ϵ satisfies the m-restriction condition for the arcs and contains only essential equalities), in such a way that any realization of t_ϵ would reduce to a realization of t, in the limit as ϵ approaches zero.

Thus, it is always possible to assume, without loss of generality, that the function t to be realized contains only essential equalities.

It should be noted here, that one can find various procedures of obtaining the perturbed function t_ϵ (one such procedure can be found in the appendix of [6]). However, these procedures are usually not worth applying in practice due to their complexity. Furthermore, in most practical applications, a net realizing a given function t, with a certain degree of tolerance in the values of t, is acceptable; therefore, after a simple inspection, the required values of t can be changed by a small quantity, so that the essential equality condition is satisfied. This approach will be further discussed in one of the following examples.

(ii) The m-restriction condition, which is required for every pair of vertices, cannot be relaxed; it can be easily seen that, unless it is satisfied, for every arc, at least one among the sets $\mathcal{S}_\mu$ will be empty and the function x in the previous theorem is not defined. This, in turn, means that, in such cases, the synthesis procedure given in the theorem cannot be applied.

As an example, consider the case where the given graph is

$$\mathcal{G} = (\{1,2,3\}, \{(1,2), (1,3), (2,1), (2,3), (3,1)\ (3,2)\})$$

and the terminal capacity function to be realized is such that

$$t((1,2)) = t((1,3)) = 2,\ t((2,1)) = 0,\ t((2,3)) = 4,\ t((3,1)) = $$
$$= t((3,2)) = 3$$

$\mathcal{S}_1$, is then empty and one can easily verify (by simply drawing the graph) that there is no way of realizing t with no capacity redundancies on the arcs.

3.1.2. The constraint function c

In the synthesis procedure described in 3.1.1., the function x, used in the definition of w_μ, is clearly, not uniquely defined (it is so, only in the case where each A_μ has only one element, that is, only in the case where there are $m = n(n-1)$ distinct values of the function t). This, in turn, suggests that possibly more than one realization of the given function t could exist. In such cases it is natural to admit the presence of constraints on the arcs of the net. Then, among the possible realizations, the required ones are those satisfying the constraints. Such constraints in practice are, for example, of the following type.

(i) The weight of an arc cannot be any real number; particularly in the case of communications nets, a limited number of channel capacities is available.

(ii) The presence of an arc in the net to be realized might be absolutely necessary, while the absence of another might be desirable.

The constraints in the synthesis of a net can be conveniently described by a "constraint function" c: $A \longrightarrow \mathbb{R}$, "grading", in some sense, the arcs, so that in a set of arcs the one with the minimum "grade" should be chosen. This will be illustrated in one of the examples which follows.

Note: The case where no constraints are present is indicated by a constant constraint function, that is, by equally "grading" all the arcs.

3.1.3. Examples

In what follows, three examples illustrating the synthesis procedure are given. In the first example, the function t contains only essential equalities and the constraint function is constant, while, in the second example, non-essential equalities are present; in the third example, the function t of the first example is considered in the presence of constraints and a realization of t, different than in the first example, is obtained.

It should be noted here that for convenience of notation, any real function g defined on the set of arcs of a finite, quasicomplete, oriented graph $\mathcal{G} = (V,A)$ with n vertices, is usually given in the form of an n x n matrix G, whose element at the i-th row and j-th column is the value of g at the arc (i,j). Observe that since no loops are present in $\mathcal{G}$, the diagonal elements of the matrix G are not specified.

In the case of the functions w, t and c, the corresponding matrices W, T and C will be called "weight (or capacity) matrix", "terminal capacity matrix" and "constraint matrix", respectively. Thus, in the following examples, the matrices T and C will be given and a matrix W "realizing" T under the constraints in C will be determined. In these examples, a more convenient notation for semicuts will be also used; for example, the semicut $\{(2,1), (2,3), (4,1), (4,3)\}$ will be denoted by 24/13.

Example 1 Construct a capacity matrix W, realizing the terminal capacity matrix

$$T = \begin{bmatrix} * & 2 & 2 & 2 \\ 3 & * & 4 & 6 \\ 3 & 7 & * & 8 \\ 3 & 5 & 4 & * \end{bmatrix}$$

under no constraints (that is, the constraint function is constant).

Synthesis procedure

Step 1 Determine t_μ, A_μ, $\mathcal{S}_\mu$ for every $\mu=1,\ldots,$ m and check if the conditions for solution are satisfied.

$$t_1 = 2, \quad A_1 = \{(1,2), (1,3), (1,4)\}, \quad \mathcal{S}_1 = \{1/234\}$$
$$t_2 = 3, \quad A_2 = \{(2,1), (3,1), (4,1)\}, \quad \mathcal{S}_2 = \{234/1\}$$
$$t_3 = 4, \quad A_3 = \{(2,3), (4,3)\}, \quad \mathcal{S}_3 = \{124/3, 24/13\}$$
$$t_4 = 5, \quad A_4 = \{(4,2)\}, \quad \mathcal{S}_4 = \{14/23, 4/123\}$$
$$t_5 = 6, \quad A_5 = \{(2,4)\}, \quad \mathcal{S}_5 = \{12/34, 2/134\}$$
$$t_6 = 7, \quad A_6 = \{(3,2)\}, \quad \mathcal{S}_6 = \{134/2, 34/12\}$$
$$t_7 = 8, \quad A_7 = \{(3,4)\}, \quad \mathcal{S}_7 = \{123/4, 23/14, 13/24, 3/124\}$$

By simple inspection now, of each triple $(t_\mu, A_\mu, \mathcal{S}_\mu)$ for $\mu = 1, \ldots, 7$, we see that t contains only essential equalities and that the m-restriction condition is satisfied for every arc. Therefore, it is possible to proceed to the synthesis of the net.

Step 2 Determine W_μ, for $\mu=1, \ldots, 7$; W_7 will be the required W.

By definition:

$$W_o = \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

(W_1) $x((1,2)) + x((1,3)) + x((1,4)) = 2 - \min\{0\} = 2$

Taking, $x((1,2)) = 0$, $x((1,3)) = 2$, $x((1,4)) = 0$, then one possible definition of the function x on A_1, is

$$W_1 = \begin{bmatrix} * & 0 & 2 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

(W_2) $x((2,1)) + x((3,1)) + x((4,1)) = 3 - \min\{0\} = 3$

Taking, $x((2,1)) = 3$, $x((3,1)) = 0$, $x((4,1)) = 0$, then one possible definition of the function x on A_2, is

$$W_2 = \begin{bmatrix} * & 0 & 2 & 0 \\ 3 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

(W_3) $x((2,3)) + x((4,3)) = 4 - \min\{2,3\} = 2$

Taking, $x((2,3)) = 0$, $x((4,3)) = 2$, then one possible definition of the function x on A_3, is

$$W_3 = \begin{bmatrix} * & 0 & 2 & 0 \\ 3 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 2 & * \end{bmatrix}$$

(W_4) $x((4,2)) = 5 - \min\{4,2\} = 3$

Then the only possible definition of the function x on A_4 is, $x((4,2)) = 3$, yielding,

$$W_4 = \begin{bmatrix} * & 0 & 2 & 0 \\ 3 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 3 & 2 & * \end{bmatrix}$$

(W_5) $x((2,4)) = 6 - \min\{2,3\} = 4$

The only possible definition of x on A_5 is, $x((2,4)) = 4$, yielding,

$$W_5 = \begin{bmatrix} * & 0 & 2 & 0 \\ 3 & * & 0 & 4 \\ 0 & 0 & * & 0 \\ 0 & 3 & 2 & * \end{bmatrix}$$

(W_6) $x((3,2)) = 7 - \min\{3,3\} = 4$

The only possible definition of x on A_6 is $x((3,2)) = 4$, yielding,

$$W_6 = \begin{bmatrix} * & 0 & 2 & 0 \\ 3 & * & 0 & 4 \\ 0 & 4 & * & 0 \\ 0 & 3 & 2 & * \end{bmatrix}$$

(W_7) $x((3,4)) = 8 - \min\{4,7,4,4\} = 4$

The only possible definition of x on A_7 is $x((3,4)) = 4$, yielding,

$$W_7 = \begin{bmatrix} * & 0 & 2 & 0 \\ 3 & * & 0 & 4 \\ 0 & 4 & * & 4 \\ 0 & 3 & 2 & * \end{bmatrix}$$

<u>Step 3</u> The net has been now synthesized; its capacity matrix is $W = W_7$. Using now the capacity matrix, give the graph representation

of the corresponding net which, clearly, is one of the (infinitely many) nets, realizing the given matrix T, under no constraints.

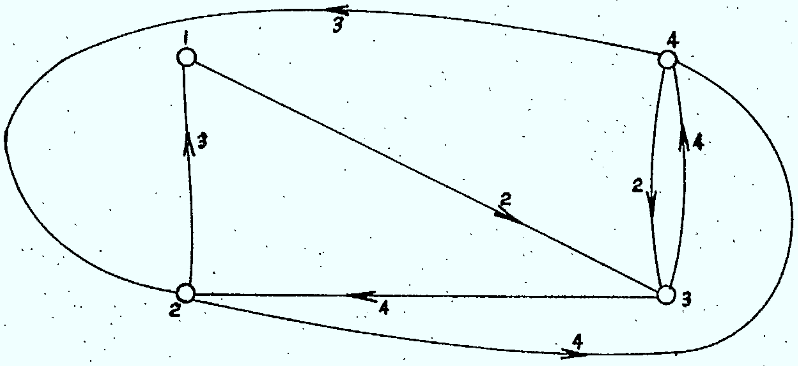

Figure 3

Example 2 Construct a capacity matrix W realizing the terminal capacity matrix

$$T = \begin{bmatrix} * & 1 & 1 & 1 \\ 4 & * & 6 & 5 \\ 4 & 6 & * & 5 \\ 4 & 6 & 6 & * \end{bmatrix}$$

under no constraints.

Synthesis procedure Observe that the given matrix T contains non-essential equalities. Indeed, the equality, $t((2,3)) = t((3,2))$ implies that the arcs (2,3) and (3,2) should be put in the same A_μ, while no semicut can contain both those arcs.

Thus, consider, instead of the matrix T, the following perturbed matrix

$$T' = \begin{bmatrix} * & 1 & 1 & 1 \\ 4 & * & 6 & 5 \\ 4 & 6+\epsilon_1 & * & 5 \\ 4 & 6+\epsilon_2 & 6+\epsilon_3 & * \end{bmatrix}$$

where, ϵ_1, ϵ_2, ϵ_3 are arbitrary, very small real numbers, such that $0 < \epsilon_1 < \epsilon_2 < \epsilon_3$.

Next, following the procedure of example 1, a capacity matrix W', realizing T' under no constraints, is constructed, and letting ϵ_1, ϵ_2, ϵ_3 tend to zero, in the matrix W', a matrix W is obtained, which is the one realizing T under no constraints.

Step 1

$t_1 = 1$, $A_1 = \{(1,2), (1,3), (1,4)\}$, $\mathcal{S}_1 = \{1/234\}$

$t_2 = 4$, $A_2 = \{(2,1), (3,1), (4,1)\}$, $\mathcal{S}_2 = \{234/1\}$

$t_3 = 5$, $A_3 = \{(2,4), (3,4)\}$, $\mathcal{S}_3 = \{23/14, 123/4\}$

$t_4 = 6$, $A_4 = \{(2,3)\}$, $\mathcal{S}_4 = \{2/134, 12/34\}$

$t_5 = 6+\epsilon_1$, $A_5 = \{(3,2)\}$, $\mathcal{S}_5 = \{3/124, 13/24\}$

$t_6 = 6+\epsilon_2$, $A_6 = \{(4,2)\}$, $\mathcal{S}_6 = \{34/12, 134/2\}$

$t_7 = 6+\epsilon_3$, $A_7 = \{(4,3)\}$, $\mathcal{S}_7 = \{4/123, 14/23, 24/13, 124/3\}$

Step 2 By definition,

$$W'_o = \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

(W'_1) $x((1,2)) + x((1,3)) + x((1,4)) = 1 - \min\{0\} = 1$

Taking, $x((1,2)) = 1$, $x((1,3)) = 0$, $x((1,4)) = 0$,

$$W'_1 = \begin{bmatrix} * & 1 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

(W'_2) $x((2,1)) + x((3,1)) + x((4,1)) = 4 - \min\{0\} = 4$

Taking , $x((2,1)) = 4$, $x((3,1)) = 0$, $x((4,1)) = 0$,

$$W'_2 = \begin{bmatrix} * & 1 & 0 & 0 \\ 4 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

(W'_3) $x((2,4)) + x((3,4)) = 5 - \min \{4,0\} = 5$

Taking, $x((2,4)) = 5, \quad x((3,4)) = 0$

$$W'_3 = \begin{bmatrix} * & 1 & 0 & 0 \\ 4 & * & 0 & 5 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

(W'_4) $x((2,3)) = 6 - \min \{9,5\} = 1$

$$W'_4 = \begin{bmatrix} * & 1 & 0 & 0 \\ 4 & * & 1 & 5 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

(W'_5) $x((3,2)) = 6 + \varepsilon_1 - \min \{0,1\} = 6 + \varepsilon_1$

$$W'_5 = \begin{bmatrix} * & 1 & 0 & 0 \\ 4 & * & 1 & 5 \\ 0 & 6+\varepsilon & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

(W'_6) $x((4,2)) = 6 + \varepsilon_2 - \min \{6 + \varepsilon_1, 7 + \varepsilon_1\} = \varepsilon_2 - \varepsilon_1$

$$W'_6 = \begin{bmatrix} * & 1 & 0 & 0 \\ 4 & * & 1 & 5 \\ 0 & 6+\varepsilon_1 & * & 0 \\ 0 & \varepsilon_2-\varepsilon_1 & 0 & * \end{bmatrix}$$

(W_7') $x((4,3)) = 6 + \epsilon_3 - \min\{\epsilon_2-\epsilon_1,\ 1+\epsilon_2-\epsilon_1,\ 5,\ 1\} = 6 + \epsilon_3 + \epsilon_1 - \epsilon_2$

$$W_7' = \begin{bmatrix} * & 1 & 0 & 0 \\ 4 & * & 1 & 5 \\ 0 & 6+\epsilon_1 & * & 0 \\ 0 & \epsilon_2-\epsilon_1 & 6+\epsilon_3+\epsilon_1-\epsilon_2 & * \end{bmatrix}$$

Step 3 From the matrix $W_7' = W'$, letting ϵ_1, ϵ_2, ϵ_3 tend to zero, the desired capacity matrix is

$$W = \begin{bmatrix} * & 1 & 0 & 0 \\ 4 & * & 1 & 5 \\ 0 & 6 & * & 0 \\ 0 & 0 & 6 & * \end{bmatrix}$$

and the corresponding net has the following graph representation

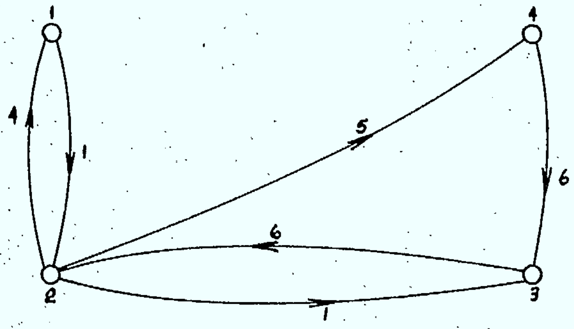

Figure 4

Example 3 Construct a capacity matrix W, realizing the matrix T under the constraints in C.

$$T = \begin{bmatrix} * & 2 & 2 & 2 \\ 3 & * & 4 & 6 \\ 3 & 7 & * & 8 \\ 3 & 5 & 4 & * \end{bmatrix}, \qquad C = \begin{bmatrix} * & 2 & 10 & 100 \\ 3 & * & 3 & 8 \\ 100 & 5 & * & 2 \\ 1 & 5 & 2 & * \end{bmatrix}$$

<u>Synthesis procedure</u> The procedure to be followed is that of example 1, with the only difference that, for the definition of the function x on the set $A_\mu (\mu = 1, \ldots, 7)$ the constraint matrix C will be taken into account.

<u>Step 1</u> By definition,

$$W_o = \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

(W1) $x((1,2)) + x((1,3)) + x((1,4)) = 2 - \min\{0\} = 2$

and, according to the matrix C, the choice must be

$$x((1,2)) = 2,\ x((1,3)) = 0,\ x((1,4)) = 0, \text{ yielding,}$$

$$W_1 = \begin{bmatrix} * & 2 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

(W2) $x((2,1)) + x((3,1)) + x((4,1)) = 3 - \min\{0\} = 3$

and, according to the matrix C, the choice of the values of x are

$$x((2,1)) = 0,\ x((3,1)) = 0,\ x((4,1)) = 3, \text{ yielding,}$$

$$W_2 = \begin{bmatrix} * & 2 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 3 & 0 & 0 & * \end{bmatrix}$$

(W3) $x((2,3)) + x((4,3)) = 4 - \min\{0,3\} = 4$

and, according to the matrix C, the choice of the values of x are

$$x((2,3)) = 0,\ x((4,3)) = 4, \text{ yielding,}$$

$$W_3 = \begin{bmatrix} * & 2 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 3 & 0 & 4 & * \end{bmatrix}$$

(W_4) $x((4,2)) = 5 - \min\{6,7\} = -1,$

$$W_4 = \begin{bmatrix} * & 2 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 3 & -1 & 4 & * \end{bmatrix}$$

(W_5) $x((2,4)) = 6 - \min\{0,0\} = 6,$

$$W_5 = \begin{bmatrix} * & 2 & 0 & 0 \\ 0 & * & 0 & 6 \\ 0 & 0 & * & 0 \\ 3 & -1 & 4 & * \end{bmatrix}$$

(W_6) $x((3,2)) = 7 - \min\{1,2\} = 6$

$$W_6 = \begin{bmatrix} * & 2 & 0 & 0 \\ 0 & * & 0 & 6 \\ 0 & 6 & * & 0 \\ 3 & -1 & 4 & * \end{bmatrix}$$

(W_7) $x((3,4)) = 8 - \min\{6,6,8,6\} = 2,$

$$W_7 = \begin{bmatrix} * & 2 & 0 & 0 \\ 0 & * & 0 & 6 \\ 0 & 6 & * & 2 \\ 3 & -1 & 4 & * \end{bmatrix}$$

Step 3 The required matrix is therefore, the matrix $W=W_7$ and the corresponding net has the following graph representation,

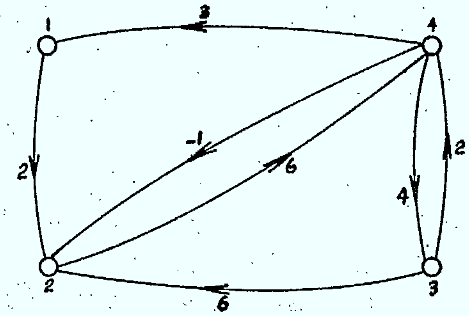

Figure 5

3.2. Synthesizing a communications net

The procedure used for the synthesis of a net "realizing" a given terminal capacity matrix T, can be used for the synthesis of communications nets. An obvious practical requirement is that the elements of the matrix T be non-negative real numbers.

However, it is possible to end up with a capacity matrix W, in which one or more elements are negative real numbers (this is exactly the case in example 3, of the previous section). Since, in this case, the corresponding net is not a communications net, one is interested to know if there is an equivalent communications net.

In what follows, the conditions under which such an equivalent communications net exists, are given and the procedure of finding such a net is illustrated in an example, using the capacity matrix W obtained in example 3 of the previous section.

3.2.1. Theorem If two nets are C_p - equivalent, then they are equivalent (that is, they have the same terminal capacity matrix).

Proof Let $(\mathcal{C},w_1)$ and $(\mathcal{C},w_2)$ be the two C_p-equivalent nets and let

$$(\mathcal{C},w_2) = (\mathcal{C},w_1+g_1+\ldots+g_k)$$

where $(\mathcal{C},g_1),\ldots,(\mathcal{C},g_k)$ are bicircuit nets.

In order to show that these two nets are equivalent, it is enough to show that for each semicut S_Z in $\mathcal{C}$, $|S_Z|_{w_1} = |S_Z|_{w_2}$ or, since for each S_Z

$$|S_Z|_{w_1+g_1+\ldots+g_k} = |S_Z|_{w_2} \quad \text{(by assumption)}$$

it is enough to show that for each S_Z in $\mathcal{C}$

$$|S_Z|_{g_1+\ldots+g_k} = 0$$

(because, $|S_Z|_{w_1+g_1+\ldots+g_k} = |S_Z|_{w_1} + |S_Z|_{g_1+\ldots+g_k}$).

It will now be shown that, for each S_Z in $\mathcal{C}$,

$$|S_Z|_{g_\lambda} = 0 \text{ , for every } \lambda = 1, \ldots, k$$

which implies that $|S_Z|_{g_1+\ldots+g_k} = 0$.

Since $(\mathcal{C},g_\lambda)$ is a bicircuit net, for each $\lambda = 1, \ldots, k$,

$$g_\lambda((i,j)) = \begin{cases} \alpha_\lambda, & \text{if } (i,j) \in C_{P_\lambda} \\ -\alpha_\lambda, & \text{if } (i,j) \in C_{\bar{P}_\lambda} \\ 0, & \text{otherwise} \end{cases}$$

where, for each $\lambda = 1, \ldots, k$, α_λ is some real number, while C_{P_λ} is a circuit in $\mathcal{C}$.

Then, for each S_Z in $\mathcal{G}$,

$$|S_Z|_{g_\lambda} = \Sigma\,\{g_\lambda((i,j))/(i,j) \in S_Z\} =$$

$$= \Sigma\,\{g_\lambda((i,j))/(i,j) \in S_Z \cap (C_{P_\lambda} \cup C_{P_{\bar\lambda}})\} =$$

$$= \Sigma\,\{g_\lambda((i,j))/(i,j) \in (S_Z \cap C_{P_\lambda}) \cup (S_Z \cap C_{P_{\bar\lambda}})\} =$$

$$= \Sigma\,\{g_\lambda((i,j))/(i,j) \in S_Z \cap C_{P_\lambda}\} + \Sigma\,\{g_\lambda((i,j))/(i,j) \in S_Z \cap C_{P_{\bar\lambda}}\}$$

(because, $(S_Z \cap C_{P_\lambda}) \cap (S_Z \cap C_{P_{\bar\lambda}}) = S_Z \cap (C_{P_\lambda} \cap C_{P_{\bar\lambda}}) = S_Z \cap \emptyset = \emptyset$)

$$= a_\lambda \cdot \text{card}(S_Z \cap C_{P_\lambda}) + (-a_\lambda) \cdot \text{car}(S_Z \cap C_{P_{\bar\lambda}})$$

$$= a_\lambda[\text{card}(S_Z \cap C_{P_\lambda}) - \text{card}(S_Z \cap C_{P_{\bar\lambda}})]$$

$$= 0$$ Q.E.D.

3.2.2. Theorem A net $(\mathcal{G},w)$ with terminal capacity function t is C_P-equivalent (and, therefore, equivalent) to a communications net, if and only if, $t((i,j))$ and $w((i,j)) + w((j,i))$ are each non-negative, for each $(i,j) \in A$.

Proof If the net $(\mathcal{G},w)$ is C_P-equivalent to a communications net, it is also equivalent to this net (see previous theorem) and, therefore, each $t((i,j))$ is non-negative, while, each sum $w((i,j)) + w((j,i))$ is also non-negative, as equal to the one of the communications net.

Next, suppose that each of the numbers $t((i,j))$ and $w((i,j)) + w((j,i))$ is non-negative for each $(i,j) \in A$. In order to show that the net $(\mathcal{G},w)$ is C_P-equivalent to a communications net, it is enough to show that, if $(\mathcal{G},w)$ contains arcs with negative weights, then it is

C_p-equivalent to a net in which the number of arcs with negative weights is less by at least one (then, repeating this step, after a finite number of steps a net is obtained which is C_p-equivalent to $(\mathcal{G},w)$ and in which every arc has non-negative weight, that is, a communications net).

To do this, it is necessary only to show that if $(i_o,j_o) \in A$ is such that $w((i_o,j_o)) < 0$, then it is possible to construct a finite sequence $(\mathcal{G},w_o), \ldots, (\mathcal{G},w_\sigma), \ldots, (\mathcal{G},w_s)$ of nets, such that

(i) $(\mathcal{G},w_\sigma)$ is C_p-equivalent to $(\mathcal{G},w)$, for each $\sigma = 0, 1, \ldots, s$.

(ii) $w_\sigma((i,j)) < 0$ implies $w_{\sigma-1}((i,j)) < 0$ for each $\sigma = 0, 1, \ldots, s$.

in which $w_s((i_o,j_o)) \geq 0$.

(property (ii) means that, if $w_{\sigma-1}((i,j)) \geq 0$, then $w_\sigma((i,j)) \geq 0$; because, otherwise, $w_{\sigma-1}((i,j))$ should be negative).

In order to show that such a sequence can be constructed, we use induction. Taking $\sigma=0$, it is easily verified that, for $w_o = w$, the net $(\mathcal{G},w_o)$ has the required properties, (i) and (ii). Suppose next that the net $(\mathcal{G},w_{\sigma-1})$ has the properties (i) and (ii); moreover, it is assumed that $w_{\sigma-1}((i_o,j_o)) < 0$, otherwise, there would be no next term in the sequence.

It now remains to show that, using $(\mathcal{G},w_{\sigma-1})$, it is possible to construct a net $(\mathcal{G},w_\sigma)$ having the properties (i) and (ii). To show this, consider a path $P \neq \{(i_o,j_o)\}$, in $(\mathcal{G},w_{\sigma-1})$, joining i_o to j_o and such that no arc in P has zero weight and at least one arc has positive weight. Addition then to $(\mathcal{G},w_{\sigma-1})$ of the bicircuit net corresponding to the pair (C_p, α), where

$C_P = P \quad \{(i_o,j_o)\}$ and $\alpha = -\min \{w((i,j))/(i,j) \in C_P,\ w((i,j)) > 0\}$ yields a net $(\mathcal{G},w_o)$ which has the properties (i) and (ii).

Finally, it should be shown that for some finite $\sigma = s$, $w_s((i_o,j_o)) \geq 0$. To show this, observe that whenever $w_{\sigma-1}((i_o,j_o)) < 0$, a path P, as defined above, must exist in $(\mathcal{G},w_{\sigma-1})$, joining is, to j_o; on the other hand, observe that the number of such paths in $(\mathcal{G},w_\sigma)$ is less by at least one than in $(\mathcal{G},w_{\sigma-1})$. Therefore, after a finite number of steps, say $\sigma = s$, no such path will exist in $(\mathcal{G},w_s)$, which means that $w_s((i_o,j_o))$ must then be non-negative, otherwise such a path should exist.

3.2.3. An example To illustrate the procedure described in the proof of the previous theorem, refer to example 3 of section 3.1.3.

Although the elements of the terminal capacity matrix T are non-negative, the capacity matrix W, realizing T under the constraint matrix C, contains one negative entry, namely $w((4,2)) = -1$.

Since the conditions required by the previous theorem are satisfied, a communications net $(\mathcal{G},w_c)$ equivalent to $(\mathcal{G},w)$ exists and to find this net, we proceed as follows.

Identifying w_o with w_1 immediately gives the net $(\mathcal{G},w_o)$ as the first net of the sequence, and it is immediately apparent that the path $P = \{(4,3),(3,2)\}$, joining the vertex 4 to the vertex 3, has the required by the previous theorem properties. Then, considering the corresponding circuit $C_P = \{(4,3),\ (3,2),\ (2,4)\}$ and the real number $\alpha = -\min \{w((i,j))/(i,j) \in C_P,\ w((i,j)) > 0\} = -4$, and adding to the net $(\mathcal{G},w_o)$ the bicircuit net corresponding to the pair $(C_p, -4)$, gives the net $(\mathcal{G},w_1)$, with graph representation

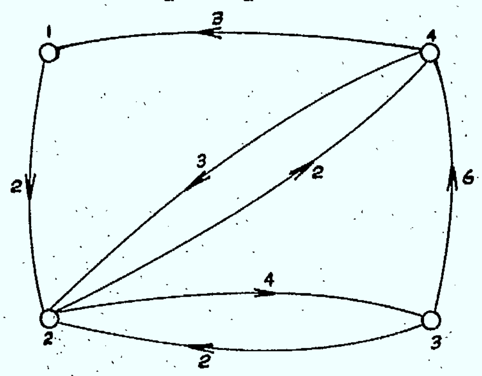

Figure 6

which is obviously the last term of the sequence, that is, a communications net $(\mathcal{G},w_c)$ equivalent to the net $(\mathcal{G},w)$.

Observe, however, that the communications net obtained does not necessarily satisfy the constraint matrix C, in the original problem.

Concluding Remarks

The general synthesis problem of finding an optimal net, in the case of a uniform cost, can be formulated as follows.

Among the nets providing given maximum flow requirements, find the one(s) with the minimum total capacity (if such optimal net(s) exist at all).

The case where such an optimal net has, at the same time, no capacity redundancies on the arcs was considered in the present work, the conditions for its existence were given and a synthesis procedure was developed and "computerized" (the computer program along with a short description is given in the appendix).

Although the emphasis was on communication networks it should be apparent that the synthesis procedure considered here deals with general nets, like economics and operations research. What remains to be done, is an appropriate interpretation of negative capacities.

It should be noted, however, that the results presented in this study, apply to systems with completely time-shared requirements, that is, to systems where time is broken up into distinct periods and during any one period there is only one flow on each channel of the system. This is by no means a restriction, since the problem of simultaneous flows in a net, is a simple application of the solution to the time-shared problem (see [3], page 299).

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