Government of Canada
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Gouvernement du Canada
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ELEMENTS OF TRANSFORMATION METHODS

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ELEMENTS OF TRANSFORMATION METHODS

Operational Calculus and the corresponding field of Integral Transformations have become more and more powerful and indispensable tools in many branches of Pure and Applied Mathematics. The mathematical procedures used originated in the work of Oliver Heaviside (1800-1925), who for the first time seems to have made a systematic use of operational methods in the solution of problems in Physics and Technology.

Since the two-sided Laplace Transform method is almost isomorphic to the operational method used by Heaviside, some writers base their transform techniques mainly on this Transform and the closely related General Fourier Transform. Laplace and Fourier Transforms are <u>integral transforms</u> of the type

$$\hat{f}(x) = \int_a^b f(u)K(x,u)\,du. \qquad (1)$$

K (x,u) is called the Kernel of the Transform. For some well-known transforms these kernels are of the following type:

$\exp(-ux)$	Fourier and Laplace Transform
$uJ_n(xu)$	Haenkel Transform
u^{x-1}	Mellin Transform
$\frac{1}{\pi(x-u)}$	Hilbert Transform This transform is a special case of a class of transforms called convolutions with kernel g(x-u), i.e.:
$g(x-u)$	Convolution Transform

The corresponding integral in a convolution transform is called a convolution product.

Singularity functions

By the introduction of singularity functions it is possible to define all the convolution integrals over an interval $(-\infty,+\infty)$ by defining the integrands in the integral transformation outside their interval of definition as identical to zero. So e.g., a function $f(t)$ defined in $0 \leq t < +\infty$ could be defined as $f(t)\ U^{+}(t)$

$$\text{where} \quad \begin{aligned} U^{+}(t) &= 0 \text{ for } t < 0 \\ U^{+}(t) &= 1 \text{ for } t \geq 0 \end{aligned} \tag{2}$$

Basically the most important singularity functions can be derived from the 'signum function', defined as

$$\begin{aligned} \text{sign } t &= -1 \quad \text{for} \quad t < 0 \\ \text{sign } t &= \ \ 0 \quad \text{for} \quad t = 0 \\ \text{sign } t &= +1 \quad \text{for} \quad t > 0 \end{aligned} \tag{3}$$

In terms of this function we may define

$$\begin{aligned} U^{+}(t) &= 1 + \tfrac{1}{2}\text{ sign } t - \tfrac{1}{2}\text{ sign}^{2}\, t \\ U^{0}(t) &= \tfrac{1}{2}\,(1 + \text{sign } t) \\ U^{-}(t) &= \tfrac{1}{2}\text{ sign } t + \tfrac{1}{2}\text{ sign}^{2}\, t \end{aligned} \tag{4}$$

We summarize the relevant functions described above in the following table

Table 1

t	$U^{-}(t)$	$U^{+}(t)$	$U^{0}(t)$	sign t	$\text{sign}^2\, t$
<0	0	0	0	-1	$+1$
$=0$	0	1	$\frac{1}{2}$	0	0
>0	1	1	1	$+1$	$+1$

If we define $|t| = 0$ for $t = 0$ we also have

$$|t| = t \text{ sign } t \qquad (5)$$

The well-known gaussian symbol $[\ ]$ defined by

$[t] \triangleq$ largest integer $\leq t$ (integral part of t)

may be written as

$$[t] = \sum_{k=-\infty}^{+\infty} U^{+}(t-k) \qquad k \in I \qquad (6)$$

The U functions and the sign function have many interesting applications, both in mathematics and technology. So we verify easily

$$f(t) \text{ sign } f(t) \equiv \text{'full wave rectifier'} \qquad (7)$$

$$f(t)\left(\frac{1+\text{sign } f(t)}{2}\right) \equiv \text{'half wave rectifier'} \qquad (8)$$

$$b_1 + (f(t) - b_1) \text{ sign}^2 (t-a_1) \qquad (9)$$

formally may be used to redefine a point $(a_1, f(a_1))$ as (a_1, b_1)

An operator of the form

$$f(t)\{U(t-a) - U(t-b)\} \qquad a < b \qquad (10)$$

will redefine f(t) outside (a,b) as zero and will give various values in a and b, depending on the choice of the U-functions. We call such an operator on f(t) a <u>truncating operator</u>. This operator makes it possible to replace integrals of the form $\int_a^b f(u)\,du$ by $\int_{-\infty}^{+\infty} f(u)\,(U(t-a) - U(t-b))\,du$

Thus all functions of real variables can formally be defined in an interval $(-\infty,+\infty)$ without loss of generality.

That this is more than a formal concept we will illustrate by the following example:

Problem: Find the volume of an n-dimensional sphere.

Solution: Let sphere have equation

$$x_1^2 + x_2^2 + \ldots + x_n^2 = r^2 = t$$

$$V_n(t) = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} U(t - \sum_{i=1}^{n} x_i^2)\, dx_1\, dx_2 \ldots dx_n$$

Use double-sided L-transform on t

$$L(V_n(t)) = \varphi_n(s) = \frac{1}{s} \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} e^{-sx_1^2}\; e^{-sx_2^2} \ldots e^{-sx_n^2}\, dx_n \ldots dx_1$$

$$= \frac{1}{s}\left(\frac{\sqrt{\pi}}{\sqrt{s}}\right)^n$$

Transforming back we obtain

$$L^{-1}\varphi_n(s) = \frac{\sqrt{\pi}^n t^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} = \frac{\sqrt{\pi}^n r^n}{\Gamma(\frac{n}{2} + 1)} = V_n$$

Taking $n = 1, 2, 3, \ldots$ we find $V_1 = 2r$; $V_2 = \pi r^2$; $V_3 = \frac{4}{3}\pi r^3$; . . .

The δ -operator

For every finite h>0 the function $\{U(t) - U(t-h)\}\frac{1}{h}$ is well defined (assuming the U-function to be specified). Essentially it is a 'pulse' with width h and height $\frac{1}{h}$, so that its area is 1. The pulse lies right of the origin of the t-axis.

Now consider more generally

$$f(t)\ \{\frac{U(t-a) - U(t-a-h)}{h}\}\ .$$

The limit of this product would not exist in the t-domain. But if we take the L-transform of this product, we get

$$\frac{1}{h}\int_a^{a+h} f(u)\ e^{-su}\ du = \frac{f(a+\theta h)}{h}\ e^{-s(a+\theta h)}h \qquad 0 \leq \theta \leq 1$$

For $h \to 0$ this integral would have the limit $f(a)\ e^{-sa}$. We therefore formally associate with this limit an operator $\delta(t-a)$ and write

$$L\ \delta(t-a)\ f(t) = f(a)\ e^{-sa} \tag{11}$$

Note that we must define

$$f(t)\ \delta(t-a) = f(a)\ \delta(t-a) \tag{12}$$

We also have

$$\int_{-\infty}^{+\infty} f(u)\ \delta(t-u)du = f(t) \tag{13}$$

In a similar way operators $\delta^{(k)}(t)$ can be introduced, corresponding to the limit of the k^{th} order differences of U-functions. It should be noted that the signum function and the U-functions are indeed functions, but the $\delta^{(k)}(t)$ symbols are to be treated as operators, though in many applications they are called 'functions'(Dirac δ -functions; impulse functions). They correspond to the inverse generalized L-transforms of s^k. So we may formally write

$$L\,\{\delta^{(k)}(t)\} = s^k \qquad k = 0,\ 1,\ 2\ \ldots \tag{14}$$

Though the theory of distributions makes the use of these 'impulse functions' no longer necessary, a formal use of the δ's as operators is still permissible, provided that all the operational rules are checked rigorously.

Other functional transforms

Besides the Laplace and Fourier Transform the Hilbert Transform is playing an important role in communications. The Hilbert Transform of a function f(t) is defined by

$$Hf(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f(u)}{t-u}\,du \tag{15}$$

Since this transform is of the convolutional type, and convolution products $f_1(t) * (f_2(t)$ in the 'object space' correspond to ordinary products $\varphi_1(s)\,\varphi_2(s)$ in the 'result space', one would expect that this transform would formally be a product of two transforms.
Using the Fourier Transform, we have indeed

$$\text{If } F(f(t)) = \alpha(\omega) - j\,\beta\,(\omega)$$

$$\text{then } FH(f(t)) = -\,j \text{ sign } \omega\ \ (\alpha(\omega) - j\beta(\omega)) \tag{16}$$

So with $\frac{1}{\pi t}$ we may associate the **Fourier** Transform $-j$ sign ω :

$$F\left(\frac{1}{\pi t}\right) = - j \text{ sign } \omega \tag{17}$$

Hilbert Transform and Fourier Transform are therefore closely related. Indeed, if we form $f(t) + jHf(t)$, we have

$$F(f(t) + jHf(t)) = (1 + \text{sign } \omega) \quad (\alpha(\omega) - j\beta(\omega)) \tag{18}$$

This Fourier Transform is zero for negative frequencies and plays an important role in sampling theorems. $f(t) + jHf(t)$ is often referred to as the analytic signal. This expression gives a natural extension to the frequency concept for an arbitrary signal. Calling $Hf(t) = g(t)$ and writing the analytic signal in the polar form we get:

$$f(t) + jg(t) = \sqrt{f^2 + g^2}\; e^{j\tan^{-1}\frac{g}{f}}$$

ω instantaneous can then be defined as:

$$\omega_{inst} = \frac{d \tan^{-1}\frac{g}{f}}{dt} = \frac{1}{1 + \frac{g^2}{f^2}} \; \frac{fg' - gf'}{f^2} = \frac{fg' - gf'}{f^2 + g^2} \tag{19}$$

Note that for $f = A \cos\omega_0$; $Hf = A \sin\omega_0 t$ and ω_{inst} reduces to ω_0 , as is immediately verified.

Of all the integral transforms available, the two-sided (bilateral) Laplace Transform and corresponding generalized Fourier Transform are probably the ones most used in applications. Since they are essentially not different, it is irrelevant which of the two is used. It is important to notice that the one-sided L-transform may be written as a two-sided L-transform by using the U function:

$$\int_0^{\infty} f(t)e^{-st}dt = \int_{-\infty}^{+\infty} f(t)U(t)e^{-st}dt \tag{20}$$

One is led to another type of Transforms by considering the L-transforms of the functions of type $f\{[x]\}$ ('entier'functions), where $[x]$ is the largest integer $\leq x$.

We find $L\{[x]\} = \dfrac{1}{s(e^s-1)}$

and more generally

$$L\left\{\binom{[x]}{r}\right\} = \frac{1}{s(e^s-1)^r} \qquad r=0,\ 1,\ 2 \tag{21}$$

Using Newton's difference formula, we get formally:

$$L\, f\{[x]\} = \frac{f(0)}{s} + \frac{\{f(1) - f(0)\}}{s(e^s - 1)} + \frac{\{f(2) - 2f(1) + f(0)\}}{s(e^s - 1)^2} + \quad \cdots \tag{22}$$

Except for factor $\frac{1}{s}$ this is formally a power series in $\dfrac{1}{(e^s - 1)}$

We also have formally

$$L\, f\{[x]\} = \frac{1}{s}\,(1-e^{-s})\left(\sum_{n=0}^{\infty} f(n)\,e^{-ns}\right) \tag{23}$$

Again we have essentially a power series in e^{-s}. This leads us to consider a type of transform where the 'object' function is a function of a discrete integer-valued argument n, which may range over the set of all integers. In interpolation and sampling problems we meet these entier functions; consider for instance $g(x) = g(x_0 + t\Delta x) = f(t)$. Taking $f([t])$ we have the function $g(x)$ sampled at intervals Δt. We will call a function like $f([t]) = f_n$ a <u>sequence</u>. We usually write n as a subscript and call it <u>index</u>. Also in many cases we consider only the argument values $n \geq 0$. Particularly we will define as Z-transform of the sequence f_n:

$$F(z) \triangleq \sum_{n=0}^{\infty} f_n z^{-n} \qquad (24)$$

i.e. a formal power series in $\frac{1}{z}$

The Z-transform is also sometimes defined in positive powers of z:

$$\tilde{F}(z) = \sum_{n=0}^{\infty} f_n z^n$$

We will choose the definition given in (24).

Both F and $\tilde{F}$ generate f_n and can be called <u>generating</u> functions for f_n. This concept of generating function is more familiar for the case where f_n is also a function of a continuous real variable t. $f_n \triangleq f_n(t)$ and $F(z)$ and $\tilde{F}(z)$ are especially used for the generation of these functions $f_n(t)$. Further extensions of these concepts are often found. A combination of Laplace and/or Fourier transform methods makes it sometimes possible to find such generating functions, as illustrated by the following example for the Laguerre polynomials:

$$L_n(t) \triangleq 1 - \binom{n}{1}\frac{t}{1!} + \binom{n}{2}\frac{t^2}{2!} - \binom{n}{3}\frac{t^3}{3!} + \dots \qquad (25)$$

$$L\{L_n(t)\} = \frac{1}{s} - \binom{n}{1}\frac{1}{s^2} + \binom{n}{2}\frac{1}{s^3} - \dots = \frac{1}{s}\left(1 - \frac{1}{s}\right)^n \qquad (26)$$

$$L\{\sum_{n=0}^{\infty} z^n L_n(t)\} = \frac{1}{s}\sum_{n=0}^{\infty} z^n \left(1 - \frac{1}{s}\right)^n = \frac{1}{s(1-z)+z}$$

$$\Rightarrow \sum_{n=0}^{\infty} z^n L_n(t) = \frac{1}{1-z}\, e^{\frac{-z}{1-z}t} \qquad (27)$$

This example illustrates how transform methods may even be formally used to find a possible answer to a problem, which is itself a transformation problem. Such heuristic techniques are often extremely powerful.

Consider for example the Bernoulli polynomials:

$$B_n(t) = B_0\, t^n + B_1 \binom{n}{1} t^{n-1} + B_2 \binom{n}{2} t^{n-2} + \ldots \tag{28}$$

For which the well known relations hold

$$\Delta B_n(t) = B_n(t+1) - B_n(t) = nt^{n-1} = Dt^n \tag{29}$$

Now the operators Δ and D are formally related by

$$\Delta = e^D - 1; \quad D = \ln(1+\Delta) = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \ldots \tag{30}$$

$$\text{So } \Delta B_n(t) = \left(\Delta - \frac{\Delta^2}{2} + \ldots + (-1)^{n-1} \frac{\Delta^n}{n} + \ldots\right) t^n \tag{31}$$

This leads formally to

$$B_n(t) = t^n - \frac{\Delta}{2} t^n + \frac{\Delta^2}{3} t^n - \ldots + \frac{(-1)^n \Delta^n}{(n+1)} t^n$$

i.e.

$$B_n(t) = t^n - \tfrac{1}{2}\{(t+1)^n - t^n\} + \frac{1}{3}\{(t+2)^n - 2(t+1)^n + t^n\} + - \ldots + \frac{(1)^n}{(n+1)}\{(t+n)^n - \binom{n}{1}(t+(n-1))^n + \ldots\} \tag{32}$$

which is perhaps not such a well known, but indeed a correct formula for generating these polynomials.

For an efficent use of transform methods it is essential to have at one's disposal a list of corresponding 'object' and 'result' functions ('dictionary') and a list of corresponding operators in 'object' and 'result' space ('grammar'). The way in which transforms then make it sometimes possible to find a solution S to a problem P is by transforming the problem P from 'object' space into a problem P^* in 'result' space. If a solution S^* in 'result' space can be found and S^* can be transformed back uniquely into S, we have solved our problem. It is interesting to note that even when rigorous 'looping' around to the solution of the problem may not be possible, a formal application of the transformation rules may not lead to the correct solution, buy may give us a good hint with respect to its format. In this way transformation methods become a heuristic tool for finding solutions to problems.

So summarizing, Transformation Methods sometimes can be used rigorously to find the correct solution to a problem, or sometimes heuristically to help find a solution.