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DEPARTMENT OF COMMUNICATIONS
OTTAWA

Probabilistic Studies of Complex Telecommunication-Computer Systems

by

Dr. John deMercado

January 26, 1971.

POLICY, PLANS AND PROGRAMMES

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NOTE

The opinions expressed in this report are entirely the author's and are not necessarily those of the Department of Communications.

INTENT

This study is an attempt to provide a pragmatic yardstick for resolving often repeated but usually semantic statements, such as "my system is more reliable than yours etc." The author has attempted to develop purely analytical techniques for analyzing, synthesizing and comparing communication-computer systems with arbitrary but specifiable reliability.

ACKNOWLEDGEMENTS

Doug Parkhill saw the need for methods for making a definitive comparison and analysis of the reliability of various communication-computer systems, and encouraged me to make a start in this direction. I wish to thank my secretary Judy Tremblay for giving up some (a lot) of her spare time to type this report.

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Abstract

In this report methods are given for obtaining the reliability function, and moments of the first time to failure for a general class of complex systems. The class of complex systems considered are all those systems that have subsystems (any number) with known (constant) failure and repair rates. This class for all practical purposes includes telecommunication systems, such as microwave systems, etc., as well as most telecommunication-computer (computer-utility) systems. In addition the methods given are also applicable to many stability problems in economic systems.

Specifically, complex systems composed of any finite number of subsystems are considered. The complex system at any time, can be in any one of $r(r \geq 1)$ acceptable states or in any one of $m(m \geq 1)$ failed states. The methods presented for the reliability modelling of such complex systems, assume a state behavior that is characterizable by a stationary Markov process (also called Markov chain) with finite-dimensional state space and a discrete time set.

It is shown that once the matrix of the constant failure and repair rates of the subsystems is known, and the state assignment is made, then it is a straightforward matter to obtain the probabilistic description of the complex system.

Introduction

It is well known [1], [2], that the reliability modelling of complex systems that operate in a repair environment, and whose subsystems have known constant failure and repair rates, can be accomplished via a linear matrix calculus and use of elements of the theory of stationary Markov processes. The methods that exist for modelling such complex systems may be summarized as follows: Let the complex system have r acceptable*) states A_i (i = 1, ..., r),which form the set A, and let all failed states be lumped into a single failed state F. Then methods exist for obtaining a time dependent reliability function R(n), defined as the probability that the complex system is in some acceptable state in A at time n. Methods also exist, which allow computation of the moments of the first time to failure, that is, the moments of the first time that the complex system passes from acceptable states in A to the single lumped failed state F. These methods all suffer from a number of obvious limitations, first of all the lumping of failed states into a single failed state conceals the relative importance of the different types of failure modes that are present in any complex system. Secondly, no techniques are provided for computing the important moments of the first time the complex system passes from specified acceptable states to specified failed states.

*) See definition 1.

In this paper, the above approach is extended to include complex systems having $m (m \geq 1)$ failed states in the set F. Methods are presented for obtaining a time dependent reliability function for such complex systems, as well as for obtaining the moments of first time to (a particular) failed state, as well as the moments of the first time to failure (any state).

The construction of a model for predicting the behavior of such a complex system poses three distinct problems. The first two are in effect specification problems. The first of these is the "state assignment" problem, that is the enumeration of the states that suffice to characterize the various operating modes of the complex system. The method for making such a state assignment will depend on the specification of the structure and operation of the given complex system. The second problem involves the determination of meaningful numerical estimates of the one step state transition probabilities *). This is the, so called "general inference" problem (3, pp 69-70) **) for Markov processes. The third problem which is the one this paper solves involves the application of techniques from the theory of stationary Markov process to develop methods for obtaining apriori, state probability functions, a reliability function, and estimates of the moments of the first time that it takes the complex system to pass from one state to another. We have assumed that the solution to the "state assignment" problem as well as the "general inference" problem is known. That is there exists a state characterization of the complex system and the matrix $|M|$ of one step state transition probabilities.

*) The one step state transition probabilities are dimensionless and are obtained by multiplying the failure or repair rates, (whichever are appropriate) by the "unit of time" (for example, 1 hour, 1 day, etc.).

**) Numbers in superscript brackets refer to the references.

In section 1, the basic definitions of the elements of the Markov model of a complex system are presented. It is shown that the matrix*) |M| of one step state transition probabilities, that is of the failure and repair rates of the subsystems, can be partitioned into four matrices |A| , | B| , |O| , |I| . In later sections it is shown that such partitioning is sufficient to use all the methods presented herein.

In section 2, it is shown that the state probability functions $\overline{s}$ (n) are obtained simply by taking the n^{th} power of the matrix |M| . Then once the set A of acceptable states is known, the time dependent reliability function R(n) is shown to be the sum of these state probability functions over the set A. Thus the reliability function R(n), is the probability that at time n, the complex system is operating acceptably.

In example (I), at the end of the paper, it is shown that another possible interpretation of R(n), in the context of a telecommunication network, is to interpret R(n) as the probability that two points i and j within a complex telecommunication network will remain connected for time n.

In section 3, the steady state transition failure probabilities P_{ij} are derived. P_{ij} is the probability that the complex system, will eventually pass from acceptable state A_i to failed state F_j. A theorem is presented which shows that the (r x m) matrix| P| , of these steady state failure probabilities is obtainable directly in terms of the matrices |B| , |I| and |A| , which are the partitions

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*) Capital letters in square brackets denote matrices; the bar on top of a letter denotes a vector. An explanation of the notation is given after the references.

of $|M|$. The concept of an evolution diagram, as introduced by Girault[6], is utilized to prove this theorem. These evolution diagrams provide a useful conceptual aid for establishing many interesting results in the theory of stationary Markov processes.

In section 4, the steady state transition probability failure functions $p_{ij}(n)$ are derived. $p_{ij}(n)$ is the probability that the complex system will pass after n units of time from acceptable state A_i to failed state F_j. A theorem is presented which shows that the $(r \times m)$ matrix $|P(n)|$ of these transition probability failure functions is expressible directly in terms of the matrices $|A|$ and $|B|$.

In section 5, a method is presented for obtaining the (pseudo) generating functions $g_{ij}(z)$ that give the time moments $\tau_{ij}(k)$ of the random variables $\tau_{ij} \equiv$ "first time from acceptable state A_i to failed state F_j". It is shown that these moments are obtained in the usual manner, that is, by differentiating the (pseudo) generating functions. A theorem is presented, which shows that the $(r \times m)$ matrix $|G(z)|$ of these generating functions is a simple linear function of the matrices $|A|$, $|B|$ and $|I|$.

In section 6 the exit probability functions $w_i(n)$ are derived. $w_i(n)$ is the probability that the complex system will pass from the successful state A_i, into any failed state in F in time n. A theorem is presented which shows that the $(r \times 1)$ column vector $\overline{W}(n)$ of the exit probability functions is a simple function of the matrices $|A|$ and $|B|$.

In section 7, a method is presented for computing the generating functions $c_i(z)$ that give the moments $\tau_i(k)$ of the random variables $\tau_i \equiv$ "first exit time from acceptable state A_i into the class F". A theorem is presented which relates the generating function $g_{ij}(z)$ of section (5) to the generating function $c_i(z)$. Another theorem is presented which shows that the (r x l) vector $\bar{C}(z)$ of these generating functions is a simple linear function of the matrices $|A|$, $|B|$ and $|I|$.

Almost all of the results presented in this paper, appear (as far as the author is aware) for the first time in the context of reliability prediction theory. They are applicable to a large class of diversified systems, including economic systems, control systems and computer-communication systems etc.

I. Preliminaries

In developing the reliability model we shall use a stationary Markov process $S(\cdot)$, defined on a discrete finite dimensional state space $A \cup F$, and a discrete time set T. The random variable $S(n)$ is called "state of complex system at time n". We will derive for each state $A_i \varepsilon A$ and $F_j \varepsilon F$, state probability functions $s_i(n)$, $s_j(n)$, defined as*)

$$s_i(n) \equiv \text{Prob} \quad \{S(n) = A_i\} \quad , \quad n \,\varepsilon T \qquad \text{---------} \; 1$$

$$s_j(n) \equiv \text{Prob} \quad \{S(n) = F_j\} \quad , \quad n \,\varepsilon T \qquad \text{---------} \; 2$$

It is well known (6,8), that if the set of states in A, are a transient class (ie, are acceptable states), and if the states in F are absorbing states (ie, are failed states), then the one step transition probabilities between the states $A \rightarrow A$, $A \rightarrow F$, $F \rightarrow F$ and $F \rightarrow A$ can be defined as follows.

$A \rightarrow A$

The one step state transition probabilities between states A_i, A_k of A denoted by a_{ik}, are the elements of a (rxr) matrix $|A|$ and are defined as

$$a_{ik} \equiv \text{Prob} \quad \{S(n+1) = A_k \mid S(n) = A_i\} \quad , \quad \begin{matrix} i=1,--,r. \\ k=1,--,r. \end{matrix} \qquad \text{------} \; 3$$

*) The subscript i, (i=1,--,r) refers to states in A and the subscript j, (j=1,--, m) refers to states in F.

$\underline{A \rightarrow F}$

The one step state transition probabilities from transient states $A_i \varepsilon A$ to absorbing (failed) states $F_j \varepsilon F$, denoted by b_{ij} are the elements of a (rxm) matrix $|B|$, and are defined as

$$b_{ij} \equiv \text{Prob} \quad \{S(n+1) = F_j \mid S(n) = A_i\}, \qquad \begin{matrix} i=1,---,r \\ j=1,---,m \end{matrix} \qquad ----- \ 4$$

$\underline{F \rightarrow F}$

The one step state transition probabilities between absorbing (failed) states F_j, F_u of F denoted by δ_{ju}, are the entries of a (mxm) unit*) matrix $|I|$, and are defined as

$$\text{Prob} \quad \{S(n+1) = F_u \mid S(n) = F_j\} = \delta_{ju} \begin{cases} = 1, j=u \\ = o, u \neq j \end{cases} \qquad ----- \ 5$$

$\underline{F \rightarrow A}$

Since transitions from failed states in F to transient (acceptable) states in A are not permitted, the one step state transition probabilities from $F_j \varepsilon F$ to $A_i \varepsilon A$ are all zero. That is they form a (mxr) null matrix $|O|$, because

$$\text{Prob} \ \{S(n+1) = A_i \mid S(n) = F_j\} = 0, \quad \begin{matrix} \text{for all } F_j \ \varepsilon \ F \\ \text{and} \\ \text{for all } A_i \ \varepsilon \ A \end{matrix} \qquad ------- \ 6$$

* The methods presented in this paper could be further generalized by allowing transitions among the failed states of F. That is by replacing the matrix $|I|$ by some general matrix.

Thus, we have that the one step transition matrix $|M|$ for the Markov processes $S(\cdot)$ with state space AUF, can be partitioned into four matrices $|A|$, $|B|$, $|I|$, $|0|$ as

$$|M| \equiv \begin{array}{r} A \rightarrow A\{ \\ \\ F \rightarrow A\{ \end{array} \left| \begin{array}{c:c} |A| & |B| \\ \hdashline |0| & |I| \end{array} \right| \begin{array}{l} \}A \rightarrow F \\ \\ \}F \rightarrow F \end{array} \qquad \text{---------} \; 7$$

The following definitions make it possible to interpret equations (3) to (7) in the context of the reliability model of a complex system having A_i, $i=1,---r$ acceptable states and F_j, $j=1,---,m$ failed states.

<u>Definition 1</u> <u>Acceptable State</u> The transient state $A_i \varepsilon A$ is called an acceptable state, if it characterizes some acceptable working mode of the complex system.

<u>Definition 2</u> <u>Failed State</u> The absorbing state $F_j \varepsilon F$ is called a failed state, if it characterizes some unsatisfactory mode of operation of the complex system.

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2. <u>State Probability & Reliability Functions</u>

Let $\overline{s(n)}$, be the $(1\times(r+m))$ vector of state probabilities defined by equations (1) and (2). Then it is well known(6, Page 56) that

$$\overline{s}(n) = \overline{s(o)} \quad |M|^n \qquad \text{----------------} \; 8$$

where $\overline{s}(o)$ is the vector of the initial (time $n=o$) state probabilities. We can now immediately define a reliability function $R(n)$ for the complex system as

<u>Definition 3</u> <u>Reliability Function R(n)</u> The reliability function $R(n)$ is the probability that time n the complex system is operating acceptably, that is, is in some acceptable state, thus

$$R(n) = \text{Prob } \{S(n) \varepsilon A\} \text{ ------------ } 9$$

alternatively then

$$R(n) = \sum_{i=1}^{r} s_i(n) \text{ ------------------- } 10$$

Thus in order to obtain $R(n)$, it is necessary only to raise the matrix $|M|$ to the n^{th} power, multiply by $\bar{s}(o)$ and then sum the elements of the set $\{s_i(n),\ i=1,---,r\}$. There are several well known methods (11,12,13) yielding closed form expressions for $|M|^n$ and therefore for $\bar{s}(n)$ and $R(n)$

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3. <u>The Steady State Transition Failure Probabilities</u>

In this section, a method is presented for computing the steady transition probability p_{ij}, $(i=1,--,r;j=1,--,m)$ that a complex system that starts in acceptable state A_i will eventually end up in a specified failed state F_j. In what follows, it will be shown that once the partition of $|M|$ has been carried out as shown in (7), it is a simple computational matter to obtain these probabilities. Formally, defining p_{ij} as

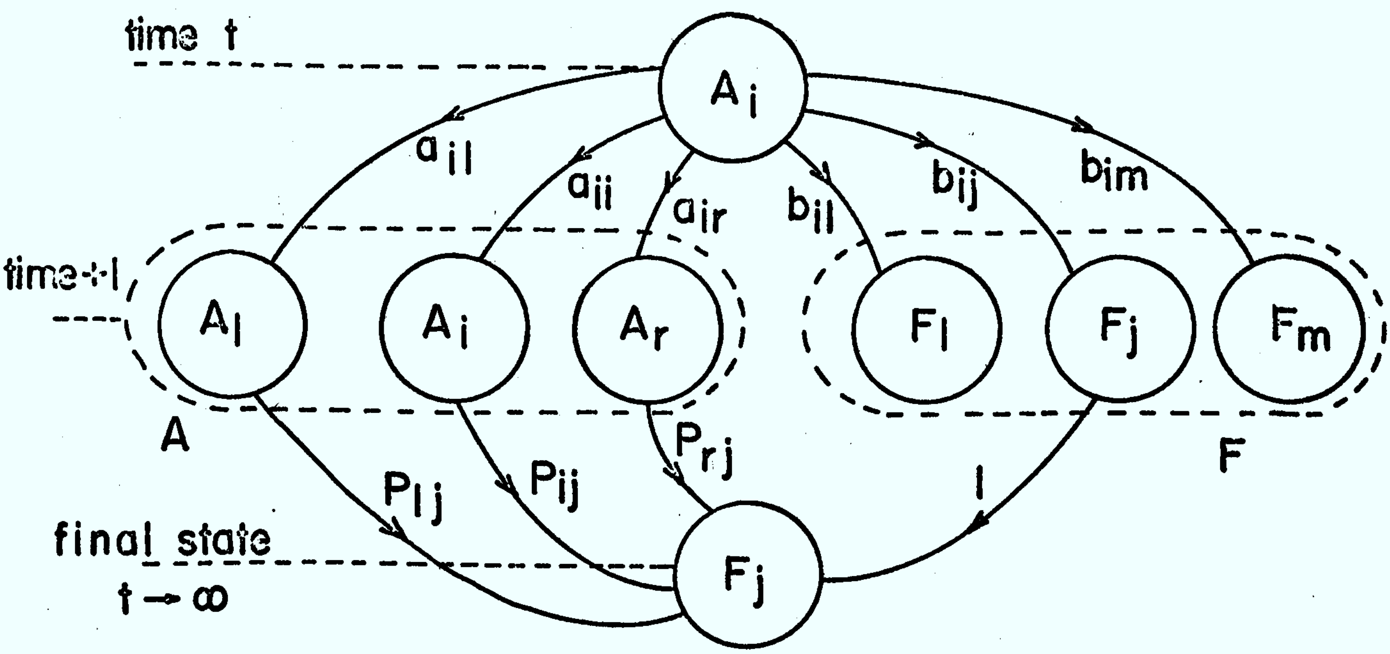
$$p_{ij} \equiv \text{Prob } \{ S(\infty) = F_j \mid S(n) = A_i \} \text{------------ } 11$$

and denoting the $(r \times m)$ matrix $|p_{ij}|$ by $|P|$, we have,

<u>Theorem 1</u> For a complex system with r acceptable states and m failed states operating in a repair environment, and having subsystems with known constant failure and repair rates (that is, with known matrix $|M|$), the (r x m) matrix $|P|$ satisfies

$$|P| = \left| |I| - |A| \right|^{-1} |B| \qquad \text{-------------- } 12$$

<u>Proof</u> The proof of this and other theorems presented herein, is facilitated by formally introducing evolution diagrams(6 pp 74-78). Consider therefore Evolution Diagram 1, which shows the eventual possible evolutions from state $A_i \varepsilon A$ to state $F_j \varepsilon F$.



<u>Evolution Diagram 1</u>

From the above diagram, summing the transmittances of the paths incident on node F_j from A_i, gives

$$p_{ij} = \sum_{k=1}^{r} a_{ik}\, p_{kj} + b_{ij}, \quad (j=1,----, m) \text{ ------ } 13$$

obviously such a diagram can be constructed for every $A_i \varepsilon A$ and every $F_j \varepsilon F$ and therefore equation (13) can be written in matrix form as

$$|P| = |A|\;|P| + |B|$$

or --------------- 14

$$\big|\, |I| - |A| \,\big|\;|P| = |B|$$

which completes the proof QED.

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4. <u>The Transition Probability Failure Functions</u>

In this section, a method is presented for computing the transition probability failure functions $p_{ij}(n)$, $i=1,---,r$; $j=1,---,m$. Specifically, $p_{ij}(n)$ is the probability that at time n, the complex system is in failed state $F_j \varepsilon F$ given that at time n=0, it was in acceptable state $A_i \varepsilon A$.

Formally,

$$p_{ij}(n) = \text{Prob}\,\{ S(t+n) = F_j \mid S(t) = A_i \}, \quad \begin{matrix} j=1,---,m \\ i=1,---,r \end{matrix} \text{ ----- } 15$$

and denoting the $(r \times m)$ matrix $|p_{ij}(n)|$ by $|P(n)|$, we have

<u>Theorem 2</u> For a complex system with r acceptable states and m failed states, and having subsystems with known constant failure and repair rates (that is with known matrix $|M|$), the $(r \times m)$ matrix $|P(n)|$ satisfies

$$|P(n)| = |A|^{n-1}|B| + |P(n-1)| \quad \text{-------------- } 16$$

<u>Proof</u>

Comparing equations (4) and (15), it is immediately apparent

$$|P(1)| \equiv |B| \quad \text{-------------------------- } 17$$

Then from (7)

$$|M|^n \equiv \left| \begin{array}{c|c} |A| & |P(n-1)| \\ \hline |0| & |I| \end{array} \right|$$

taking the n^{th} power of $|M|$, using (17), we find

$$|P(n)| = \Big|\, |I| + |A| + |A|^2 + \text{--} + |A|^{n-1} \Big| |B| \quad \text{----- } 18$$

which can also be written as (16)

QED

Comments

(a) $p_{ij}(n)$, is the probability that the complex system will pass from acceptable state A_i to failed state F_j in n units of time. Thus letting τ_{ij} be the (pseudo) random variable "time taken to go from state A_i to state F_j", we have that $p_{ij}(n)$ is the probability distribution function of τ_{ij}; that is

$$p_{ij}(n) = \text{Prob}\ \{ \tau_{ij} = n \} \quad \text{-----------------} \quad 19$$

(b) Since $|P| = \lim_{n \to \infty} |P(n)|$, from (18) we find

$$|P| = \sum_{n=0}^{\infty} |A|^n\ |B|\ ; \qquad |A|^0 \equiv |I|$$

this is an infinite geometric series whose sum is

$$|P| = \left| |I| - |A| \right|^{-1} |B|$$

which is equation (12) as obtained previously

5. Moments of the First Time to Failed State

In this section expressions are derived for the (pseudo) generating functions $g_{ij}(z)$ for the moments $\tau_{ij}(k)$, k=1, ,--n, of the (pseudo) random variables τ_{ij}, These random variables are defined as, $\tau_{ij} \equiv$ "first time from acceptable state A_i to failed state F_j".

These moments are the moments of the first time the complex system passes from state $A_i \varepsilon A$ to failed state $F_j \varepsilon F$.

Since the discrete time approach is being used, it is standard practice to define the generating function $g_{ij}(z)$ for these moments in terms of its one sided z-transform. That is, the generating function $g_{ij}(z)$ is defined as

$$g_{ij}(z) = \sum_{n=1}^{\infty} z^n \; p_{ij}(n) \qquad \text{------------------} \; 20$$

Definition 4. Moments of First Time to Failed State

The moments $\tau_{ij}(k)$, k=1,...n, of the first time to failed state, are defined as the moments of the first time the complex system passes from acceptable state $A_i \varepsilon A$ to failed state $F_j \varepsilon F$. These moments are obtained from the generating function (20) in the conventional way

$$\tau_{ij}(k) = \frac{d^k}{dz^k}(g_{ij}(z))\Bigg|_{z=1}, \; k = 1,2,\ldots n \qquad \text{-------} \; 21$$

The following theorem shows that it is possible to obtain the (pseudo) generating functions $g_{ij}(z)$ in terms of matrices $|A|$ and $|B|$, without the need for evaluating infinite series of the form (20).

Letting $|G(z)|$ be the $(r \times m)$ matrix of the generating functions $|g_{ij}(z)|$, we have

<u>Theorem 3</u>. Let $|\tau(k)|$ be the $(r \times m)$ matrix of the k^{th}, $k=1,---,n$ moments $\tau_{ij}(k)$. For a complex system with r acceptable and m failed states and having subsystems with known constant failure and repair rates, (that is with known matrix $|M|$), the moments $|\tau(k)|$ are

$$|\tau(k)| = \left.\frac{d^k |G(z)|}{dz^k}\right|_{z=1} \qquad k = 1,2,....n$$

where

$$|G(z)| = z \left| |I| - z |A| \right|^{-1} |B| \qquad ----------- 22$$

where $|A|$ $|I|$ and $|B|$ are the partitions of the $|M|$ matrix, as given in equation (7)

<u>Proof</u> (See reference 9, page 16)

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6. The Exit Probability Functions

In this section equations are derived for the exit probability functions $w_i(n)$, defined as

$$w_i(n) = \text{Prob} \{S(n+t) \in F \mid S(t) = A_i\} \qquad ------- 23$$

$$= \sum_{j=1}^{m} \text{Prob} \{S(n+t) = F_j \mid S(t) = A_i\} \qquad -- 24$$

Thus, $w_i(n)$, is the probability that the complex system will pass from acceptable state A_i, into the set F in n units of time. Obviously comparing (15) and (24), we have

$$w_i(n) = \sum_{j=1}^{m} p_{ij}(n) \quad \text{------------------} \quad 25$$

Equation (25) states that $w_i(n)$ is the sum of the probabilities $p_{ij}(n)$ on the set F.

Letting $\overline{W}(n)$ be the (r x 1) column vector of the exit probability functions, we have

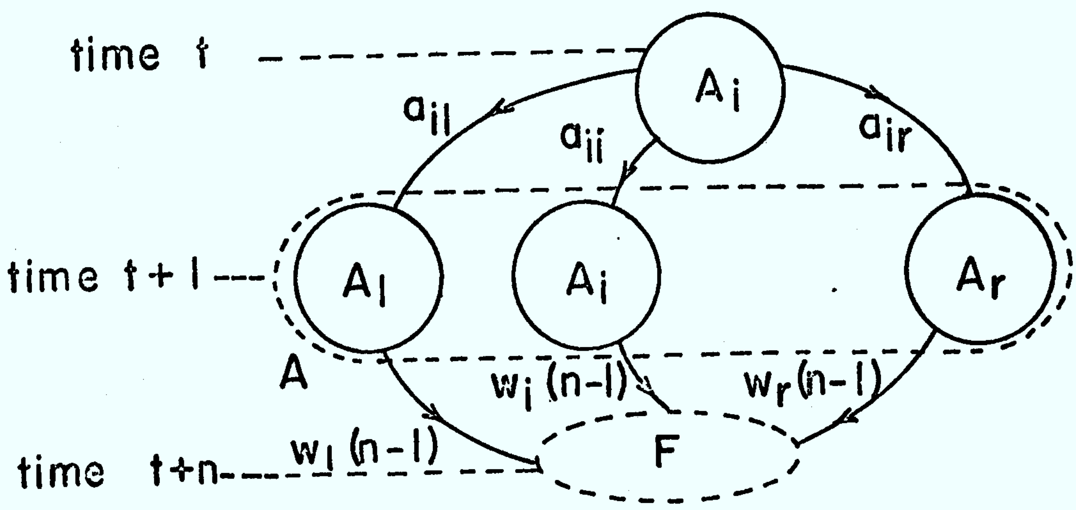
<u>Theorem 4</u> The exit probability functions equation (23), satisfy

$$w_i(1) = \sum_{j=1}^{m} b_{ij}, \qquad i=1,---,r \quad \text{-------} \quad 26$$

$$\overline{W}(n) = |A| \; \overline{W}\,(n-1) \quad \text{-----------------} \quad 27$$

<u>Proof</u>

Result (26) follows by equating (4) and (24) after letting n=1, in (25). To prove (27) consider Evolution Diagram 2.



Evolution Diagram 2

The above diagram enumerates the possible evolutions in n steps from any state $A_i \in A$ to the class F of failed states. Then, summing the transmittances of the paths incident on F, gives for each of the r states $A_i \in A$, an expression for $w_i(n)$, namely

$$w_i(n) = \sum_{k=1}^{m} a_{ik} \, w_k(n-1), \quad i=1,---,r$$

QED.

7. Moments of the First Exit Time From Acceptable Class A

In this section equations are derived for the generating functions $c_i(z)$ for the moments $\tau_i(k)$, $k=1,---,n$ of the random variables $\tau_i \equiv$ "first exit time from acceptable state A_i into failed class F".

Obviously $w_i(n)$ is the probability distribution function of this random variable τ_i, that is

$$w_i(n) = \text{Prob}\{\tau_i = n\}$$

The moments, $\tau_i(k)$, $k=1,---,n$ are the moments of the first time the complex system passes from state $A_i \in A$ into the class of failed states F. In the reliability literature, these moments are called moments of the first time to failure *).

Since the discrete time approach is being used, we again define the generating function $c_i(z)$ in terms of its one sided z transform. The generating function $c_i(z)$ is therefore

$$c_i(z) = \sum_{n=1}^{\infty} z^n \cdot w_i(n) \qquad \text{-----------------} \quad 28$$

Definition 5. Moments of the First Time to Failure

The moments of the first time to failure, are defined as the moments of the first time the complex system passes from acceptable state A_i to any failed state in F. These moments are obtained from the generating function (28) as

---

*) In particular, the first moment, is the mean time to first failure.

$$\tau_i(k) = \frac{d^k}{dz^k}(c_i(z))\Big|_{z=1} \qquad k=1,2, \ldots n \quad i=1,\ldots r$$

The following theorem establishes the relationship between the generating functions $g_{ij}(z)$ and $c_i(z)$ or, equivalently, the relationship between $\tau_{ij}(k)$ and $\tau_i(k)$.

<u>Theorem 5</u> The generating functions $g_{ij}(z)$ and $c_i(z)$ are related as

$$c_i(z) = \sum_{j=1}^{m} g_{ij}(z) \quad \text{------------------- } 29$$

or equivalently

$$\tau_i(k) = \sum_{j=1}^{m} \tau_{ij}(k) \quad \text{------------------- } 30$$

<u>Proof</u>

Substituting (25) into (28) gives

$$\tau_i(z) = \sum_{n=1}^{\infty} \sum_{j=1}^{m} z^n p_{ij}(n) \quad \text{---------- } 31$$

Interchanging the order of summation in (31)

$$c_i(z) = \sum_{j=1}^{m} \sum_{n=1}^{\infty} z^n p_{ij}(n) \quad \text{------------- } 32$$

substituting (20) into (32) we obtain (29). Equation (30) follows, by definition.

QED.

Comment

Equation (29) can be computed directly once (21) has been evaluated, or directly, in terms of the matrices $|A|$ and $|B|$, as given in theorem 6 below.

Letting $\overline{C}(z)$ be the $(r \times 1)$ column vector of the generating functions $c_i(z)$, we have

Theorem 6 Let $\overline{\tau}(k)$ be the $(r \times 1)$ vector of the moments $\tau_i(k)$. For a complex system operating in a repair environment and having r acceptable and m failed states and known matrix $|M|$, these moments are

$$\overline{\tau}(k) = \frac{d^k}{dz^k}\left(\overline{C}(z)\right)\Bigg|_{z=1}$$

with

$$\overline{C}(z) = z\,\Big|\,|I| - z\,|A|\,\Big|^{-1}\,\overline{B}' \qquad \text{-----------} \quad 33$$

where *)

$$\overline{B}' = |b'_1, \ldots\ldots; b'_r|^{\#}$$

and

$$b'_i = \sum_{j=1}^{m} b_{ij}, \qquad i=1,---,r$$

Proof (See reference 9, page 22)

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*) The # in $|- - -|^{\#}$, means "transpose".

Example I

This example illustrates how the theory presented in the paper can be used to obtain a probabilistic description of the complex systems (a) and (b) below. It will be shown, that the reliability modelling of these two different systems leads to identical analytical expressions for their reliability functions, moments of the first time to failure, etc.

(a) The Telecommunication Network (Figure I)

(b) The Parallel Standby System (Figure 2)

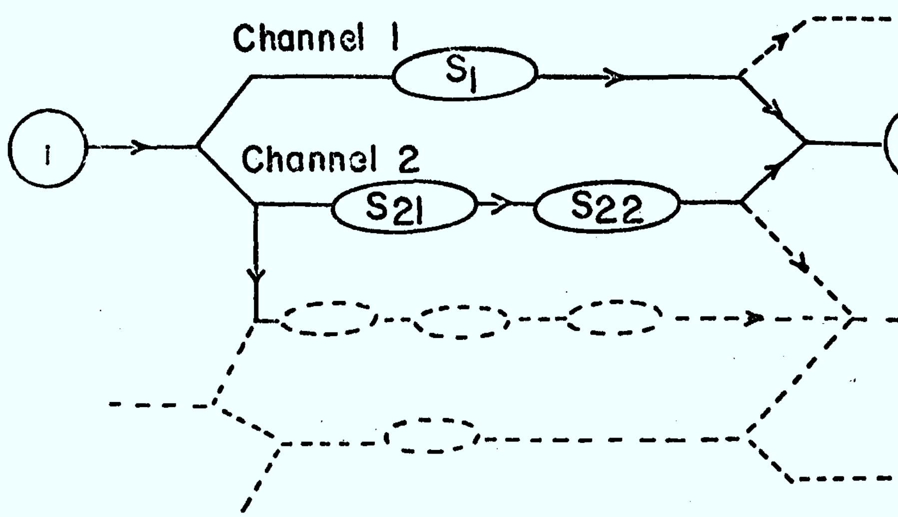

Figure I

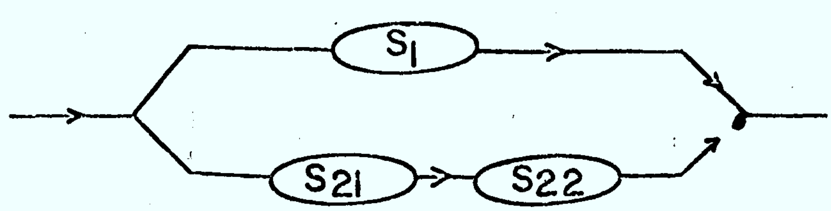

Figure 2

(a) State/Word Description of the Telecommunication Network (Figure 1)

The network consists of two channels between i and j. Channel 1, is in continuous use, and channel 2 provides a standby path, that is always available for use whenever channel 1 fails. For the purposes of this example, we assume channel 1, fails whenever system S_1 fails and channel 2 fails whenever systems S_{21} or S_{22} fail. The operation of the network requires that whenever S_1 fails, channel 1 is closed down and channel 2 immediately begins to handle traffic. Repair crews may not be immediately available to start repairing S_1 and there is therefore usually a delay before work starts. The network is then considered to be in a failed state when either system S_{21} or S_{22} fails before work has been started on S_1 or before work has been completed on S_1.

The reliability function R(n), is then the probability that the transmission path will exist between points i and j in the network (Figure 1) for time n.

From the above verbal description, it is possible to make the state assignment shown in Figure (3). In keeping with the notation used above, the acceptable states are labelled A_i, i=1,2,3 and the failed states are labelled F_j, j=1,2,3,4.

State Assignment

State	Word Description
A_1	Both channel 1 and channel 2 are operating, that is systems S_1 and S_{21} and S_{22} are operating.
A_2	S_1 fails (that is channel 1, closes down and channel 2, immediately begins to operate.
A_3	Service begins on system S_1 of channel 1.
F_1	System S_{21} of channel 2 fails before service to system S_1 begins.
F_2	System S_{22} of channel 2 fails before service to S_1 begins.
F_3	System S_{21} fails before service to system S_1 is complete.
F_4	System S_{22} fails before service to system S_1 is complete.

Figure 3

(b) State/Word Description of the Parallel Standby System (Figure 2)

The parallel standby system (Figure 2), consists of an on line system S_1, and a standby system S_2. System S_2 acts as a perfect spare, that is, S_2 cannot fail while it is on standby. Although system S_2 performs the same functions as system S_1, it fails if its subsystems S_{21} or S_{22} fail. The operation of the system requires that on failure of S_1, the system S_2 immediately goes on line. Repairs crews may not be immediately available to start repairing S_1 and there is usually a delay before work starts. The system is considered to be in a failed state when either S_{21} or S_{22} fails before work has been started on S_1 or before work has been completed on S_1.

The two systems S_1 and S_2 are not identical and therefore have different failure and repair rates.

From the above description, it is possible to make the state assignment shown in Figure(4). In keeping with the notation used above, the acceptable or working states are labelled A_i, $i=1,2,3$ and the failed states are labelled F_j, $j=1,2,3,4$.

State Assignment

State	Word Description
A_1	Both systems S_1 and S_2 are operating
A_2	S_1 fails and S_2 immediately goes on line
A_3	Service begins on S_1
F_1	S_{21} fails before service to S_1 begins
F_2	S_{22} fails before service to S_1 begins
F_3	S_{21} fails before service to S_1 is complete
F_4	S_{22} fails before service to S_1 is complete

Figure 4

Obviously, since both the systems (a) and (b) have equivalent state assignments (See Figures 3 and 4), then they will have the same transition Graph (Figure 5) and $|M|$ matrix (Figure 6). The methods presented can now be applied to find R(n) etc.

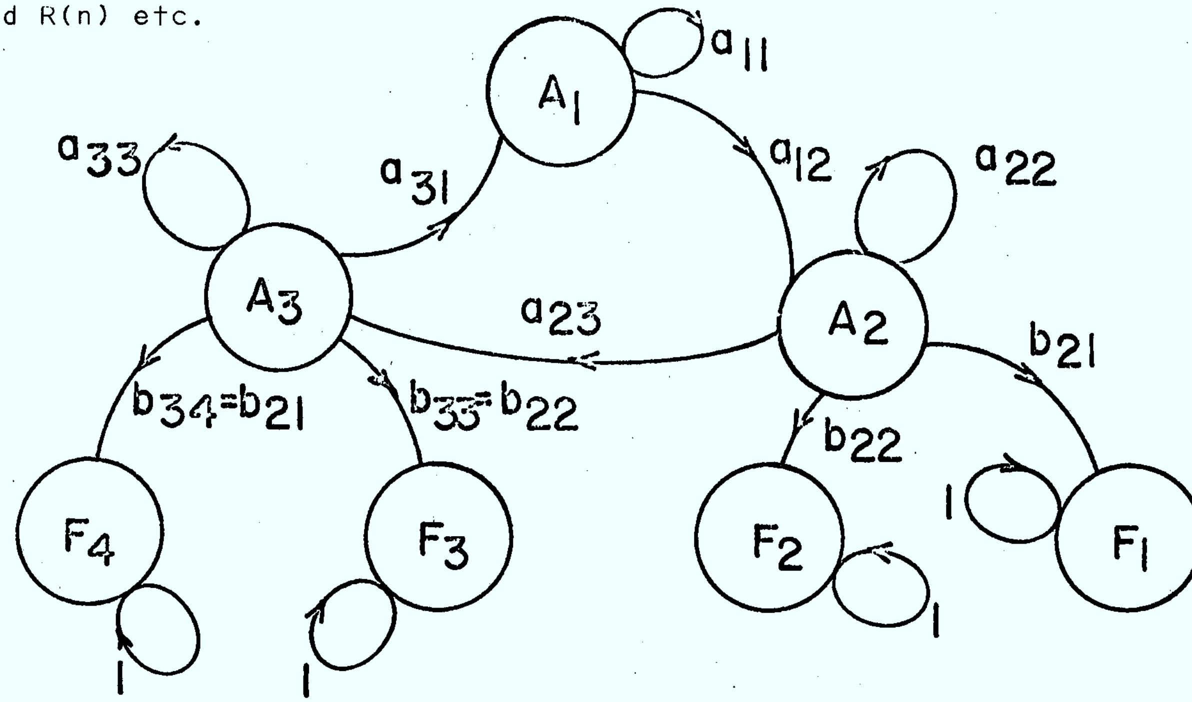

Figure 5

The $|M|$ matrix can be copied directly from Figure (5), it is

$$
[M] = \begin{array}{c|c}
\text{time } t \searrow \text{ time } t+1 & \begin{array}{ccccccc} A_1 & A_2 & A_3 & F_1 & F_2 & F_3 & F_4 \end{array} \\
\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \end{array} &
\left[\begin{array}{ccc|cccc}
a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 \\
0 & a_{22} & a_{23} & b_{21} & b_{22} & 0 & 0 \\
a_{31} & 0 & a_{33} & 0 & 0 & b_{33} & b_{34} \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

$[A]$ (rows A_1–A_3, columns A_1–A_3), $[B]$ (rows A_1–A_3, columns F_1–F_4), $[O]$ (rows F_1–F_4, columns A_1–A_3), $[I]$ (rows F_1–F_4, columns F_1–F_4)

Figure 6

Example 2 [10]

Consider the following simplified version (Figure 7) of the telecommunication network (Figure I). This network operates in exactly the same way as the network of (Figure I), except that system S_2 is not decomposed into two systems S_{21} and S_{22}.

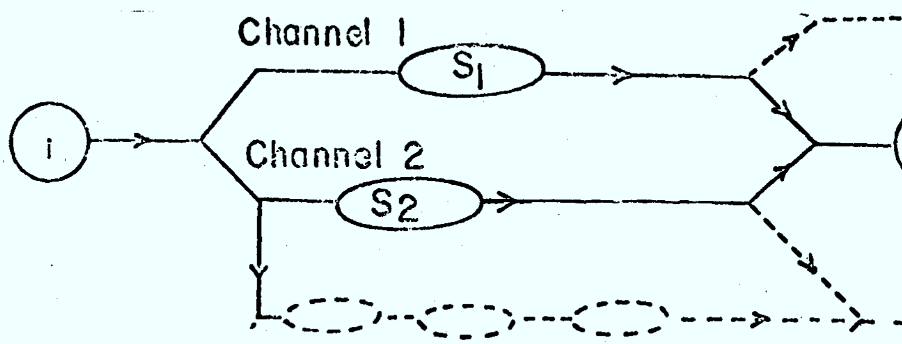

Figure 7

The state/word assignment for this system is shown in Figure (8).

State Assignment

State	Word Description
A_1	Both S_1 and S_2 provide a path from i to j
A_2	S_1 fails and the only path is provided by S_2. Repairs to S_1 are not yet started.
A_3	Repairs to S_1 start. S_2 is still providing the connections between i and j
F_1	S_2 fails before repairs to S_1 have begun.
F_2	S_2 fails before repairs to S_1 are completed.

Figure 8

For ease of numerical computation we assume that the failure and repair rates of S_1 and S_2 are equal and are respectively

$$\lambda = .002/\text{hr.}$$

$$\mu = .004/\text{hr.}$$

Likewise we define a delay rate ρ as

$$\rho = \frac{1}{\text{average time before repairs begin}} = .2/\text{hr.}$$

Taking the one step transition probabilities as characterizing the state behaviour of the system for one hour,*) we can draw the transition graph (Figure 9) and from it obtain the $|M|$ matrix (Figure 10).

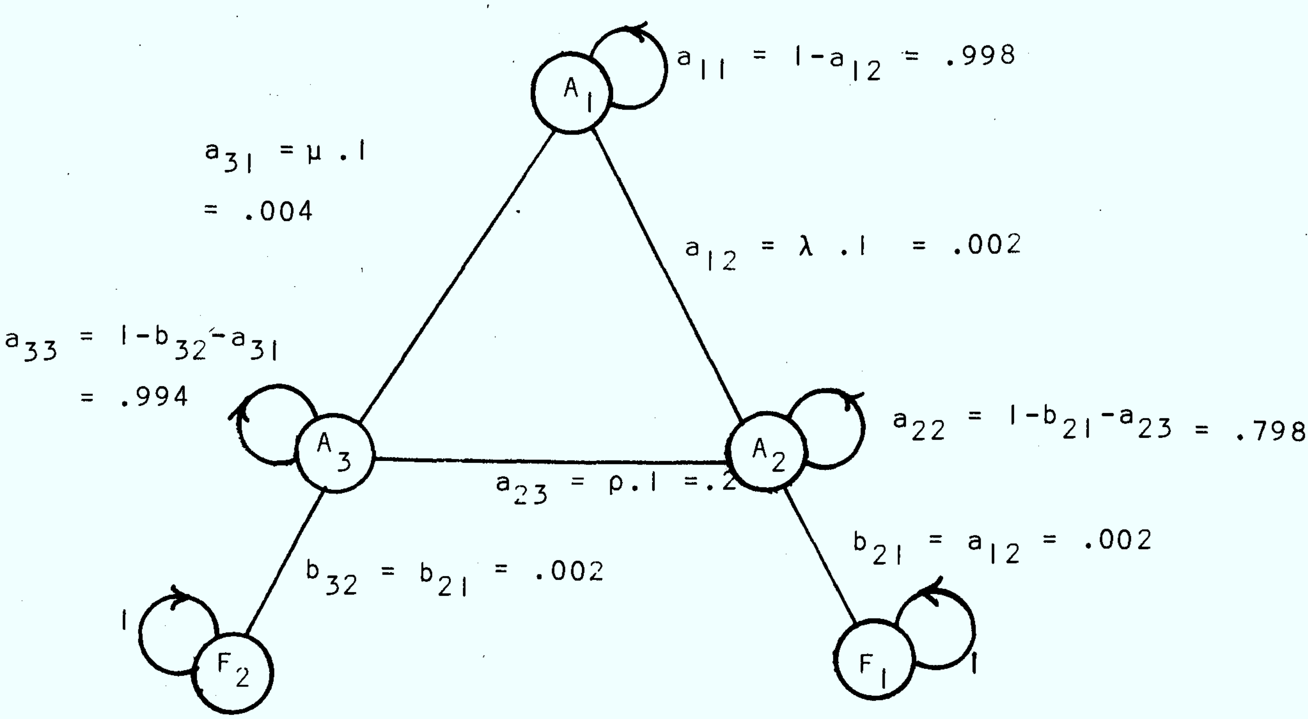

Figure 9

*) See footnote on Page 3.

$$
|M| \equiv
\begin{array}{c|c|c|c||c|c|}
 & A_1 & A_2 & A_3 & F_1 & F_2 \\
\hline
A_1 & .998 & .002 & 0 & 0 & 0 \\
\hline
A_2 & 0 & .798 & .2 & .002 & 0 \\
\hline
A_3 & .004 & 0 & .994 & 0 & .002 \\
\hline\hline
F_1 & 0 & 0 & 0 & 1 & 0 \\
\hline
F_2 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}
\equiv
\begin{array}{c|c}
|A| & |B| \\
\hline
|0| & |I|
\end{array}
$$

Figure 10

Now the equations (12) and (22) are

$$|P| = \big||I| \;\; = \;\; |A|\big|^{-1} \; |B| \qquad \text{----------------- } 12$$

$$|G(z)| = z \; \big||I| \; - z|A|\big|^{-1} \; |B| \equiv z|N(z)||B| \qquad \text{-------}22$$

Performing the matrix inversion in (22) gives $|N(z)|$

$$|N(z)| = \frac{1}{1-2.8z+2.6z^2-.8z^3}
\begin{vmatrix}
(1-.8z)\,(1-z\,), & .002z(1-z), & -\,.0004z^2 \\
.0008z^2\,, & (1-z)\,(1-z), & .2z(1-z) \\
.004z(1-.8z), & .000008z^2\,, & (1-z)\times(1-8z)
\end{vmatrix}$$

comparing (12) and (22) it is obvious that

$$|P| = |N(z)|\;|B|\Big|_{z=1}$$

Thus,

$$|P| = \begin{array}{c|c|c|} & F_1 & F_2 \\ \hline A_1 & .03 & .97 \\ \hline A_2 & .03 & .97 \\ \hline A_3 & .02 & .98 \\ \hline \end{array}$$

From $|P|$, it is seen, that the small delay (ρ = .2/hr.) of average duration 5 hours, before repairs start to S_1 does not seriously affect the long run operating performance of the overall network. Specifically it is seen that failure will usually occur to state F_2 rather than F_1.

SUMMARY

It has been shown, that the reliability modelling of complex systems with many failed states presents no particular problem, once the matrix $|M|$ of failure and repair rates of the component systems of the complex system is known.

The only computations required are simple linear operations on Matrices [11,12,13,14] and thus all the methods presented are ideally suited for digital computer computation.

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Explanation of Notation

$S(\cdot)$	Stationary Markov process characterizing the state of the complex system.
$\lvert M \rvert$	The $(r + m)$ square matrix of the one step transition probabilities.
ε	Set inclusion.
$\cup$	Set union.
A	The set of r acceptable states.
A_i	The acceptable state $A_i \ \varepsilon \ A$, $i = 1, \ldots, r$.
a_{ik}	The one step transition probability between acceptable states A_i and A_k.
$\lvert A \rvert \equiv \lvert a_{ik} \rvert$	The r square matrix of the one step transition probabilities between the r acceptable states.
F	The class containing the m failed states.
F_j	The failed state $F_j \ \varepsilon \ F$, $j = 1, \ldots, m$.
$\lvert I \rvert$	The m square matrix of the one step transition probabilities between the m failed states. This is by definition the unit $(m \times m)$ matrix.
b_{ij}	The one step transition probability between acceptable state A_i and failed state F_j.
$\lvert B \rvert \equiv \lvert b_{ij} \rvert$	The $(r \times m)$ matrix of the one step transition probabilities from the class A of acceptable states to the class F of failed states.
$\lvert 0 \rvert$	The $(m \times r)$ null matrix.
$s_i(n)$	The probability that at time n, the complex system is in state A_i.
$s_j(n)$	The probability that at time n, the complex system is in state F_j.

$\bar{s}(n)$	The $1\times(r+m)$ vector of the state probability functions.
$R(n)$	The reliability function. $R(n)$ is the probability that at time n, the complex system is operating acceptable, that is, in some acceptable state in A.
p_{ij}	The steady state transition failure probability. This is the probability of going eventually from acceptable state A_i to failed state F_j.
$\lvert P\rvert \equiv \lvert p_{ij}\rvert$	The $(r \times m)$ matrix of steady state transition failure probabilities.
$p_{ij}(n)$	The transition probability failure function. This is the probability that the complex system goes from acceptable state A_i to failed state F_j in n units of time.
$\lvert P(n)\rvert \equiv \lvert p_{ij}(n)\rvert$	The $(r \times m)$ matrix of the transition probability failure functions.
τ_{ij}	The random variable "first time the state of the system is failed state F_j given that the network initially started in acceptable state A_i".
$\tau_{ij}(k)$	The k^{th} moment of τ_{ij}; $k = 1, 2, \ldots$
$\lvert \tau(k)\rvert \equiv \lvert\tau_{ij}(k)\rvert$	The $(r \times m)$ matrix of the k^{th} moments of τ_{ij}.
$g_{ij}(z)$	The generating function for the moments $\tau_{ij}(k)$.
$\lvert G(z)\rvert \equiv \lvert g_{ij}(z)\rvert$	The $(r \times m)$ matrix of the above generating functions.
$w_i(n)$	The probability that after n units of time the complex system will be in some failed state in F, given that it initially started in acceptable state A_i. This is called the exit probability function.

$\overline{W}(n)$	The $(r \times 1)$ column vector of the exit probability function.
τ_i	The random variable "first exit time from acceptable state A_i into the failed class F".
$\tau_i(k)$	The k^{th} moment of τ_i; $k = 1, 2, \ldots$
$\overline{\tau}(k)$	The $(r \times 1)$ column vector of the k^{th} moments of τ_i.
$c_i(z)$	The generating function for the moments $\tau_i(k)$.
$\overline{C}(z)$	The $(r \times 1)$ vector of the above generating functions.

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