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FUNCTIONAL GRAPHS OF INTERCONNECTED SYSTEMS

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FUNCTIONAL GRAPHS OF INTERCONNECTED SYSTEMS

by

R. Jakubowski* and M. Krieger**

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ABSTRACT

This research report presents the functional graph representation of a complex system whose elements are dynamic systems. The properties of these complex systems are discussed in detail. Finally, a method for obtaining a supervisory program based on the interconnections of the dynamic systems is developed.

RÉSUMÉ

Cette étude donne une représentation de graphes fonctionnels d'un système complexe dont les éléments constituants sont des systèmes dynamiques. Les auteurs traitent de façon détaillée des particularités de ces systèmes complexes. Finalement, ils proposent une méthode d'élaboration d'un programme de surveillance fondé sur les interconnexions des systèmes dynamiques.

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1. INTRODUCTION

In a previous paper [1], we considered the functional graph representation of a dynamic system, together with its digital computer simulation. In this paper, we are presenting a method for deriving the functional graph representation of a complex system, $\mathcal{G}$, whose elements are dynamic systems as defined previously [1], and we shall refer to them as subsystems or subgraphs.

<u>List of Symbols</u>:

$\{a_1, a_2, \ldots, a_n\}$	:	a set of elements $a_1, a_2, \ldots, a_n$
$\in$	:	is an element of
$\cap$	:	intersection
$\cup$	:	union
$\subset$	:	is contained in
$\setminus$	:	set-theoretic difference
$\emptyset$	:	empty set
$\langle a_1, a_2, \ldots, a_n \rangle$	:	an ordered n-tuple
$\Longrightarrow$	:	implies
$\exists$	:	existential quantifier
$\forall$	:	universal quantifier
$\vee$	:	disjunction
$\wedge$	:	conjunction

Our basic purpose is to present a second hierarchy of computations, which will give an ordering by which each subsystem has to be considered. In line with this, we shall define some types of terms as in [1], using subgraphs instead of functional relations as basic elements.

2. CONNECTION OF SUBGRAPHS

The interconnection of subgraphs is achieved by introducing a special interpretation of zero-argument functional branches and by providing a partitioning of the nodes of the subgraphs. We now consider a specific subgraph $G^{(\ell)}$ defined as follows:

<u>Definition 2-1</u>

A functional subgraph $G^{(\ell)}$ is a system of relations of the following form:

$$G^{(\ell)} = \langle Z^{(\ell)}, \; F^{(\ell)}, \; R^{(\ell)} \rangle \qquad \ldots \qquad (2-1)$$

where

$Z^{(\ell)}$ are the elements of the ℓ^{th} system of relations;

$F^{(\ell)}$ is the set of functional relations of the ℓ^{th} subsystem; and

$R^{(\ell)}$ is the set of non-functional relations of the ℓ^{th} subsystem.

In this definition, $F_i^{(\ell)}$ and $R_{i,\delta}^{(\ell)}$ are, respectively, functional and non-functional relations as defined in the previous paper [1]. Furthermore, the set of indices of subgraph $G^{(\ell)}$ we denote by $I^{(\ell)}$.

To connect two subgraphs requires the inclusion of additional inputs in each subgraph. These additional inputs are represented by special types of source nodes which we associate with zero-argument functional branches. Considering the set of indices of these zero-argument functional branches, we define a partition of the set of indices $I^{(\ell)}$ of subgraph $G^{(\ell)}$ as follows:

Definition 2-2

The set of indices $I^{(\ell)}$ of $G^{(\ell)}$ is partitioned into two subsets of mutually exclusive indices $\mathcal{A}^{(\ell)}$ and $\mathcal{B}^{(\ell)}$ defined as follows:

$$\mathcal{A}^{(\ell)} = \{i \epsilon I^{(\ell)} : a_{i,1}^{(\ell)} = 0;\ i \neq 0\} \qquad \ldots \qquad (2\text{-}2)$$

$$\mathcal{B}^{(\ell)} = I^{(\ell)} \setminus \mathcal{A}^{(\ell)} \cup \{0\} \qquad \ldots \qquad (2\text{-}3)$$

where $a_{i,1}^{(\ell)}$ is the first element of the characteristic $A_i^{(\ell)}$ of the functional relation $F_i^{(\ell)}$ and denotes the number of its arguments.

In other words, $\mathcal{B}^{(\ell)}$ denotes the set of indices of $G^{(\ell)}$ which is associated with non-zero-argument functional branches. $\mathcal{A}^{(\ell)} \cup \{0\}$ denotes the set of indices associated with all the zero-argument functional branches in $G^{(\ell)}$. Thus, each subgraph $G^{(\ell)}$ can include two types of zero-argument functional branches, namely Type-1, a single zero-argument functional branch $F_0^{(\ell)}$ representing the constant source node as introduced in [1], and Type-2, the zero-argument functional branches associated with the additional source nodes that are the inputs of $G^{(\ell)}$. The set of indices of the Type 2 zero-argument functional branches is given by $\mathcal{A}^{(\ell)}$, which may be empty.

Definition 2-3

Subgraph $G^{(k)}$ is said to be connected to subgraph $G^{(\ell)}$ if, and only if,

$$\underset{\substack{\langle j, i \rangle \\ j \epsilon \mathcal{B}^{(k)};\ i \epsilon \mathcal{A}^{(\ell)}}}{\exists} y_j^{(k)} = y_i^{(\ell)} \qquad \ldots \qquad (2\text{-}4)$$

In other words, $y_j^{(k)}$ are the source nodes of subgraph $G^{(k)}$ that supply the additional inputs to subgraph $G^{(\ell)}$. For bookkeeping purposes, the parameters of the zero-argument functional branches $F_i^{(\ell)}$ associated with the nodes $y_i^{(\ell)}$ of Equation 2-4 are denoted as follows:

$p_{i,1}^{(\ell)}$ = 0, as it is a source node;

$p_{i,2}^{(\ell)}$ = k, the index of the subgraph connected to $G^{(\ell)}$;

$p_{i,3}^{(\ell)}$ = j, the index of the source node in the subgraph $G^{(k)}$.

For further clarification, the following notation is introduced:

$$C_{k,\ell} = \{j \in \mathcal{B}^{(k)} : \forall_j \exists_{i \in \mathcal{A}^{(\ell)}} y_j^{(k)} = y_i^{(\ell)}\} \quad \ldots \quad (2\text{-}5)$$

Thus, the set $C_{k,\ell}$ is the set of indices associated with the source nodes of $G^{(k)}$ that are inputs of $G^{(\ell)}$.

Using this notation, Definition 2-3 can be restated as follows: "Subgraph $G^{(k)}$ is said to be connected to subgraph $G^{(\ell)}$ if, and only if, $C_{k,\ell} \neq \emptyset$."

Definition 2-4

A sequence of subgraphs $G^{(\ell_1)}$, $G^{(\ell_2)}, \ldots,$ $G^{(\ell_p)}$ is said to form a circuit, denoted by the character $\mathcal{f} = \{\ell_1, \ell_2, \ldots, \ell_p\}$, if, and only if,

$$\forall_{\ell_k,\ k=1,2,\ldots,p-1} C_{\ell_k, \ell_{k+1}} \neq \emptyset \quad \ldots \quad (2\text{-}6)$$

In other words, in this sequence of subgraphs every $G^{(\ell_k)}$ is connected to $G^{(\ell_{k+2})}$ for k = 1, 2, ..., p-1.

Definition 2-5

A sequence of subgraphs $G^{(\ell_1)}$, $G^{(\ell_2)}$ ... $G^{(\ell_p)}$, is said to form a closed circuit if, and only if, they form a circuit and $C_{\ell_p, \ell_1} \neq \emptyset$.

Definition 2-6

The set of all characters associated with the closed circuits in the complex system $\mathcal{G}$ is called the composition N of $\mathcal{G}$.

$$N = \{\mathcal{f} : \mathcal{f} \text{ is the character of a closed circuit in } \mathcal{G}\} \quad \ldots \quad (2\text{-}7)$$

3. COMPLEX FUNCTIONAL GRAPHS

In this section, we shall consider a complex system $\mathcal{S}$ as represented by interconnected subgraphs $G^{(\ell)}$. Here, a subgraph $G^{(\ell)}, \ell \in J$, is as defined in the previous section, and J is the set of indices of all the subgraphs in the complex system. To manipulate such a complex system, we define a binary relation, W, representing a connection and a hierarchical graph, H, corresponding to the interconnected system.

<u>Definition 3-1</u>

The binary relation $kW\ell$, $k, \ell \in J$, is said to hold if, and only if, $C_{k,\ell} \neq \emptyset$. Thus, the set of ordered pairs W is given by:

$$W = \{\langle k, \ell \rangle : kW\ell\} \qquad \ldots \qquad (3\text{-}1)$$

<u>Definition 3-2</u>

The hierarchical graph, H, of a complex system is given by the system of relations:

$$H = \langle J, W \rangle \qquad \ldots \qquad (3\text{-}2)$$

where

J, the set of indices of the subgraphs, represents the elements of this system of relations; and

W is the binary relation defined by Equation 3-1.

To obtain a graphical representation of H, the elements of J are represented by nodes, and relation W by a directed line. This is shown schematically in Figure 3-1.

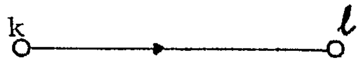

Figure 3-1. Graphical representation of $kW\ell$.

Definition 3-3

By the environment of $G^{(\ell)}$, we mean the set of indices Π_ℓ such that

$$\Pi_\ell = \{ k \in J : k W \ell \} \qquad \ldots \qquad (3\text{-}3)$$

or, by definition 2-3,

$$\Pi_\ell = \{ p_{i,2}^{(\ell)} : i \in \mathcal{A}^{(\ell)} \}. \qquad \ldots \qquad (3\text{-}4)$$

In other words, Π_ℓ is the set of indices associated with the subgraphs connected to $G^{(\ell)}$. Note that this is an extension of the term enclosure of a sink node defined in Reference [1].

Definition 3-4

A cut of zero order of a hierarchical graph $H = \langle J, W \rangle$ is a set of nodes $\mathcal{K}^{(0)}$ such that:

1. $\mathcal{K}^{(0)} \subset J$
2. $\exists_{\ell \in J} \mathcal{A}^{(\ell)} = \phi \Longrightarrow \ell \in \mathcal{K}^{(0)}$
3. $N \neq \phi \Longrightarrow \forall_{f \in N} f \cap \mathcal{K}^{(0)} \neq \phi$

A cut of zero order $\mathcal{K}^{(0)}$ of a hierarchical graph $H = \langle J, W \rangle$ is an extension of the cut of zero order $\mathcal{F}^{(0)}$ of a functional graph $G = \langle Z, F, R \rangle$, as defined in Reference [1]. A set $\mathcal{K}^{(0)}$ includes all source nodes and at least one node from each closed circuit. Note that a source node ℓ of H represents a subgraph $G^{(\ell)}$ that has no additional inputs, and a closed circuit in $\mathcal{G}$ corresponds to a closed circuit in H.

Definition 3-5

A cut of order n of the hierarchical graph $H = \langle J, W \rangle$, for a given cut of zero order $\mathcal{K}^{(0)}$, is the set of indices $\mathcal{K}^{(n)}$ such that:

$$\mathcal{K}^{(n)} = \{ k : k \in J \setminus \bigcup_{j=0}^{n-1} \mathcal{K}^{(j)} \wedge \Pi_k \subset \bigcup_{j=0}^{n-1} \mathcal{K}^{(j)} \} \qquad \ldots \qquad (3\text{-}5)$$

In other words, a cut of order n corresponds to the set of nodes of the hierarchical graph which is the image of the union of all lower-order cuts under the relation W.

Example 3-1 -

Find the higher-order cuts of the hierarchical graph given in Figure 3E-1.

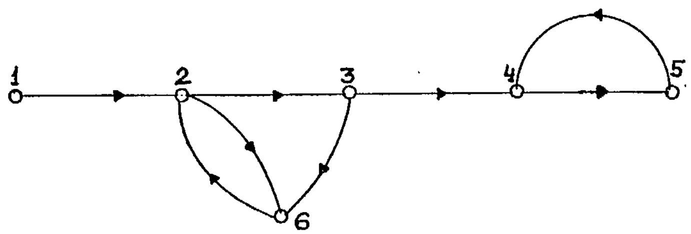

Figure 3E-1. The hierarchical graph of Example 3-1.

As a first step, we need to choose a cut of zero order for this graph. The composition N of this graph is:

$$N = \{\{2,6\},\ \{2,3,6\},\ \{4,5\}\} \qquad \ldots \qquad (3\text{-}6)$$

Thus, a possible $\mathcal{K}^{(0)}$ is

$$\mathcal{K}^{(0)} = \{1,2,5\} \qquad \ldots \qquad (3\text{-}7)$$

To define the higher-order cuts for this $\mathcal{K}^{(0)}$, we first determine all the environments Π_i:

$$\Pi_1 = \emptyset \qquad \Pi_2 = \{1,6\}$$
$$\Pi_3 = \{2\} \qquad \Pi_4 = \{3,5\}$$
$$\Pi_5 = \{4\} \qquad \Pi_6 = \{2,3\}$$

For deriving $\mathcal{K}^{(1)}$, the following two conditions need to be satisfied simultaneously:

(a) $k \in J \backslash \mathcal{K}^{(0)} \Longrightarrow k \in \{3,4,6\}$

(b) $\Pi_k \subset \mathcal{K}^{(0)} \Longrightarrow k \in \{1,3\}$

Thus,

$$\mathcal{K}^{(1)} = \{3\} \qquad \ldots \qquad (3\text{-}8)$$

For deriving $\mathcal{K}^{(2)}$, the following two conditions need to be satisfied simultaneously:

(a) $k \in J \backslash \mathcal{K}^{(0)} \cup \mathcal{K}^{(1)} \Longrightarrow k \in \{4,6\}$

(b) $\Pi_k \subset \mathcal{K}^{(0)} \cup \mathcal{K}^{(1)} \Longrightarrow k \in \{1,3,4,6\}$

Thus,

$$\mathcal{K}^{(2)} = \{4, 6\}. \qquad \ldots \qquad (3\text{-}9)$$

Since $\mathcal{K}^{(0)} \cup \mathcal{K}^{(1)} \cup \mathcal{K}^{(2)} = J$, there are no higher-order cuts.

Note that in this example we derived the higher-order cuts from basic definitions. In Appendix A a general algorithm to derive this by a computer is given.

At this point, we shall introduce additional definitions to characterize the subgraphs $G^{(\ell)}$ of a complex system $\mathcal{G}$. The purpose of these definitions is to simplify the description of the computational procedure as described in the next section.

Definition 3-6

A subgraph $G^{(\ell)}$ is said to be proper if, and only if, $\mathcal{A}^{(\ell)} = \emptyset$.

In other words, $G^{(\ell)}$ is a subgraph that has no additional inputs, corresponding to a source node in the hierarchical graph H. Note that a proper subgraph $G^{(\ell)}$ has a single zero-argument functional branch $F_o^{(\ell)}$, and it is the type of functional graph presented in [1].

Definition 3-7

A subgraph $G^{(\ell)}$ is said to be complete if, and only if, $\mathcal{A}^{(\ell)} \neq \emptyset$ and

$$\forall_{k \in \Pi_\ell} C_{k,\ell} \subset \mathcal{F}_k^{(0)} \qquad \ldots \qquad (3\text{-}10)$$

where $\mathcal{F}_k^{(0)}$ is the cut of zero order of subgraph $G^{(k)}$.

In other words, the indices which correspond to the additional inputs of $G^{(\ell)}$ are included in zero-order cuts of the subgraphs connected to it. That is, the inputs of $G^{(\ell)}$ correspond to variables, in the other subgraphs, that are computed first.

Definition 3-8

A subgraph $G^{(\ell)}$ is said to belong to class $\mathcal{C}^{(p)}$ if, and only if, $\ell \in \mathcal{K}^{(p)}$.

By this definition, the subgraphs of a complex system are partitioned into disjoint classes according to the higher-order cuts of its hierarchical graph. Furthermore, on the basis of these classes, one can obtain an ordering of the subgraphs.

Definition 3-9

A class $\mathcal{C}^{(q)}$ is said to precede class $\mathcal{C}^{(p)}$ if, and only if, $q < p$.

Note that, to have a meaningful ordering, the corresponding cuts $\mathcal{K}^{(q)}$ and $\mathcal{K}^{(p)}$ are computed with respect to the same zero-order cut $\mathcal{K}^{(0)}$.

4. DIGITAL MODEL AND COMPUTATIONAL PROCEDURE

The digital model $\hat{\mathcal{G}}$ of a complex system is the system obtained from $\mathcal{G}$ by transforming each subgraph $G^{(\ell)} \in \mathcal{G}$ into its corresponding digital model $\hat{G}^{(\ell)}$. The digital model of a subgraph is as defined in Reference [1].

To derive the required sequence of computations for simulating a complex system $\mathcal{G}$, we extend the approach previously needed to derive the sequence of computations of the digital model $\hat{G}$. Here, we need to define an ordering by which the different subgraphs $G^{(\ell)}$ are simulated. As seen in [1], to simulate a particular subgraph $G^{(\ell)}$, at each integration step it is required to have all of its inputs specified. Thus, in complex systems we need to define a computation cycle. First, we execute an integration step for the class of subgraphs whose inputs are known at the start of each cycle, and then we sequentially execute an integration step for the class of subgraphs whose inputs are dependent on the previous classes of subgraphs. If this can be done for all the subgraphs, we say that the given system $\hat{\mathcal{G}}$ is computable. In this sense, not every digital model $\hat{\mathcal{G}}$ is computable, and to show this we introduce the following theorem.

Theorem 4-1

If in a complex system $\mathcal{G}$, there exists a zero-order cut $\mathcal{K}^{(0)}$ that includes only indices of proper and complete subgraphs, then $\hat{\mathcal{G}}$ is computable.

Proof -

If we consider the partitioning of the subgraphs according to equivalence classes $\mathcal{C}^{(p)}$ (Definition 3-8), the following can be seen:

1. The inputs of all $G^{(\ell)} \in \mathcal{C}^{(0)}$ are known at the start of each computation cycle, since they are either proper or complete subgraphs (see Definitions 3-6 and 3-7).
2. The inputs of all $G^{(\ell)} \in \mathcal{C}^{(p)}$ are determined by the subgraphs $G^{(\ell)} \in \bigcup_{k=0}^{p-1} \mathcal{C}^{(k)}$ (see Definitions 3-5 and 3-9).
3. The union of all equivalence classes includes all the subgraphs of $\mathcal{G}$.

Now if we choose the classes of the computation cycle to be $\mathcal{C}^{(0)}$, $\mathcal{C}^{(1)}, \ldots, \mathcal{C}^{(r)}$, we satisfy all the requirements for $\hat{\mathcal{G}}$ to be computable.

Corollary 4-1

If every closed circuit of the complex system $\mathcal{G}$ includes at least one complete subgraph, then $\hat{\mathcal{G}}$ is computable.

Proof -

If every closed circuit of $\mathcal{G}$ includes at least one complete subgraph, by Definition 3-4 it is possible to choose a $\mathcal{K}^{(0)}$ that satisfies the requirements of Theorem 4-1.

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If, in a complex system $\mathcal{G}$, the requirements of Corollary 4-1 are not satisfied, then we must introduce isolating blocks [1] [2] in suitable subgraphs to make the required subgraph complete.

To summarize: for a given complex system, $\mathcal{G}$, we have two hierarchies of computations, the first being the computational procedure required for each subgraph $G^{(\ell)}$, which is as defined in [1], and the second being the order in which each subgraph is simulated. To do this we define a $\mathcal{K}^{(0)}$ satisfying the requirements of Theorem 4-1 (if necessary, we include the above-mentioned isolating blocks). Using this $\mathcal{K}^{(0)}$, we specify its equivalence classes as the classes of a computation cycle. This second hierarchy of computations corresponds to a supervisory program.

5. REFERENCES

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2. Steel, G.H., "Programming of digital computers for transient studies in control systems", Internat. J. of Electrical Engineering /Education, Vol. 3, 1965, pp. 261-278.

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APPENDIX A

Computation of Higher-Order Cuts

In this appendix, we present an algorithm for finding the higher-order cuts $\mathcal{K}^{(n)}$ of a hierarchical graph H for a given cut of zero order $\mathcal{K}^{(0)}$. For this purpose, we introduce the following notation and theorem:

Let $S^{(n)}$ be a string of p binary characters of the form

$$S^{(n)} = s_1^{(n)} s_2^{(n)} \ldots\ldots\ldots s_p^{(n)}$$

where p is the number of elements in J, and n = 1, 2, . . . r, with r being the maximal-order cut of H. The characters of this string are defined as follows:

$$s_k^{(n)} = \begin{cases} 1 & \text{if } k \in \bigcup_{i=0}^{n-1} \mathcal{K}^{(i)} \\ 0 & \text{otherwise} \end{cases} \qquad \ldots \qquad (1)$$

Theorem 1

An index $k \in \mathcal{K}^{(n)}$ if, and only if, $s_k^{(n)} = 0$ and $\forall_{j \in \Pi_k} s_j^{(n)} = 1$.

Proof -

From the definition of higher-order cuts, $k \in \mathcal{K}^{(n)}$ if, and only if, the following two conditions are satisfied simultaneously:

a) $k \in J \setminus \bigcup_{i=0}^{n-1} \mathcal{K}^{(i)} \Longrightarrow s_k^{(n)} = 0$

b) $\Pi_k \subset \bigcup_{i=0}^{n-1} \mathcal{K}^{(i)} \Longrightarrow \forall_{j \in \Pi_k} s_j^{(n)} = 1$

On the basis of the above notation and Theorem 1, the algorithm for finding all the higher-order cuts of H for a given $\mathcal{K}^{(o)}$ is derived; this is shown in Figure A-1.

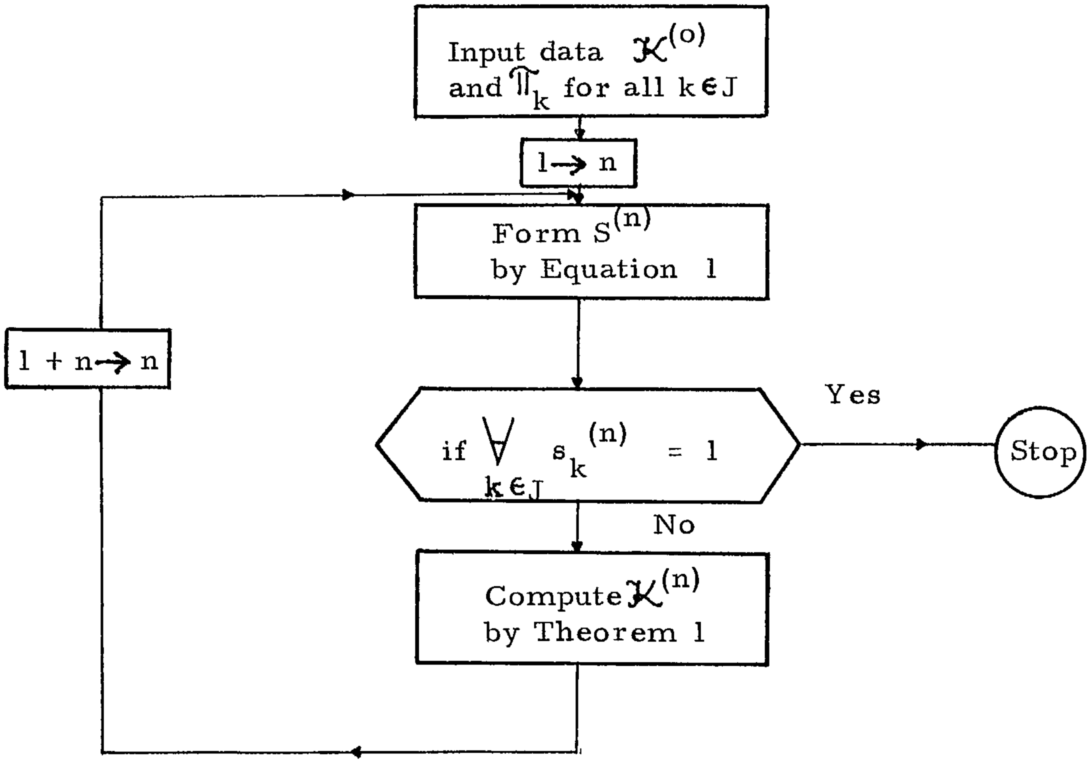

Figure A-1. Algorithm for finding higher-order cuts.

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