Report

on

2. ADAPTIVE KALMAN FILTERING WITH APPLICATIONS
TO ORBITAL VERSION OF UNIFIED STATE MODEL

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CONTENTS

ADAPTIVE ESTIMATION OF THE STATES OF A SYNCHRONOUS-ORBIT SATELLITE WITH UNKNOWN PRIOR STATISTICS

Abstract

A sequential estimator for suboptimal adaptive estimation of the unknown a priori state and observation noise statistics simultaneously with the system state, is applied to the estimation of the states of a synchronous orbit satellite.

Some tracking data supplied by the CRC have been used as real observations. Results of simulation are presented, using this approach with Altman's unified state model of the orbital trajectory and the corresponding obervation model.

Introduction

A well-known limitation of the application of the Kalman-Bucy filter to real world problem is the assumption of known a priori statistics for the stochastic errors in both the state and observation process [1]. This approach leads to a nonadaptive filter and although estimation performance may be satisfactory over some global operating region, it will be inferior to that obtained when a priori statistics are known locally as a function of time. Therefore, in the presence of unknown system disturbance it may be desirable to adaptively estimate the a priori statistics simultaneously with the system state [2].

Often it is desirable to estimate the prior statistics from actual operating records and use these statistics in implementation of the optimum estimation algorithm.

A sequential estimator was derived for suboptimal adaptive estimation of the state and observation noise statistics simultaneously with the system-state [3].

Results of simulation indicate that the adaptive sequential estimator gives fairly accurate estimates of the states.

Suboptimum discrete adaptive estimation algorithm

The process is assumed to be presented by the linear vector difference equation

$$x(k) = \phi(k, k-1)\, x(k-1) + \Gamma(k-1)\, w(k-1) \tag{1}$$

Also, linear measurements are assumed to be available in the form

$$z(k) = H(k)\, x(k) + v(k) \tag{2}$$

The stochastic processes are uncorrelated with one another and assumed to be gaussian sequences with means and covariance matrices

$$E\{w(k)\} = q(k) \qquad E\{v(k)\} = r(k)$$

$$\mathrm{cov}\,\{w(j), w(k)\} = Q(k)\, \delta_{jk}$$

$$\mathrm{cov}\,\{v(j), v(k)\} = R(k)\, \delta_{jk} \tag{3}$$

The filtered estimate $\hat{x}(k|k)$ which results from minimization of a loss function

$$E\{||x(k) - \hat{x}(k|)||^2 \mid z(1), z(2), \ldots z(k)\} \tag{4}$$

may be determined from sequential solution of

$$\hat{x}(k|k) = [I - K(k)\, H(k)]\, \hat{x}(k|k-1) + K(k)\, [z(k) - r(k)] \tag{5}$$

$$\hat{x}(k|k-1) = \phi(k,k-1)\, \hat{x}(k-1|k-1) + \Gamma(k-1)\, q(k-1) \tag{6}$$

$$K(k) = P(k|k-1)\, H'(k)\, [H(k)\, P(k|k-1)\, H'(k) + R(k)]^{-1} \tag{7}$$

To determine the error covariance matrix

$$P(k|k-1) = \phi(k,k-1)\, P(k-1|k-1)\, \phi'(k,k-1) + \Gamma(k-1)\, Q(k-1)\, \Gamma'(k-1) \tag{8}$$

$$P(k|k) = [I - K(k)\, H(k)]\, P(k|k-1) \tag{9}$$

This is the well known Kalman filter for the discrete case.

If the prior means and variances for noise $r(k)$ and $R(k)$ and plant noise $q(k)$ and $Q(k)$ are constant but unknown or random, there are many circumstances in which estimation or identification of these variances may result in great reduction in estimation error variance over that which result form some guessed incorrect value.

An algorithm to implement the adaptive filter is given by "Sage & Husa" in [3]. This algorithm was applied to the linearized system for state estimation of the synchronous orbit satellite.

Simulation Results

Some tracking data supplied by the CRC on a magnetic tape include range plus az-el information.

The tape was read on a CDC/6400 computer, it includes 2 files, each consisting of 39 records. Only the second file gives the range and az-el readings. After transforming these readings into engineering units, they were used as data with the SAF algorithm (suboptimum adaptive filter).

Keeping in mind that the system is nonlinear and the algorithm is for linear systems only, the extended Kalman filter is employed to reduce the effects of nonlinearities which requires that the a priori state error estimation $\hat{x}(k|k-1)$ be set to zero.

Also the matrices $\phi(k,k-1)$ and $H(k)$ consist of partial derivatives which are evaluated on the current estimate of the state derived from the nonlinear equations of motion. Finally $z(k)$ is the difference between the nonlinear true and predicted observation vectors.

Using the data given we can generate the coloured noise v from

$$x(k+1) = \phi x(k) + w(k)$$

$$z(k) = Hx(k) + v(k)$$

and then calculate the variance R.

Starting with the true noise statistics the results obtained prove the convergence of the filter.

The error in the state estimates are very small and decreasing for 156 iterations. Also we notice that the trace of [P] is almost constant. The number of points read from one data file (39 records) was 156 data points. So for 156 iterations the estimator give the following results.

The trace of [P] was constant and trace [P] = -2.3877.

The following is the error in state estimation written every 20 iterations:

STATE ERROR ESTIMATES

$(c-\hat{c})$	$(R_{f1}-\hat{R}_{f1})$	$(R_{f2}-\hat{R}_{f2})$	$(e_1-\hat{e}_1)$	$(e_2-\hat{e}_2)$	$(e_3-\hat{e}_3)$	$(e_4-\hat{e}_4)$
-0.410×10^{-5}	-0.755×10^{-8}	-0.410×10^{-5}	0	0	-0.101×10^{-8}	0.127×10^{-5}
-0.716×10^{-6}	-0.1319×10^{-8}	-0.716×10^{-6}	0	0	-0.176×10^{-9}	0.223×10^{-6}
-0.180×10^{-7}	-0.331×10^{-10}	-0.180×10^{-7}	0	0	-0.443×10^{-11}	0.562×10^{-8}
0.328×10^{-7}	0.601×10^{-10}	0.329×10^{-7}	0	0	0.809×10^{-11}	-0.102×10^{-7}
0.133×10^{-7}	0.237×10^{-10}	0.133×10^{-7}	0	0	0.327×10^{-11}	-0.415×10^{-8}
0.479×10^{-8}	0.786×10^{-11}	0.479×10^{-8}	0	0	0.118×10^{-11}	-0.149×10^{-8}
0.192×10^{-8}	0.250×10^{-11}	0.192×10^{-8}	0	0	0.472×10^{-12}	-0.597×10^{-9}
0.449×10^{-10}	-0.942×10^{-12}	0.468×10^{-10}	0	0	0.113×10^{-13}	-0.145×10^{-10}

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STATE ESTIMATION FOR NONLINEAR SYSTEMS

The research and results obtained in the area of nonlinear filtering have been growing during the past few years. The first definitive contributions were those of Kalman and Bucy [1], [2] which dealt with the optimal estimation of the state variables of a linear dynamical system. These ideas were later extended to the estimation of the states of nonlinear dynamical systems using the so-called first-order filter, or extended Kalman filter [3], [4], [5]. In all of these papers (and many others) different techniques (ie.g. least-squares, maximum-likelihood, etc.) have been used for deriving filter equations. Most of these techniques, at one stage or another employ a Taylor series expansion, neglect second and higher order terms, and use linearized equations to compute the conditional error covariance matrix and the filter gain.

Another method of approach is based on the determination of the exact equations satisfied by the conditional probability density functions and conditional expectations. This approach uses the stochastic It_o calculus and the results indicate that the optimal filter cannot be realized by a finite-dimensional system. However, the exact equations can be approximated to derive suboptimal finite dimensional filters [6], [7], [8]. Some references (orbit estimation) claim considerable improvement in the performance of second-order filters.

The extended Kalman filter was applied to the problem of state estimation for synchronous-orbit satellites and the present research aims to developing a realistic mathematical model for optimal nonlinear estimation.

These approaches are now under investigation.

1. Suboptimal finite dimensional filter by retaining second-order terms.
2. Use of high-order weighting functions via invariant imbedding.
3. Innovations approach to nonlienar filtering.

1. Second-order Filter

Mainly two problems are considered here. First a suboptimal filter for continuous-time nonlinear system was suggested by Athans, Wishner and Bertolini [9]. The primary motivation for this work was provided by problems arising in the estimation of the state variables from discrete-time radar observations. In this approach the state and output nonlinearities are taken into account by simply retaining second-order terms in the usual Taylor series expansions. The results of this approach are summarized below.

Plant:

$$\dot{x}(t) = f(x(t)) \qquad x(t_o) = x_o \quad \text{(state eq.)}$$

$$y(t) = h(x(t)) \qquad \text{(output eq.)}$$

Observations:

$$Z_k = h(x_k) + v_k$$

State Estimate at Observation Time: $t_k = \hat{x}_k$

$$\hat{e}_k = x_k - \hat{x}_k$$

$$\Sigma_k = E\{\hat{e}_k \hat{e}_k'\}$$

State Estimate at t:, $t_k < t < t_{k+1} = w(t)$

$$e(t) = x(t) - w(t)$$

$$S(t) = E\{e(t)e'(t)\}$$

Basic Assumption:

$\hat{e}_k$ and $e(t)$ gaussion, zero-mean

Starting Conditions:

$$\hat{x}_o = w(t_o) = E\{x_o\} = x_o$$

$$\Sigma_o = S(t_o) = E\{(x_o - x_o)(x_o - x_o)'\}$$

Continuous Time Filter: $t_k < t < t_{k+1}$

$$\dot{w}(t) = f(w(t)) + \frac{1}{2} \sum_{i=1}^{n} \phi_i \ \mathrm{tr}\ [F_i(w(t))S(t)]$$

$$w(t_k) = \hat{x}_k$$

$$S(t) = A(w(t))S(t) + S(t)A'(w(t))$$

$$S(t_k) = \Sigma_k$$

Update at $t = t_{k+1}$

$$\hat{x}_{k+1} = w_{k+1} + G_{k+1}\ [z_{k+1} - h(w_{k+1})] - \pi_{k+1}$$

$$\pi_{k+1} = \frac{1}{2} G_{k+1} \sum_{j=1}^{m} \phi_j \ \mathrm{tr}\ [D_j(w_{k+1})S_{k+1}]$$

$$G_{k+1} = S_{k+1}\ C'\ (w_{k+1})\ [C(w_{k+1})\ S_{k+1}\ C'(w_{k+1}) + R_{k+1} + L_{k+1}]^{-1}$$

$$\Sigma_{k+1} = S_{k+1} - S_{k+1}\ C'(w_{k+1})[C(w_{k+1})S_{k+1}\ C'(w_{k+1}) + R_{k+1} + L_{k+1}]^{-1} \cdot C(w_{k+1})S_{k+1}$$

$$(L_{k+1})_{ij} = \frac{1}{2}\ \mathrm{tr}\ [D_i(w_{k+1})\ S_{k+1}\ D_j(w_{k+1})\ S_{k+1}]$$

Where

$$[A(u)]_{\alpha\beta} \triangleq \frac{\partial f_\alpha}{\partial u_\beta}$$

$$[F_i(u)]_{\alpha\beta} \triangleq \frac{\partial^2 f_i}{\partial u_\alpha \partial u_\beta}$$

$$[C(u)]_{\alpha\beta} \triangleq \frac{\partial h_\alpha}{\partial u_\beta}$$

$$[D_j(u)]_{\alpha\beta} \triangleq \frac{\partial^2 h_j}{\partial u_\alpha \partial u_\beta}$$

$$\text{cov}\ [V_k;\ V_j] = R_k\ S_{kj}$$

It has been demonstrated by a specific example [9] that the use of second order filter yields on the average superior performance whenever the nonlinearities are significant.

The second algorithm was suggested by Mahalanabis [10]. This is for noisy nonlinear discrete time systems and is based on a statistical quadrization of the nonlinear functions.

First the functions $f(x_k)$ and $h(x_k)$ are approximated by second order polynomials where

$$x_{k+1} = f(x_k) + u_k$$

$$y_k = h(x_k) + v_k$$

and $E\{u_k\ u_j\ T] = Q_k\ \delta_{kj}$

$$E\{V_k\ V_j\ T] = R_k\ \delta_{kj}$$

$$E\{x_o\} = \hat{x}_o$$

$$E\{(x_o - \hat{x}_o)(x_o - \hat{x}_o)^T] = P_o$$

Assume that the function $f(x_k)$ is expanded in the form

$$f(x_k) = a_k + B_k\ \tilde{x}_{k/k} + \frac{1}{2}\sum_{i=1}^{n} e_i\ \tilde{x}^T_{k/k}\ C^i_k \tilde{x}_{k/k} + \epsilon(x_k)$$

where $\epsilon(x_k)$ = error of approximation.

Now the vector a_k and the matrices B_k and C^i_k are selected so as to minimize $E_k\{\epsilon^T(x_k)\ \epsilon(x_k)\}$ for the best quadratic approximation of $f(x_k)$ in the statistical sense.

Similarly $h(x_k)$ is given by

$$h(x_k) = d_k + M_k \tilde{x}_{k/k-1} + \frac{1}{2} \sum_{j=1}^{m} e_j \tilde{x}^T_{k/k-1} N^j_k \tilde{x}_{k/k-1} + \varepsilon'(x_k)$$

The sequential filter derived in [10] is given by:

$$\hat{x}_{k/k} = \hat{x}_{k/k-1} + K_k \tilde{y}_{k/k-1}$$

$$\hat{x}_{k/k-1} = a_{k-1} + \frac{1}{2} \sum_{i=1}^{n} e_i \, tr\,(C^i_{k-1} P_{k-1/k-1})$$

$$\hat{E}_{k-1} \{y_k\} = d_k + \frac{1}{2} \sum_{j=1}^{m} e_j \, tr\,(N^j_k P_{k/k-1})$$

$$K_k = P_{k/k-1} M^T_k [M_k P_{k/k-1} M^T_k + \frac{1}{2} \sum_{i,j=1}^{m} e_i e^T_j$$

$$tr\,(N^i_k P_{k/k-1} N^j_k P_{k/k-1}) + R_k]^{-1}$$

$$P_{k/k} = (I - K_k M_k) P_{k/k-1}$$

$$P_{k/k-1} = B_{k-1} P_{k-1/k-1} B^T_{k-1} + Q_{k-1} + \frac{1}{2} \sum_{i,j=1}^{n} e_i e^T_j \, tr$$

$$(C^i_{k-1} P_{k-1/k-1} C^j_{k-1} P_{k-1/k-1})$$

given

$$\hat{x}_{o/-1} = \hat{x}_o \quad P_{o/-1} = P_o$$

with given numerical examples it was shown that this algorithm leads to better estimates than either the Taylor's series based second order filter or the first order filter.

2. A Nonlinear Filter with Higher-order Weighting Functions via Invariant Imbedding [1]

The approach chosen here gives a sequential higher-order approximated optimal non-linear filter which estimates the state variables of a continuous-time noisy non-linear dynamical system from noisy non-linear observations. No statistical assumptions are required concerning both the input and measurement noise.

The system is represented by the nonlinear differential equation

$$\dot{x}(t) = f(x(t),t) + w(t)$$

and the observations are

$$z(t) = h(x(t),t) + v(t)$$

The problem is to estimate the current state at t_f, using the least squares criterion so that the following integral is minimized

$$J = \int_{t_o}^{t_f} \{k_1(t)(z(t) - h(x(t),t))^2 + k_2(t)\ (x(t) - f(x(t),t))^2\}\ dt$$

The result is an approximate solution for the optimal filtering where three terms are retained in an infinite expansion of a nonlinear function. Details are given in reference [11].

By giving examples simulated results indicate the effectiveness of this filter compared to the first and second order filters.

3. Innovations Approach

This approach was reported several years ago in a series of papers by Kailath [12], Frost and Kailath [13] and Fugisaki et al. [14].

The innovations method resolves the problem into two parts:

(i) The data process y is transformed into a white noise process ν called the innovations process.

(ii) The optimal estimator is determined as a functional of the innovations process.

Despite the advantages of this method it does not immediately yield any simpler and practically usable nonlinear estimator.

The formula for the optimal nonlinear estimate is given by:

$$\hat{x}(t/\zeta) = E\{x(t) \mid \nu(s), \quad 0 \le s < \zeta\}$$

$$= \int_0^{\zeta} E\{x(t)\nu'(s) \mid \nu(\sigma), 0 \le \sigma < s\}\, \nu(s)\} \, ds$$

Using the innovations theory with the stochastic approximation techniques an algorithm is derived for numerical computation of the innovations processes and consequently the system state of the nonlinear dynamical systems [15].

This concept will be applied to the orbit determination problem for the first time.

Now we are in the process of applying these three approaches to the unified state model for satellite orbit determination and in the near future we will develop a new nonlinear filtering approach based on the invariant imbedding technique combined with stochastic approximations.

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A NON-LINEAR FILTER WITH HIGHER-ORDER FUNCTIONS VIA INVARIANT IMBEDDING

1. Introduction

In engineering and physical sciences there occur many two-point or multipoint boundary-value problems. Since these problems usually are nonlinear, they are accompanied by various analytical and numerical difficulties. Analytically there is no general proof for the existence and uniqueness of the solutions. Numerically we possess no convenient technique for obtaining the numerical solutions on modern digital computers. These numerical difficulties are caused by the fact that not all the conditions are given at one point. To obtain the missing condition a trial and error iterative procedure is generally used. But this procedure is not suited to modern digital computers. Furthermore for a large number of problems (e.g., orbit determination) the starting or guessed missing condition must be very close to the correct and yet unknown condition before the procedure will converge.

The invariant imbedding approach is a useful technique for solving such problems.

2. Invariant imbedding and nonlinear filtering

The invariant imbedding concept is used to derive some useful results in nonlinear filtering and state estimation problems. Since Wiener's pioneering work [1] on the theory of optimal filtering and prediction, many extensions and new developments have been made in this field.

First the work of Kalman and Bucy [2] is concerned with the estimation of state variables for linear systems. Later Cox [3] treated the estimation problem by dynamic programming. Bryson and Frazier [4] treated the nonlinear version of this problem. Detchmendy and Sridhar [5] applied the invariant imbedding approach to nonlinear filtering problems.

Since the invariant imbedding approach is different from the usual classical approach, several advantages have been gained. First, the present approach is applicable to a wide variety of nonlinear problems. Second, a sequential estimation scheme is obtained which makes it suitable for on-line or real time simulations. Also, no statistical assumptions will be made concerning the noise or disturbances, which is more often the case in practice. The commonly used least squares criterion is employed to obtain the optimal estimates.

3. Formulation of the estimation problem

Consider a system whose dynamic behavior can be represented by the differential equation

$$\frac{dx}{dt} = f(x, t) \tag{1}$$

The state, x, is being observed starting at an initial time $t_o = 0$ and continuing to the present time t_f. Owing to the presence of noise or measurement errors, the observed state, z, of the system does not represent the true state. Let

$$z(t) = x(t) + (\text{meas. or observation errors}) \tag{2}$$

The problem is to estimate the current state of the system at t_f, using the classical least squares criterion, so that the following integral is minimized:

$$J = \int_{t_o}^{t_f} (x(t) - z(t))^2 \, dt \tag{3}$$

The problem can be stated differently as follows: On the basis of the observation $z(t)$, $0 \leq t \leq t_f$, estimate the unknown condition

$$x(t_f) = c \tag{4}$$

so that J is minimized.

4. The Invariant Imbedding Approach

Define the new variable

$$y(t) = \int_{o}^{t} (x(t) - z(t))^2 \, dt$$

or

$$\frac{dy}{dt} = (x(t) - z(t))^2 \tag{5}$$

$$y(t_f) = J \tag{6}$$

Consider now the general problem

$$\frac{dx}{dt} = f(x, t)$$

$$\frac{dy}{dt} = (x(t) - z(t))^2$$

with the given condition

$$x(a) = c \tag{7}$$

with $0 \leq t \leq a$. In other words, the missing final condition $y(t_f)$ is to be obtained by considering a family of processes with different final points a.

Define $$r(c, a) = y(a) ; \tag{8}$$

by invarinat imbedding, an expression for r in terms of c and a is obtained and called the "Invariant Imbedding equation"

$$f(c,a) \frac{\partial r(c,a)}{\partial c} + \frac{\partial r(c,a)}{\partial a} = (c - z(a))^2 \tag{9}$$

The derivation is in reference [6] and is based on Taylor's series expansion up to the second order.

This equation (9) gives $y(t_f)$ but is not easy to solve as the initial $r(c,o)$ or $y(0)$ is not known, but the aim of the estimation problem is to minimize $y(t_f)$, so we do not need to solve (9) directly.

5. The optimal estimates

The problem is to obtain a series of values of c which minimize the cost function $r(c,a)$ for a series of final terminal points, a.

Let these minimizing values be denoted by $e(a)$ which are the optimal estimates of x, so

$$\frac{\partial r(e,a)}{\partial c} = r_c(e, a) = 0 \tag{10}$$

The total differential of (10) is

$$r_{cc}(e, a)\, de + r_{ca}(e, a)\, da = 0 \tag{11}$$

or

$$\frac{de}{da} = - \frac{r_{ca}(e, a)}{r_{cc}(e, a)} \tag{12}$$

Once the function r, r_{cc}, r_{ca} are determined, eqn. (12) can be solved knowing the initial condition

$$e(0) = \text{best estimate of the } x(0) \;;$$

however the computational procedure is not simple. It involves the solution of the partial differential equation (9) which is

$$f(c,a)\; r_c(c,a) + r_a(c,a) = (c - z(a))^2$$

Differentiating (9) w.r.t. c, we obtain

$$f\, r_{cc} + r_c\, f_c + r_{ac} = 2(c - z(a)) \tag{13}$$

or

$$-\frac{r_{ac}}{r_{cc}} = f + \frac{r_c}{r_{cc}}\, f_c + \frac{2(z(a) - c)}{r_{cc}} \tag{14}$$

Substituting (10) into (14), we obtain

$$-\frac{r_{ac}(e,a)}{r_{cc}(e,a)} = f(e,a) + \frac{2(z(a) - e)}{r_{cc}(e,a)} \tag{15}$$

Combining (12) and (15)

$$\frac{de}{da} = f(e,a) + \frac{2(z(a) - e)}{r_{cc}(e,a)}$$

or

$$\frac{de}{da} = f(e,a) + q(a)\;(z(a) - e) \tag{16}$$

where

$$q(a) = \frac{2}{r_{cc}(e,a)} \tag{17}$$

Equation (16) gives the desired optimal estimate. The function $q(a)$ is considered as a weighting function and should be obtained.

6. Equation for the weighting function

Differentiate equation (17)

$$\frac{dq}{da} = -2 \frac{r_{ccc}(de/da) + r_{cc}'a}{r_{cc}^2} \tag{18}$$

Using (16) and (18)

$$\frac{dq}{da} = -\frac{q^2}{2} \quad r_{ccc} [f + (z(a) - e) q] + r_{cc} a \tag{19}$$

Differentiate (13) w.r.t. c

$$f\ r_{ccc} + r_c\ f_{cc} + r_{acc} = 2(1 - f_c\ r_{cc}) \tag{20}$$

Substitute (18) and (17) into (20) we obtain

$$r_{acc} = 2\ (1 - \frac{2}{q}\ f_c) - f_{rccc}$$

so equation (19) becomes

$$\frac{dq}{da} = 2\ f_c\ q - q^2 - \frac{q^3}{2}\ (z(a) - e)\ r_{ccc} \tag{21}$$

For many practical situations, the function $r(c,a)$ can be approximated by the equation

$$r(c,a) \approx P_0(a) + P_1(a)\ c + P_2(a)\ c^2 \tag{22}$$

in the neighborhood of the optimal estimate $e(a)$. In this case r_{ccc} is negligible and

$$\frac{dq}{da} = 2\ f_c\ (e,a)\ q(a) - q^2(a) \tag{23}$$

7. Summary

The least squares filter with first weighting function (second-order filter) is given by

$$\frac{de}{da} = f(e,a) + (z(a) - e)\, q(a)$$

$$\frac{dq}{da} = 2\, f_e\, (e,a)\, q(a) - q^2(a)$$

with initial conditions

$$e(0) \text{ and } q(0) \text{ given.}$$

8. Generalization for the multivariable system

The above results can be generalized to systems with dynamics represented by M differential equations

$$\frac{dx}{dt} = f(x,t)$$

where x and f are M-dimensional vectors. The observation model is

$$z(t) = h(x,t) + \text{measurement errors;}$$

z and h are m-dimensional vectors.

Following the same derivations as for a scalar case, the filter is given by the following equations

$$\frac{de}{dt} = f(e,t) + q(t)\, h_e^T(e,t)\, [z(t) - h(e,t)]$$

$$\begin{aligned}\frac{dq}{dt} = {} & f_e(e,t)\, q(t) + q(t)\, [f_e(e,t)]^T \\ & + q(t)\, \{h_{ee}(e,t)\, [z(t) - h(e,t)]\}\, q \\ & - q(t)\, [h_e(e,t)]^T\, h_e(e,t)\, q(t)\end{aligned}$$

The first equation represents M diff. equations. The second represents M^2 diff. equations where

$$f_e = \begin{bmatrix} \frac{\partial f_1}{\partial e_1} & \cdots & \frac{\partial f_1}{\partial e_M} \\ \vdots & & \\ \frac{\partial f_M}{\partial e_1} & \cdots & \frac{\partial f_M}{\partial e_M} \end{bmatrix}$$

is an (M x M) matrix ;

$$h_e = \begin{bmatrix} \frac{\partial h_1}{\partial e_1} & \cdots & \frac{\partial h_1}{\partial e_M} \\ \vdots & & \\ \frac{\partial h_m}{\partial e_1} & \cdots & \frac{\partial h_m}{\partial e_M} \end{bmatrix}$$

is an (m x M) matrix ;

$$h_{ee}[z-h] = \begin{bmatrix} h^T_{e_1 e_1}(z-h) & h^T_{e_1 e_2}(z-h) & \cdots & h^T_{e_1 e_M}(z-h) \\ \vdots & & & \\ h^T_{e_M e_1}(z-h) & & \cdots & h^T_{e_M e_M}(z-h) \end{bmatrix}$$

is an (M x M) matrix where the elements are scalar or inner products of the vectors h_{eiej} and [z-h].

9. Orbit determination of synchronous-orbit satellite using the second order filter

Using the U.S.M. (unified state model) to represent the orbital trajectory dynamics [7]

$$\frac{d}{dt}\begin{bmatrix} C \\ R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} 0 & -p \\ \cos\lambda & -(1+p)\sin\lambda \\ \sin\lambda & (1+p)\cos\lambda \end{bmatrix}\begin{bmatrix} a_{e_1} \\ a_{e_2} \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & w_3 & 0 & w_1 \\ -w_3 & 0 & w_1 & 0 \\ 0 & -w_1 & 0 & w_3 \\ -w_1 & 0 & -w_3 & 0 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$

where

$$p = \frac{c}{v_{e2}}$$

$$v_{e2} = c - \sin\lambda\ R_1 + \cos\lambda\ R_2$$

$$\sin\lambda = 2\ e_3e_4/(e_3^2 + e_4^2)$$

$$\cos\lambda = e_4^2 - e_3^2/(e_3^2 + e_4^2)$$

$$w_1 = a_{e3}/v_{e2}$$

$$w_3 = C\ v_{e2}^2/p$$

and a_i are the perturbing accelerations $[a] = \Sigma\ a_i$

Taking into account only zonal and tesseral harmonics for perturbations, the dynamic state equation

$$\frac{dx}{dt} = f(x,\ t)$$

is given by the following first differential equations.

$$\frac{dC}{dt} = \frac{12\ R_e^2\ J}{\mu^3}\ C^5\ v_{e2}^3\ (e_1e_3 - e_2e_4)(e_2e_3 + e_1e_4)$$

$$\frac{dR_1}{dt} = \frac{R_e^2 \quad J}{\mu^3} \quad C^4 \quad v_{e2}^4 \; [24(e_1 e_3 - e_2 e_4)(e_2 e_3 + e_1 e_4) \quad e_3 e_4$$

$$1 + \frac{c}{v_{e2}} \; - \; 1.5 \quad (1-12(e_1 e_3 - e_2 e_4)^2)(e_4^2 - e_3^2)]/(e_3^2 + e_4^2)$$

$$\frac{dR_2}{dt} = \frac{R_e^2 \; J}{\mu^3} \; C^4 \quad v_{e2}^4 \; [-12(e_1 e_3 - e_2 e_4)(e_2 e_3 + e_1 e_4)(e_4^2 - e_3^2)$$

$$1 + \frac{c}{v_{e2}} \; - \; 3(1-12(e_1 e_3 - e_2 e_4)^2) \quad e_3 e_4]/(e_3^2 - e_4^2)$$

$$\frac{de_1}{dt} = \frac{-3 \; R_e^2 \; J}{\mu^3} \quad C^4 \quad v_{e2}^3 \; (e_1 e_3 - e_2 e_4)(1-2e_1^2 - we_2^2) e_4$$

$$+ \frac{c}{2\mu} \quad e_2 \quad v_{e2}^2$$

$$\frac{de_2}{dt} = \frac{3 \; R_e^2 \; J}{\mu^3} \quad C^4 \quad e_3 \quad v_{e2}^3 \; (e_1 e_3 - e_2 e_4)(1-2e_1^2 - 2e_2^2)$$

$$- \frac{1}{2\mu} \quad C \quad e_1 \quad v_{e2}^2$$

$$\frac{de_3}{dt} = \frac{-3 \; R_e^2 \; J}{\mu^3} \quad C^4 \quad e_2 \quad v_{e2}^3 \; (e_1 e_3 - e_2 e_4)(1-2e_1^2 - 2e_2^2)$$

$$+ \frac{1}{2\mu} \quad C \quad e_4 \quad v_{e2}^2$$

$$\frac{de_4}{dt} = \frac{-3\ R_e^2\ J}{\mu^3}\ C^4\ e_1\ v_{e2}^3\ (e_1 e_3 - e_2 e_4)(1 - 2e_1^2 - 2e_2^2)$$

$$- \frac{1}{2\mu}\ C\ e_3\ v^2_{\ e2}$$

These equations give the 7 order vector [8] as a nonlinear function of the states.

The observation model in this case is given as

$$\begin{bmatrix} \gamma \\ h \\ A \end{bmatrix} = \begin{bmatrix} \sin^{-1} \frac{z}{h} \\ (x^2 + y^2 + z^2)^{1/2} \\ \tan^{-1} \frac{x}{y} \end{bmatrix}$$

where (x, y, z) are the cartesian coordinates and (γ, h, A) are the polar coordinates giving (elevation, range, Azimuth) as observations. To transform this nonlinear vector as a function of the states we obtained (x,y,z) as functions of (C, R_1, R_2, e_1, e_2, e_3, e_4)

$$x = \mu \sqrt{(e_4^2 + e_3^2)}\ (e_2 e_3 + e_1 e_4)\ /\ C\ v_{e2} \sqrt{(e_1^2 + e_2^2)}$$

$$y = \mu \sqrt{(e_4^2 + e_3^2)}\ (e_1 e_3 - e_2 e_4)\ /\ C\ v_{e2} \sqrt{(e_1^2 + e_2^2)}$$

$$z = R_e$$

where

R_e = Earth's radius.

Now having the model in the form

$$x(t) = f(x,\ t)$$

$$z(t) = h(x,\ t) + (\text{meas. errors})$$

to apply the nonlinear second order filter previously derived we obtained the first and second derivatives required for the filter. Mathematical details are not given here, but after several manipulations the matrices f_e, h_e, h_{ee} are derived to use for the filter.

10. Simulation Results

The second order filter derived using the invariant imbedding approach was applied to the unified state model for orbit determination.

The initial conditions of the states [x(1) ... x(7)] are given by:

$C = 3.074656$ km/sec

$R_{f1} = R_{f2} = 0$ km/sec

e_1, e_2, e_3, e_4 are calculated using

$i = 0.9^\circ$

$\Omega = 240^\circ$

and $\nu = -\Omega/2$

The data [z] are provided by the CRC. The file containing the data covers 24 hours of tracking data. The range h is provided at the rate of 6 samples/min., elevation and azimuth vary at a slower rate every 40 or 80 sec. sampling interval.

Data included in 4 records of the tape were chosen for simulation.

The program used for integration of nonlinear differential equations is DVOGER based on Gear's subroutine [8].

The results shown on the following page are the estimated states C, R_{f1}, ..., e_4. We note that C, e_1, e_3, e_4 are almost the initial or optimal estimates, as for R_{f1}, R_{f2} and e_2, there is an error in the estimation that increases slightly during this period of time (750 sec. for the 3 records considered).

Transformation from USM State Vector to Classical Elements

Euler angles i and Ω are given as:

$$\cos i = 1-2(e_1^2 + e_2^2)$$

$$\cos \Omega = (e_1e_4 - e_2e_3)/ (e_1^2 + e_2)(e_3^2 + e_4^2)$$

The orbit is defined by

$$e = R/C$$

where $R = (R_{f1}^2 + R_{f2}^2)^{1/2}$

$$a = \mu/(C^2 - R^2)$$

and $\phi = \lambda - x$

where $\cos \lambda = (e_4^2 - e_3^2)/(e_3^2 + e_4^2)$

$$\cos x = R_{f2}/R$$

Using these equations we get the following results.

	Initial Estimate	Final Estimate
i	0.9°	0.899°
Ω	240°	240.12°
e	0	2.186×10^{-6}
a	4.216 km	$\approx$ 4.216 km
ϕ	120°	$\approx 120^\circ$

We can see that the classical elements hardly vary during this simulation as the error in the estimation is almost negligible.

This case gives almost optimal state estimation because the system input noise is assumed to be zero, and no assumption is made about the observation noise, which is included with the data. Further examples are being tried with different initial conditions to ascertain whether this method will work equally well in all cases.