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MODELING OF ENERGY DISSIPATION (DAMPING) IN FLEXIBLE SPACE STRUCTURES

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PREFACE

Acknowledgments

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Parts of Sections 6 and 7 are based on the doctoral dissertation of David Golla (in preparation).

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Units and Spelling

This report uses S.I. units and North American spelling.

SUMMARY

Damping is a critical phenomenon in the control of large flexible space structures. Just because it is "small" does not mean it is "unimportant". Section 4 of this report shows, using several contexts, that virtually every important output from a flexible structure subject to excitation is modulated by the factor $1/\zeta$, where ζ is a characteristic damping factor. Therefore, to assume $\zeta = 0.001$ when it is in reality 0.01 is to introduce an error of $\doteq$ 1000% in all subsequent calculations, design, and simulations. Such a profound error is quite unnecessary. The physical principles of energy dissipation are known; the numerical methods (chiefly the finite element method) for undamped structures exist; the computer power is available. All that appears to be lacking is a zeal for the detailed calculation of damping characteristics to match the current zeal for calculating natural frequencies. A fruitful approach would involve both analysis and test results. Current practice is often to eschew both of these in favour of mere speculation.

The one fact about structural dissipation that is agreed to by all concerned is that the damping effect is not pronounced, but 'light'. In this report it is shown that this important qualitative characteristic can lead to a number of quantitative conclusions of practical significance. These include the following:

(i) The differences between the mode shapes (eigenvectors) of an undamped structure and those of the same structure when lightly damped are insignificant for practical purposes.

(ii) The differences between the eigenvalues for an undamped structure and those of the same structure when lightly damped are crucially significant for practical purposes.

(iii) The crucial differences just mentioned are accounted for in a natural way by the "linear-viscous modal damping factors" ζ_α inserted in the modal equations of motion. The "off-diagonal", "damping coupling" terms are not important, and such terms can justifiably be omitted for most purposes.

(iv) A "linear-viscous damping factor" added to each modal equation of motion does not imply the assumption of linear-vis viscous damping. Provided the damping is "light", both hysteretic and viscoelastic damping (and presumeably other reasonable damping models) also lead to an *effective* linear-viscous damping factor.

(v) For "light" damping, the only remaining problem is therefore to calculate the effective linear-viscous damping factors. This requires that the appropriate local structural dissipation parameters be known, and that an effective technique (e.g., the finite element method) be available to calculate the overall structural modal damping factors from these local parameters. Although work is currently proceeding apace to this objective, much remains to be done.

If the damping is not "light", the details of the dissipation model become more apparent. Cross-coupling terms cannot be neglected, and the inclusion of "equivalent linear-viscous damping factors" is no longer adequate. Indeed, as the damping becomes more substantial, the undamped modes (and their associated coordinates) become less relevant. "Damped modal coordinates" lead to a more efficient formulation in the sense that a lesser number of "damped modal coordinates" is required to represent the structure to a given degree of accuracy. One approach to such coordinates is derived in this report.

The last main theme to be noted, in some respects the most important one, is this: not only should structural damping be calculated accurately, it should be designed adequately. "Structural damping" is virtually as important as "structural stiffness". It is easier to control a well-damped structure than to control an inadequately damped structure that is slightly stiffer. Yet the proper compromise between damping and stiffness is seldom made. Another tradeoff seldom broached exists between "passive" damping (i.e., structural damping) and "active" damping. Most control strategies for large space structures boil down to simple rate-feedback control. Why this is often considered to be a more "advanced" approach is not altogether clear. It is especially ironic that virtually all the structural disturbances being "controlled" are in fact caused by the control system itself. One wonders if "passive rate feedback" (structural damping) might not be an important part of the solution to the problem of "controlling" large space structures. It is reliable and failure proof, requires neither on-board control nor ground control, consumes no power, dedicates no microprocessors, and does not cause "modal spillover". In fact, it tends to remove spillover. Discrete damping devices, judiciously located, could remove energy from selected troublesome modes, and distributed damping (composite materials?) would add damping to all modes, thereby simplifying the control system and eliminating spillover.

TABLE OF PRINCIPAL SYMBOLS

Note: Symbols used only locally are defined when introduced.

Roman Symbols

a_ρ	- residue at p-th pole in viscoelastic damping model
b_ρ	- p-th pole in viscoelastic damping model
$\underline{\mathcal{B}}$	- input distribution matrix
$\underline{C}_{xy}$	- covariance matrix between $\underline{x}(t)$ and $\underline{y}(t)$; = $<\underline{x}(t+\tau)\underline{y}^T(t)>$
$d_{\alpha\beta}$	- element of $\underline{\mathcal{D}}$
$\hat{d}_{\alpha\beta}$	- element of $\underline{\hat{\mathcal{D}}}$
$\underline{\mathcal{D}}$	- damping matrix (for all coordinates)
$\underline{\mathcal{D}}_e$	- damping matrix (for elastic coordinates)
$\underline{\hat{\mathcal{D}}}$	- $\underline{E}^T\underline{\mathcal{D}}\underline{E}$
$\underline{\hat{\mathcal{D}}}_e$	- $\underline{E}_e^T\underline{\mathcal{D}}_e\underline{E}_e$
$\underline{e}_\alpha$	- undamped eigenvector (mode shape) for mode α
E	- total mechanical energy
$\underline{E}$	- undamped modal matrix (columns are $\underline{e}_\alpha$) for all coordinates
$\underline{E}_e$	- undamped modal matrix for elastic coordinates
f	- force
$\underline{\mathit{f}}$	- column of f_α
f_α	- generalized force associated with physical coordinate q_α
$h_{\alpha\beta}$	- response of coordinate α to impulse at coordinate β
$\mathcal{h}_{\alpha\beta}$	- element of $\underline{\mathcal{H}}$
$\hat{\mathcal{h}}_{\alpha\beta}$	- element of $\underline{\hat{\mathcal{H}}}$
$\underline{\mathcal{H}}$	- hysteretic damping matrix (all physical coordinates)
$\underline{\mathcal{H}}_e$	- hysteretic damping matrix (elastic physical coordinates)
$\underline{\hat{\mathcal{H}}}$	- $\underline{E}^T\underline{\mathcal{H}}\underline{E}$
$\underline{\hat{\mathcal{H}}}_e$	- $\underline{E}_e^T\underline{\mathcal{H}}_e\underline{E}_e$

j	-	$j^2 = -1$
$\underline{K}$	-	stiffness matrix (associated with physical coordinates)
$\underline{K}_e$	-	stiffness matrix (elastic coordinates only)
$\underline{\ell}_i$	-	left eigenvectors $(i = 1,\ldots,2n)$
$\underline{L}$	-	rows of $\underline{L}$ are $\underline{\ell}_i^H$
M	-	number of substructures
$\underline{M}$	-	mass matrix (associated with physical coordinates)
$\underline{M}_e$	-	mass matrix (elastic coordinates only)
n	-	number of physical coordinates (degrees of freedom)
$2n$	-	number of state variables
p	-	number of poles in viscoelastic model
q	-	a physical coordinate (displacement)
$\underline{q}$	-	a matrix of coordinates (displacements)
$\underline{q}_r$	-	rigid-body (absolute) displacements
$\underline{q}_e$	-	elastic (relative) displacements
$\underline{r}_i$	-	right eigenvectors $(i = 1,\ldots,2n)$
$\underline{R}$	-	columns of $\underline{R}$ are $\underline{r}_i$
s	-	Laplace variable (complex)
t	-	time
T	-	kinetic energy
$\underline{u}$	-	column of input variables
V	-	potential energy

Greek Symbols

γ_α	-	generalized force associated with mode α
$\underline{\gamma}$	-	column of γ_α
δ_{ij}	-	(or $\delta_{\alpha\beta}$) unity if $i = j$; zero if $i \neq j$

$\delta(t)$	-	zero if $t \neq 0$; $\int f(t)\delta(t)dt = f(0)$
ζ_α	-	viscous damping factor associated with mode α
ζ_α	-	*equivalent* viscous damping factor associated with mode α
ζ	-	in the Common Theory, all $\zeta_\alpha = \zeta$
η_α	-	coordinate associated with undamped mode α
$\underline{\eta}$	-	column of η_α
λ_i	-	eigenvalue
$\underline{\Lambda}$	-	diagonal matrix of the λ_i
μ	-	material static stiffness parameter
τ	-	time delay in random-variable correlation
τ	-	dummy time variable in integration
$\underline{\Phi}_{xy}$	-	spectral density function: $\underline{\Phi}_{xy}(\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau}\underline{C}_{xy}(\tau)d\tau$
ω	-	frequency of sinusoidal excitation
ω_α	-	natural frequency of vibration mode α
$\underline{\Omega}$	-	diagonal matrix of the ω_α

Subscripts

i,j	-	on state variables; range 1 to 2n
α,β	-	on modal coordinates; range 1 to n
ρ	-	on poles in viscoelastic model; range 1 to p

Special Notations

$\underline{1}$	-	unit matrix (dimension should be clear from context)
$\underline{1}_\alpha$	-	column matrix of 0's (whose length should be clear from context) except for a 1 in the α-th position
$(\cdot)^H$	-	Hermitian of $(\cdot)$; $\equiv (\cdot)^{*T} \equiv (\cdot)^{T*}$

$\delta(\cdot)$	- first variation in $(\cdot)$
$<\cdot>$	- expected value of $(\cdot)$; ensemble average of $(\cdot)$
$(\cdot)^*$	- complex conjugate: $(x + jy)^* \equiv x - jy$
$\mathrm{sgn}(x)$	- $\lvert x \rvert / x$, $x \neq 0$; undefined for $x = 0$
$(\bar{\cdot})$	- Laplace transform of $(\cdot)$
$\lambda\{\underline{A}\}$	- eigenvalue(s) of the matrix $\underline{A}$

1. INTRODUCTION

The modeling of energy dissipation in dynamical systems of engineering interest has received much less attention historically than it deserves. One might cite fluid mechanics as an example. A large and elegant theory of fluid mechanics has been evolved based on the inviscid (zero dissipation) approximation. This theory permits the 'potential flow' assumption with all the mathematical benefits conferred by that assumption. Unfortunately, inviscid theory cannot be used to make one of the most fundamentally important calculations in fluid mechanics--the drag on a body immersed in a fluid flow. Furthermore, without the seemingly arbitrary Kutta hypothesis (zero velocity difference between the upper and lower surfaces at a sharp trailing edge--a hypothesis that is, in fact, valid because of viscosity!) one cannot calculate lift either. A totally inviscid flow (i.e., no viscosity and no Kutta hypothesis) mathematically predicts zero lift and zero drag on an airfoil at an angle of attack. This is hardly a stunning achievement for a fluid mechanical theory.

An example that is likely to be of greater interest to readers of this report concerns the stability of rigid-body rotations. Euler's celebrated result is that a rigid body spinning about either its major or minor axis of inertia is stable; an intermediate-axis spin is unstable. It took dynamicists more than a century and the stunning object lesson provided by the first U.S. satellite, Explorer I, before it was realized that Euler's theory does not apply to real bodies. The theory is mathematically exact, of course, but omits a critical characteristic of physical bodies--energy dissipation. With dissipation taken into account, the minor-axis spin is in fact unstable.

History appears to be repeating itself yet again in the modeling of flexible space structures. Control systems for flexible spacecraft have in the past been designed with a great deal of attention to the modeling of vibration modes (the irresistible attraction of a conservation-based theory again) and with virtually no attention given to the modeling of energy dissipation. This weakness was overcome in the *Hermes* program by a ground-testing program to determine experimentally the damping characteristics of the *Hermes* solar array. However, as flexible space structures become larger and larger, ground testing may become virtually impractical. There is therefore an urgent need for reliable and practical methods of modeling damping in large space structures.

This technical report attempts to review, in a succinct fashion, the most important facts about damping in space structures and to offer a brief review of the practical analytical techniques that are already available for dealing with this problem. The current 'theory' most often used is caricatured in Section 2. This 'analytical cartoon' might be humorous were it not for the serious deficiencies underlying this 'theory'; indeed, it is to highlight these deficiencies that Section 2 is written, Section 3 then focuses on the 'light damping' assumption. It is explained that all the 'elastic mode' calculations that neglect damping are not in vain, provided a *proper* damping model is subsequently incorporated. [Parenthetically, the same is true of the inviscid (potential) theory of fluid flow referred to a moment ago. Inviscid theory is still universally used to calculate important quantities in fluid dynamics, including lift (when the dissipation-motivated Kutta condition is imposed!); all the same, inviscid theory cannot be used to calculate drag. Extensive experience with aerodynamic calculations seems to indicate that it is important for the analyst to know when to include dissipation and when to ignore it. Also indicated is this maxim: when including dissipation, take it seriously and model it accurately.]

Section 4 is a technical essay on why damping is important for large space structures and why those responsible for the design of control systems for flexible spacecraft should take structural damping seriously. Some popular misconceptions are also identified. Section 5 provides a treatment of *linear viscous damping*--the most common type (more precisely, the type most commonly *assumed*). Sections 6 and 7 go on to deal with *linear hysteretic damping* and *linear viscoelastic damping*. The report concludes with some summarizing remarks in Section 8.

2. THE COMMON 'THEORY'

The remarks made in this section are intended to create the impression that the mathematical modeling of dissipative mechanisms in space structures has been inadequate in the past. In view of the history of dissipation modeling in engineering mechanics, alluded to briefly in the introduction, this state of affairs should not be surprising. Part of the problem is that the task of modeling damping is *not easy*. Viewed from a sufficiently distant perspective, the current modeling of flexible structures as linear-elastic systems requires only

(i) one outstanding constitutive assumption--strain proportional to stress;

(ii) one outstanding geometrical assumption--deflections are first-order infinitesimals;

(iii) one outstanding numerical method--the 'finite element' method.

The rest is details.

The first two assumptions are made to facilitate a linear theory. The finite element method enables the analyst to extract numerical data from his theory in spite of spatially varying constitutive parameters, complex substructural interconnections, and arbitrary structural boundaries. Energy dissipation is an unwelcome intruder because it violates assumption (i) and thereby eliminates the applicability of a substantial body of mathematical, physical, and numerical theory.

To obtain a deeper understanding of energy dissipation, one must refer to the science of materials. Within material science, a great deal is known about energy dissipation. However, most designers of control systems for modern spacecraft are not conversant with the germane particulars from material science. (This may well be true of most spacecraft structural analysts as well.) A more profound problem is that the models used by material scientists are cumbersome or intractable when applied, not to an element dV of uniform material, but to a typical space structure. It is probably fair to say that any methodology that is not amenable to treatment by some reasonably straightforward modification to the finite element method is doomed to obscurity as far as spacecraft modeling is concerned.

These barriers to accurate structural modeling may be surmounted in the 1980's, but only with considerable effort and imaginative thinking. It is interesting to note that Graham (Ref. 1), in an extensive review in 1973 that attempted to bridge the gap between material science and structural modeling practice, stated: "A fairly general opinion among engineering dynamicists today is that a greater effort must be made to place the treatment of material damping on a more substantial analytic basis". In spite of this "general opinion" in 1973, the last decade has hardly seen a quantum improvement. In fact, to judge from the recent technical literature, the only theory currently being applied is the Common Theory described in the remainder of this section.

We choose as our 'undamped system' the linear, time-invariant differential system represented by

$$\underline{M}\ddot{\underline{q}} + \underline{K}\underline{q} = \underline{f} \tag{2.1}$$

where

$$\underline{M}^T = \underline{M} > 0 \; ; \qquad \underline{K}^T = \underline{K} \geq 0 \tag{2.2}$$

If rigid-body modes are excluded by physical constraints, then $\underline{K} > 0$. A gyroscopic matrix term could also be included in (2.1), and this would enhance the applicability of the following discussion to certain flexible spacecraft. Such a term will be omitted in this report because it makes more extensive some of the details to follow and thereby detracts from the principal points to be made. It can be stated, however, that *the ideas expressed below are applicable to flexible gyroscopic systems*. The system matrices $\underline{M}$, $\underline{K}$ and $\underline{f}$ are typically calculated with the aid of finite element methods, although other procedures are also sensible in particular situations.

2.1 Phase 1 - The Elusive 'Damping Matrix', $\underline{D}$

The first phase of the Common Theory is to observe that the dissipation-free system (2.1) can be converted to a dissipative system by the addition of a $\underline{\mathcal{D}\dot{q}}$ term, as follows:

$$\underline{M\ddot{q}} + \underline{\mathcal{D}\dot{q}} + \underline{Kq} = \underline{f} \tag{2.3}$$

where

$$\underline{\mathcal{D}}^T = \underline{\mathcal{D}} \geq 0 \tag{2.4}$$

The Theory is much simplified by the assumption that $\underline{\mathcal{D}} > 0$, but unless the spacecraft is damped with respect to an inertial reference frame this assumption is not valid. [Exceptions: Eddy-current damping and magnetic-hysteresis damping are examples of weak external damping; if these were included, $\underline{\mathcal{D}} > 0$ for the attitude motion equations.] In any case, the system energy

$$E = \tfrac{1}{2}\underline{\dot{q}}^T\underline{M\dot{q}} + \tfrac{1}{2}\underline{q}^T\underline{Kq} \tag{2.5}$$

is reduced, in the absence of external influences ($\underline{f} \equiv \underline{0}$), at the rate

$$\dot{E} = -\tfrac{1}{2}\underline{\dot{q}}^T\underline{\mathcal{D}\dot{q}} \leq 0 \tag{2.6}$$

and upon this fact rests the success of the $\underline{\mathcal{D}\dot{q}}$ term in (2.3).

In spite of the notoriety of the $\underline{\mathcal{D}\dot{q}}$ term, it is exceedingly rare for anyone to actually calculate $\underline{\mathcal{D}}$ for flexible structures. Textbooks (and too many research papers) observe sagely that 'if $\underline{\mathcal{D}}$ is proportional either to $\underline{M}$ or to $\underline{K}$, the transformation that simultaneously diagonalizes $\underline{M}$ and $\underline{K}$ will also diagonalize $\underline{\mathcal{D}}$.' Let $\underline{E}$ be the matrix of (undamped) eigenvectors that simultaneously diagonalizes $\underline{M}$ and $\underline{K}$. Then

$$\underline{E}^T\underline{ME} = \underline{1} \; ; \qquad \underline{E}^T\underline{KE} = \underline{\Omega}^2 \tag{2.7}$$

where $\underline{\Omega}$ is the diagonal matrix of natural frequencies (rigid-body modes included). Then, if

$$\underline{\mathcal{D}} = c_1\underline{M} + c_2\underline{K} \tag{2.8}$$

it follows that

$$\underline{E}^T \underline{\mathcal{D}} \underline{E} = c_1 \underline{1} + c_2 \underline{\Omega}^2 \tag{2.9}$$

In fact, one can write

$$\underline{E}^T \underline{\mathcal{D}} \underline{E} = \mathrm{diag}\{2\zeta_1\omega_1, \ldots, 2\zeta_n\omega_n\} \tag{2.10}$$

where

$$\zeta_\alpha = \tfrac{1}{2}(c_1/\omega_\alpha + c_2\omega_\alpha) \qquad (\alpha=1,\ldots,n) \tag{2.11}$$

with ω_α the natural frequency of vibration of mode α, and n the number of degrees of freedom (the number of coordinates in $\underline{q}$). The diagonalization of $\underline{\mathcal{D}}$ indicated by (2.9) is much to be desired because the transformation

$$\underline{q} = \underline{E}\underline{\eta}$$

which converts the *undamped* system (2.1) to

$$\ddot{\eta}_\alpha + \omega_\alpha^2 \eta_\alpha = \gamma_\alpha \qquad (\alpha=1,\ldots,n) \tag{2.12}$$

where

$$\underline{\gamma} = \underline{E}^T \underline{f} \tag{2.13}$$

converts the *damped* system (2.3) to

$$\ddot{\eta}_\alpha + 2\zeta_\alpha\omega_\alpha\dot{\eta}_\alpha + \omega_\alpha^2\eta_\alpha = \gamma_\alpha \qquad (\alpha=1,\ldots,n) \tag{2.14}$$

Now virtually all the benefits of modal analysis that make (2.12) attractive apply still to (2.14).

These benefits are, however, based on the proportionality assumption (2.8) and this assumption is not based on any rigorous theory of structural damping. An important exception--discrete viscous dampers--will be discussed in Section 5.

2.2 The Critical Role of the 'Knowledgable Person'

The second phase of the Common Theory is to dispense with $\underline{\mathcal{D}}$ altogether and proceed directly to the real objective--the modal coordinate motion equations (2.14). After a detailed and sophisticated calculation of $\underline{M}$ and $\underline{K}$, one arrives at the undamped equations (2.12). Then a damping term, $2\zeta_\alpha \omega_\alpha \dot{\eta}_\alpha$, is arbitrarily added to each equation to produce the form (2.14). (This procedure is not applied to modal equations for rigid-body modes; they are left intact, in the form $\ddot{\eta}_\alpha = \gamma_\alpha$.)

Now comes the swindle: the modal damping factors ζ_α *are not calculated*. Their values are simply *assigned* by a Knowledgeable Person. This is not to criticize the K.P., who is usually reluctant to make these numerical assignments, especially in the amount of time allocated. Reliable damping-factor information on structures is hard to come by, especially for space structures (see, however, Section 2.4). Not only is theory avoided; there is no handbook either. In the end, the Knowledgeable Person must simply make a Guess.

2.3 The 'Knowledgeable Person' Makes His 'Guess'

After realizing that there is no way to escape his designation as Knowledgeable Person, the K.P. goes through a thought process that is essentially as follows. First, since he does not know what any of the ζ_α actually are, he has no basis for choosing them to be different. Thus he makes

Decision No. 1 - All the modal damping factors are equal:

$$\zeta_\alpha = \zeta \qquad (\alpha=1,\ldots,n) \tag{2.15}$$

With this assumption, one on which he can hardly be faulted, his task has been reduced by a factor of 1/n. It remains to pick the value of ζ. This is not easy. All he knows for certain is that ζ is positive, but 'small'. He knows ζ is positive because to assume otherwise would be to violate the second law of thermodynamics; he knows it is 'small' because all observations of space structures (and of similar ground-based structures) confirm that the damping is 'small' (see Section 2.5). It might then seem that any 'small' number could be chosen with essentially equivalent validity. But this is not so. The Knowledgeable Person is aware (or should be aware) of the points made in Section 4 of this report. To pick $\zeta = 0.005$ is to pick ζ 400% greater than $\zeta = 0.001$, and if ζ is really 0.001, the K.P. does not wish to be guilty of a 400%

error in his Guess. In fact, the K.P. is in a bind. If he chooses ζ to be too small, he is placing a much greater load on the control system designer to provide the missing rate feedback, a challenge that those who earn their living designing control systems are eager to accept. (He also--almost with a wave of his hand--makes necessary perhaps hundreds of thousands of dollars worth of additional computation to confirm the control system design by simulation. Better perhaps to spend a few thousand dollars to model passive damping!) On the other hand, if the K.P. assumes ζ to be too large, an even more unwelcome consequence is possible: the attitude control system may not be capable of stabilizing the spacecraft and a multi-million-dollar mission may be lost. On balance, the K.P. is forced to be 'conservative', that is, he is forced to choose ζ somewhat smaller that he really thinks it is likely to be. Thus the Knowledgeable Person makes

Decision No. 2 - *The value of ζ is*

$$\zeta = 0.005 \tag{2.16}$$

The K.P. knows that damping factors in space tend to be in the range

$$0.01 \leq \zeta \leq 0.02$$

and the value chosen, $\zeta = 0.005$, gives him a certain safety margin. A larger safety margin would be available by choosing $\zeta = 0.001$; but the K.P. knows that no freely vibrating space structure has ever been observed to take $1000/2\pi$ periods to damp down to $1/e$ of its initial vibration amplitude and so, rather than be replaced by a More Knowledgeable Person, he chooses $\zeta = 0.005$.

2.4 Reflections on the 'Common Theory'

It is evident that the Common Theory of damping for flexible space structures is not a 'theory' at all. It is instead an exercise in basic engineering judgment. The author has on file dozens (perhaps hundreds) of papers and company reports by leading organizations in which the above procedure is followed, although never with such candor. No justification whatever is given for the selection of the ζ_α; they are simply *assigned*.

An interesting mental experiment is the following: choose a structural vibrations specialist and show him an engineering drawing of the structure. Tell him the materials from which the structure is made and give him, in addition, only the following numerical data:

(i) the mass of the structure;

(ii) the characteristic length of the structure.

Now ask him to estimate all the natural frequencies for the important modes of the structure as accurately as possible. If this situation sounds absurd, it should be realized that this is precisely analogous to the position in which the Knowledgeable Person of Sections 2.2 and 3.3 finds himself when asked to estimate the modal damping factors. Obviously, this procedure for natural frequencies would not be tolerated.

Why, then, is the analogous procedure so widespread for damping factors? Some argue that damping factors are not all that important, provided they are small. This point of view is attacked in Section 4. Others may say that they are virtually impossible to calculate. This assertion will probably not stand up to careful scrutiny. In the opinion of the author, if the effort over the last fifteen years to calculate the ζ_α had been as intense as the effort to calculate the ω_α, a substantial body of engineering practice would by this time have evolved, permitting the calculation of the ζ_α to within (say) 10%, instead of the current Guess to within 900%.

The real obstacles to the development of such procedures appear to include the following ones.

<u>Obstacle 1</u> - *There is a widely held belief that ζ does not matter because it is 'small'.*

<u>Obstacle 2</u> - *Controls engineers are happy to see ζ remain small, because it elevates the criticality of their discipline.*

(Hundreds of papers on control of flexible spacecraft assume the structural damping to be *zero*, thus virtually guaranteeing that 'spillover' from the neglected modes will be a problem.)

Obstacle 3 - While natural frequencies can be calculated solely on the basis of <u>static</u> test data--mass properties and stiffness properties--damping factors inherently require data from dynamic tests.

The first two obstacles are rather artificial and can be eliminated through an increased awareness of the nature and function of passive structural damping. The third obstacle is more fundamental and can be overcome only through major research and development programs.

2.5 Flight Results on Structural Damping

Not only is there no comprehensive engineering procedure for modeling structural damping; there is very little data available from flight results either. Some data is available for the FRUSA solar array (Ref. 2) indicating a damping factor of about 2% in the primary mode. For the Hermes satellite (Ref. 3), which had a somewhat similar solar array, the damping factors were about 1% - 2% depending on the mode. Recent on-orbit results for OSO-9 (Ref. 4) also imply damping factors $\leq$ 1%. It should be noted that these damping factors, already greater than 1%, were occurring naturally in the structure; no attempt was made to design damping into them. In the case of Hermes, it is also noteworthy that the values measured in space were often significantly greater than those measured in ground tests in vacuum. It is suspected that joint freeplay, much reduced in ground tests by gravity-induced tension, may have contribute extra damping under the free-fall conditions of space.

3. THE 'LIGHT DAMPING' ASSUMPTION

It has been observed that the level of energy dissipation naturally occurring in space structures is 'small'. This fact can be used to advantage in obtaining basic analytical results. Specifically, it will be shown in this section that, provided the level of damping is sufficiently low (less than a few percent), the eigenvectors of the damped system are, for all practical pruposes, the eigenvectors of the undamped system (the vibration modes). The only really important effect of light damping is the slow exponential decay of vibration amplitude, and this will be shown to depend only on the diagonal elements of the modal damping matrix.

3.1 Jacobi's Formula for Perturbation of Eigenvalues and Eigenvectors

One point of departure is Jacobi's formulas for the first-order perturbations in the eigenvalues and eigenvectors of a general real matrix $\underline{A}$ due to first-order change in $\underline{A}$. The latter change in $\underline{A}$ is denoted $\delta\underline{A}$. Let the eigenvalues and eigenvectors of $\underline{A}$ be denoted

$$(\lambda_i, \underline{r}_i) \qquad i=1,\ldots,2n \tag{3.1}$$

(The reason for the notation $\underline{r}_i$ for the eigenvector $\underline{r}_i$ will become clear presently.) In general, $(\lambda_i, \underline{r})$ may be complex. For the application in mind, we prefer to let $\underline{A}$ be a $2n \times 2n$ matrix and to let n denote, as before, the number of coordinates in the problem. If the number of coordinates in a structural model is n, the number of state variables is 2n. Then

$$\underline{A}\underline{r}_i = \lambda_i \underline{r}_i \qquad (i=1,\ldots 2n) \tag{3.2}$$

This set of equations can be combined into a single $2n \times 2n$ matrix equation

$$\underline{AR} = \underline{R\Lambda} \tag{3.3}$$

where

$$\underline{R} = [\underline{r}_1 \cdots \underline{r}_{2n}] \tag{3.4}$$

is the *modal matrix* or *eigenmatrix* for $\underline{A}$, and $\underline{\Lambda}$ is the diagonal matrix of eigenvalues

$$\underline{\Lambda} = \text{diag}\{\lambda_1, \ldots, \lambda_{2n}\} \tag{3.5}$$

The form (3,3)-(3.5) implies the assumption of 2n linearly independent eigenvectors, a valid assumption for structures except in the trivial case of uncoupled structures.

The modal relationship (3.3) can be rewritten

$$\underline{L}\underline{A} = \underline{\Lambda}\underline{L} \tag{3.6}$$

where

$$\underline{L} = \underline{R}^{-1} \tag{3.7}$$

There is no difficulty about whether $\underline{R}^{-1}$ exists because it has already been assumed that the eigenvectors $\underline{r}_1,\ldots,\underline{r}_{2n}$ are linearly independent. If, now, we let the rows of $\underline{L}$ be $\underline{\ell}_i^H$,

$$\underline{L} = \begin{bmatrix} \underline{\ell}_i^H \\ \vdots \\ \underline{\ell}_{2n}^H \end{bmatrix} \tag{3.8}$$

then (3.6) can be decomposed into the 2n relations

$$\underline{\ell}_i^H \underline{A} = \lambda_i \underline{\ell}_i^H \qquad (i=1,\ldots 2n) \tag{3.9}$$

The significance of the symbols $\underline{r}_i$ and $\underline{\ell}_i$ can now be discerned: $\underline{r}_i$ is the "right eigenvector" of $\underline{A}$, and $\underline{\ell}_i$ is the "left eigenvector" of $\underline{A}$. The right and left eigenvectors satisfy the orthonormality conditions

$$\underline{\ell}_i^H \underline{r}_j = \delta_{ij} \qquad (i,j = 1,\ldots 2n) \tag{3.10}$$

as is obvious from (3.7), when written as $\underline{L}\underline{R} = \underline{1}$.

The way is now prepared to consider the "eigenconsequences" of small perturbations $\delta\underline{A}$ in $\underline{A}$. From (3.2),

$$(\underline{A} + \delta\underline{A})(\underline{r}_i + \delta\underline{r}_i) = (\lambda_i + \delta\lambda_i)(\underline{r}_i + \delta r_i) \tag{3.11}$$

To first order, this relationship can be written

$$\underline{A}\underline{r}_i + \underline{A}\delta\underline{r}_i + (\delta\underline{A})\underline{r}_i = \lambda_i\underline{r}_i + \lambda_i\delta\underline{r}_i + (\delta\lambda_i)\underline{r}_i \tag{3.12}$$

When (3.2) is subtracted, one arrives at the following first-order result:

$$\underline{A}\delta\underline{r}_i + (\delta\underline{A})\underline{r}_i = \lambda_i\delta\underline{r}_i + (\delta\lambda_i)\underline{r}_i \qquad (i=1,\ldots,2n) \tag{3.13}$$

On the basis of this equation we can find the small changes in λ_i and $\underline{r}_i$ due to small changes in $\underline{A}$.

First, premultiply (3.13) by $\underline{\ell}_j^H$. Upon noting (3.9) and (3.10), we have

$$\underline{\ell}_i(\delta\underline{A})\underline{r}_j = 0 \qquad (i \neq j) \tag{3.14}$$

$$\delta\lambda_i = \underline{\ell}_i^H(\delta\underline{A})\underline{r}_i \qquad (i=1,\ldots,2n) \tag{3.15}$$

The latter equation expresses the change in the eigenvalue λ_i caused by the change in the matrix $\underline{A}$.

Next, we find the change $\delta\underline{r}_i$ in the eigenvectors $\underline{r}_i$. With no loss in generality, the condition

$$\underline{\ell}_i^H\delta\underline{r}_i = 0 \qquad (i=1,\ldots,2n) \tag{3.16}$$

is imposed on $\delta\underline{r}_i$. This condition may seem arbitrary, or even wrong, at first glance. It is needed, however, to make $\delta\underline{r}_i$ unambiguous. Suppose $\underline{r}_i + \delta\underline{a}_i$ is an eigenvector of $\underline{A} + \delta\underline{A}$. Then $(\underline{r}_i + \delta\underline{a}_i)w$ is also an eigenvector of $\underline{A} + \delta\underline{A}$, where w is any complex constant. The condition (3.16) resolves the ambiguity of the factor w. Thus

$$\delta\underline{r}_i = (\underline{r}_i + \delta\underline{a}_i)w - \underline{r}_i \tag{3.17}$$

and (3.16) fixes w to be

$$w = (1 + \underline{\ell}_i^H\delta\underline{a}_i)^{-1} \tag{3.18}$$

To derive the desired expression for $\delta\underline{r}_i$, note that

$$\underline{A} = \sum_{k=1}^{2n} \lambda_k\underline{r}_k\underline{\ell}_k^H \tag{3.19}$$

This relation simply expresses $\underline{A}$ in terms of its eigenvalues and eigenvectors, and clearly satisfies (3.2) and (3.9). Then (3.13) can be rewritten thus:

$$\sum_{k=1}^{2n} \lambda_k \underline{r}_k \underline{\ell}_k^H \delta \underline{r}_i + (\delta \underline{A})\underline{r}_i = \lambda_i \delta \underline{r}_i + (\delta \lambda_i)\underline{r}_i \tag{3.20}$$

Now, premultiply by $\underline{\ell}_j^H (j \neq i)$, and note the orthonormality conditions (3.10), to obtain

$$\underline{\ell}_j^H \delta \underline{r}_i = \frac{\underline{\ell}_j^H (\delta \underline{A}) \underline{r}_i}{\lambda_i - \lambda_j} \qquad (i \neq j) \tag{3.21}$$

If (3.21) is premultiplied by $\underline{r}_j$ and summed over all j (except j = i), the result is

$$\delta \underline{r}_i = \left[\sum_{\substack{j=1 \\ j \neq i}}^{2n} \frac{\underline{r}_j \underline{\ell}_j^H}{\lambda_i - \lambda_j} \right] (\delta \underline{A}) \underline{r}_i \qquad (i=1,\ldots,2n) \tag{3.22}$$

because from $\underline{RL} = \underline{1}$ we learn that

$$\sum_{j=1}^{2n} \underline{r}_j \underline{\ell}_j^H = \underline{1} \tag{3.23}$$

and $\overbrace{\underline{\ell}_i^H \delta \underline{r}_i}^{= 0}$ by agreement. Equation (3.22) is the relationship sought; it expresses the change in the right eigenvector $\underline{r}_i$ caused by the change in the matrix $\underline{A}$. Note also that *distinct eigenvalues* have been assumed.

For the sake of completeness, it is remarked that the change in the left eigenvectors due to $\delta \underline{A}$ can be shown to be

$$\delta \underline{\ell}_i^H = \underline{\ell}_i^H (\delta \underline{A}) \left[\sum_{\substack{j=1 \\ j \neq i}}^{2n} \frac{\underline{r}_j \underline{\ell}_j^H}{\lambda_i - \lambda_j} \right] \qquad (i=1,\ldots 2n) \tag{3.24}$$

using a similar derivation.

3.2 Perturbations in System Eigenvalues Due to Light Damping

With the above results in hand, we turn now to the real objective--finding the perturbations in the undamped modal parameters caused by small structural damping. The undamped vibrating system represented by

$$\underline{M}\ddot{\underline{q}} + \underline{K}\underline{q} = \underline{0} \tag{3.25}$$

can also be represented by

$$\underline{q}(t) = \underline{E}\underline{\eta}(t) = \sum_{\alpha=1}^{n} \underline{e}_\alpha \eta_\alpha(t) \tag{3.26}$$

where

$$\ddot{\eta}_\alpha + \omega_\alpha^2 \eta_\alpha = 0 \qquad (\alpha=1,\ldots,n) \tag{3.27}$$

and $\underline{e}_\alpha$, the columns of $\underline{E}$, are the undamped modal vectors. They satisfy the orthonormality relations

$$\underline{e}_\alpha^T \underline{M} \underline{e}_\beta = \delta_{\alpha\beta}; \qquad \underline{e}_\alpha^T \underline{K} \underline{e}_\beta = \omega_\alpha^2 \delta_{\alpha\beta} \tag{3.28}$$

which is just another way of writing (2.7). In this section, we consider the elastic modes only, $\omega_\alpha^2 > 0$.

Now, (3.29) can be rewritten

$$\begin{bmatrix} \underline{\eta} \\ \dot{\underline{\eta}} \end{bmatrix}^{\bullet} = \begin{bmatrix} \underline{0} & \underline{1} \\ -\underline{\Omega}^2 & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{\eta} \\ \dot{\underline{\eta}} \end{bmatrix} \tag{3.29}$$

and so we have the following correspondences to the theory of Section 3.1:

$$\underline{A} = \begin{bmatrix} \underline{0} & \underline{1} \\ -\underline{\Omega}^2 & \underline{0} \end{bmatrix} \tag{3.30}$$

$$\underline{r}_\alpha = \begin{bmatrix} \underline{1}_\alpha \\ j\omega_\alpha \underline{1}_\alpha \end{bmatrix}; \qquad \underline{r}_{n+\alpha} = \begin{bmatrix} \underline{1}_\alpha \\ -j\omega_\alpha \underline{1}_\alpha \end{bmatrix} \tag{3.31}$$

$$\underline{\ell}_\alpha^H = [\tfrac{1}{2}\underline{1}_\alpha^T \quad -\tfrac{1}{2}j\omega_\alpha^{-1}\underline{1}_\alpha^T] \; ; \qquad \underline{\ell}_{n+\alpha}^H = [\tfrac{1}{2}\underline{1}_\alpha^T \quad \tfrac{1}{2}j\omega_\alpha^{-1}\underline{1}_\alpha^T] \tag{3.32}$$

$$\lambda_\alpha = j\omega_\alpha \; ; \qquad \lambda_{n+\alpha} = -j\omega_\alpha \tag{3.33}$$

for $\alpha = 1,\ldots,n$. Here, $\underline{1}_\alpha$ is a n x 1 column of 0's, except for a 1 in position α.

Now we consider the damping terms to be perturbations to the above (undamped) system. The damped system is

$$\underline{M}\ddot{\underline{q}} + \underline{\mathcal{D}}\dot{\underline{q}} + \underline{K}\underline{q} = 0 \tag{3.34}$$

where $\underline{\mathcal{D}}$ is 'small'. The undamped eigenvectors $\underline{e}_\alpha$ are still a vector basis in which to express $\underline{q}(t)$. Thus we still use (3.29), except that η_α no longer satisfies (3.27), but

$$\ddot{\eta}_\alpha + \sum_{\beta=1}^{n} \hat{d}_{\alpha\beta}\dot{\eta}_\beta + \omega_\alpha^2 \eta_\alpha = 0 \tag{3.35}$$

instead, where $\hat{d}_{\alpha\beta}$ are the elements of $\hat{\underline{\mathcal{D}}}$, and

$$\hat{\underline{\mathcal{D}}} = \underline{E}^T \underline{\mathcal{D}E} \tag{3.36}$$

Furthermore, we define

$$\zeta_\alpha = \tfrac{1}{2}\hat{d}_{\alpha\alpha}/\omega_\alpha \tag{3.37}$$

as the damping factor for mode α. Note, however, that we are *not* assuming that $\hat{d}_{\alpha\beta} = 0$ for $\alpha \neq \beta$.

What are the changes in the eigenvalues due to $\underline{\mathcal{D}}$? According to (3.15), we can use (3.30)-(3.33) above, provided we observe that

$$\delta\underline{A} = \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{0} & -\hat{\underline{\mathcal{D}}} \end{bmatrix} \tag{3.38}$$

in order to be consistent with (3.35). The result is

$$\delta\lambda_\alpha = \delta\lambda_{n+\alpha} = -\zeta_\alpha\omega_\alpha \qquad (\alpha=1,\ldots,n) \tag{3.39}$$

From this simple result we can come to the following important conclusions.

<u>*Conclusion 1*</u> - *For lightly damped structures, the change to the undamped eigenvalue* $j\omega_\alpha$ *caused by damping depends only on the diagonal elements of* $\underline{\mathcal{D}}$*, not on the off-diagonal elements.*

Conclusion 2 - *For lightly damped structures, the change to the undamped eigenvalues* $j\omega_\alpha$ *caused by damping is a small shift into the left plane, of magnitude* $\zeta_\alpha\omega_\alpha$. *There is no change in the imaginary part of the eigenvalue.*

3.3 Perturbations in System Eigenvectors Due to Light Damping

Next, we consider the change in the eigenvectors caused by light damping. We write (3.22) as

$$\delta\underline{r}_\alpha = \left[\frac{\underline{r}_{n+\alpha}\underline{\ell}^H_{n+\alpha}}{\lambda_\alpha - \lambda_{n+\alpha}} + \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n} \left(\frac{\underline{r}_\beta\underline{\ell}^H_\beta}{\lambda_\alpha - \lambda_\beta} + \frac{\underline{r}_{n+\beta}\underline{\ell}^H_{n+\beta}}{\lambda_\alpha - \lambda_{n+\beta}} \right) \right] (\delta\underline{A})\underline{r}_\alpha \qquad (3.40)$$

for $\alpha = 1,\ldots,n$, and $\delta\underline{r}_{n+\alpha} = (\delta\underline{r}_\alpha)^*$. Using (3.31) - (3.33) and (3.38), one finds

$$\delta\underline{r}_\alpha = -\tfrac{1}{2}j\zeta_\alpha \begin{bmatrix} \underline{1}_\alpha \\ -j\omega_\alpha\underline{1}_\alpha \end{bmatrix} + j\omega_\alpha \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n} \frac{\hat{d}_{\alpha\beta}}{\omega_\alpha^2 - \omega_\beta^2} \begin{bmatrix} \underline{1}_\beta \\ j\omega_\alpha\underline{1}_\beta \end{bmatrix} \qquad (3.41)$$

for $\alpha = 1,\ldots,n$. There is the suggestion in (3.41) that the damping must be especially light when the frequencies are clustered (nearly indistinct) if our 'light damping' theory is to remain valid.

3.4 Perturbations to Undamped Motion Due to Light Damping

In the absence of damping, the general free motion of the structure consists of a linear superposition of the (undamped) vibration modes, as shown in (3.26):

$$\underline{q}(t) = \sum_{\alpha=1}^{n} \underline{e}_\alpha \eta_\alpha(t) \qquad (3.42)$$

where

$$\eta_\alpha(t) = c_{1\alpha}\cos\omega_\alpha t + c_{2\alpha}\sin\omega_\alpha t \qquad (3.43)$$

and the constants $c_{1\alpha}$ and $c_{2\alpha}$ are determined from initial conditions.

To find the effect of light damping on this solution, we re-examine the undamped solution (3.42) - (3.43) in another light by setting

$$\underline{q}(t) = Re\left[\sum_{\alpha=1}^{n} c_\alpha \underline{q}_\alpha e^{\lambda_\alpha t}\right] \tag{3.44}$$

where $\underline{q}_\alpha$ are the eigenvectors in physical coordinates, λ_α are the eigenvalues and c_α are a set of complex constants determined from initial conditions. Although we allow c_α, $\underline{q}_\alpha$ and λ_α to be complex, $\underline{q}(t)$ itself is, of course, real.

The eigenvector in physical coordinates, $\underline{q}_\alpha$, is related to the eigenvector in undamped modal coordinates, $\underline{\eta}_\alpha$, by

$$\underline{q}_\alpha = \underline{E\eta}_\alpha \tag{3.45}$$

However, from (3.31),

$$\underline{\eta}_\alpha = \underline{1}_\alpha \tag{3.46}$$

so that

$$\underline{q}_\alpha = \underline{e}_\alpha \tag{3.47}$$

Also, $\lambda_\alpha = j\omega_\alpha$. So (3.44) is identical to (3.42) - (3.43), with the identification that

$$c_\alpha = c_{1\alpha} - jc_{2\alpha} \tag{3.48}$$

For the lightly damped system we can still write the solution as a sum of modes, as given in (3.44). But now

$$\underline{q}_\alpha = \underline{E\eta}_\alpha = \underline{E}\left[\underline{1}_\alpha - \tfrac{1}{2} j\zeta_\alpha \underline{1}_\alpha + \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{n} \frac{\hat{d}_{\alpha\beta}}{\omega_\alpha^2 - \omega_\beta^2}\underline{1}_\beta\right]$$

as indicated in (3.41). In other words the 'lightly-damped mode shapes' $\underline{q}_\alpha$ can be expressed in terms of the undamped mode shapes $\underline{e}_\alpha$ as follows:

$$\underline{q}_\alpha = (1 - \tfrac{1}{2}j\zeta_\alpha)\underline{e}_\alpha + j\omega_\alpha \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n} \frac{\hat{d}_{\alpha\beta}}{\omega_\alpha^2 - \omega_\beta^2}\underline{e}_\beta \qquad (3.49)$$

Speaking loosly, a 1% damping results in a 1% change in the undamped eigenvector, at 90^o phase.

The general free motion of a lightly damped structure is then of the form (3.44), with $\underline{q}_\alpha$ given by (3.49), and

$$\lambda_\alpha = j\omega_\alpha - \zeta_\alpha\omega_\alpha \qquad (3.50)$$

as indicated in (3.39). It follows that

$$Re\{\underline{q}_\alpha e^{\lambda_\alpha t}\} = e^{-\zeta_\alpha\omega_\alpha t}\left[\underline{e}_\alpha \cos\omega_\alpha t + \left(\tfrac{1}{2}\zeta_\alpha\underline{e}_\alpha - \omega_\alpha \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{n} \frac{\hat{d}_{\alpha\beta}}{\omega_\alpha^2 - \omega_\beta^2}\underline{e}_\beta\right)\sin\omega_\alpha t\right] \qquad (3.51)$$

The expression for $Im\{\underline{q}_\alpha e^{\lambda_\alpha t}\}$ is the same except for

$$\{\cos\omega_\alpha t, \sin\omega_\alpha t\} \rightarrow \{\sin\omega_\alpha t, \ -\cos\omega_\alpha t\} \qquad (3.52)$$

The following two conclusions concerning the effect of light structural damping on the free motion are now clear:

Conclusion 3 - *The motion is no longer purely oscillatory, but is slowly damped. The envelope for mode* α *is* $\exp(-\zeta_\alpha\omega_\alpha t)$.

Conclusion 4 - *The damped mode shapes are essentially the same as the undamped mode shapes, but with small,* 90^o*-phase components proportional to the damping.*

From a practical standpoint, the slow expoential damping action is crucially important, while the small change in eigenvector is unimportant. So long as the structural damping is of the order of 1% or less, there seems little justification for keeping the 1% out-of-phase term in (3.51). In other words, for all practical purposes, the general free motion of a lightly damped structure is given by

$$\underline{q}(t) = \sum_{\alpha=1}^{n} \underline{e}_\alpha (c_{1\alpha}\cos\omega_\alpha t + c_{2\alpha}\sin\omega_\alpha t)e^{-\zeta_\alpha\omega_\alpha t} \qquad (3.53)$$

where the $\underline{e}_\alpha$ are the undamped vibration mode shapes, the ω_α are the undamped vibration frequencies, and the ζ_α are calculated from

$$2\zeta_\alpha\omega_\alpha = \hat{d}_{\alpha\alpha} \equiv \underline{e}_\alpha^T \mathcal{D} \underline{e}_\alpha \tag{3.54}$$

The following conclusions also follow:

<u>*Conclusion 5*</u> - *Effort spent in calculating undamped mode shapes and frequencies is entirely valid. It is these modal quantities that are used in (3.53).*

<u>*Conclusion 6*</u> - *Only the <u>diagonal</u> elements of the modal damping matrix $\underline{\hat{\mathcal{D}}}$ have an important effect on the lightly damped motion.*

3.5 <u>A Criterion for Diagonal Dominance</u>

In connection with (3.51), a criterion for diagonal dominance in the modal damping matrix $\underline{\hat{\mathcal{D}}}$ can be inferred. If

$$4 \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{n} \frac{\omega_\alpha^2}{\omega_\alpha^2 - \omega_\beta^2} \hat{d}_{\alpha\beta}\underline{e}_\beta << \hat{d}_{\alpha\alpha}\underline{e}_\alpha \qquad (\alpha=1,\ldots,n) \tag{3.55}$$

then the effects of the off-damped elements in $\underline{\hat{\mathcal{D}}}$ on the (small) change in the eigenvector are negligible. Diagonal dominance permits us to write (3.51) as

$$Re\{\underline{q}_\alpha e^{\lambda_\alpha t}\} \doteq \underline{e}_\alpha(\cos\omega_\alpha t + \tfrac{1}{2}\zeta_\alpha \sin\omega_\alpha t)e^{-\zeta_\alpha\omega_\alpha t} \tag{3.56}$$

which has the attraction of simplicity.

However, the main thrust of this section is not the 'diagonal dominance' assumption, but the 'light damping' assumption. With light damping, the motion depends, for all practical purposes, only on the diagonal elements of $\underline{\hat{\mathcal{D}}}$, whether $\underline{\hat{\mathcal{D}}}$ is diagonally dominant or not.

4. IMPORTANCE OF STRUCTURAL DAMPING

The objective of this section is to demonstrate the following two assertions:

Assertion 1 - *Even though structural damping is 'small', it is very important.*

Assertion 2 - *In comparing values of the modal damping factor ζ for lightly damped structures, it is the value of ζ compared to 0 that is important, not the value of ζ compared to 1.*

There are several relatively simple ways to verify these assertions. In this section, we shall consider the following situations:

(i) resonant response to sinusoidal inputs;
(ii) steady-state response to random inputs;
(iii) bounded response to bounded inputs;
(iv) open-loop modal cost analysis;
(v) stability regions for a selected control system.

In all cases the importance of the damping factor is quite evident.

4.1 Response to Sinusoidal Inputs

One of the most obvious demonstrations of the importance of ζ is the well-known result that the response to a sinusoidal input, whose frequency coincides with a natural frequency of vibration, is proportional to ζ^{-1}. That is, the resonant response amplitude goes as ζ^{-1}. For mode α,

$$\ddot{\eta}_\alpha + 2\zeta_\alpha\omega_\alpha\dot{\eta}_\alpha + \bar{\omega}_\alpha^{\,2}\eta_\alpha = \gamma_\alpha(t) \qquad (4.1)$$

and, if

$$\gamma_\alpha = \gamma_{\alpha o}\cos\bar{\omega}t \equiv Re\{\gamma_{\alpha o}e^{j\omega t}\} \qquad (4.2)$$

then the steady-state response may be written

$$\eta_\alpha(t) = Re\{\eta_{\alpha o}(\omega)e^{j\omega t}\} \qquad (4.3)$$

where $\eta_{\alpha o}$ is complex.

Substitution of (4.3) into (4.1) shows that

$$\eta_{\alpha o}(\omega) = \frac{\gamma_{\alpha o}}{(\omega_\alpha^2 - \omega^2) + 2j\zeta_\alpha\omega_\alpha\omega} \tag{4.4}$$

The *amplitude* of response is

$$|\eta_{\alpha o}(\omega)| = \frac{\gamma_{\alpha o}}{[(\omega_\alpha^2 - \omega^2)^2 + 4\zeta_\alpha^2\omega_\alpha^2\omega^2]^{\frac{1}{2}}} \tag{4.5}$$

As is well-known, this function is largest at resonance, i.e., when $\omega = \omega_\alpha$. At resonance,

$$|\eta_{\alpha o}(\omega_\alpha)| = \frac{\gamma_{\alpha o}}{2\zeta_\alpha\omega_\alpha^2} \tag{4.6}$$

If ζ_α is assumed to be 0.001 when it is in reality 0.01, the resonant response will be overestimated by 900%.

For lightly damped structures the response of the other modal coordinates (η_β, $\beta \neq \alpha$) will be dwarfed by the mode in resonance. Hence

$$\underline{q}(t) \doteq \underline{e}_\alpha\eta_\alpha(t) \tag{4.7}$$

and the amplitudes of vibration of the physical coordinates are also determined by $1/\zeta_\alpha$.

4.2 Response to Random Inputs

Lest it be thought that the above conclusion depends on the assumption of sinusoidal excitation, the analysis will now be generalized to include stationary random inputs.

Let $\gamma_\alpha(t)$, α=1,...,n , be a set of stationary random inputs with zero mean:

$$\langle\gamma_\alpha(t)\rangle = 0 \qquad (\alpha=1,\ldots,n) \tag{4.8}$$

where $\langle\cdot\rangle$ means *expected value of* $(\cdot)$. These inputs are characterized by their *covariance matrix*

$$\underline{C}_{\gamma\gamma}(\tau) = \langle \underline{\gamma}(t + \tau)\underline{\gamma}^T(t)\rangle \tag{4.9}$$

The assumptions of stationarity and ergodicity permit us to relate expected values to time averages:

$$\underline{C}_{\gamma\gamma}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \underline{\gamma}(t + \tau)\underline{\gamma}^T(t)dt \tag{4.10}$$

An equivalent representation is through the *spectral density matrix* $\underline{\Phi}_{\gamma\gamma}$, defined as the Fourier transform of $\underline{C}_{\gamma\gamma}$:

$$\underline{\Phi}_{\gamma\gamma}(\omega) = \int_{-\infty}^{\infty} \underline{C}_{\gamma\gamma}(\tau)e^{-j\omega\tau}d\tau \tag{4.11}$$

We denote by $h_{\alpha\beta}(\omega)$ the frequency response of mode α to a sinusoidal input to mode β. Thus

$$h_{\alpha\alpha}(\omega) = \frac{1}{(\omega_\alpha^2 - \omega^2) + 2j\zeta_\alpha\omega_\alpha\omega} \tag{4.12}$$

$$h_{\alpha\beta}(\omega) = 0 \qquad\qquad (\beta \neq \alpha) \tag{4.13}$$

[Alternatively, $h_{\alpha\alpha}(\omega)$ can be regarded as the Fourier transform of the impulse response. The response of (4.1) to

$$\gamma_\alpha(t) = \delta(t)$$

is

$$\eta_\alpha(t) = \frac{1}{\omega_\alpha} e^{-\zeta_\alpha\omega_\alpha t} \sin\omega_\alpha t \tag{4.14}$$

for 'light' damping (i.e., dropping ζ_α^2 terms). The Fourier transform of (4.14) is indeed (4.12) within the 'light' damping approximation.]

Let

$$\underline{H}(\omega) = \{h_{\alpha\beta}(\omega)\} \tag{4.15}$$

a diagonal matrix of frequency response functions. Then, from linear system theory, the spectral density matrix for the outputs, $\underline{\Phi}_{\eta\eta}$, is related to the spectral density matrix for the inputs, $\underline{\Phi}_{\gamma\gamma}$, by the following relation:

$$\underline{\Phi}_{\eta\eta}(\omega) = \underline{H}(\omega)\underline{\Phi}_{\gamma\gamma}(\omega)\underline{H}^H(\omega) \tag{4.16}$$

where $(\cdot)^H$ denotes the *Hermitian operation* (complex-conjugate transpose).

The elements of $\underline{\Phi}_{\eta\eta}$ are

$$\{\underline{\Phi}_{\eta\eta}\}_{\alpha\beta} \equiv \phi_{\eta_\alpha\eta_\beta} = h_{\alpha\alpha}h^*_{\beta\beta}\phi_{\gamma_\alpha\gamma_\beta} \tag{4.17}$$

where $(\cdot)^*$ denotes the complex congugate.

The basic results are now in place to calculate the average energy of the system. The total energy is the sum of the kinetic energy and the potential energy:

$$E(t) = \tfrac{1}{2}\underline{\dot{q}}^T\underline{M}\underline{\dot{q}} + \tfrac{1}{2}\underline{q}^T\underline{K}\underline{q} \tag{4.18}$$

Or, in terms of modal coordinates, with $\underline{q} = \underline{E}\underline{\eta}$,

$$E(t) = \tfrac{1}{2}\underline{\dot{\eta}}^T\underline{\dot{\eta}} + \tfrac{1}{2}\underline{\eta}^T\underline{\Omega}^2\underline{\eta} \tag{4.19}$$

Thus the mean (expected) value of the energy is

$$<E(t)> = \tfrac{1}{2}\sum_{\alpha=1}^{n}[<\dot{\eta}_\alpha^2> + \omega_\alpha^2<\eta_\alpha^2>] \tag{4.20}$$

Fortunately, there is a simple relationship between the mean of η_α^2 and spectral density for η_α. According to Ref. 5, p. 338,

$$<\eta_\alpha^2> = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_{\eta_\alpha\eta_\alpha}(\omega)d\omega \tag{4.21}$$

and, from the same reference, p.339,

$$<\dot{\eta}_\alpha^2> = \frac{1}{2\pi}\int_{-\omega}^{\infty}\omega^2\phi_{\eta_\alpha\eta_\alpha}(\omega)d\omega \tag{4.22}$$

By inserting (4.17) in (4.21) and (4.22), and the latter two equations in (4.20), one can express the mean energy of the system in terms of $\underline{\Phi}_{\gamma\gamma}$:

$$\langle E(t)\rangle = \frac{1}{4\pi}\sum_{\alpha=1}^{n}\int_{-\infty}^{\infty}(\omega^2+\omega_\alpha^2)|h_{\alpha\alpha}(\omega)|^2\phi_{\gamma_\alpha\gamma_\alpha}(\omega)d\omega \tag{4.23}$$

Or, from (4.12),

$$\langle E(t)\rangle = \frac{1}{4\pi}\sum_{\alpha=1}^{n}\int_{-\infty}^{\infty}\frac{\omega^2+\omega_\alpha^2}{(\omega_\alpha^2-\omega^2)^2+4\zeta_\alpha^2\omega_\alpha^2\omega^2}\phi_{\gamma_\alpha\gamma_\alpha}(\omega)d\omega \tag{4.24}$$

In general, these integrals would need to be integrated by numerical techniques.

However, even without integrating, it is clear that the denominators in (4.24) are singular when $\omega = \omega_\alpha$, $(\alpha=1,\ldots,n)$, and that the energy mean will be very sensitive to the values of the modal damping factors, ζ_α. This is not surprising in view of the fact that the role of damping is to remove energy from the system.

4.3 Bounded Response to Bounded Inputs

Another means of showing the critical roles of the damping factors ζ_α is to show their function in relating the bounded response to a bounded disturbance. We begin again with (4.1), and assume that the input has a known limit on its magnitude:

$$|\gamma_\alpha(t)| \leq \gamma_{\alpha,max} \tag{4.25}$$

The modal coordinate $\eta_\alpha(t)$ is known to have the solution

$$\eta_\alpha(t) = \omega_\alpha^{-1}\int^{t}\exp[-\zeta_\alpha\omega_\alpha(t-\tau)]\sin[\omega_\alpha(t-\tau)]\gamma_\alpha(\tau)d\tau \tag{4.26}$$

after transients have died away. The integral solution (4.26) is based on the well-known convolution property of the input $\gamma_\alpha(t)$ with the impulse response function (4.14). It should also be pointed out that the expression (4.26) assumes light damping.

After transients have died out, we have, from (4.25) and

(4.26), the following bound on $\eta_\alpha(t)$:

$$|\eta_\alpha(t)| \leq \omega_\alpha^{-1} \int^t |\exp[-\zeta_\alpha\omega_\alpha(t-\tau)]| |\sin[\omega_\alpha(t-\tau)]| |\gamma_\alpha(\tau)| d\tau$$

$$\leq \omega_\alpha^{-1}\gamma_{\alpha,\max} \int^t \exp[-\zeta_\alpha\omega_\alpha(t-\tau)] d\tau$$

$$\leq \frac{\gamma_{\alpha,\max}}{\zeta_\alpha\omega_\alpha^2} \tag{4.27}$$

According to this bound, if $\zeta_\alpha = 0.01$ the modal coordinate is restricted an order of magnitude more than if $\zeta_\alpha = 0.001$.

A bound on each physical coordinate can be obtained as follows:

(1) $\quad ||\underline{q}||^2 \equiv \underline{q}^T\underline{q} = \underline{\eta}^T(\underline{E}^T\underline{E})\underline{\eta}$

(2) $\quad \leq \lambda_{\max}\{\underline{E}^T\underline{E}\} \, ||\underline{\eta}||^2$

(3) $\quad = \lambda_{\min}^{-1}\{\underline{M}\} \, ||\underline{\eta}||^2$

(4) $\quad \equiv \lambda_{\min}^{-1}\{\underline{M}\}\underline{\eta}^T\underline{\eta}$

(5) $\quad \leq \lambda_{\min}^{-1}\{\underline{M}\} \sum_{\alpha=1}^{n} \frac{\gamma_{\alpha,\max}^2}{\zeta_\alpha^2\omega_\alpha^4}$

(6) $\quad < \frac{n\lambda_{\min}^{-1}\{\underline{M}\}}{(\zeta_\alpha^2\omega_\alpha^4)_{\min}} (\underline{\gamma}^T\underline{\gamma})_{\max}$

(7) $\quad = \frac{n\lambda_{\min}^{-1}\{\underline{M}\}}{(\zeta_\alpha^2\omega_\alpha^4)_{\min}} \, ||\underline{f}^T\underline{M}^{-1}\underline{f}||_{\max}$

(8) $\quad \leq \frac{n\lambda_{\min}^{-1}\{\underline{M}\}}{(\zeta_\alpha^2\omega_\alpha^4)_{\min}} \lambda_{\max}\{\underline{M}^{-1}\} \, ||\underline{f}||_{\max}^2$

(9) $$= \frac{n}{(\zeta_\alpha^2\omega_\alpha^4)_{min}} \|\underline{\delta}\|^2_{max} \qquad (4.28)$$

A few words of explanation should be helpful:

(i) the notation $\lambda\{\underline{A}\}$ means "an eigenvalue of $\underline{A}$".

(ii) $\lambda_{max}\{\underline{A}\}$ means "the maximum eigenvalue of $\underline{A}$", and $\lambda_{min}\{\underline{A}\}$ means "the minimum eigenvalue of $\underline{A}$".

(iii) at line (2) it was recognized that the eigenvalues of $\underline{E}^T\underline{E}$ satisfy

$$\begin{aligned} 0 &= \det[\underline{E}^T\underline{E} - \lambda\underline{1}] \\ &= \det[\underline{E}^T(\underline{1} - \lambda\underline{E}^{-T}\underline{E}^{-1})\underline{E}] \\ &= \det \underline{E}^T\det[\underline{1} - \lambda(\underline{E}\underline{E}^T)^{-1}]\det \underline{E} \\ &= \det \underline{E}^T\det[\underline{1} - \lambda\underline{M}]\det \underline{E} \end{aligned} \qquad (4.29)$$

because $\underline{M} = (\underline{E}\underline{E}^T)^{-1}$ follows from $\underline{E}^T\underline{M}\underline{E} = \underline{1}$. However, $\det \underline{E}^T = \det \underline{E} \neq 0$, whence the eigenvalues of $\underline{E}^T\underline{E}$ satisfy

$$\begin{aligned} 0 &= \det[\underline{1} - \lambda\underline{M}] \\ &= \det \underline{M}^{-1}\det[\underline{M}^{-1} - \lambda\underline{1}] \end{aligned} \qquad (4.30)$$

Or, since $\det \underline{M}^{-1} \neq 0$, the eigenvalues of $\underline{E}^T\underline{E}$ are the eigenvalues of $\underline{M}^{-1}$, i.e. they are the reciprocals of the eigenvalues of $\underline{M}$.

(iv) at line (4), the bound (4.2) was inserted.

(v) at line (6), the fact that

$$\underline{\gamma}^T\underline{\gamma} = (\underline{E}^T\underline{\delta})^T(\underline{E}^T\underline{\delta}) = \underline{\delta}^T\underline{E}\underline{E}^T\underline{\delta} = \underline{\delta}^T\underline{M}^{-1}\underline{\delta} \qquad (4.31)$$

was observed.

(vi) at line (8), it was noted that, since the eigenvalues of $\underline{M}^{-1}$ are the reciprocals of the eigenvalues of $\underline{M}$, the maximum eigenvalue of $\underline{M}^{-1}$ is the reciprocal of the minimum eigenvalue of $\underline{M}$.

Thus if the input $\underline{\delta}(t)$ is bounded,

$$\| \underline{f}(t) \| \le f_{max} \tag{4.32}$$

the response, in physical coordinates, will be bounded according to

$$\| \underline{q}(t) \| \le n \left[\min_{\alpha=1}^{n} \{ \zeta_\alpha \omega_\alpha \} \right]^{-1} f_{max} \tag{4.33}$$

In particular, if the conventional assumption

$$\zeta_1 = \zeta_2 = \ldots = \zeta_n = \zeta$$

is made [see "Decision No. 1", Equation (2.15) of the Common Theory],

$$\| \underline{q}(t) \| \le \frac{n f_{max}}{\zeta \omega_1^2} \tag{4.34}$$

The critical nature of ζ is readily apparent from (4.34).

4.4 Open-Loop Modal Cost Analysis

In Ref. 6, Appendix E, it is shown that the relative importance of a mode can be assessed on a quantitative basis of the following is known:

(i) the natural frequency ω_α of the mode;

(ii) the damping factor ζ_α of the mode;

(iii) the degree to which the mode is excited;

(iv) the degree to which the mode matters to the quadratic performance function.

Some elucidation of the latter two ideas is required.

To determine how much mode α is excited, it is assumed that

$$\underline{M\ddot{q}} + \underline{D\dot{q}} + \underline{Kq} = \underline{f} = \underline{Bu} \tag{4.35}$$

Then, if the ith component of $\underline{u}$ contains only an impulsive signal,

$$u_i(t) = \delta(t - t_i) \qquad (t_i > 0) \tag{4.36}$$

the response of mode α is

$$\eta_\alpha(t) = (\hat{b}_{\alpha i}/\omega_\alpha)\exp[-\zeta_\alpha\omega_\alpha(t - t_i)]\sin[\omega_\alpha(t - t_i)] \tag{4.37}$$

where $\hat{b}_{\alpha i}$ is the ith element in the row

$$\underline{\hat{b}}_\alpha^T = \underline{e}_\alpha^T \underline{B} \tag{4.38}$$

(The expression (4.37) assumes 'light damping'.)

Then, to determine how much this modal excitation matters, we assume a vector of outputs to be regulated has been defined:

$$\underline{y}(t) = \underline{Pq}(t) \equiv \underline{\hat{P}\eta}(t) \tag{4.39}$$

where $\underline{\hat{P}} = \underline{PE}$. These outputs are then combined in a weighted sum-of-squares to produce a single scalar measure of the seriousness of the perturbations at time t:

$$y_Q^2(t) = \underline{y}^T \underline{Qy} \tag{4.40}$$

Finally, it is assumed that the objective of the control system is to minimize

$$V = \int_o^\infty y_Q^2(t)dt \tag{4.41}$$

It is shown in Ref. 6, Appendix E, that

$$V = \sum_{\alpha=1}^{n} V_\alpha \tag{4.42}$$

where

$$V_\alpha = \frac{(\underline{e}_\alpha^T \underline{\hat{Q}} \underline{e}_\alpha)\hat{d}_{\alpha i}^2}{4\zeta_\alpha\omega_\alpha^3} \tag{4.43}$$

where $\underline{\hat{Q}} = \underline{P}^T\underline{QP}$. The formula (4.43) assumes that an impulse is applied (at $t = t_i > 0$) only to the ith input u_i. If impulses are applied to *all* the inputs, $\hat{b}_{\alpha i}^2$ must be replaced by

$$\hat{b}_{\alpha i}^2 \rightarrow \sum_{i=1}^{m} \hat{b}_{\alpha i}^2 \tag{4.44}$$

where m is the number of inputs (the dimension of $\underline{u}$). However, we note that the sum on the right side of (4.44) can be written

$$\sum_{i=1}^{m} \hat{b}_{\alpha i}^2 \equiv \underline{\hat{b}}_{\alpha} \underline{\hat{b}}_{\alpha} = \underline{e}_{\alpha}^T \mathcal{B}\mathcal{B}^T \underline{e}_{\alpha} \tag{4.45}$$

and thus the expression for the *modal cost* V_α, from (4.43), becomes

$$V_\alpha = \frac{(\underline{e}_\alpha^T \hat{Q} \underline{e}_\alpha)(\underline{e}_\alpha^T \mathcal{B}\mathcal{B}^T \underline{e}_\alpha)}{4\zeta_\alpha \omega_\alpha^3} \tag{4.46}$$

Once again we see that, "other things being equal"--where by "other things" we mean the modal natural frequency, the modal excitation, and the importance of the mode in the error criterion $y_Q(t)$--the "cost" associated with mode α is inversely proportional to ζ_α.

4.5 Closed-Loop Stability

When the control feedback loops are closed it is intuitively obvious that the distances of the open-loop poles from the imaginary axis (the stability boundary) are critical. These distances are $\zeta_\alpha \omega_\alpha$ $(\alpha = 1,\ldots,n)$. Even if the control-system designer has complete confidence in his design, and has no worries that the "controlled modes" might get dangerously close to, or even cross, the imaginary axis, the problem of 'spillover' is still present--what does the control system do to the 'unmodeled' modes, i.e., to the poles at $-\zeta_\alpha \omega_\alpha + j\omega_\alpha$, $\alpha > n$? The stability margin for these poles is directly proportional to ζ_α.

In Ref. 7 a straightforward but relatively realistic control problem was analyzed. As shown in Fig. 4.1, the attitude of a 'flexible spacecraft' is controlled by

(i) a sensor with a time lag:

$$\bar{\theta}_s(s) = \frac{k_s}{s + \omega_s} \bar{\theta}(s) \tag{4.47}$$

where $\theta(t)$ is the actual attitude angle and $\theta_s(t)$ is the attitude reported by the sensor;

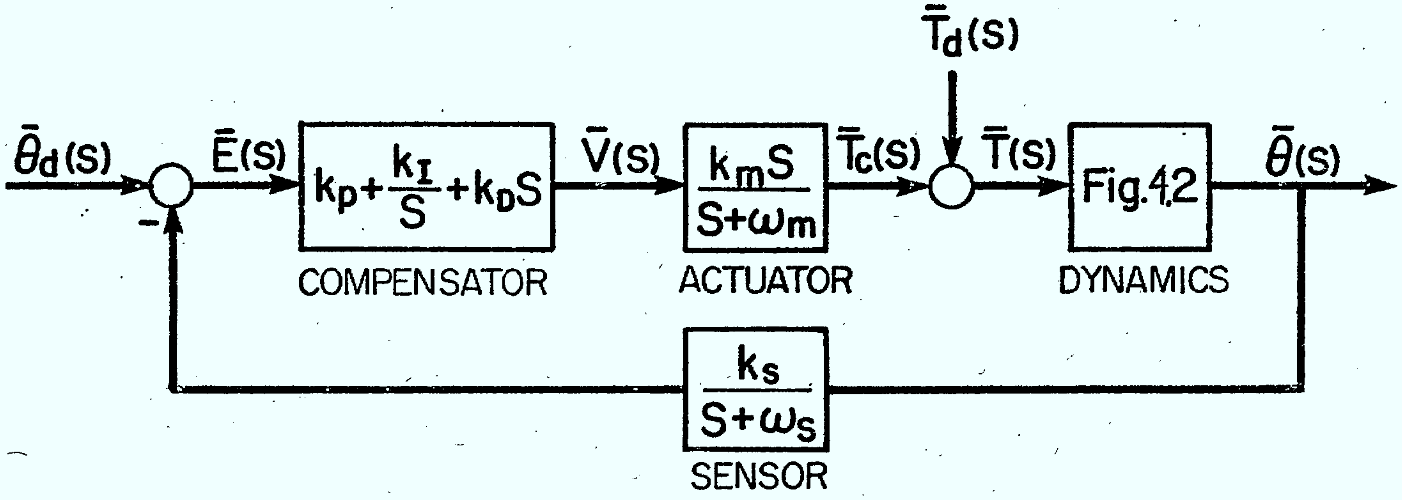


Fig. 4.1: A Simple One-Axis Control System for a Flexible Spacecraft

(ii) a (reaction-wheel) motor with transfer function

$$T_c(s) = \frac{k_m s}{s + \omega_m} \bar{v}(s) \tag{4.48}$$

where $T_c(t)$ is the motor torque generated by the command voltage $v(t)$;

(iii) a control law (or 'compensator') whose transfer function is

$$\bar{v}(s) = -(k_P + k_D s + k_I/s)\bar{\theta}_s(s) \tag{4.49}$$

corresponding to proportional, derivative, and integral feedback.

(iv) the 'flexible satellite' is represented by the transfer function

$$\bar{\theta}(s) = \frac{1}{s^2 I_e(s)} \bar{T}_c(s)$$

where

$$\frac{1}{s^2 I_e(s)} = \frac{1}{Is^2} + \sum_{\alpha=1}^{n} \frac{(k_\alpha/I)}{s^2 + 2\zeta_\alpha\omega_\alpha s + \omega_\alpha^2} \tag{4.50}$$

and I is the moment of inertia of the satellite. The 'modal' gain' k_α applies to (unconstrained) mode α. See Fig. 4.2.

For this 'spacecraft', the unconstrained-modal parameters $(\omega_\alpha, \zeta_\alpha, k_\alpha)$ were calculated from their constrained-modal counterparts $(\Omega_\alpha, Z_\alpha, K_\alpha)$.

It is known (Ref. 8) that

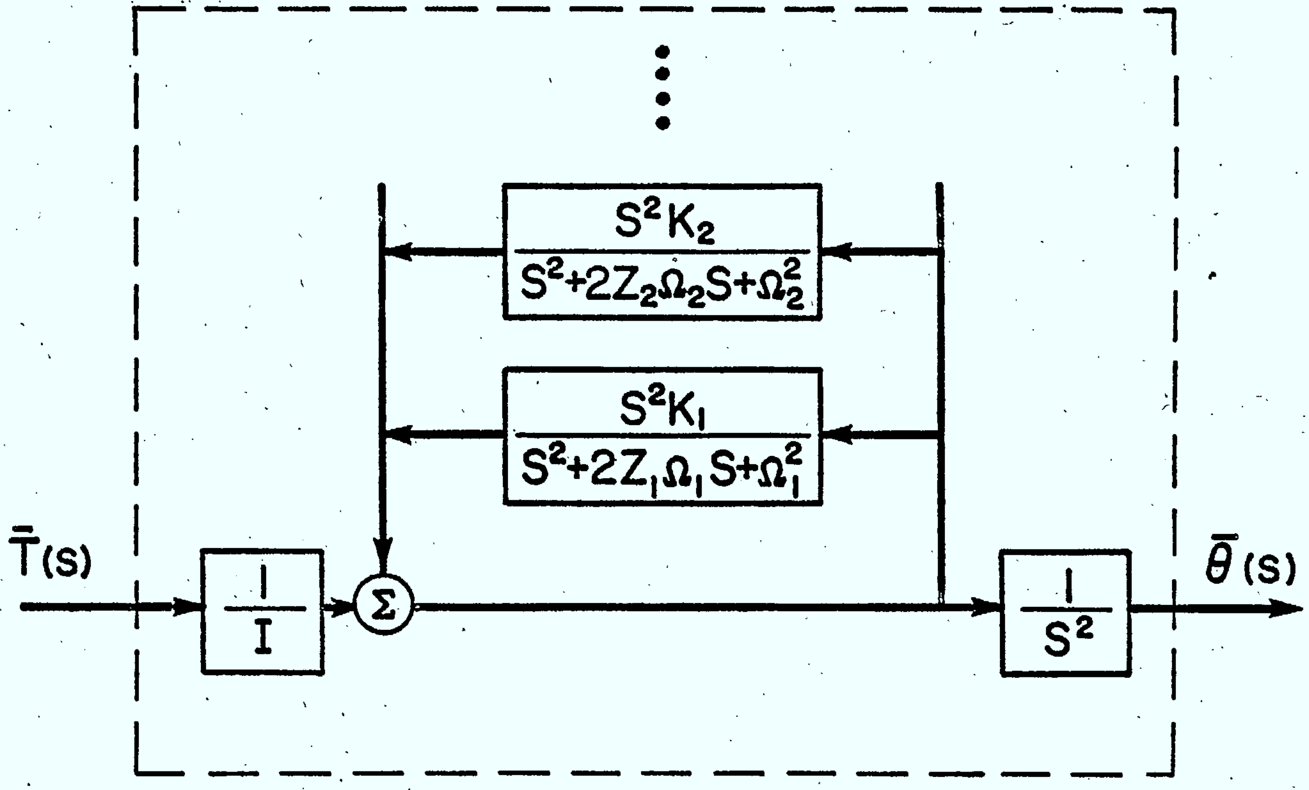
$$\sum_{\alpha=1}^{\infty} K_\alpha = I_f/I$$

where I_f is the moment of inertia of the elestic portion of the spacecraft. The constrained-modal parameters were chosen to be

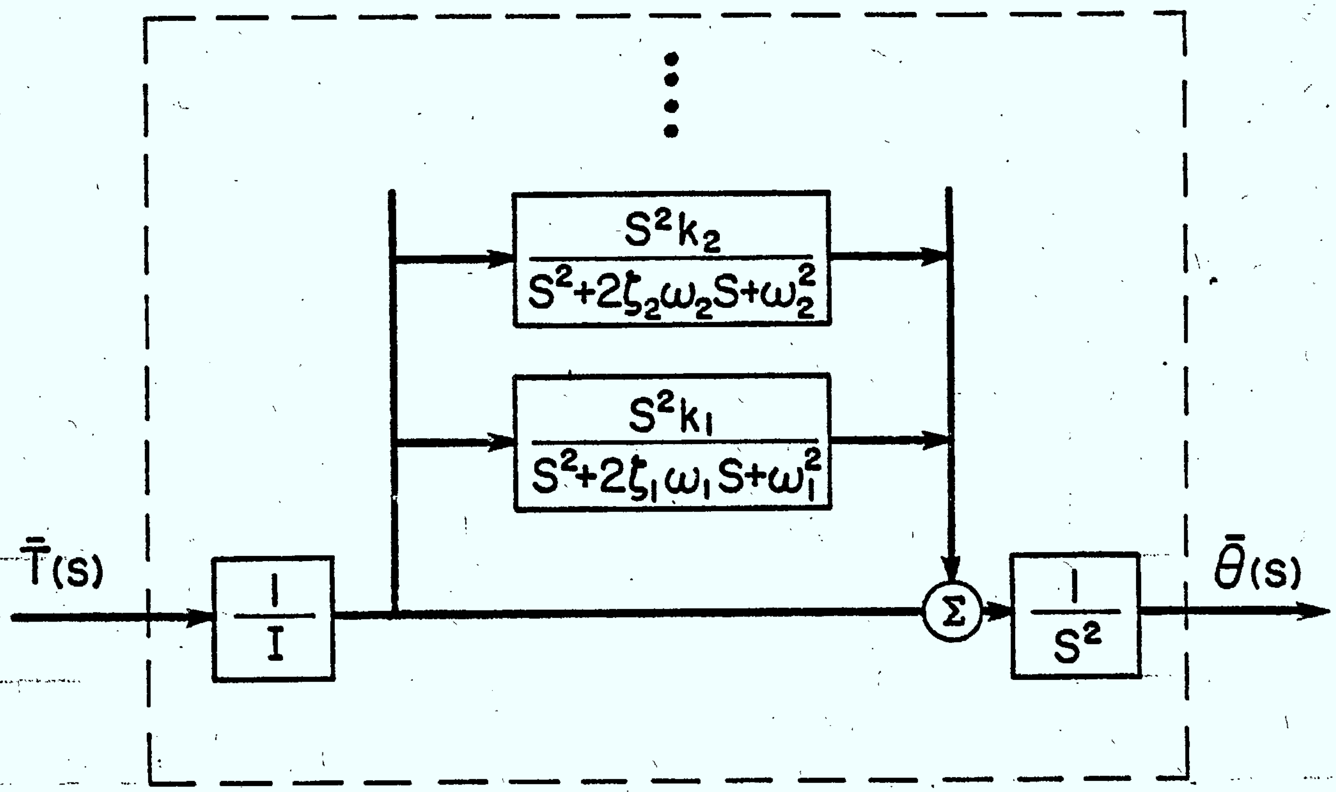
$$\Omega_\alpha = \alpha^2 \Omega_1$$

$$K_\alpha \Omega_\alpha^2 \quad \text{independent of } \alpha$$

$$K_\alpha = 90(I_f/I)/(\alpha^4\pi^4)$$



(a) Constrained Appendage Modes



(b) Unconstrained Spacecraft Modes

Fig. 4.2: Two Equivalent Dynamical Representations

$$Z_\alpha = Z_1 \quad (\text{all } \alpha)$$

Thus Ω_1 is a measure of the degree of flexibility, I_f/I is a measure of the size of the flexible appendage ($I_f/I \leq 1$), and Z_1 is a measure of the degree of passive dissipation.

Other details of the subject study can be found in Ref. 7. Some results are shown in Fig. 4.3. This stability diagram shows the significance of structural flexibility relative to control system bandwidth, as measured by Ω_1/ω_c, where ω_c is the 'phase-crossover' frequency (i.e., the value of ω for which the phase angle for the open-loop transfer function is -180^0). For the particular control system chosen, the system is always stable for $\Omega_1 > \omega_c$. Also shown in Fig. 4.3 are the effects of I_f/I (size of flexible appendage) and Z_1 (degree of passive damping). As might be expected, as I_f/I increases, matters get worse.

The main point of this discussion, however, is the effect of passive damping, as measured by the size of Z_1. When $Z_1 = 0$, all cases where $\Omega_1 < \omega_c$ are unstable. When $Z_1 = 0.001$, many of these unstable cases disappear. When $Z_1 = 0.01$, many more disappear; only the largest, most flexible appendages now lead to instability. Finally, for $Z_1 = 0.05$, no unstable cases exist. Obviously the stability of the closed-loop system is strongly dependent on the level of passive damping.

4.6 Summary

In summary, the two assertions made at the beginning of this section have been demonstrated in several ways. Damping, even though small, is critically important. Furthermore, the occurrence again and again of the factor $1/\zeta_\alpha$ in the expressions derived above--for example, in (4.6), (4.24), (4.27), (4.34), and (4.46)--shows that it is not a matter of indifference whether ζ_α is 0.01 or 0.001. Quite simply, there is an order-of-magnitude error in assuming one of these values if the other is the true value.

Would any project manager accept an estimate of ω_1 that was accurate to within only a factor of 10? Of course not. Yet project managers *do* accept estimates of ζ_1 that are accurate to within only a

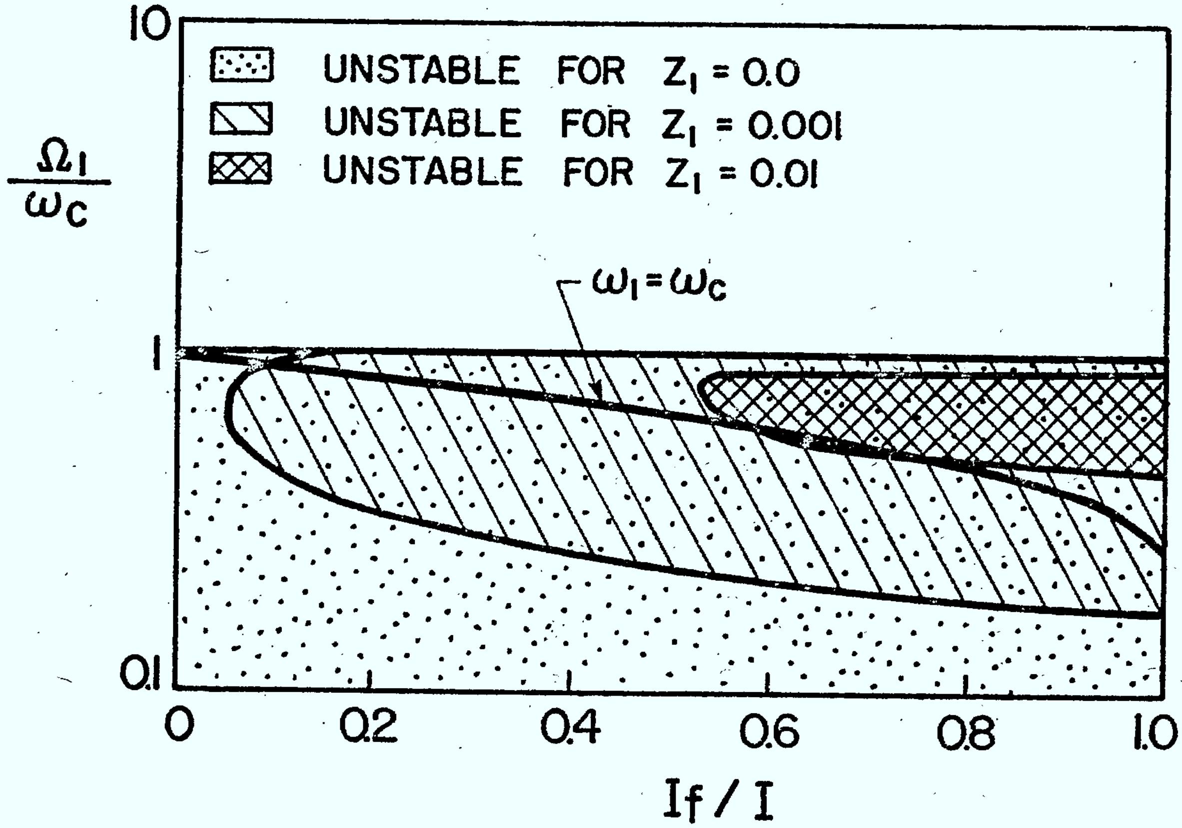


Fig. 4.3: Stability Diagram Emphasizing Importance of Damping

factor of 10. Whatever the explanation of this inconsistency, it does not constitute sound engineering practice. Damping factors may always be more elusive than natural frequencies, but proper analytical procedures can be expected to make substantial reductions in their factor-of-10 uncertainty. Such procedures can be expected to require considerable time and effort to be evolved. It is hoped that the discussion presented in the remainder of this report will at least be a step in the right direction.

5. LINEAR VISCOUS DAMPING

No aspect of structural modeling is *exactly* linear and the same can be said of energy dissipation mechanisms. We *assume* the stress-strain law of the material to be linear; we *assume* displacements to be small with respect to characteristic vehicle dimensions; we *assume* that 'small angles remain small', and so on. The strong motivation for these linearizing assumptions is that analysis is possible; the strong justification for them is that they are valid within the usual operating regimes of normal spacecraft. We shall assume in this report that damping is also linear.

For a system with one degree of freedom represented by the coordinate $q(t)$ and possessing mass m, let the damping force be $f_d(t)$. To assume linear damping is to assume a relationship between $f_d(t)$ and $\dot{q}(t)$ of the form

$$f_d(t) = -\int_{-\infty}^{t} h_d(t-\tau)\dot{q}(\tau)d\tau \tag{5.1}$$

Or, in the domain of the Laplace variable s,

$$\bar{f}_d(s) = -s\bar{h}_d(s)\bar{q}(s) \tag{5.2}$$

where an overbar denotes a Laplace-transformed variable, and $\bar{h}_d(s)$ is the Laplace transform of $h_d(t)$. The kinetic energy of the system is, of course, $T = \frac{1}{2}m\dot{q}^2$, so the rate of change in T due to the force f_d is

$$\dot{T} = m\dot{q}\ddot{q} = \dot{q}f_d \tag{5.3}$$

Energy is extracted whenever

$$\dot{q}f_d < 0 \tag{5.4}$$

One special case of linear damping is (so-called) linear *viscous* damping, in which

$$\bar{h}_d(s) \equiv c_d = \text{constant} \tag{5.5}$$

In other words

$$h_d(t) = c_d\delta(t) \tag{5.6}$$

and

$$f_d = -c_d\dot{q} \qquad \leftrightarrow \qquad \bar{f}_d = -sc_d\bar{q} \tag{5.7}$$

It is with this type of linear damping that this section (Section 5) is concerned.

5.1 Discrete Linear-Viscous Dampers

Many types of mechanical devices have been developed that have the linear-viscous characteristic mentioned above in (5.7). Although an extensive review of such devices is beyond the scope of this report, the following comments should be helpful:

(i) Because the device has been designed to have the linear-viscous characteristic (5.7), the use of (5.7) is especially accurate. In particular, the objections to (5.7) that apply when (5.7) is used as a model for *material* damping (see Section 6.1 below) do not apply to such devices.

(ii) There is a rich history of the use of such devices in spacecraft attitude stabilization and control. All gravity-gradient satellites, all spin-stabilized satellites, and all duel-spin satellites flown to date have included a 'damper' as an essential piece of attitude control hardware.

(iii) It may seem surprising in view of the last remark, but the development of passive dampers in connection with the control of 'large flexible spacecraft' has been vanishingly small. Most persons apparently assume that the extraction of unwanted mechanical energy must be accomplished almost exclusively with active control methods.

(iv) Discrete linear-viscous dampers provide a design option that is attractive in the following respects: a wide variety of damper designs is possible; they can be placed at crucial points in the structure (i.e., to selectively damp the most important modes); and the damping strength can be selected at will.

(v) When control analysts speak (as they often do) of 'co-located sensors and actuators' the sensors usually turn out to be rate sensors. This means that these analysts are implementing, by active feedback control, the linear-viscous damping characteristic (5.7). Yet, for some reason, the passive-damper alternative is not considered. It is not clear (at least to this writer) why this is the case. In the end, it may simply reflect the experience of the analyst.

The importance of the development in this section (Section 5) rests partially on the credibility of discrete dampers as an important possibility for large space structure control. However, the modeling of general structural damping by a viscous model is also the motivation for much of the sequel.

5.2 Absolute and Relative Displacements

Let us assume that a model for the undamped flexible spacecraft is available in the form

$$\underline{M}\ddot{\underline{q}} + \underline{K}\underline{q} = \underline{f} \tag{5.8}$$

The most powerful current method for constructing such a model is the finite element method, which assumes that the elements of $\underline{q}$--the coordinates of the model--are *absolute* displacements, i.e., displacements with respect to an inertial reference frame. They may be translational displacements, or rotational displacements or, in some cases, higher-order displacements that a rigid body cannot have [e.g., $q_1 = z_0$, a 'translational' displacement; $q_2 = (dz/dx)_0$, a 'slope', and thus a 'rotational' displacement; $q_3 = (d^2z/dx^2)_0$, a coordinate associated with a 'higher-order' displacement for a flexible body]. The point is that $\{q_1, q_2, q_3, \ldots\}$ are as seen by an *inertial* observer.

However, in a discussion of internal damping, it is much more convenient to use *relative* displacements wherever possible, because it is these relative displacements that are directly damped. Thus we partition $\underline{q}$ with

$$\underline{q} = \begin{bmatrix} \underline{q}_r \\ \underline{q}_e \end{bmatrix} \tag{5.9}$$

where $\underline{q}_r$ denotes the displacements that are absolute and associated with rigid-body motion, and $\underline{q}_e$ denotes relative displacements that could not exist unless the vehicle were flexible. For a vehicle consisting of a single elastic body there are 6 coordinates in $\underline{q}_r$, 3 for absolute translation and 3 for absolute rotation. If the vehicle has internal rigid-body degrees of freedom, there are extra degrees of freedom in $\underline{q}_r$ that are relative (not absolute) coordinates but that do not create any strain energy. The gimbal angles at the reflector constitute an example of such coordinates for MSAT.

Although it is not proved here, it is important to note that it is *always possible* to replace a set of absolute coordinates (as may naturally occur in a finite element model) by a set of coordinates of the form (5.9), where $\underline{q}_r$ contains 6 (absolute) rigid coordinates and possibly other (relative) rigid coordinates associated with articulation, and $\underline{q}_e$ contains the (relative) elastic coordinates. Thus we may with no loss in generality assume that

$$\underline{M} = \begin{bmatrix} \underline{M}_r & \underline{M}_{re} \\ \underline{M}_{re}^T & \underline{M}_e \end{bmatrix} \quad ; \qquad \underline{K} = \begin{bmatrix} \underline{O} & \underline{O} \\ \underline{O} & \underline{K}_e \end{bmatrix} \tag{5.10}$$

The partitions in (5.10) correspond to those in (5.9). The consequence of using relative coordinates wherever possible that $\underline{K}$ has the much simplified form shown in (5.10) and, moreover,

$$\underline{K}_e > 0 \tag{5.11}$$

because only 'elastic' coordinates have been included in $\underline{q}_e$.

The simplification in the form of $\underline{K}$ evident in (5.10) applies also to linear viscous damping. Assuming the only form of damping present is caused by internal relative motion, the most general linear viscous damping term that can be added to (5.8) is a $\underline{\mathcal{D}}\dot{\underline{q}}$ term, where

$$\underline{\mathcal{D}} = \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{0} & \underline{\mathcal{D}}_e \end{bmatrix} \tag{5.12}$$

Thus the damped system has the form

$$\underline{M}\ddot{\underline{q}} + \underline{\mathcal{D}}\dot{\underline{q}} + \underline{K}\underline{q} = \underline{\delta} \tag{5.13}$$

where $\underline{\mathcal{D}}$ is of the form (5.12). If $\underline{\mathcal{D}}$ is based on discrete linear viscous dampers, then $\underline{\mathcal{D}}_e$ has the form

$$\underline{\mathcal{D}}_e = \text{diag}\{0,\ldots,0,d_a,0,..0,d_b,0,\ldots,0\} \tag{5.14}$$

where discrete dampers are located at the points corresponding to certain internal relative coordinates. The $\underline{\mathcal{D}}_e$ is positive semi-definite, $\mathcal{D} \geq 0$, and $\{d_a,d_b,\ldots\}$ are the damping constants of the discrete dampers. In spite of the fact that $\underline{\mathcal{D}}_e$ is only *semi*-definite, it is possible to have (indeed it would be singular not to have) *pervasive* damping of the internal coordinates $\underline{q}_e$. [The condition for pervasiveness is that the outputs $\underline{\mathcal{D}}_e\dot{\underline{q}}_e$ make the system $\underline{M}_e\ddot{\underline{q}}_e + \underline{K}_e\underline{q}_e = \underline{0}$ observable.]

If, however, $\underline{\mathcal{D}}_e$ is to represent structural damping as well,

$$\underline{\mathcal{D}}_e > 0 \tag{5.15}$$

since no internal motion can occur within a structure without causing energy dissipation.

To return to discrete dampers, as represented by the damping matrix (5.12) and (5.14), one important point so obvious it may be overlooked is that $\underline{\mathcal{D}}$ is *accurately known*. Thus, in addition to providing an adequate level of damping, discrete dampers remove uncertainty as to what the damping properties of the system in fact are. If the discrete dampers are not too large, the 'light damping' theory of Section 3 can be applied. If one or more of the discrete dampers is sufficiently large, the damping will no longer be 'light' in all modes.

5.3 'Damped' Modes

If the damping is not 'light', as considered in Section 3, the alternative of 'damped' modes should be considered. In these modes, the eigenvectors of the *damped* system are used as the set of basis functions in which to expand the physical displacements, rather than the eigenvectors of the *undamped* system (the 'vibration modes'). A brief treatment of 'damped modes' is now given.

We begin with the system

$$\underline{M}\ddot{\underline{q}} + \underline{\mathcal{D}}\dot{\underline{q}} + \underline{K}\underline{q} = \underline{f}(t) \qquad (5.16)$$

with $\underline{K} > 0$ to exclude rigid-body modes. Following the classic paper of Foss (Ref. 9), one can re-organize (5.16) thus:

$$\underline{A}\dot{\underline{z}} = \underline{B}\underline{z} + \underline{w}(t) \qquad (5.17)$$

where

$$\underline{A} = \begin{bmatrix} \underline{0} & \underline{M} \\ \underline{M} & \underline{\mathcal{D}} \end{bmatrix}; \qquad \underline{B} = \begin{bmatrix} \underline{M} & \underline{0} \\ \underline{0} & -\underline{K} \end{bmatrix} \qquad (5.18)$$

$$\underline{z} = \begin{bmatrix} \dot{\underline{q}} \\ \underline{q} \end{bmatrix}; \qquad \underline{w} = \begin{bmatrix} \underline{0} \\ \underline{f} \end{bmatrix} \qquad (5.19)$$

Note that $\underline{A}^T = \underline{A}$, $\underline{B}^T = \underline{B}$. The eigenvalue-eigenvector problem associated with the homogeneous version of (5.17) is [set $\underline{z} = \underline{z}_i \exp(\lambda_i t)$]:

$$\lambda_i \underline{A}\underline{z}_i = \underline{B}\underline{z}_i \qquad (i = 1,\ldots,2n) \tag{5.20}$$

In general, $(\lambda_i, \underline{z}_i)$ may be real or complex. If a particular $(\lambda_i, \underline{z}_i)$ is complex, there must be another $(\lambda_j, \underline{z}_j)$ that is the complex conjugate of $(\lambda_i, \underline{z}_i)$. If the damping were zero, of course, the damped modes would revert to undamped modes:

$$\lambda_\alpha = j\omega_\alpha \, ; \qquad \lambda_{n+\alpha} = -j\omega_\alpha \qquad (\alpha = 1,\ldots,n) \tag{5.21}$$

$$\underline{z}_\alpha = \begin{bmatrix} j\omega_\alpha \underline{e}_\alpha \\ \underline{e}_\alpha \end{bmatrix} ; \qquad \underline{z}_{n+\alpha} = \begin{bmatrix} -j\omega_\alpha \underline{e}_\alpha \\ \underline{e}_\alpha \end{bmatrix} \qquad (\alpha = 1,\ldots,n) \tag{5.22}$$

[Compare with (3.30) - (3.33).] As it is, with $\underline{\mathcal{D}} > 0$, the special forms (5.21) - (5.22) no longer obtain. What *is* true, however, is that the eigenvectors $\underline{z}_i$ must be of the form

$$\underline{z}_i = \begin{bmatrix} \lambda_i \underline{q}_i \\ \underline{q}_i \end{bmatrix} \qquad (i = 1,\ldots,2n) \tag{5.23}$$

in order to satisfy (5.20).

The orthogonality conditions for the 'damped' modes $\underline{z}_\alpha$ are found in the following manner. First, let $i \rightarrow j$ in (5.20) and premultiply by $\underline{z}_i^H$, the Hermitian of $\underline{z}_i$:

$$\lambda_j \underline{z}_i^H \underline{A}\underline{z}_j = \underline{z}_i^H \underline{B}\underline{z}_j \tag{5.24}$$

(Actually, Foss in Ref. 9 uses simple transposition on $\underline{z}_i$ instead of the Hermitian operation; the present writer is of the opinion that a better formulation results in the latter case.) Next, form the Hermitian of (5.20) and post-multiply by $\underline{z}_j$:

$$\lambda_i^* \underline{z}_i^H \underline{A}\underline{z}_j = \underline{z}_i^H \underline{B}\underline{z}_j \tag{5.25}$$

Now, substract (5.25) from (5.24):

$$(\lambda_j - \lambda_i^*)\underline{z}_i^H \underline{A}\underline{z}_j = 0 \tag{5.26}$$

We shall assume the λ_i occur in complex conjugate pairs (although it requires only a modest extension to the theory to include real λ_i). It is unlikely that any form of passive damping will be so intense that a mode is 'overdamped' ($\zeta_i > 1$). The λ_i can be ordered as follows:

$$\{\lambda_i\} = \{\lambda_\alpha, \lambda_{n+\alpha}\} \qquad (i = 1,\ldots,2n) \qquad (\alpha = 1,\ldots,n) \tag{5.27}$$

where

$$\lambda_{n+\alpha} = \lambda_\alpha^* \tag{5.28}$$

and all the λ_α $(i = 1,\ldots,n)$ have *positive* imaginary parts. (And therefore all the $\lambda_{n+\alpha} = \lambda_\alpha^*$ have negative imaginary parts.) Similarly, from (5.20),

$$\underline{z}_{n+\alpha} = \underline{z}_\alpha^* \qquad (\alpha = 1,\ldots,n) \tag{5.29}$$

The $4n^2$ equations (5.26) then reduce to the $2n^2$ equations

$$(\lambda_\beta - \lambda_\alpha^*)\underline{z}_\alpha^H A \underline{z}_\beta = 0 \tag{5.30}$$

$$(\lambda_\beta - \lambda_\alpha)\underline{z}_\alpha^T A \underline{z}_\beta = 0 \tag{5.31}$$

for $\alpha,\beta = 1,\ldots,n$. Now λ_β can never be equal to λ_α^* since they are on opposite sides of the real axis. Therefore

$$\underline{z}_\alpha^H A \underline{z}_\beta = 0 \qquad (\alpha,\beta = 1,\ldots,n) \tag{5.32}$$

Furthermore, from (5.31), for distinct eigenvalues we have

$$\underline{z}_\alpha^T A \underline{z}_\beta = 0 \qquad (\alpha \neq \beta) \tag{5.33}$$

These last two equations are the basic orthogonality conditions.

Furthermore, it follows from (5.24) that

$$\underline{z}_\alpha^H B \underline{z}_\beta = 0 \qquad (\alpha,\beta = 1,\ldots,n) \tag{5.34}$$

$$\underline{z}_\alpha^T B \underline{z}_\beta = 0 \qquad (\alpha = \beta) \tag{5.35}$$

These are the auxiliary orthogonality conditions.

In spite of their compact from, (5.32) - (5.35) are not as informative to the structural dynamicist as the expanded form available by substitution of (5.18) and (5.23). The orthogonality conditions in terms of the damped mode shapes $\underline{q}_\alpha$ are then

$$(\lambda_\alpha^* + \lambda_\beta)\underline{q}_\alpha^H \underline{M} \underline{q}_\beta + \underline{q}_\alpha^H \underline{\mathcal{D}} \underline{q}_\beta = 0 \qquad (\alpha,\beta = 1,\ldots,n) \qquad (5.36)$$

$$(\lambda_\alpha + \lambda_\beta)\underline{q}_\alpha^T \underline{M} \underline{q}_\beta + \underline{q}_\alpha^T \underline{\mathcal{D}} \underline{q}_\beta = 0 \qquad (\alpha \neq \beta) \qquad (5.37)$$

and the auxiliary orthogonality conditions are

$$\lambda_\alpha^* \lambda_\beta \underline{q}_\alpha^H \underline{M} \underline{q}_\beta = \underline{q}_\alpha^H \underline{K} \underline{q}_\beta \qquad (\alpha,\beta = 1,\ldots,n) \qquad (5.38)$$

$$\lambda_\alpha \lambda_\beta \underline{q}_\alpha^T \underline{M} \underline{q}_\beta = \underline{q}_\alpha^T \underline{K} \underline{q}_\beta \qquad (\alpha \neq \beta) \qquad (5.39)$$

It is not difficult to show that, when $\underline{\mathcal{D}} = \underline{O}$, and thus $\lambda_\alpha = j\omega_\alpha$, $\underline{q}_\alpha = \underline{e}_\alpha$, the above orthogonality conditions are in accord with

$$\underline{e}_\alpha^T \underline{M} \underline{e}_\beta = 0 = \underline{e}_\alpha^T \underline{K} \underline{e}_\beta \qquad (\alpha \neq \beta) \qquad (5.40)$$

as they should be.

The normality conditions for the $\underline{q}_\alpha$ are now chosen. Motivated by the undamped case, for which $\underline{e}_\alpha^T \underline{M} \underline{e}_\alpha = 1$ is the natural normality condition, we choose

$$\underline{q}_\alpha^H \underline{M} \underline{q}_\alpha = 1 \qquad (\alpha = 1,\ldots,n) \qquad (5.41)$$

to be the normality conditions. It is interesting to note from (5.36) and (5.38), when $\alpha = \beta$, that

$$\underline{q}_\alpha^H \underline{\mathcal{D}} \underline{q}_\alpha = -2\mathcal{Re}\{\lambda_\alpha\} \qquad (5.42)$$

$$\underline{q}_\alpha^H \underline{K} \underline{q}_\alpha = |\lambda_\alpha|^2 \qquad (5.43)$$

The whole point of this exercise, of course, is to develop a set of damped modes in terms of which the general forced motion governed by (5.13) can be expanded. To this end, we set

$$\underline{z}(t) = Re\{\sum_{\beta=1}^{n} \underline{z}_\beta \xi_\beta(t)\} \tag{5.44}$$

in (5.17), premultiply by $\underline{z}_\alpha^T$, and use the orthogonality conditions. The result is

$$(\underline{z}_\alpha^T A \underline{z}_\alpha)\dot{\xi}_\alpha = (\underline{z}_\alpha^T B \underline{z}_\alpha)\xi_\alpha + \underline{z}_\alpha^T \underline{w} \tag{5.45}$$

Now, from (5.20),

$$\underline{z}_\alpha^T B \underline{z}_\alpha = \lambda_\alpha \underline{z}_\alpha^T A \underline{z}_\alpha \tag{5.46}$$

and, from (5.19) and (5.23),

$$\underline{z}_\alpha^T \underline{w} = \underline{q}_\alpha^T \underline{f}$$

Therefore the modal equations of motion are

$$\dot{\xi}_\alpha = \lambda_\alpha \xi_\alpha + \underline{q}_\alpha^T \underline{f} / (\underline{z}_\alpha^T A \underline{z}_\alpha) \tag{5.47}$$

for $\alpha = 1,\ldots,n$. Note that ξ_α is complex, and that

$$\underline{z}_\alpha^T A \underline{z}_\alpha = 2\lambda_\alpha \underline{q}_\alpha^T \mathcal{M} \underline{q}_\alpha + \underline{q}_\alpha^T \mathcal{D} \underline{q}_\alpha \tag{5.48}$$

For numerical work, it may be necessary to write the single complex equation (5.47) as a pair of real differential equations by setting $\xi_\alpha = \xi_{\alpha R} + j\xi_{\alpha I}$:

$$\begin{aligned} \dot{\xi}_{\alpha R} &= \sigma_\alpha \xi_{\alpha R} - \omega_\alpha \xi_{\alpha I} + Re\{\underline{q}_\alpha^T \underline{f} / (\underline{z}_\alpha^T A \underline{z}_\alpha)\} \\ \dot{\xi}_{\alpha I} &= \omega_\alpha \xi_{\alpha R} + \sigma_\alpha \xi_{\alpha I} + Im\{\underline{q}_\alpha^T \underline{f} / (\underline{z}_\alpha^T A \underline{z}_\alpha)\} \end{aligned} \tag{5.49}$$

and $\lambda_\alpha = \sigma_\alpha + j\omega_\alpha$. Then

$$\underline{q}(t) = \sum_{\beta=1}^{n} [\underline{q}_{\beta R} \xi_{\beta R}(t) - \underline{q}_{\beta I} \xi_{\beta I}(t)] \tag{5.50}$$

is the final expression for the forced motion in terms of damped modes.

When the structure is only 'lightly' damped, the complexity of damped modes can be avoided. To a very good approximation, the theory of Section 3 can be used and the undamped mode shapes suffice.

5.4 Damping Synthesis from Substructues

In many spacecraft the 'structure' can be decomposed into 'substructures' in a natural way. In the case of MSAT, for example, such substructures are the antenna reflector, the support tower and the solar array. We assume that the elastic coordinates $\underline{q}_e$ are internal, relative coordinates, and that $\underline{q}_e$ can be further partitioned into substructural contributions thus:

$$\underline{q}_e = \text{col}\{\underline{q}^1, \underline{q}^2, \ldots, \underline{q}^M\} \tag{5.51}$$

The coordinates $\underline{q}^m$ are associated with substructure m, m = 1,...,M. By an appropriate choice of coordinates it can be arranged that

$$\underline{K}_e = \text{block diag}\{\underline{K}^1, \underline{K}^2, \ldots, \underline{K}^M\} \tag{5.52}$$

with the partitions in (5.52) matching the partitions in (5.51).

We now choose the elastic damping matrix $\underline{\mathcal{D}}_e$ to be

$$\underline{\mathcal{D}}_e = \text{block diag}\{\underline{\mathcal{D}}^1, \underline{\mathcal{D}}^2, \ldots, \underline{\mathcal{D}}^M\} \tag{5.53}$$

[See (5.12) and (5.13).] In other words, $\underline{\mathcal{D}}_e$ is chosen to be block diagonal, with each block being proportional to the corresponding stiffness block. The rationale for this choice runs as follows: an element of the stiffness matrix, k_{ij} is nonzero if the stiffness of structure offers a resistance at coordinate q_j to a force in the direction of q_i; if the structure offers zero *static* resistance at q_j to a force at q_i, how can it offer any *dynamic* (i.e., damping) resistance at q_j to a force at q_i? If we agree that the answer to this question is "It can't", then the form (5.53) follows immediately.

Perhaps a simple example will help to clarify this concept. In Fig. 5.1 is shown a straightforward physical system whose coordinates

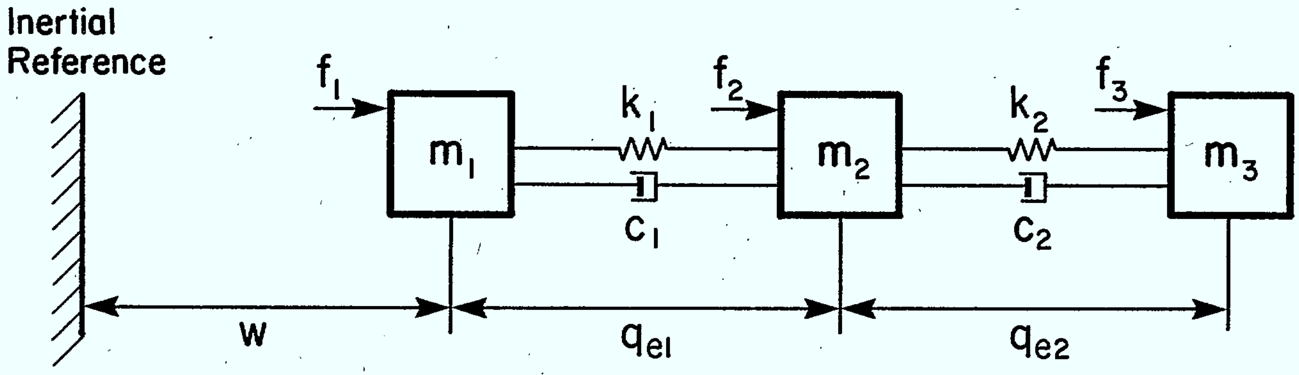


Fig. 5.1: Simple "Vehicle" Consisting of a "Main Bus" m_1 and Two "Substructures", m_2 and m_3.

are chosen to be $\{w, q_{e1}, q_{e2}\}$ as shown. Note that w is an absolute coordinate, while q_{e1} and q_{e2} are relative (internal) coordinates. The 'main structure' is m_1 (rigid for simplicity here), and m_2 and m_3 represent 'substructures'. It is elementary to derive the following motion equations for the system of Fig. 5.1:

$$\underline{M}\ddot{\underline{q}} + \underline{D}\dot{\underline{q}} + \underline{K}\underline{q} = \underline{f} \tag{5.54}$$

where

$$\underline{M} = \begin{bmatrix} \underline{M}_r & \underline{M}_{re} \\ \underline{M}_{re}^T & \underline{M}_e \end{bmatrix} \qquad \underline{D} = \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{0} & \underline{D}_e \end{bmatrix} \tag{5.55}$$

$$\underline{K} = \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{0} & \underline{K}_e \end{bmatrix}; \quad \underline{q} = \begin{bmatrix} \underline{q}_r \\ \underline{q}_e \end{bmatrix}; \quad \underline{f} = \begin{bmatrix} \underline{f}_r \\ \underline{f}_e \end{bmatrix} \tag{5.56}$$

In this case $\underline{q}_r$ is a single coordinate, w, and $\underline{q}_e$ consists of q_{e1} and q_{e2}. Also, $\underline{M}_r$ is a 1 x 1 matrix,

$$\underline{M}_r = m_1 + m_2 + m_3 \tag{5.57}$$

and $\underline{f}_r$ is a 1 x 1 force input,

$$\underline{f}_r = f_1 + f_2 + f_3 \tag{5.58}$$

The momentum matrix is

$$\underline{M}_{re} = [m_2 + m_3 \quad m_3] \tag{5.59}$$

because the (relative) momentum associated with the coordinates q_{e1} and q_{e2} is

$$(m_2 + m_3)\dot{q}_{e1} + m_3\dot{q}_{e2}$$

To continue with the findings of the 'elementary derivation', we can make the following identifications:

$$\underline{M}_e = \begin{bmatrix} m_2 + m_3 & m_3 \\ m_3 & m_3 \end{bmatrix}; \quad \underline{K}_e = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \tag{5.60}$$

$$\underline{\mathcal{D}}_e \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}; \quad \underline{f}_e = \begin{bmatrix} f_2 + f_3 \\ f_3 \end{bmatrix} \tag{5.61}$$

The stiffness matrix $\underline{K}_e$ is in the partitioned form (5.52); there are two substructures, each with one coordinate.

The point of this simple example is to note that $\underline{\mathcal{D}}_e$ is automatically partitioned to match $\underline{K}_e$. Suppose, on the contrary, that there were elements c_{12} and $c_{21} = c_{12}$ off the diagonal in $\underline{\mathcal{D}}_e$. This would imply that the motion of m_2 relative to m_1 causes a force $-c_{12}\dot{q}_{e1}$ on m_3; even more mysterious is the force $-c_{12}(\dot{q}_{e2} - \dot{q}_{e1})$ implied to be acting on m_2. These forces simply do not exist, and we conclude that $c_{12} = 0$. More generally, when $\underline{K}_e$ partitions in the form (5.52) by substructure, $\underline{\mathcal{D}}_e$ partitions in the same manner into the form (5.53).

On the surface, this idea may not seem too exciting because all that has been done is to show that many of the elements of $\underline{\mathcal{D}}_e$ are 0's. However, if initially *none* of the elements of $\underline{\mathcal{D}}_e$ are known, a determination that many of these elements are (exactly) zero is in fact a major step forward. To be specific, suppose there were 10 substructures, each with 20 elastic coordinates. Then $\underline{\mathcal{D}}_e$ is 200 x 200 and has 40,000 elements. Because of symmetry only ½(200)(201) = 20,100 of these elements are independent. The partitioned form (5.53) informs us that 18,000 of these elements are 0's. About 90% of the $\underline{\mathcal{D}}_e$ matrix has thus been determined exactly!

If $\underline{\mathcal{D}}_e$ is indeed partitioned according to (5.53), the modal damping matrix $\underline{\hat{\mathcal{D}}}$ has a much more explicit form than just

$$\underline{\hat{\mathcal{D}}} = \underline{E}^T \underline{\mathcal{D}} \underline{E} \tag{5.62}$$

The modal matrix $\underline{E}$ can also be partitioned in accordance with (5.55) - (5.56):

$$\underline{E} = \begin{bmatrix} \underline{E}_r & \underline{E}_{re} \\ \underline{O} & \underline{E}_e \end{bmatrix} \tag{5.63}$$

(The $\underline{0}$ partition occurs because the elastic coordinates $\underline{q}_e$ cannot participate in the rigid-body modes.) Thus, from (5.62),

$$\hat{\underline{\mathcal{D}}} = \begin{bmatrix} \underline{0} & \underline{0} \\ \underline{0} & \hat{\underline{\mathcal{D}}}_e \end{bmatrix} \tag{5.64}$$

where

$$\hat{\underline{\mathcal{D}}}_e = \underline{E}_e^T \underline{\mathcal{D}}_e \underline{E}_e \tag{5.65}$$

Next, we further partition $\underline{E}_e$ to match the substructural coordinates, as in (5.51)

$$\underline{E}_e = \begin{bmatrix} \underline{E}_e^1 \\ \vdots \\ \underline{E}_e^M \end{bmatrix} \tag{5.66}$$

The columns of $\underline{E}_e^m$ indicate the participation of the coordinates in substructure m in the eigenvectors for the spacecraft modes. Finally, from (5.53) and (5.66),

$$\hat{\underline{\mathcal{D}}}_e = \sum_{m=1}^{M} \underline{E}_e^{mT} \underline{\mathcal{D}}^m \underline{E}_e^m \tag{5.67}$$

Equations (5.64) and (5.67) constitute the special form of the spacecraft modal damping matrix in terms of substructural damping matrices.

There remains, of course, the vexing question of what the substructural damping matrices $\underline{\mathcal{D}}^1,\ldots,\underline{\mathcal{D}}^M$ might be. Three possibilities are considered here. The first of these is to carry out substructure tests to determine the $\underline{\mathcal{D}}^m$ experimentally. The second is for the knowledgeable Person of Section 2 to make a Guess at what the substructural modal damping factors are. In these 'substructure modes', all coordinates are constrained except for the $\underline{q}^m$ of the structure in question. Thus, for the modes of substructure m,

$$\begin{gathered} \underline{q}_r \equiv \underline{0} \\ \underline{q}^1 = \underline{q}^2 = \ldots = \underline{q}^{m-1} = \underline{q}^{m+1} = \ldots = \underline{q}^M \equiv \underline{0} \end{gathered} \tag{5.68}$$

The equations of motion for these modes are

$$\underline{M}^m\ddot{\underline{q}}^m + \underline{\mathcal{D}}^m\dot{\underline{q}}^m + \underline{K}^m\underline{q}^m = \underline{0} \tag{5.69}$$

where $\underline{M}^m$ is the appropriate block partition on the diagonal of $\underline{M}_e$. Let the modal matrix for these modes be $\underline{E}^m$. This modal matrix contains only the coordinates $\underline{q}^m$, and must be distinguished from the $\underline{E}_e^m$ used in (5.66); the latter gives the participation of the coordinates $\underline{q}^m$ in the overall spacecraft modes. If the modal damping factors for the substructure modes in substructure m are denoted $\{\zeta_1^m,\ldots,\zeta_m^m\}$, let

$$\underline{Z}_m = \mathrm{diag}\{\zeta_1^m,\ldots,\zeta_m^m\} \tag{5.70}$$

Then

$$\underline{\mathcal{D}}^m = 2\underline{M}^m\underline{E}^m\underline{Z}^m\underline{\Omega}^m\underline{E}^{mT}\underline{M}^m \tag{5.71}$$

where

$$\underline{\Omega}^m = \mathrm{diag}\{\omega_1^m,\ldots,\omega_m^m\} \tag{5.72}$$

is the diagonal matrix of natural frequencies of vibration for the (constrained) substructure m. Equation (5.71) is correct because

$$\underline{E}^{mT}\underline{\mathcal{D}}^m\underline{E}^m = 2\underline{Z}^m\underline{\Omega}^m \tag{5.73}$$

as it should be (recall that $\underline{E}^{mT}\underline{M}^m\underline{E}^m = \underline{1}$). On this basis, $\hat{\underline{\mathcal{D}}}_e$ becomes, according to (5.67),

$$\hat{\underline{\mathcal{D}}}_e = 2\sum_{m=1}^{M}\underline{E}_e^{mT}\underline{M}^m\underline{E}^m\underline{Z}^m\underline{\Omega}^m\underline{E}^{mT}\underline{M}^m\underline{E}_e^m \tag{5.74}$$

The matrix multiplications are straightforward on a computer.

The third and final possibility is to choose

$$\underline{\mathcal{D}}^m = \alpha_m\underline{K}^m \tag{5.75}$$

The reasoning behind this assumption is to apply the same logic coordinate-by-coordinate as was applied substructure-by-substructure in (5.52) and (5.53); see also the simple example of Fig. 5.1. Only one damping parameter is needed per substructure, namely α_m. This is the method used by Dynacon in earlier MSAT modeling work (Ref. 6). The advantage of (5.75) is that it gives a good deal of 'structure' to $\underline{D}_e$ with only a few parameters needed. The disadvantage is that it probably tends to over-estimate the damping of the highest modes. The latter statement can be explained by letting all the substructural damping parameters be equal: $\alpha_1 = \alpha_2 = \ldots = \alpha_M = \alpha_d$. Then, from (5.6)

$$\hat{\underline{\mathcal{D}}}_e = \sum_{m=1}^{M} \alpha_m \underline{E}_e^{mT} \underline{K}^m \underline{E}_e^m$$

$$= \alpha_d \sum_{m=1}^{M} \underline{E}_e^{mT} \underline{K}^m \underline{E}_e^m = \alpha_d \underline{\Omega}^2 \tag{5.76}$$

(The diagonal elements of $\underline{\Omega}$ are the overall spacecraft natural frequencies.) Comparing this to

$$\hat{\underline{\mathcal{D}}}_e = 2\underline{Z\Omega} \tag{5.77}$$

where $\underline{Z}$ is a diagonal matrix of spacecraft mode damping factors, we see that

$$\underline{Z} = \tfrac{1}{2}\alpha_d \underline{\Omega} \tag{5.78}$$

That is

$$\zeta_\alpha = \tfrac{1}{2}\alpha_d \omega_\alpha \tag{5.79}$$

and so the damping factors tend to increase with frequency.

6. LINEAR HYSTERETIC DAMPING

Let us return briefly to the elementary discussion at the beginning of Section 5. A 'system' that has a single degree of freedom with coordinate $q(t)$ was considered. The rate of energy decrease is $\dot{T} = \dot{q}f_d$, where f_d is the damping force. If f_d is strictly *viscous* damping, i.e., $f_d = -c_d\dot{q}$, then

$$\dot{T} = -c_d\dot{q}^2 \tag{6.1}$$

Now, suppose q is forced by some external agency to execute simple harmonic motion at frequency ω:

$$q = q_o \cos\omega t \tag{6.2}$$

The energy extracted by the damper in one cycle is

$$\begin{aligned}\Delta T &= -\int_o^{2\pi/\omega} \dot{T}\, dt \\ &= (\pi/2) c_d \omega q_o^2 \end{aligned} \tag{6.3}$$

As expected, the loss is proportional to the damping factor c_d and, as with all energy-related quantities, it is proportional to the square of the amplitude. Attention is specifically drawn here to the presence of the coefficient ω in (6.3): the energy lost per cycle with linear viscous damping is proportional to the frequency of oscillation.

Unfortunately, *the loss factors for materials (and joints as well) are not experimentally found to be proportional to frequency.*

6.1 The Linear Hysteretic Damping Law

The dependence of loss factor on frequency for real materials and real joints, as experimentally determined, is not simple, but a reasonably accurate model is that *the loss factor is independent of frequency.* While not exact, this simple model is much more realistic than the viscous-damping alternative which leads, as we have seen above, to a frequency-proportional loss factor characteristic.

How should the damping law be changed to provide a frequency-independent loss factor? In the frequency domain, the linear-viscous characteristic is

$$\bar{f}_d(s) = -c_d s \bar{q}(s) \tag{6.4}$$

where overbars denote Laplace transforms. In particular, the frequency response function is

$$\bar{f}_d(j\omega) = -j c_d \omega \bar{q}(j\omega) \tag{6.5}$$

To remove the frequency-proportional characteristic, we replace ω in this manner:

$$c_d\omega \rightarrow h\,\mathrm{sgn}(\omega) \tag{6.6}$$

where the sgn function is

$$\mathrm{sgn}(\omega) = \begin{cases} +1\,, & \omega > 0 \\ -1\,, & \omega < 0 \\ \text{undefined for} & \omega = 0 \end{cases} \tag{6.7}$$

The proportionality to ω is thus removed, although the sgn(ω) is retained to ensure that the force is in fact dissipative. Equation (6.6) is the elementary linear-hysteretic damping law.

For structures represented by not one coordinate, but many, the same ideas apply. With linear viscous damping, we have, from (5.13),

$$[\underline{K} - \omega^2\underline{M} + j\omega\underline{\mathcal{D}}]\bar{\underline{q}}(j\omega) = \bar{\underline{\mathfrak{z}}}(j\omega) \tag{6.8}$$

where as, with linear hysteretic damping,

$$[\underline{K} - \omega^2\underline{M} + j\,\mathrm{sgn}(\omega)\,\underline{\mathcal{H}}]\bar{\underline{q}}(j\omega) = \bar{\underline{\mathfrak{z}}}(j\omega) \tag{6.9}$$

The hysteretic damping matrix $\underline{\mathcal{H}}$ is symmetric and positive semi-definite,

$$\underline{\mathcal{H}}^T = \underline{\mathcal{H}} \geq 0 \tag{6.10}$$

It is important to note that the hysteretic damping law is formulated in the ω-domain, not the time domain. It is, strictly speaking, valid only for sinusoidal oscillations and its usefulness in the time domain is therefore limited. On the other hand, it is especially appropriate to frequency-domain techniques (e.g., Nyquist analysis).

For flexible spacecraft, we also know that $\underline{\mathcal{H}}$ can be partitioned into its 'rigid' and 'elastic' partitions:

$$\underline{\mathcal{H}} = \begin{bmatrix} \underline{O} & \underline{O} \\ \underline{O} & \underline{\mathcal{H}}_e \end{bmatrix} \tag{6.11}$$

This is identical to the partitioning pointed out in (5.12) for $\underline{D}$, the viscous damping matrix. Since all deforming structures dissipate energy,

$$\underline{H}_e > 0 \tag{6.12}$$

i.e., $\underline{H}_e$ is positive definite.

6.2 'Light' Hysteretic Damping

We shall for the remainder of this section dispense with the rigid-body modes (which are undamped anyway), and consider the system

$$[\underline{K} - \omega^2 \underline{M} + j\mathrm{sgn}(\omega)\underline{H}]\bar{\underline{q}}(j\omega) = \bar{\underline{f}}(j\omega) \tag{6.13}$$

with $\underline{K} > 0$, $\underline{H} > 0$. If *un*damped modal coordinates $\underline{\eta}(t)$ are introduced into (6.13),

$$\underline{q} = \underline{E\eta} \tag{6.14}$$

we have

$$[\underline{\Omega}^2 - \omega^2 \underline{1} + j\,\mathrm{sgn}\,(\omega)\underline{\hat{H}}]\bar{\underline{\eta}}(j\omega) = \bar{\underline{\gamma}}(j\omega) \tag{6.15}$$

where $\underline{\Omega}$ contains the undamped natural frequencies, $\underline{\gamma} = \underline{E}^T\underline{f}$, and

$$\underline{\hat{H}} = \underline{E}^T\underline{HE} \tag{6.16}$$

is the hysteretic damping matrix for modal coordinates.

The model (6.15) has two disadvantages:

(i) It is in the frequency domain and does not lend itself to a time-domain interpretation. How, for example, would one carry out numerical simulations with this model?

(ii) The modal equations for η_α are coupled by $\underline{\hat{H}}$. This does not present a problem for numerical simulation work but does prohibit many analytical techniques that are useful in the design process.

Both of these disadvantages can be removed with the aid of the 'light' damping assumption.

If the system is lightly damped, we know that the system response is essentially the undamped system response, except at the resonances. Therefore, except in the neighbourhood of $\omega = \omega_\alpha$, $\alpha = 1,\ldots,n$, it matters not whether the damping is viscous, hysteretic, or even present at all--so long as it is 'light' damping. With viscous damping,

$$\underline{\bar{\eta}}(j\omega) = [\underline{\Omega}^2 - \omega^2\underline{1} + j\omega\underline{\hat{\mathcal{D}}}]^{-1}\,\underline{\bar{\gamma}}(j\omega) \tag{6.17}$$

and, for *light* viscous damping, we know from Section 3.2 (Conclusion 1) that only the diagonal elements of $\underline{\hat{\mathcal{D}}}$ influence the motion. In fact, with the light damping assumption, (6.17) becomes

$$\bar{\eta}_\alpha(j\omega) = \frac{\bar{\gamma}_\alpha(j\omega)}{(\omega_\alpha^2 - \omega^2) + j\omega\hat{d}_{\alpha\alpha}} \tag{6.18}$$

[Compare with Equation (4.4).] When ω is not near ω_α (including, in particular, when ω is near other resonances, ω_β, $\beta \neq \alpha$),

$$\bar{\eta}_\alpha(j\omega) \doteq \frac{\bar{\gamma}_\alpha(j\omega)}{\omega_\alpha^2 - \omega^2} \qquad (\omega \text{ not near } \omega_\alpha) \tag{6.19}$$

and, at $\omega = \omega_\alpha$,

$$\bar{\eta}_\alpha(j\omega_\alpha) = \frac{\bar{\gamma}_\alpha(j\omega_\alpha)}{j\omega_\alpha\hat{d}_{\alpha\alpha}} \tag{6.20}$$

Now, consider light hysteretic damping, as governed by (6.15):

$$\underline{\bar{\eta}}(j\omega) = [\underline{\Omega}^2 - \omega^2\underline{1} + j\,\mathrm{sgn}(\omega)\,\underline{\hat{H}}]^{-1}\,\underline{\bar{\gamma}}(j\omega) \tag{6.21}$$

Since $\underline{\hat{H}}$ is small, (6.21) will automatically produce (6.19) when ω is not near ω_α. At $\omega = \omega_\alpha$, (6.21) becomes

$$\bar{\eta}_\alpha(j\omega_\alpha) = \frac{\bar{\gamma}_\alpha(j\omega_\alpha)}{j\hat{h}_{\alpha\alpha}} \tag{6.22}$$

(The sgn function is not necessary here because $\omega_\alpha > 0$ by convention.)

A comparison of (6.20) and (6.23) leads to the following two important conclusions:

Conclusion 1 - *There is no important difference between (light) viscous damping and (light) hysteretic damping provided*

$$\hat{d}_{\alpha\alpha} = \hat{h}_{\alpha\alpha}/\omega_\alpha \tag{6.23}$$

Conclusion 2 - *'Effective linear viscous damping coefficients' ζ_α can be used in the modal motion equations, even when the damping is hysteretic, provided the damping is light and provided the ζ_α are chosen to be*

$$\zeta_\alpha = \tfrac{1}{2}\hat{h}_{\alpha\alpha}/\omega_\alpha^2 \tag{6.24}$$

On the basis of these conclusions, the advantages of modal uncoupling and a time-domain representation can be retained even with hysteretic damping. If the hysteretic damping matrix $\underline{H}$ is known. $\hat{\underline{H}}$ is found from (6.16), and the equivalent viscous damping factors ζ_α computed from (6.24). Another version of (6.24) is

$$\zeta_\alpha = \tfrac{1}{2}\underline{e}_\alpha^T \underline{H} \underline{e}_\alpha/\omega_\alpha^2 \tag{6.25}$$

where $\underline{e}_\alpha$ are the undamped vibration mode shapes. If $\zeta_\alpha << 1$, $\alpha = 1,..,n$, then the hysteretic damping is indeed 'light'. The modal motion equations are

$$\ddot{\eta}_\alpha + 2\zeta_\alpha\omega_\alpha\dot{\eta}_\alpha + \omega_\alpha^2\eta_\alpha = \gamma_\alpha(t) \tag{6.26}$$

in the time domain, with ζ_α calculated from (6.25).

6.3 Biggs' Method

Biggs' method does for hysteretic damping what (5.75) does for viscous damping. Hysteretic damping matrices for each substructure, $\underline{H}^m (m = 1,...,M)$ are selected based on

$$\underline{H}^m = \varepsilon_m \underline{K}^m \qquad (m = 1,...M) \tag{6.27}$$

and

$$\underline{H}_e = \text{block diag}\{\underline{H}^1,\ldots\underline{H}^M\} \tag{6.28}$$

Then

$$\hat{\underline{H}}_e = \sum_{m=1}^{M} \underline{E}_e^{mT}\underline{H}^m\underline{E}_e^m \tag{6.29}$$

which is analogous to (5.67) for viscous damping and, in view of (6.25), this becomes

$$\hat{\underline{H}}_e = \sum_{m=1}^{M} \varepsilon_m\underline{E}_e^{mT}\underline{K}^m\underline{E}_e^m \tag{6.30}$$

One special case is to pick

$$\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_M = \varepsilon_d \tag{6.31}$$

for which (6.30) becomes

$$\hat{\underline{H}}_e = \varepsilon_d \sum_{m=1}^{M} \underline{E}_e^{mT}\underline{K}^m\underline{E}_e^m = \varepsilon_d\underline{\Omega}^2 \tag{6.32}$$

The equivalent viscous damping factors in this special case are, from (6.24),

$$\zeta_\alpha = \tfrac{1}{2}\varepsilon_d \qquad (\alpha = 1,\ldots,n) \tag{6.33}$$

From this we learn that *assuming all viscous damping factors to be equal is equivalent to assuming a hystertic damping matrix* $\underline{H}_e$ *proportional to the stiffness matrix* $\underline{K}_e$.

An alternative representation to (2.12) is (Ref. 10):

$$\zeta_\alpha = \frac{1}{2}\,\frac{\sum_{m=1}^{M}\varepsilon_m V_{m\alpha}}{\sum_{m=1}^{M} V_{m\alpha}} \tag{6.34}$$

where $V_{m\alpha}$ is the maximum stored strain potential energy in the mth substructure when oscillating in mode α.

6.4 Kana's Method

Kana's method is similar to Biggs' method except that substructure kinetic energies are used in place of substructure potential energies in (6.34):

$$\zeta_\alpha = \frac{1}{2} \frac{\sum_{m=1}^{M} \varepsilon_m T_{m\alpha}}{\sum_{m=1}^{M} T_{m\alpha}} \tag{6.35}$$

where $T_{m\alpha}$ is the maximum kinetic energy in the m-th substructure when oscillating in mode α.

In comparing Biggs' formula (6.34) with Kana's formula (6.35), one should note that

$$\sum_{m=1}^{M} T_{m\alpha} = \sum_{m=1}^{M} V_{m\alpha} = \frac{1}{2}\omega_\alpha^2 \tag{6.36}$$

However,

$$T_{m\alpha} \neq V_{m\alpha} \tag{6.37}$$

in general, so the two formulas do not give the same answer. It is not clear which formula is preferrable.

7. LINEAR VISCOELASTIC DAMPING

To introduce linear viscoelastic damping we return to the simple one-coordinate model at the beginning of Section 5. For viscous damping, the transfer function between the speed $\dot{q}$ and the damping force f_d was just $-c_d$. That is,

$$\bar{f}_d = -c_d s\bar{q} \tag{7.1}$$

7.1 Linear Viscoelastic Damping Law

For viscoelastic damping, we take

$$\bar{f}_d = -\bar{h}_d(s)s\bar{q} \tag{7.2}$$

and let $\bar{h}_d(s)$ be a more complicated relation than $\bar{h}_d \equiv c_d$. As suggested in Ref. 11, we take

$$\bar{h}_d(s) = \mu_o \left[c + \sum_{\rho=1}^{p} \frac{a_\rho}{s + b_\rho} \right] \tag{7.3}$$

where μ_o is a static stiffness parameter (e.g., Young's modulus). Clearly viscous damping is a special case of (7.3) with $c_d = \mu_o c$ and the a_ρ all zero.

There is no difficulty in transforming (7.2) and (7.3) to the time domain. In fact, the damping force is

$$f_d(t) = -\int_{-\infty}^{t} h_d(t-\tau)\dot{q}(\tau)d\tau \tag{7.4}$$

where

$$h_d(t) = \mu_o[c\delta(t) + \sum_{\rho=1}^{p} a_\rho \exp(-b_\rho t)] \tag{7.5}$$

In other words,

$$f_d(t) = -\mu_o[c\dot{q} + \sum_{\rho=1}^{p} a_\rho \int_{-\infty}^{t} \exp[-b_\rho(t-\tau)]\dot{q}(\tau)d\tau] \tag{7.6}$$

However, this integral expression is not as useful as the 'augmented' set of differential equations given below.

7.2 Viscoelastic Damping in Structures

The basic, undamped structural model

$$\underline{M}\ddot{\underline{q}} + \underline{K}\underline{q} = \underline{f} \tag{7.7}$$

we now write as

$$\underline{M}\ddot{\underline{q}} + \mu_o\underline{K}\underline{q} = \underline{f} \tag{7.8}$$

where μ_o represents a structural stiffness parameter (Young's modulus say). If there is more than one such parameter, the relation $\underline{K} = \mu_o\underline{K}$

can be extended accordingly. This may be especially necessary for structures fabricated from composite materials.

Laplace transformed, (7.8) becomes

$$[s^2\underline{M} + \mu_o\underline{K}]\bar{\underline{q}}(s) = \bar{\underline{f}}(s) \tag{7.9}$$

Now, for viscoelastic damping, we amend (7.9) to the form

$$[s^2\underline{M} + \mu(s)\underline{K}]\bar{\underline{q}}(s) = \bar{\underline{f}}(s) \tag{7.10}$$

where

$$\mu(s) = \mu_o \left[1 + c\,s + \sum_{\rho=1}^{p} \frac{a_\rho s}{s + b_\rho} \right] \tag{7.11}$$

This is consistent with (7.3). In particular, for sinusoidal excitation, let

$$\mu(j\omega) = \mu_R(\omega) + j\mu_I(\omega) \tag{7.12}$$

where

$$\mu_R(\omega) = \mu_o \left[1 + \omega^2 \sum_{\rho=1}^{p} \frac{a_\rho}{b_\rho^2 + \omega^2} \right] \tag{7.13}$$

$$\mu_I(\omega) = \mu_o\omega \left[c + \sum_{\rho=1}^{p} \frac{a_\rho b_\rho}{b_\rho^2 + \omega^2} \right] \tag{7.14}$$

The 'loss factor' is defined as

$$\mathrm{L.F.}(\omega) = \frac{\mu_I(\omega)}{\mu_R(\omega)} \tag{7.15}$$

For the sake of comparison, note that for simple viscous damping,

$$\mathrm{L.F.}(\omega) = c\omega \tag{7.16}$$

and for hysteretic damping

$$\mathrm{L.F.}(\omega) = (h/\mu_o)\mathrm{sgn}(\omega) \tag{7.17}$$

[Compare with (6.6).] For viscoelastic damping, $\mu_R(\omega)$ is called the *storage modulus* (an even function of ω) and $\mu_I(\omega)$ is called the *loss modulus* (an odd function of ω). By measuring the loss factor over the frequency range of interest, a quite accurate damping model can be constructed (i.e., the coefficients a_ρ, b_ρ and c determined).

7.3 Viscoelastic Damping in the Time Domain

By defining auxiliary state variables it is possible to convert the viscoelastic damping model (7.10) & (7.11) to a time-domain representation. First, transform (7.10) to undamped modal coordinates by setting $\underline{q} = \underline{E}\underline{\eta}$:

$$[s^2\underline{1} + \{\mu(s)/\mu_o\}\underline{\Omega}^2]\bar{\underline{\eta}}(s) = \bar{\underline{\gamma}}(s) \tag{7.18}$$

Because of our simplifying assumption that there is only *one* stiffness parameter, μ_o, these equations are uncoupled. For more complicated situations there might be one or more stiffness parameters for each finite element. To develop a theory for that case is obviously a major task; although beyond the scope of this report, such a development should be made. In any case, (7.18) consists of the following n uncoupled equations

$$(s^2 + \{\mu(s)/\mu_o\}\omega_\alpha^2)\bar{\eta}_\alpha(s) = \bar{\gamma}_\alpha(s) \qquad (\alpha = 1,\ldots,n) \tag{7.19}$$

The usual state variables $\{\eta_\alpha, \dot{\eta}_\alpha\}$ must be augmented to include the states associated with viscoelastic damping. As with any set of state variables, the choice of these extra state variables is not unique, so long as there are p of them. We choose as our set of state variables for mode α the set

$$\underline{x}_\alpha = \mathrm{col}\{\eta_\alpha, \dot{\eta}_\alpha, \xi_{\alpha 1}, \ldots, \xi_{\alpha p}\} \tag{7.20}$$

where

$$\dot{\xi}_{\alpha\rho} + b_\rho \xi_{\alpha\rho} = a_\rho \dot{\eta}_\alpha \qquad (\rho = 1,\ldots,p) \tag{7.21}$$

and $\alpha = 1,\ldots,n$. Then

$$\bar{\xi}_{\alpha\rho} = \frac{a_\rho s \bar{\eta}_\alpha}{s + b_\rho} \tag{7.22}$$

and (7.19) can be re-written

$$(s^2 + c\omega^2 s + \omega_\alpha^2)\bar{\eta}_\alpha + \sum_{\rho=1}^{p} \omega_\alpha^2 \bar{\xi}_{\alpha\rho} = \bar{\gamma}_\alpha \tag{7.23}$$

The extra viscoelastic terms are clearly evident

To write these equations in state form in the time domain, one needs merely to combine the time-domain equivalents of (7.20) - (7.23):

$$\dot{\underline{x}}_\alpha = \underline{A}_\alpha \underline{x}_\alpha + \underline{\Gamma}_\alpha \gamma_\alpha \tag{7.24}$$

where

$$\underline{A}_\alpha = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ -\omega_\alpha^2 & -c\omega_\alpha^2 & -\omega_\alpha^2 & \cdots & -\omega_\alpha^2 \\ 0 & a_1 & -b_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a_p & 0 & \cdots & -b_p \end{bmatrix} \tag{7.25}$$

$$\underline{\Gamma}_\alpha = [\,0 \quad 1 \quad 0 \quad \cdots \quad 0\,]^T \tag{7.26}$$

It is evident that the number of state variables in the model has been substantially increased, from 2n to (2 + p)n. However, modern computer capacity and recently developed techniques for model reduction can be expected to produce tractable models.

7.4 Eigenvalue Perturbations

The characteristic equation for mode α is, from (7.19),

$$\lambda_\alpha^2 + \{\mu(\lambda_\alpha)/\mu_o\}\omega_\alpha^2 = 0 \tag{7.27}$$

where, from (7.11),

$$\mu(\lambda_\alpha)/\mu_o = 1 + (2\zeta_\alpha/\omega_\alpha)\lambda_\alpha + \sum_{\rho=1}^{p} \frac{a_\rho \lambda_\alpha}{\lambda_\alpha + b_\rho} \tag{7.28}$$

and we have set

$$c = 2\zeta_\alpha/\omega_\alpha \tag{7.29}$$

If $\zeta_\alpha = 0$ and $a_\rho = 0$ $(\rho = 1,\ldots,p)$, i.e., if there were no damping, $\lambda_\alpha = j\omega_\alpha$. To find the *small* perturbation $\delta\lambda_\alpha$ due to the *small* parameters ζ_α and a_ρ $(\rho = 1,\ldots,p)$, we set

$$\lambda_\alpha = j\omega_\alpha + \delta\lambda_\alpha$$

in (7.27) and keep only first-order terms in small quantities. The result is

$$\delta\lambda_\alpha = -\zeta_\alpha\omega_\alpha - \tfrac{1}{2}\omega_\alpha^2 \sum_{\rho=1}^{p} \frac{a_\rho}{b_\rho + j\omega_\alpha} \tag{7.30}$$

Note that both the real and imaginary parts of λ_α are perturbed by viscoelastic damping.

Another way to state the result (7.30) is that the *effective viscous damping factor* is

$$\zeta_{\alpha,\mathrm{eff}} = \zeta_\alpha + \tfrac{1}{2}\omega_\alpha \sum_{\rho=1}^{p} \frac{a_\rho b_\rho}{b_\rho^2 + \omega_\alpha^2} \tag{7.31}$$

and that the *effective natural frequency* is

$$\omega_{\alpha,\mathrm{eff}} = \omega_\alpha \left[1 + \tfrac{1}{2} \sum_{\rho=1}^{p} \frac{a_\rho}{1 + (b_\rho/\omega_\alpha)^2} \right] \tag{7.32}$$

These equations provide handy design formulas for the $(\zeta_\alpha, \omega_\alpha)$ parameters used in the Common Theory of Section 2, provided the physical parameters c, a_ρ, b_ρ are known.

8 CONCLUDING REMARKS

An attempt has been made to review the analytical concepts that underlie the modeling of energy dissipation for large space structures. It has been repeatedly argued that the modeling of structural damping should be given the same priority as the modeling of structural stiffness. Up to now, damping has not been given this priority by the designers of spacecraft attitude control systems, and Section 4 was devoted to arguing that there is no justification whatever for such neglect. Loosely speaking, the damping factors ζ_α should be known with the same accuracy as the natural frequencies ω_α.

The Common Theory (a sarcastic appellation) was described in Section 2. Briefly, one assumes the structure is altogether undamped until the very last line in the dynamical analysis, and then a single linear viscous damping term is added to each modal differential equation of motion; moreover, the value of the modal damping factors are all taken to be equal to the same value, a value Guessed At by a Knowledgeable Person. In spite of its apparent naiveté, however, much of the subsequent analysis in this report shows that the Common Theory can in fact be quantitatively defended in several respects. Specifically, it has been shown that, *provided the structural damping is light*,

(i) nonviscous damping models can be used to produce *equivalent* viscous damping coefficients. Thus the addition of a term $2\zeta_\alpha\omega_\alpha\dot{\eta}_\alpha$ to a modal equation *does not mean that one is necessarily assuming the structural damping mechanism to be viscous.* One may be assuming *hysteretic* damping and using the equivalent viscous damping factors given by (6.24). Or, one may be assuming *viscoelastic* damping and using the equivalent viscous damping factors given by (7.31).

(ii) off-diagonal damping coupling terms in the modal motion equations, even if present, *make no first-order contribution to the system eigenvalues* and *their small contribution to the system eigenvectors is not of practical significance.*

Therefore, *assuming light structural damping*, the inclusion of a 'linear viscous damping term' in each modal equation cannot really be faulted. The off-diagonal terms do not really matter, and the 'viscous damping' may just refer to an '*equivalent* viscous damping'.

Where the Common Theory deserves to be attacked is in the assignment of the equivalent viscous damping factors ζ_α. Instead of being Guessed At, they should be Calculated. Such calculation requires at least the following two things:

(i) that fundamental material damping parameters be known (in the same way that Young's modulus is known). For example, the viscoelastic parameters c, a_ρ, b_ρ must be numerically available if viscoelastic damping is to be used.

(ii) a methodology for incorporating damping calculations within the finite element method. Otherwise damping will always be an 'add-on' of lesser accuracy.

If the damping is not judged to be 'light', then the modal-damping-factor viewpoint should be abandoned, as should the attention paid to undamped modes, which cease to be relevant. Many of the considerations in this report are still useable, however. The hysteretic damping model in (6.9) still applies, as does the viscoelastic damping model in (7.10). If viscous damping is defensible, the 'damped modes' of Section 5.3 can be used.

Last but not least, the opportunity to design damping into the system should not be ignored. This may take the form either of discrete dampers, see (5.14), or distributed material damping. The former are particularly valuable when aimed at a few troublesome modes; the latter affects all modes and is an effective prescription for avoiding 'spillover' problems associated with the higher, unmodeled, uncontrolled modes.

9. REFERENCES

1. Graham, W.B., "Material Damping and its Role in Linear Dynamic Equations", UTIAS Review No. 36, March 1973

2. Wolff, G. and A. Wittmann, "The Flight of FRUSA", AIAA Paper 72-510, Electric Propulsion Conf., Bethesda, Md., April 1972.

3. Garg, S.C., P.C. Hughes, R.A. Millar and F.R. Vigneron, "Flight Results on Structural Dynamics from Hermes", J. Spacecraft and Rockets, Vol. 16, No. 2, March-April 1979, pp. 81-87.

4. Slafer, L.I., "On-Orbit Evaluation of the Control System/Structural Mode Interactions on OSO-8", J. Guid. and Control, Vol. 3, No. 3, May-June 1980, pp. 203-209.

5. Papoulis, A., "Probability, Random Variables and Stochastic Processes", McGraw-Hill, 1965.

6. Hughes, P.C., "MSAT Structural Flexibility and Control Assessment", Dynacon Report MSAT-1, March 1981.

7. Hughes, P.C., and T.M. Abdel-Rahman, "Stability of Proportional-Plus-Derivative-Plus Integral Control of Flexible Spacecraft", J. Guidance and Control, Vol. 2, No. 6, Nov-Dec 1979, pp. 479-503.

8. Hughes, P.C., "Modal Identities for Elastic Bodies, with Application to Vehicle Dynamics and Control," Jour. of Applied Mechanics, Vol. 47, No. 1, March 1980, pp. 177-184.

9. Foss, K.A., "Coordinates which Uncouple Equations of Motion of Damped Linear Dynamical Systems", J. Appl. Mech., Sept. 1958, pp. 361-364.

10. Ottens, H.H., "Mathematical Formulation of Damping for Structural Response Analysis", AGARD Conf. Proc. 277, April 1979.

11. Biot, M.A., "Variational Principles in Irreversible Thermodynamics with Application to Viscoelasticity", Phys. Rev., Vol. 97, March 1955.