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GENERIC STRUCTURAL MODELING of the DEMONSTRATION MSAT SPACECRAFT

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FOREWORD

Acknowledgements

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Units and Spelling

This report uses S.I. units and North American spelling.

Summary

It is assumed that DEMSAT consists of seven substructures (Figs. 1.1 and 1.2): a main bus, two solar-cell arrays, two dish antenna reflectors, and two towers to support these dishes. All substructures but the main bus are allowed to be flexible, and their dynamical models are presumed to be specified either in terms of mass and stiffness matrices or in terms of modal data. This report shows how to combine the seven substructural models into an overall structural model for the spacecraft.

1. INTRODUCTION

The purpose of this report is to explain how to combine the substructural models for the various configurational elements of the Demonstration MSAT (DEMSAT for short) into a single structural dynamics model. It is important to note that the substructural models themselves are not derived in detail in this report, but are defined in generic terms. Thus if, for example, one were to have a Harris dish as one reflector, a wrap-rib dish as the other reflector, two correspondingly different tower structures, and two non-identical solar array panels, this report shows how to synthesize the models of each of these elements into a single model.

A typical DEMSAT configuration is shown in Fig. 1.1. The dynamicist looks at the morphology of DEMSAT and sees it as shown in Fig. 2.1. We shall use linear structural models exclusively.

Section 2 of this report introduces appropriate reference frames, discusses the forces and torques on each substructure, and introduces the symbols needed to specify completely the mass properties of the spacecraft. Sections 3,4,5 are concerned, respectively, with the kinetic energy, potential energy, and virtual work expressions required to derive the motion equations from Hamilton's principle.

The actual equations of motion (and the associated coefficient matrices) are presented in Section 6, and Section 7 explains what one should do if the substructural models are given in terms of modal coordinates instead of discrete coordinates.

The reader who wishes to broaden his range of dynamic expertise may wish to compare the Lagrangian approach developed here with the vectorial formulation applied to a somewhat similar configuration in Ref. 1. This comparison is guided by the suggestions made in Appendix A.

2. BASIC CONSIDERATIONS

The basic configuration for DEMSAT is discussed now in its most general terms. It will be made clear which substructures are assumed rigid,

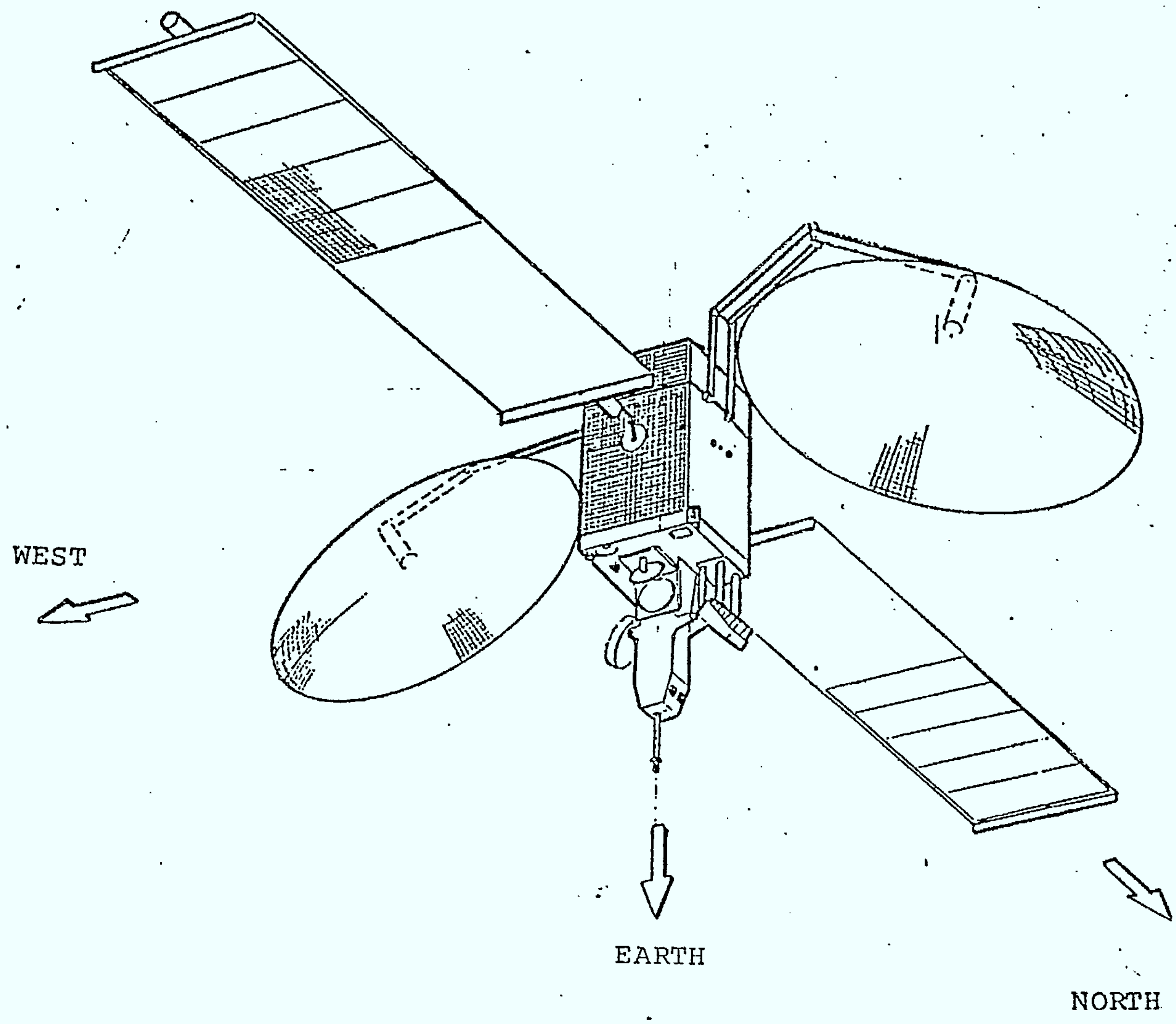

Fig. 1.1: A Typical Demonstration MSAT (DEMSAT) Design

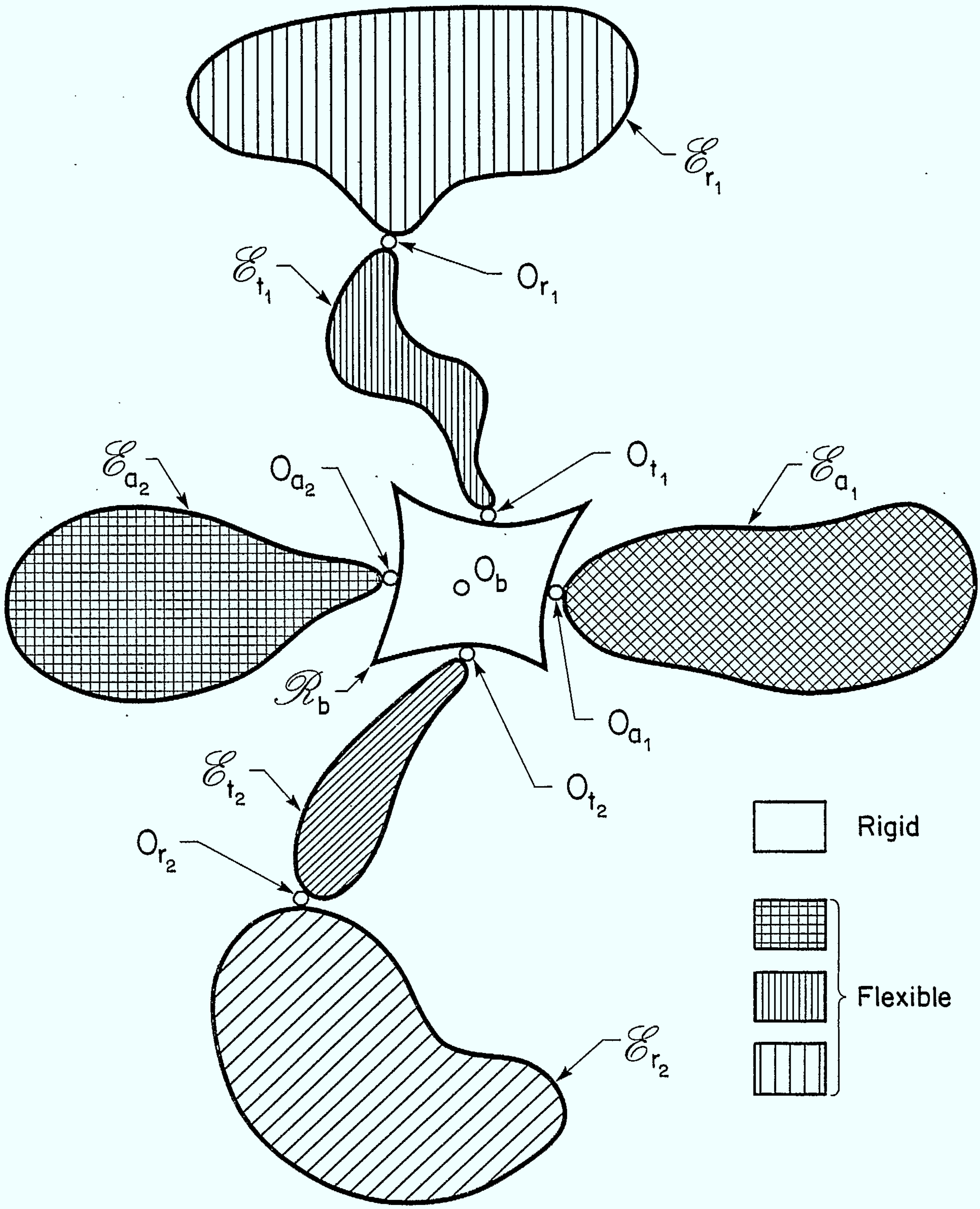

Fig. 2.1: DEMSAT Morphology

and which flexible. Appropriate reference frames, one for each substructure, are introduced. Details of the force (and hence torque) distribution over the spacecraft are clarified, and the inertia distribution for the chosen spacecraft model is defined.

2.1 DEMSAT Substructures

The morphology assumed for DEMSAT is shown in Fig. 2.1. It is assumed the DEMSAT comprises seven bodies as follows:

Bus:	$\mathcal{R}_b$	Rigid
Two Solar Arrays:	$\mathcal{E}_{a_1}$, $\mathcal{E}_{a_2}$	Elastic
Two Towers:	$\mathcal{E}_{t_1}$, $\mathcal{E}_{t_2}$	Elastic
Two Reflector Dishes:	$\mathcal{E}_{r_2}$, $\mathcal{E}_{r_2}$	Elastic

The connection points between these bodies $\{O_{a_1}, O_{a_2}, O_{t_1}, O_{t_2}, O_{r_1}, O_{r_2}\}$, are also shown in Fig. 2.1. Note that O_b is an arbitrary reference point in the bus. (The rotational equations for DEMSAT will be written with respect to O_b.)

The fixed displacements between these reference points are shown in Fig. 2.2. Note in particular that

$$\underset{\rightarrow}{r}_{br_n} = \underset{\rightarrow}{r}_{bt_n} + \underset{\rightarrow}{r}_{tr_n} \qquad (n = 1,2) \qquad (2.1)$$

2.2 Reference Frames

Eight reference frames are used in this report. Seven of these are explicit (Fig. 2.2) and one--the inertial frame $\mathcal{F}_i$--is implied. Each substructure is assigned its own frame so that each substructural model is completely independent. The absolute displacements (i.e., the displacements with respect to $\mathcal{F}_i$) of the connection points shown in Fig. 2.2, expressed in the reference frames shown in Fig. 2.3, are as follows:

absolute displacement of O_b, expressed in $\mathcal{F}_b$, is $\underline{w}_b$

absolute displacement of O_{a_n}, expressed in $\mathcal{F}_{a_n}$, is $\underline{w}_{a_n}$ $(n = 1,2)$

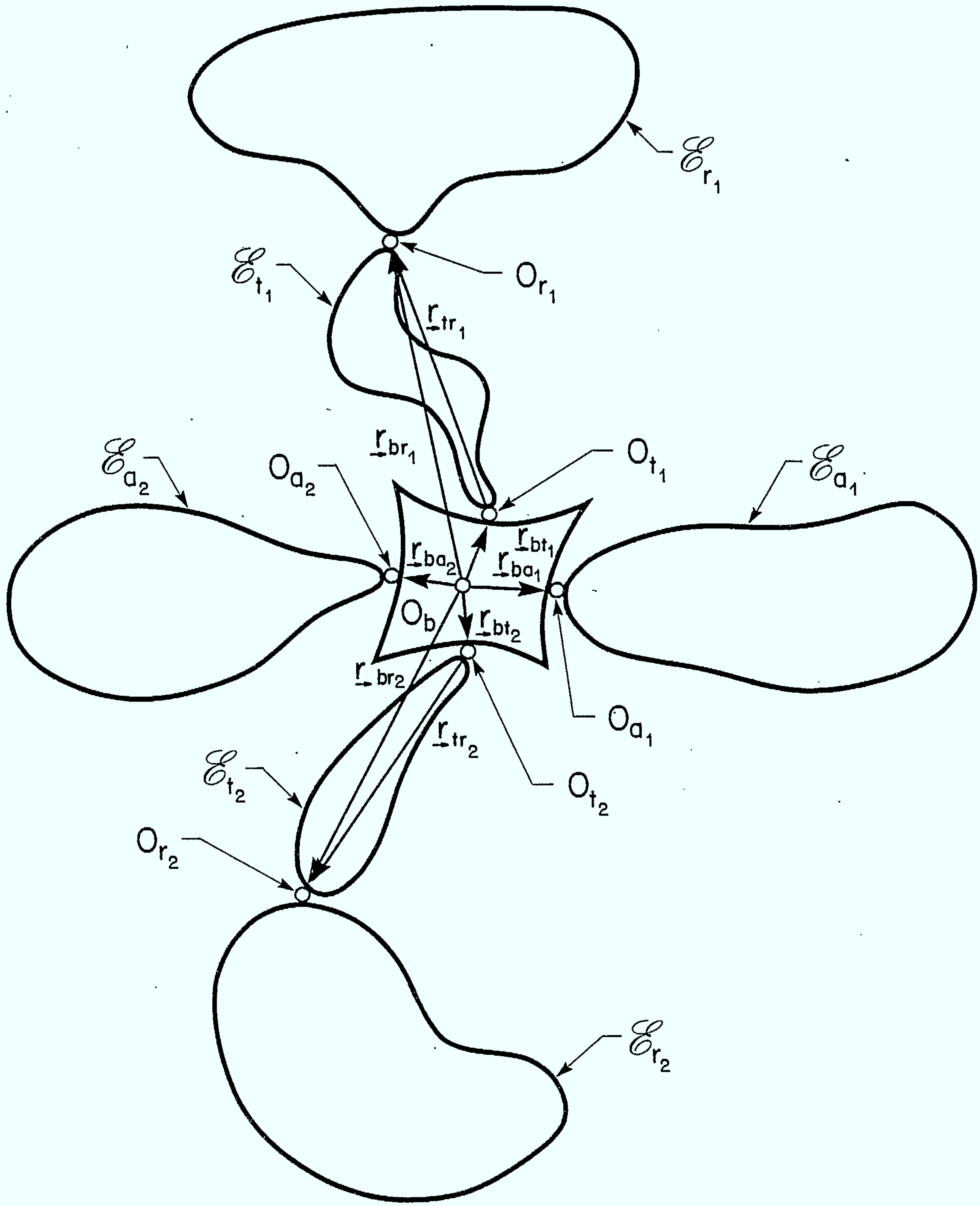

Fig. 2.2: Displacements Between Reference Points

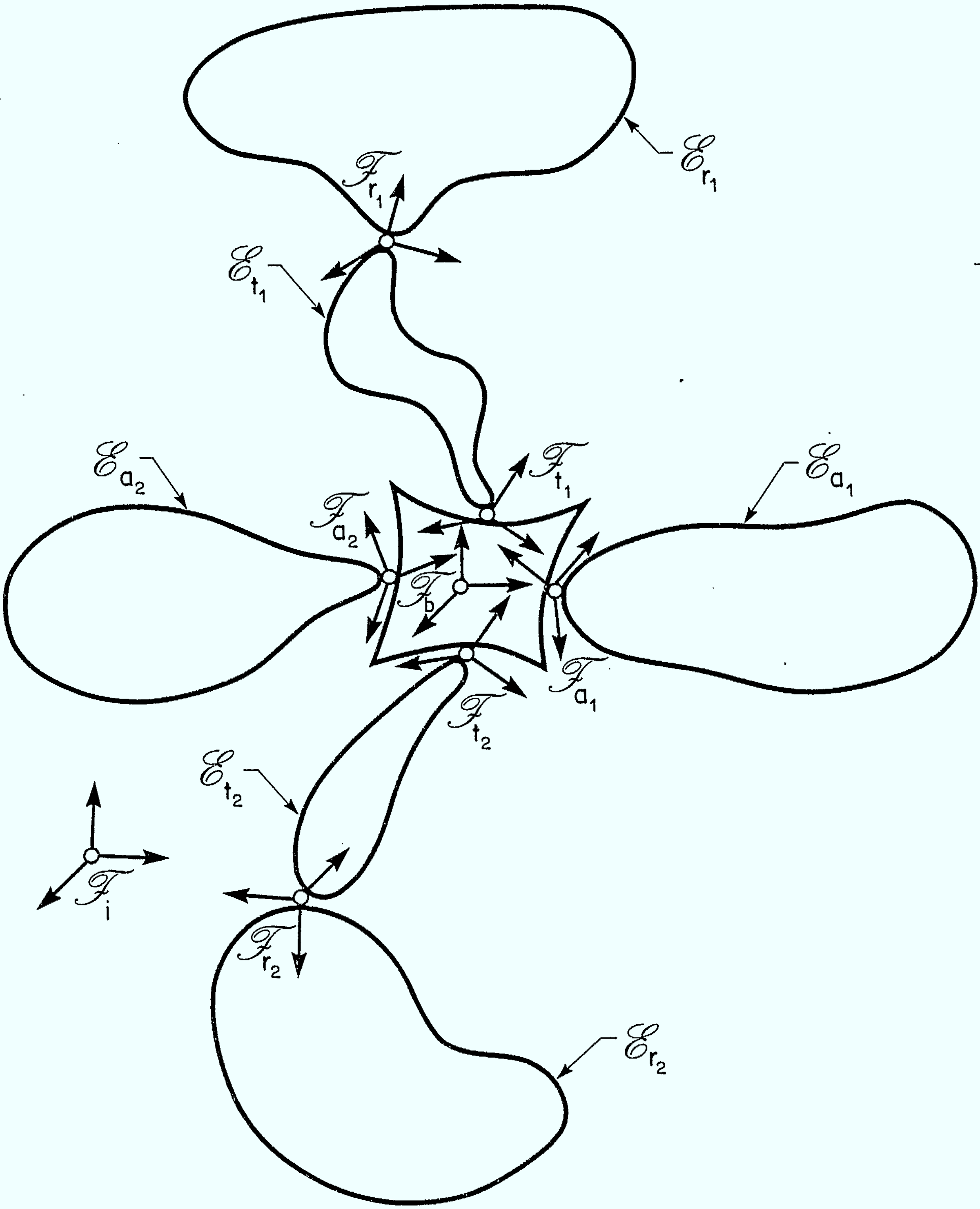

Fig. 2.3: Reference Frames Used

absolute displacement of O_{t_n}, expressed in F_{t_n}, is $\underline{w}_{t_n}$ $\qquad (n = 1,2)$

absolute displacement of O_{r_n}, expressed in F_{r_n}, is $\underline{w}_{r_n}$ $\qquad (n = 1,2)$

These absolute displacements are not assumed to be necessarily small; however, their first and second time derivatives (i.e., the absolute velocities and accelerations) *are* assumed to be first-order infinitesimals.

Similarly, the absolute rotations of the reference frames shown in Fig. 2.3 are not assumed to be small, and a general rotation-matrix notation will be employed to indicate the relationships between vector components written in these various frames. (For example, $\underline{C}_{a_1 b}$ converts components in F_b to components in F_{a_1}.) The *rates* of rotation of these reference frames are, however, taken to be small. Thus, for example, we have the following small (absolute) angular accelerations:

angular acceleration of F_b, expressed in F_b, is $\ddot{\underline{\theta}}_b$

angular acceleration of F_{a_n}, expressed in F_{a_n}, is $\ddot{\underline{\theta}}_{a_n}$ $\qquad (n = 1,2)$

angular acceleration of F_{t_n}, expressed in F_{t_n}, is $\ddot{\underline{\theta}}_{t_n}$ $\qquad (n = 1,2)$

angular acceleration of F_{r_n}, expressed in F_{r_n}, is $\ddot{\underline{\theta}}_{r_n}$ $\qquad (n = 1,2)$

[It must be understood that the question of whether the reference frames F are for the undeformed or deformed spacecraft does not arise in linear analysis. See the comments on p.4 of Ref. 1.]

2.3 Force and Torque Distribution

We shall assume that all the forces and torques on the spacecraft due to external or environmental influences arise from a distribution of force per unit volume, and we denote this distribution by $\hat{\vec{f}}(\vec{r},t)$. If there is a distributed torque distribution (rare) or a surface force distribution, or point forces or torques, or some other variant that seems superficially to be different from the formulation used here, a reasonably straightforward modification to the analysis below can be made. (The details of these modifications are omitted for brevity.)

A subscript will denote the reference frame in which $\hat{\underset{\rightarrow}{f}}$ is expressed. Thus, over R_b, $\hat{\underset{\rightarrow}{f}}$ is written $\hat{\underline{f}}_b$. The net forces on the constituent bodies are as follows:

force on R_b, expressed in F_b, is $\underline{f}_a$

force on E_{a_n}, expressed in F_{a_n}, is $\underline{f}_{a_n}$ $(n = 1,2)$

force on E_{t_n}, expressed in F_{t_n}, is $\underline{f}_{t_n}$ $(n = 1,2)$

force on E_{r_n}, expressed in F_{r_n}, is $\underline{f}_{r_n}$ $(n = 1,2)$

Thus, in terms of $\hat{\underset{\rightarrow}{f}}$,

$$\begin{aligned}
&\underline{f}_b(t) \triangleq \int \hat{\underline{f}}_b(\underline{r}_b,t)dV_b \\
&\underline{f}_{a_n}(t) \triangleq \int \hat{\underline{f}}_{a_n}(\underline{r}_{a_n},t)dV_{a_n} \qquad (n = 1,2) \\
&\underline{f}_{t_n}(t) \triangleq \int \hat{\underline{f}}_{t_n}(\underline{r}_{t_n},t)dV_{t_n} \qquad (n = 1,2) \\
&\underline{f}_{r_n}(t) \triangleq \int \hat{\underline{f}}_{r_n}(\underline{r}_{r_n},t)dV_{r_n} \qquad (n = 1,2)
\end{aligned} \tag{2.2}$$

Here, we have used the following definitions of displacement vectors:

position in R_b, relative to O_b, expressed in F_b, is $\underline{r}_b$

position in E_{a_n}, relative to O_{a_n}, expressed in F_{a_n}, is $\underline{r}_{a_n}$ $(n = 1,2)$

position in E_{t_n}, relative to O_{t_n}, expressed in F_{t_n}, is $\underline{r}_{t_n}$ $(n = 1,2)$

position in E_{r_n}, relative to O_{r_n}, expressed in F_{r_n}, is $\underline{r}_{r_n}$ $(n = 1,2)$

In a similar manner, the torques on the constituent bodies from external sources are as follows:

torque on $\mathcal{R}_b$, about O_b, expressed in $\mathcal{F}_b$, is $\underline{g}_b$

torque on $\mathcal{E}_{a_n}$, about O_{a_n}, expressed in $\mathcal{F}_{a_n}$, is $\underline{g}_{a_n}$ $\quad (n = 1,2)$

torque on $\mathcal{E}_{t_n}$, about O_{t_n}, expressed in $\mathcal{F}_{t_n}$, is $\underline{g}_{t_n}$ $\quad (n = 1,2)$

torque on $\mathcal{E}_{r_n}$, about O_{r_n}, expressed in $\mathcal{F}_{r_n}$, is $\underline{g}_{r_n}$ $\quad (n = 1,2)$

where, in terms of $\underset{\rightarrow}{\hat{f}}$,

$$
\begin{aligned}
&\underline{g}_b(t) \triangleq \int \underline{r}_b^{\times}\underline{\hat{f}}_b(\underline{r}_b, t)dV_b \\
&\underline{g}_{a_n}(t) \triangleq \int \underline{r}_{a_n}^{\times}\underline{\hat{f}}_{a_n}(\underline{r}_{a_n}, t)dV_{a_n} && (n = 1,2) \\
&\underline{g}_{t_n}(t) \triangleq \int \underline{r}_{t_n}^{\times}\underline{\hat{f}}_{t_n}(\underline{r}_{t_n}, t)dV_{t_r} && (n = 1,2) \\
&\underline{g}_{r_n}(t) \triangleq \int \underline{r}_{r_n}^{\times}\underline{\hat{f}}_{r_n}(\underline{r}_{r_n}, t)dV_{r_n} && (n = 1,2)
\end{aligned}
\qquad (2.3)
$$

This completes a specification of the forces and torques due to the disturbed force field $\underset{\rightarrow}{\hat{f}}$.

2.4 Attachment Point Vectors

Eight attachment point vectors are shown in Fig. 2.2. The convention will be adopted that $\underset{\rightarrow}{r}_{pq}$ is expressed in $\mathcal{F}_p$ (for arbitrary p, q). For example, $\underline{r}_{bt_2}$ contains the components of $\underset{\rightarrow}{r}_{bt_2}$ expressed in $\mathcal{F}_b$. In particular, the vector relations (2.1) become, in component form:

$$\underline{r}_{br_n} = \underline{r}_{bt_n} + \underline{C}_{bt_n}\underline{r}_{tr_n} \qquad (n = 1,2) \qquad (2.4)$$

This contraction will be needed several times in the sequel.

2.5 Inertia Distribution

The following symbols represent the masses of the constituent bodies:

mass of R_b is m_b

mass of E_{a_n} is m_{a_n} $(n = 1,2)$

mass of E_{t_n} is m_{t_n} $(n = 1,2)$

mass of E_{r_n} is m_{r_n} $(n = 1,2)$

The first moments of inertia ($\int \underset{\rightarrow}{r}\, dm$) for these constituent bodies are as follows:

first moment of inertia of R_b, about O_b, expressed in F_b is $\underline{c}_b$

first moment of inertia of E_{a_n}, about O_{a_n}, expressed in F_{a_n}, is $\underline{c}_{a_n}$ $(n = 1,2)$

first moment of inertia of E_{t_n}, about O_{t_n}, expressed in F_{t_n}, is $\underline{c}_{t_n}$ $(n = 1,2)$

first moment of inertia of E_{r_n}, about O_{r_n}, expressed in F_{r_n}, is $\underline{c}_{r_n}$ $(n = 1,2)$

Similarly the second moment-of-inertia matrices are

moment-of-inertia-matrix for R_b, about O_b, expressed in F_b, is $\underline{J}_b$

moment-of-inertia-matrix for E_{a_n}, about O_{a_n}, expressed in F_{a_n}, is $\underline{J}_{a_n}$ $(n = 1,2)$

moment-of-inertia-matrix for E_{t_n}, about O_{t_n}, expressed in F_{t_n}, is $\underline{J}_{t_n}$ $(1 = 1,2)$

moment-of-inertia-matrix for E_{r_n}, about O_{r_n}, expressed in F_{r_n}, is $\underline{J}_{r_n}$ $(1 = 1,2)$

Using these definitions, and the additional defintions

total spacecraft mass is m

total first moment of inertia, about O_b, in F_b, is $\underline{c}$

total moment-of-inertia matrix, about O_b, in F_b, is $\underline{J}$

we have (using the most general parallel-axis theorems)

$$m = m_b + \sum_{n=1}^{2} (m_{a_n} + m_{t_n} + m_{r_n}) \tag{2.5}$$

$$\underline{c} = \underline{c}_b + \sum_{n=1}^{2} [(m_{a_n} \underline{r}_{ba_n} + \underline{C}_{ba_n} \underline{c}_{a_n})$$

$$+ (m_{t_n} \underline{r}_{bt_n} + \underline{C}_{bt_n} \underline{c}_{t_n}) \tag{2.6}$$

$$+ (m_{r_n} \underline{r}_{br_n} + \underline{C}_{br_n} \underline{c}_{r_n})]$$

$$\underline{J} = \underline{J}_b + \sum_{n=1}^{2} [(\underline{C}_{ba_n} \underline{J}_{a_n} \underline{C}_{a_n b} - \underline{C}_{ba_n} \underline{c}_{a_n}^{\times} \underline{C}_{a_n b} \underline{r}_{ba_n}^{\times} - \underline{r}_{ba_n}^{\times} \underline{C}_{ba_n} \underline{c}_{a_n}^{\times} \underline{C}_{a_n b} - m_{a_n} \underline{r}_{ba_n}^{\times} \underline{r}_{ba_n}^{\times})$$

$$+ (\underline{C}_{bt_n} \underline{J}_{t_n} \underline{C}_{t_n b} - \underline{C}_{bt_n} \underline{c}_{t_n}^{\times} \underline{C}_{t_n b} \underline{r}_{bt_n}^{\times} - \underline{r}_{bt_n}^{\times} \underline{C}_{bt_n} \underline{c}_{t_n}^{\times} \underline{C}_{t_n b} - m_{t_n} \underline{r}_{bt_n}^{\times} \underline{r}_{bt_n}^{\times}) \tag{2.7}$$

$$+ (\underline{C}_{br_n} \underline{J}_{r_n} \underline{C}_{r_n b} - \underline{C}_{br_n} \underline{c}_{r_n}^{\times} \underline{C}_{r_n b} \underline{r}_{br_n}^{\times} - \underline{r}_{br_n}^{\times} \underline{C}_{br_n} \underline{c}_{r_n}^{\times} \underline{C}_{r_n b} - m_{r_n} \underline{r}_{br_n}^{\times} \underline{r}_{br_n}^{\times})]$$

With these 'basic considerations' now complete, a foundation has been laid for deriving the important dynamical properties of the system.

3. KINETIC ENERGY

Preparatory to the use of Hamilton's principle in the derivation of motion equations, an expression is now derived for the total system kinetic energy.

3.1 Displacements and Velocity Distribution in $\mathcal{R}_b$

Let us denote by $\underline{d}_b(\underline{r}_b, t)$ the total displacement of an element of mass dm_b in $\mathcal{R}_b$. Thus, in terms of earlier definitions,

$$\underline{d}_b(\underline{r}_b, t) = \underline{w}_b(t) - \underline{r}_b^{\times} \underline{\theta}_b(t) \tag{3.1}$$

Furthermore, let $\underline{v}_b(\underline{r}_b, t)$ denote the velocity distribution. Within the confines of linear theory and our present assumptions, we have

$$\underline{v}_b(\underline{r}_b, t) = \dot{\underline{d}}_d(\underline{r}_b, t) = \dot{\underline{w}}_b(t) - \underline{r}_b^{\times} \dot{\underline{\theta}}_b(t) \tag{3.2}$$

This enables us to calculate the kinetic energy of $\mathcal{R}_b$:

$$T_b = \tfrac{1}{2} \int \underline{v}_b(\underline{r}_b,t)^T \underline{v}_b(\underline{r}_b,t)dm_b \tag{3.3}$$

$$= \tfrac{1}{2} \begin{bmatrix} \dot{\underline{w}}_b \\ \dot{\underline{\theta}}_b \end{bmatrix}^T \begin{bmatrix} m_b\underline{1} & -\underline{c}_b^{\times} \\ \underline{c}_b^{\times} & \underline{J}_b \end{bmatrix} \begin{bmatrix} \dot{\underline{w}}_b \\ \dot{\underline{\theta}}_b \end{bmatrix} \tag{3.4}$$

In the following paragraphs, a similar calculation is made for the other substructures in the spacecraft.

3.2 Displacements and Velocity Distribution in $\mathcal{E}_{a_n}$

For simplicity, the subscript n $(n = 1,2)$ is omitted from the symbol 'a' in this subsection. It is understood that the equations derived apply to $n = 1,2$. We denote the displacement distribution by

$$\underline{d}_a(\underline{r}_a,t) = \underline{w}_a(t) - \underline{r}_a^{\times}\underline{\theta}_a(t) + \underline{u}_a(\underline{r}_a,t) \tag{3.5}$$

The first two terms in (3.5) are obviously the 'rigid' contributions to displacement. They would occur even if $\mathcal{E}_a$ were rigid (cf. Eq. (3.1). The extra term, $\underline{u}_a(\underline{r}_a,t)$, signifies the small elastic displacement distribution. This is the term that arises on account of structural flexibility.

We shall assume that the elastic displacement distribution $\underline{u}_a(\underline{r}_a,t)$ has been expressed as a superpostion of 'shape functions,' $\{\underline{\psi}_{1a}(\underline{r}_a), \underline{\psi}_{2a}(\underline{r}_a),\ldots.\}$. With each of these shape functions (in the spirit of Rayleigh and Ritz) is associated a coordinate, so that

$$\underline{u}_a(\underline{r}_a,t) = \sum_j \underline{\psi}_{ja}(\underline{r}_a)q_{ja}(t) \tag{3.6}$$

The possibility that the shape functions $\underline{\psi}_{ja}$ are vibration mode shapes, in which case q_{ja} become 'modal coordinates,' is not excluded; see Section 7. In any case, a notational simplification to (3.6) can be made if we condense all the shape functions $\underline{\psi}_{ja}$ into a single (rectangular) matrix $\underline{\Psi}_a$, and the associated coordinates q_{ja} into a single column matrix, as follows:

$$\underline{\Psi}_a(\underline{r}_a) \triangleq [\underline{\psi}_{1a} \quad \underline{\psi}_{2a} \quad \ldots] \tag{3.7}$$

$$\underline{q}_a(t) \triangleq [q_{1a} \quad q_{2a} \quad \ldots]^T \tag{3.8}$$

Then

$$\underline{u}_a(\underline{r}_a,t) = \underline{\Psi}_a(\underline{r}_a)\underline{q}_a(t) \tag{3.9}$$

and the displacement distribution (3.5) becomes

$$\underline{d}_a(\underline{r}_a,t) = \underline{w}_a(t) - \underline{r}_a^X\underline{\theta}_a(t) + \underline{\Psi}_a(\underline{r}_a)\underline{q}_a(t) \tag{3.10}$$

whence the velocity distribution is

$$\underline{v}_a(\underline{r}_a,t) = \dot{\underline{w}}_a(t) - \underline{r}_a^X\dot{\underline{\theta}}_a(t) + \underline{\Psi}_a(\underline{r}_a)\dot{\underline{q}}_a(t) \tag{3.11}$$

It is now possible to calculate the kinetic energy of each of the solar arrays:

$$T_a = \tfrac{1}{2}\int \underline{v}_a(\underline{r}_a,t)^T\underline{v}_a(\underline{r}_a,t)dm_a \tag{3.12}$$

$$= \tfrac{1}{2}\begin{bmatrix} \dot{\underline{w}}_a \\ \dot{\underline{\theta}}_a \\ \dot{\underline{q}}_a \end{bmatrix}^T \begin{bmatrix} m_a\underline{1} & -\underline{c}_a^X & \underline{P}_a \\ \underline{c}_a^X & \underline{J}_a & \underline{H}_a \\ \underline{P}_a^T & \underline{H}_a^T & \underline{M}_{aa} \end{bmatrix} \begin{bmatrix} \dot{\underline{w}}_a \\ \dot{\underline{\theta}}_a \\ \dot{\underline{q}}_a \end{bmatrix} \tag{3.13}$$

where the momenta and angular momenta associated with the coordinates q_{ja} are defined to be

$$\underline{P}_a \triangleq \int \underline{\Psi}_a(\underline{r}_a)dm_a \tag{3.14}$$

$$\underline{H}_a \triangleq \int \underline{r}_a^X\underline{\Psi}_a(\underline{r}_a)dm_a \tag{3.15}$$

and where

$$\underline{M}_{aa} = \int \underline{\Psi}_a^T\underline{\Psi}_a dm_a \tag{3.16}$$

3.3 Displacements and Velocity Distribution in $\mathcal{E}_{t_n}$

In exactly the same manner as followed in the last section, the displacement distribution, velocity distribution, and kinetic energy for either of the towers $\mathcal{E}_{t_n}$ $(n=1,2)$ can be written:

$$\underline{d}_t(\underline{r}_r,t) = \underline{w}_t(t) - \underline{r}_t^{\times}\underline{\theta}_t(t) + \underline{\Psi}_t(\underline{r}_t)\underline{q}_t(t) \tag{3.17}$$

$$\underline{v}_t(\underline{r}_t,t) = \dot{\underline{w}}_t(t) - \underline{r}_t^{\times}\dot{\underline{\theta}}_t(t) + \underline{\Psi}_t(\underline{r}_t)\dot{\underline{q}}_t(t) \tag{3.18}$$

$$T_t = \frac{1}{2}\begin{bmatrix}\dot{\underline{w}}_t \\ \dot{\underline{\theta}}_t \\ \dot{\underline{q}}_t\end{bmatrix}^T \begin{bmatrix} m_t\underline{1} & -\underline{c}_t^{\times} & \underline{P}_t \\ \underline{c}_t^{\times} & \underline{J}_t & \underline{H}_t \\ \underline{P}_t^T & \underline{H}_t^T & \underline{\mathcal{M}}_{tt}\end{bmatrix}\begin{bmatrix}\dot{\underline{w}}_t \\ \dot{\underline{\theta}}_t \\ \dot{\underline{q}}_t\end{bmatrix} \tag{3.19}$$

where

$$\underline{P}_t \triangleq \int \underline{\Psi}_t dm_t \tag{3.20}$$

$$\underline{H}_t \triangleq \int \underline{r}_t^{\times}\underline{\Psi}_t dm_t \tag{3.21}$$

$$\underline{\mathcal{M}}_{tt} \triangleq \int \underline{\Psi}_t^T\underline{\Psi}_t dm_t \tag{3.22}$$

Note that $\underline{\mathcal{M}}_{tt}$ is in script notation, indicating that it will form part of the final mass matrix of the whole system (this will be proven as the derivation evolves).

A key difference between $\mathcal{E}_t$ and $\mathcal{E}_a$ is that another structure is attached to $\mathcal{E}_t$ while none is attached to $\mathcal{E}_a$. Specifically, with reference to Fig. 2.2, we require a special notation for the displacement and velocity of O_{r_n} $(n=1,2)$. To this end, we note from (3.17) that

$$\underline{w}_r = \underline{C}_{rt}(\underline{w}_t - \underline{r}_{tr}^{\times}\underline{\theta}_t + \underline{\delta}) \tag{3.23}$$

$$\underline{\theta}_r = \underline{C}_{rt}(\underline{\theta}_t + \underline{\alpha}) \tag{3.24}$$

where

$$\underline{w}_r(t) = \underline{C}_{rt}\underline{d}_t(\underline{r}_{tr}, t) \tag{3.25}$$

$$\underline{\theta}_r(t) = \tfrac{1}{2}\underline{C}_{rt}\underline{\nabla}^X\underline{d}_t(\underline{r}_t, t)\Big|_{\underline{r}_t = \underline{r}_{tr}} \tag{3.26}$$

$$\underline{\delta}(t) = \underline{\Psi}_t(\underline{r}_{tr})\underline{q}_t(t) \tag{3.27}$$

$$\underline{\alpha}(t) = \tfrac{1}{2}\underline{\nabla}^X\underline{\Psi}_t(\underline{r}_t)\Big|_{\underline{r}_t = \underline{r}_{tr}}\cdot\underline{q}_t(t) \tag{3.28}$$

and the symbol $\underline{\nabla}$ denotes differentiation with respect to $\underline{r}_t$.

However, it is also true (cf. Eq. (3.1)) that

$$\underline{w}_t = \underline{C}_{tb}(\underline{w}_b - \underline{r}_{bt}^X\underline{\theta}_b) \tag{3.29}$$

$$\underline{\theta}_t = \underline{C}_{tb}\underline{\theta}_b \tag{3.30}$$

Therefore, making use of (2.4) and the properties of rotation matrices, we can show that

$$\underline{w}_r = \underline{C}_{rb}\underline{w}_b - \underline{C}_{rb}\underline{r}_{br}^X\underline{\theta}_b + \underline{C}_{rt}\underline{\delta} \tag{3.31}$$

$$\underline{\theta}_r = \underline{C}_{rb}\underline{\theta}_b + \underline{C}_{rt}\underline{\alpha} \tag{3.32}$$

At this point it is helpful to stipulate that the six coordinates represented by $\underline{\delta}(t)$ and $\underline{\alpha}(t)$ are six of the coordinates in $\underline{q}_t(t)$. This leads to appropriate partitioning of earlier relations:

$$\underline{q}_t \triangleq \mathrm{col}\{\underline{\delta}, \underline{\alpha}, \underline{q}_i\} \tag{3.33}$$

$$\underline{P}_t \triangleq [\underline{P}_\delta \quad \underline{P}_\alpha \quad \underline{P}_i] \tag{3.34}$$

$$\underline{H}_t \triangleq [\underline{H}_\delta \quad \underline{H}_\alpha \quad \underline{H}_i] \tag{3.35}$$

$$\mathcal{M}_{tt} = \begin{bmatrix} \underline{M}_{\delta\delta} & \underline{M}_{\delta\alpha} & \underline{\mathcal{M}}_{\delta i} \\ \underline{M}^T_{\delta\alpha} & \underline{M}_{\alpha\alpha} & \underline{\mathcal{M}}_{\alpha i} \\ \underline{\mathcal{M}}^T_{\delta i} & \underline{\mathcal{M}}^T_{\alpha i} & \underline{\mathcal{M}}_{ii} \end{bmatrix} \tag{3.36}$$

Note that four of the matrix partitions in (3.36) are not in script notation. This is because (as we shall see) additional terms have to be added to each of them to form the corresponding matrix partition in the overall mass matrix for the spacecraft.

3.4 Displacements and Velocity Distribution in $\mathcal{E}_{r_n}$

The reflector $\mathcal{E}_r$ bears the same relation to the tower $\mathcal{E}_t$ as the array $\mathcal{E}_a$ does to the bus $\mathcal{R}_b$: it is a terminal flexible body. The displacement distribution, velocity distribution, and kinetic energy for either of the reflectors $\mathcal{E}_{r_n}$ $(n = 1,2)$ are analogous to those written in Section 3.2 for the arrays. Thus

$$\underline{d}_r(\underline{r}_r,t) = \underline{w}_r(t) - \underline{r}_r^{\times}\underline{\theta}_r(t) + \underline{\Psi}_r(r_r)\underline{q}_r(t) \tag{3.37}$$

$$\underline{v}_r(\underline{r}_r,t) = \dot{\underline{w}}_r(t) - \underline{r}_r^{\times}\dot{\underline{\theta}}_r(t) + \underline{\Psi}_r(\underline{r}_r)\dot{\underline{q}}_r(t) \tag{3.38}$$

$$T_r = \tfrac{1}{2}\begin{bmatrix} \dot{\underline{w}}_r \\ \dot{\underline{\theta}}_r \\ \dot{\underline{q}}_r \end{bmatrix}^T \begin{bmatrix} m_r\underline{1} & -\underline{c}_r^{\times} & \underline{P}_r \\ \underline{c}_r^{\times} & \underline{J}_r & \underline{H}_r \\ \underline{P}_r^T & \underline{H}_r^T & \underline{\mathcal{M}}_{rr} \end{bmatrix} \begin{bmatrix} \dot{\underline{w}}_r \\ \dot{\underline{\theta}}_r \\ \dot{\underline{q}}_r \end{bmatrix} \tag{3.39}$$

where

$$\underline{P}_r \triangleq \int \underline{\Psi}_r dm_r \tag{3.40}$$

$$\underline{H}_r \triangleq \int \underline{r}_r^{\times}\underline{\Psi}_r dm_r \tag{3.41}$$

$$\underline{\mathcal{M}}_{rr} \triangleq \int \underline{\Psi}_r^T\underline{\Psi}_r dm_r \tag{3.42}$$

3.5 Total Kinetic Energy

The coordinates used so far in the analysis are collected into a (total) coordinate vector $\underline{q}_T$:

$$\underline{q}_T \triangleq \text{col}\{\underline{w}_b,\underline{\theta}_b,\underline{w}_{a_1},\underline{\theta}_{a_1},\underline{q}_{a_1},\underline{w}_{a_2},\underline{\theta}_{a_2},\underline{q}_{a_2},$$

$$\underline{w}_{t_1},\underline{\theta}_{t_1},\underline{\delta}_1,\underline{\alpha}_1,\underline{q}_{i_1},\underline{w}_{t_2},\underline{\theta}_{t_2},\underline{\delta}_2,\underline{\alpha}_2,\underline{q}_{i_2},$$

$$\underline{w}_{r_1},\underline{\theta}_{r_1},\underline{q}_{r_1},\underline{w}_{r_2},\underline{\theta}_{r_2},\underline{q}_{r_2}\} \tag{3.43}$$

Associated with $\underline{q}_T$ is a (total) mass matrix $\underline{M}_T$, as inferred from the total kinetic energy T:

$$T = \tfrac{1}{2}\underline{\dot{q}}_T^T\underline{M}_T\underline{\dot{q}}_T \tag{3.44}$$

where

$$T = T_b + \sum_{n=1}^{2}(T_{a_n} + T_{t_n} + T_{r_n}) \tag{3.45}$$

Based on (3.4) for T_b, (3.13) for T_{a_n}, (3.19) and (3.36) for T_{t_n}, and (3.39) for T_{r_n}, we conclude that $\underline{M}_T$ is of the following block-diagonal form:

$$\underline{M}_T \triangleq \text{diag}\{\underline{M}_{Tb},\underline{M}_{Ta_1},\underline{M}_{Ta_2},\underline{M}_{Tt_1},\underline{M}_{Tt_2},\underline{M}_{Tr_1},\underline{M}_{Tr_2}\} \tag{3.46}$$

where $\underline{M}_{Tb}$ is the mass matrix in (3.4), $\underline{M}_{Ta_n}$ is the mass matrix in (3.13) (with the extra subscript n added), $\underline{M}_{Tr_n}$ is the mass matrix in (3.39) (with the extra subscript n added), and $\underline{M}_{Tt_n}$ is given by

$$\underline{M}_{Tt_n} \triangleq \begin{bmatrix} m_t\underline{1} & -\underline{c}_t^\times & \underline{P}_\delta & \underline{P}_\alpha & \underline{P}_i \\ \underline{c}_t^\times & \underline{J}_t & \underline{H}_\delta & \underline{H}_\alpha & \underline{H}_i \\ \underline{P}_\delta^T & \underline{H}_\delta^T & \underline{M}_{\delta\delta} & \underline{M}_{\delta\alpha} & \underline{M}_{\delta i} \\ \underline{P}_\alpha^T & \underline{H}_\alpha^T & \underline{M}_{\delta\alpha}^T & \underline{M}_{\alpha\alpha} & \underline{M}_{\alpha i} \\ \underline{P}_i^T & \underline{H}_i^T & \underline{M}_{\delta i}^T & \underline{M}_{\alpha i}^T & \underline{M}_{ii} \end{bmatrix} \tag{3.47}$$

with the extra subscript n added to all the partitions.

However, the coordinates in $\underline{q}_T$, as expressed in (3.43), are not independent. In fact, some of the constraints among them have already been recognized in (3.29) - (3.32). To these we add

$$\underline{w}_a = \underline{C}_{ab}(\underline{w}_b - \underline{r}_{ba}^{\times}\underline{\theta}_b) \tag{3.48}$$

$$\underline{\theta}_a = \underline{C}_{ab}\underline{\theta}_b \tag{3.49}$$

These 'constraints', or 'compatibility conditions' are the symbolic way of uniting the otherwize disparate bodies to form the spacecraft shown in Figs. 2.1 - 2.3. These conditions may be unified into a single matrix equation:

$$\underline{q}_T = \underline{T}\underline{q} \tag{3.50}$$

where $\underline{q}$ contains the *independent* coordinates,

$$\underline{q} \triangleq \mathrm{col}\{\underline{w}_b, \underline{\theta}_b, \underline{q}_{a_1}, \underline{q}_{a_2}, \underline{\delta}_1, \underline{\alpha}_1, \underline{q}_{i_1}, \underline{\delta}_2, \underline{\alpha}_2, \underline{q}_{i_2}, \underline{q}_{r_1}, \underline{q}_{r_2}\} \tag{3.51}$$

and $\underline{T}$ is tabulated in Table 3.1.

The final expression for spacecraft kinetic energy (and the mass matrix implied by that expression) is found by inserting (3.50) into (3.44):

$$T = \tfrac{1}{2}(\underline{T}\dot{\underline{q}})^T \underline{M}_T (\underline{T}\dot{\underline{q}})$$

$$= \tfrac{1}{2}\dot{\underline{q}}^T \underline{M}\dot{\underline{q}} \tag{3.52}$$

where

$$\underline{M} \triangleq \underline{T}^T \underline{M}_T \underline{T} \tag{3.53}$$

The partitioned form of $\underline{M}$ is shown in Table 3.2, and expressions for these partitioned elements are given in Table 3.3. (The reader who sets out to verify Tables 3.1 - 3.3 would be well advised to acquire some *large* pieces of paper!) It is the final expression (3.52) that will figure most directly in the motion equations.

Table 3.1

Partitioning of the Contraction Transformation Matrix, $\underline{T}$

$$
\underline{T} = \left[\begin{array}{cc|c|c|ccc|ccc|c|c}
\underline{1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\hline
\underline{C}_{a_1b} & -\underline{C}_{a_1b}\underline{r}^{x}_{ba_1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{C}_{a_1b} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \underline{1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\hline
\underline{C}_{a_2b} & -\underline{C}_{a_2b}\underline{r}^{x}_{ba_2} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{C}_{a_2b} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \underline{1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\hline
\underline{C}_{t_1b} & -\underline{C}_{t_1b}\underline{r}^{x}_{bt_1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{C}_{t_1b} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\hline
\underline{C}_{t_2b} & -\underline{C}_{t_2b}\underline{r}^{x}_{bt_2} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{C}_{t_2b} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{1} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{1} & \underline{0} & \underline{0} \\
\hline
\underline{C}_{r_1b} & -\underline{C}_{r_1b}\underline{r}^{x}_{br_1} & \underline{0} & \underline{0} & \underline{C}_{rt_1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{C}_{r_1b} & \underline{0} & \underline{0} & \underline{0} & \underline{C}_{rt_1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{1} & \underline{0} \\
\hline
\underline{C}_{r_2b} & -\underline{C}_{r_2b}\underline{r}^{x}_{br_2} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{C}_{rt_2} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{C}_{r_2b} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{C}_{rt_2} & \underline{0} & \underline{0} & \underline{0} \\
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{1}
\end{array}\right]
$$

Table 3.2

Partitioning of the Mass Matrix, $\underline{M}$ $(= \underline{T}^T \underline{M}_T \underline{T})$

$$\underline{M} = \begin{bmatrix}
\underline{M}_{ww} & \underline{M}_{w\theta} & \underline{M}_{wa_1} & \underline{M}_{wa_2} & \underline{M}_{w\delta_1} & \underline{M}_{w\alpha_1} & \underline{M}_{wi_1} & \underline{M}_{w\delta_2} & \underline{M}_{w\alpha_2} & \underline{M}_{wi_2} & \underline{M}_{wr_1} & \underline{M}_{wr_2} \\
 & \underline{M}_{\theta\theta} & \underline{M}_{\theta a_1} & \underline{M}_{\theta a_2} & \underline{M}_{\theta\delta_1} & \underline{M}_{\theta\alpha_1} & \underline{M}_{\theta i_1} & \underline{M}_{\theta\delta_2} & \underline{M}_{\theta\alpha_2} & \underline{M}_{\theta i_2} & \underline{M}_{\theta r_1} & \underline{M}_{\theta r_2} \\
 & & \underline{M}_{aa_1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
 & & & \underline{M}_{aa_2} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
 & & & & \underline{M}_{\delta\delta_1} & \underline{M}_{\delta\alpha_1} & \underline{M}_{\delta i_1} & \underline{0} & \underline{0} & \underline{0} & \underline{M}_{\delta r_1} & \underline{0} \\
 & & & & & \underline{M}_{\alpha\alpha_1} & \underline{M}_{\alpha i_1} & \underline{0} & \underline{0} & \underline{0} & \underline{M}_{\alpha r_1} & \underline{0} \\
 & & & & & & \underline{M}_{ii_1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
 & & & & & & & \underline{M}_{\delta\delta_2} & \underline{M}_{\delta\alpha_2} & \underline{M}_{\delta i_2} & \underline{0} & \underline{M}_{\delta r_2} \\
 & & \text{symmetric} & & & & & & \underline{M}_{\alpha\alpha_2} & \underline{M}_{\alpha i_2} & \underline{0} & \underline{M}_{\alpha r_2} \\
 & & & & & & & & & \underline{M}_{ii_2} & \underline{0} & \underline{0} \\
 & & & & & & & & & & \underline{M}_{rr_1} & \underline{0} \\
 & & & & & & & & & & & \underline{M}_{rr_2}
\end{bmatrix}$$

Table 3.3

Expressions for Partitioned Elements in $\underline{M}$

(see Table 3.2)

$$\underline{M}_{ww} = m\underline{1}$$

$$\underline{M}_{w\theta} = -\underline{c}^{x}$$

$$\underline{M}_{w\delta_n} = \underline{C}_{bt_n}(\underline{P}_{\delta_n} + m_{r_n}\underline{1}) \qquad (n = 1,2)$$

$$\underline{M}_{w\alpha_n} = \underline{C}_{bt_n}(\underline{P}_{\alpha_n} - \underline{C}_{tr_n}\underline{c}^{x}_{rn}\underline{C}_{rt_n}) \qquad (n = 1,2)$$

$$\underline{M}_{wa_n} = \underline{C}_{ba_n}\underline{P}_{a_n} \qquad (n = 1,2)$$

$$\underline{M}_{wi_n} = \underline{C}_{bt_n}\underline{P}_{i_n} \qquad (n = 1,2)$$

$$\underline{M}_{wr_n} = \underline{C}_{br_n}\underline{P}_{r_n} \qquad (n = 1,2)$$

$$\underline{M}_{\theta\theta} = \underline{J}$$

$$\underline{M}_{\theta a_n} = \underline{C}_{ba_n}\underline{H}_{a_n} + r^{x}_{ba_n}\underline{C}_{ba_n}\underline{P}_{a_n} \qquad (n = 1,2)$$

$$\underline{M}_{\theta\delta_n} = \underline{C}_{bt_n}\underline{H}_{\delta_n} + \underline{r}^{x}_{bt_n}\underline{C}_{bt_n}\underline{P}_{\delta_n} + \underline{C}_{br_n}\underline{c}^{x}_{r_n}\underline{C}_{rt_n} + m_{r_n}\underline{r}^{x}_{br_n}\underline{C}_{bt_n} \qquad (n = 1,2)$$

$$\underline{M}_{\theta\alpha_n} = \underline{C}_{bt_n}\underline{H}_{\alpha_n} + \underline{r}^{x}_{bt_n}\underline{C}_{bt_n}\underline{P}_{\alpha_n} + \underline{C}_{br_n}\underline{J}_{r_n}\underline{C}_{rt_n} - \underline{r}^{x}_{br_n}\underline{C}_{br_n}\underline{c}^{x}_{r_n}\underline{C}_{rt_n} \qquad (n = 1,2)$$

$$\underline{M}_{\theta i_n} = \underline{C}_{bt_n}\underline{H}_{i_n} + \underline{r}^{x}_{bt_n}\underline{C}_{bt_n}\underline{P}_{i_n} \qquad (n = 1,2)$$

$$\underline{M}_{\theta r_n} = \underline{C}_{br_n}\underline{H}_{r_n} + \underline{r}^{x}_{br_n}\underline{C}_{br_n}\underline{P}_{r_n} \qquad (n = 1,2)$$

$$\underline{M}_{\delta\delta_n} = \underline{M}_{\delta\delta_n} + m_{r_n}\underline{1}$$

$$\underline{M}_{\delta\alpha_n} = \underline{M}_{\delta\alpha_n} - \underline{C}_{tr_n}\underline{c}^{x}_{r_n}\underline{C}_{rt_n}$$

$$\underline{M}_{\delta r_n} = \underline{C}_{tr_n}\underline{P}_{r_n}$$

$$\underline{M}_{\alpha\alpha_n} = \underline{M}_{\alpha\alpha_n} + \underline{C}_{tr_n}\underline{J}_{r_n}\underline{C}_{rt_n}$$

$$\underline{M}_{\alpha r_n} = \underline{C}_{tr_n}\underline{H}_{r_n}$$

4. POTENTIAL ENERGY

The second type of energy needed to form the Lagrangian utilized by Hamilton's principle is the potential energy (associated with conservative force fields). The only type of potential energy considered here is the potential energy corresponding to elastic stored strain energy. Any other (external) force fields will be included in the virtual work terms expounded upon in Section 5.

4.1 Strain Energy Stored in $\mathcal{R}_b$

Because the bus is assumed to be rigid, the elastic strain energy stored in it is zero:

$$V_b = 0 \tag{4.1}$$

4.2 Strain Energy Stored in $\mathcal{E}_{a_n}$

By looking at the coordinates associated with the solar arrays $\mathcal{E}_{a_n}$, it is evident that only $\underline{q}_{a_n}$ involves elastic deformations. Therefore

$$V_{a_n} = \tfrac{1}{2}\underline{q}_{a_n}^T \underline{\mathcal{K}}_{aa_n} \underline{q}_{a_n} \qquad (n = 1,2) \tag{4.2}$$

(The question of how to calculate $\underline{\mathcal{K}}_{aa_n}$, or what the coordinates $\underline{q}_{a_n}$ may in fact be, is not, of course, the subject of this report.)

4.3 Strain Energy Stored in $\mathcal{E}_{t_n}$

The coordinates involving elastic deformations in $\mathcal{E}_{t_n}$ are contained in $\underline{q}_{t_n}$. However, $\underline{q}_{t_n}$ is, in turn, subdivided into $\{\underline{\delta}_n, \underline{\alpha}_n, \underline{q}_{i_n}\}$, as shown in (3.33). Accordingly, we can form a partitioned stiffness matrix $\underline{\mathcal{K}}_{tt_n}$ that plays the same role for $\mathcal{E}_{t_n}$ as does $\underline{\mathcal{K}}_{aa_n}$ for $\mathcal{E}_{a_n}$ in (4.2):

$$V_t = \tfrac{1}{2}\begin{bmatrix} \underline{\delta} \\ \underline{\alpha} \\ \underline{q}_i \end{bmatrix}^T \begin{bmatrix} \underline{K}_{\delta\delta} & \underline{K}_{\delta\alpha} & \underline{K}_{\delta i} \\ \underline{K}^T_{\delta\alpha} & \underline{K}_{\alpha\alpha} & \underline{K}_{\alpha i} \\ \underline{K}^T_{\delta i} & \underline{K}^T_{\alpha i} & \underline{K}_{ii} \end{bmatrix} \begin{bmatrix} \underline{\delta} \\ \underline{\alpha} \\ \underline{q}_i \end{bmatrix} \qquad (4.3)$$

(subscript n implied).

4.4 Strain Energy Stored in E_{r_n}

Finally, for E_{r_n}, we have

$$V_{r_n} = \tfrac{1}{2}\underline{q}^T_{r_n}\underline{K}_{rr_n}\underline{q}_{r_n} \qquad (4.4)$$

as the expression for stored elastic strain energy.

4.5 Total Potential Energy

With the definition (3.43) for the aggregate $\underline{q}_T$ of all coordinates for the individual constituent bodies, and the definition (3.51) for the final set of coordinates $\underline{q}$ associated with the system consisting of these bodies linked to form a spacecraft, it is possible to show that the total potential energy V, defined by

$$V = V_b + \sum_{n=1}^{2}(V_{a_n} + V_{t_r} + V_{r_n}) \qquad (4.5)$$

is given by

$$V = \tfrac{1}{2}\underline{q}^T\underline{Kq} \qquad (4.6)$$

where $\underline{K}$ is shown in partitioned form in Table 4.1.

5. VIRTUAL WORK

The most basic expression for virtual work (needed for Hamilton's

Table 4.1

Partitioning of the Stiffness Matrix, $\underline{K}$

$$
\underline{K} = \begin{bmatrix}
\underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
 & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
 & & \underline{K}_{aa_1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
 & & & \underline{K}_{aa_2} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
 & & & & \underline{K}_{\delta\delta_1} & \underline{K}_{\delta\alpha_1} & \underline{K}_{\delta i_1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
 & & & & & \underline{K}_{\alpha\alpha_1} & \underline{K}_{\alpha i_1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
 & & & & & & \underline{K}_{ii_1} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{0} \\
 & \text{symmetric} & & & & & & \underline{K}_{\delta\delta_2} & \underline{K}_{\delta\alpha_2} & \underline{K}_{\delta i_2} & \underline{0} & \underline{0} \\
 & & & & & & & & \underline{K}_{\alpha\alpha_2} & \underline{K}_{\alpha i_2} & \underline{0} & \underline{0} \\
 & & & & & & & & & \underline{K}_{ii_2} & \underline{0} & \underline{0} \\
 & & & & & & & & & & \underline{K}_{rr_1} & \underline{0} \\
 & & & & & & & & & & & \underline{K}_{rr_2}
\end{bmatrix}
$$

principle) is

$$\delta W \triangleq \int \hat{\underset{\rightarrow}{f}}(\underset{\rightarrow}{r},t)^T \delta\underset{\rightarrow}{d}(\underset{\rightarrow}{r},t)dV \qquad (5.1)$$

It is the inner product between the applied force $\hat{\underset{\rightarrow}{f}}$ and the virtual displacement $\delta\underset{\rightarrow}{d}$, integrated over the spacecraft.

5.1 Virtual Work for R_b

Combining (2.2a), (2.3a), and (3.1), it can be seen that

$$\delta W_b = \int \hat{\underline{f}}_b(\underline{r}_b,t)^T[\delta\underline{w}_b(t) - \underline{r}_b^X\delta\underline{\theta}_b(t)]dV_b \qquad (5.2)$$

$$= \underline{f}_b^T\delta\underline{w}_b + \underline{g}_b^T\delta\underline{\theta}_b \qquad (5.3)$$

This expression is what we should have expected.

5.2 Virtual Work for E_{a_n}

For an elastic body such as E_a, on the other hand, we have

$$\delta W_a = \int \hat{\underline{f}}_a(\underline{r}_a,t)^T[\delta\underline{w}_a(t) - \underline{r}_a^X\delta\underline{\theta}_a(t) + \underline{\Psi}_a(\underline{r}_a)\delta\underline{q}_a(t)]dV_a \qquad (5.4)$$

$$= \underline{f}^T\delta\underline{w}_a + \underline{g}_a^T\delta\underline{\theta}_a + \underline{\delta}_a^T\delta\underline{q}_a \qquad (5.5)$$

where

$$\underline{\delta}_a \triangleq \int \underline{\Psi}_a^T(\underline{r}_a)\hat{\underline{f}}_a(\underline{r}_a,t)dV_a \qquad (5.6)$$

and (2.2b) and (2.3b) have been used. The above expressions have an extra subscript n understood.

5.3 Virtual Work for E_{tn}

Next, consider one of the elastic towers, E_t. Again using the definition (5.1) we have

$$\delta W_t = \int \underline{\hat{f}}_t(\underline{r}_t,t)^T[\delta\underline{w}_t(t) - \underline{r}_t^{\times}\delta\underline{\theta}_t(t) + \underline{\Psi}_t(\underline{r}_t)\delta\underline{q}_t(t)]dV_t \tag{5.7}$$

$$= \underline{f}^T\delta\underline{w}_t + \underline{g}_t^T\delta\underline{\theta}_t + \underline{\mathit{f}}_f^T\delta\underline{q}_t \tag{5.8}$$

where

$$\underline{\mathit{f}}_t \triangleq \int \underline{\Psi}^T(\underline{r}_t)\underline{\hat{f}}_t(\underline{r}_t,t)dV_t \tag{5.9}$$

However, on account of the partitioning we have used for $\underline{q}_t$--see (3.33)--we must partition $\underline{\mathit{f}}_t$ likewise:

$$\underline{\mathit{f}}_t \triangleq \mathrm{col}\{\underline{\mathit{f}}_\delta, \underline{\mathit{f}}_\alpha, \underline{\mathit{f}}_i\} \tag{5.10}$$

whereupon (5.8) becomes

$$\delta W_t = \underline{f}_t^T\delta\underline{w}_t + \underline{g}_t^T\delta\underline{\theta}_t + \underline{\mathit{f}}_\delta^T\delta\underline{\delta} + \underline{\mathit{f}}_\alpha^T\delta\underline{\alpha} + \underline{\mathit{f}}_i^T\delta\underline{q}_i \tag{5.11}$$

As usual, this latter expression holds true for both towers.

5.4 Virtual Work for E_{r_n}

In an identical manner, the virtual work for E_r is

$$\delta W_r = \underline{f}_r^T\delta\underline{w}_r + \underline{g}_r^T\delta\underline{\theta}_r + \underline{\mathit{f}}_r^T\delta\underline{q}_r \tag{5.12}$$

where

$$\underline{\mathit{f}}_r \triangleq \int \underline{\Psi}_r^T(\underline{r}_r)\underline{\hat{f}}_r(\underline{r}_r,t)dV_r \tag{5.13}$$

and these relations hold for both reflectors $(n = 1,2)$.

5.5 Total Virtual Work

The total virtual work is found from

$$\delta W = \delta W_b + \sum_{n=1}^{2} (\delta W_{a_n} + \delta W_{t_n} + \delta W_{r_n}) \tag{5.14}$$

which can be succinctly expressed as

$$\delta W = \underline{f}_T^T \delta \underline{q}_T \tag{5.15}$$

where $\underline{q}_T$ is the 'total' coordinate vector introduced earlier in (3.43), and $\underline{f}_T$ is the associated column of generalized forces:

$$\begin{aligned} \underline{f}_T = \mathrm{col}\{&\underline{f}_b, \underline{g}_b, \underline{f}_{a_1}, \underline{g}_{a_1}, \underline{\mathcal{f}}_{a_1}, \underline{f}_{a_2}, \underline{g}_{a_2}, \underline{\mathcal{f}}_{a_2}, \\ &\underline{f}_{t_1}, \underline{g}_{t_1}, \underline{\mathcal{f}}_{\delta_1}, \underline{\mathcal{f}}_{\alpha_1}, \underline{\mathcal{f}}_{i_1}, \underline{f}_{t_2}, \underline{g}_{t_2}, \underline{\mathcal{f}}_{\delta_2}, \underline{\mathcal{f}}_{\alpha_2}, \underline{\mathcal{f}}_{i_2}, \\ &\underline{f}_{r_1}, \underline{g}_{r_1}, \underline{\mathcal{f}}_{r_1}, \underline{f}_{r_2}, \underline{g}_{r_2}, \underline{\mathcal{f}}_{r_2}\} \end{aligned} \tag{5.16}$$

However, in the set of coordinates $\underline{q}_T$, the substructures have not been united to form the spacecraft. The compatibility conditions needed to bring about this unison are summarized in the single contracting transformation

$$\underline{q}_T = \underline{T}\underline{q} \tag{5.17}$$

as noted in (3.50). The (partitioned) elements of $\underline{T}$ are tabulated in Table 3.1. Note that the coordinates in $\underline{q}$, as listed in (3.51), are the coordinates for the spacecraft with the substructures all mutually 'attached.' An alternate expression for δW, and a more desirable one, is found by combining (5.15) and (5.17) to form

$$\delta W = \underline{\mathcal{f}}^T \delta \underline{q} \tag{5.18}$$

where

$$\underline{\mathcal{f}} \triangleq \underline{T}^T \underline{f}_T \tag{5.19}$$

The elements of $\underline{\mathcal{f}}$ are given (in partitioned form) in Table 5.1. Note that the first two (partitioned) elements are the total force $\underline{f}$, defined by

Table 5.1

Partitioning of the Generalized Force Matrix, $\underline{\mathscr{f}}$

$$\underline{\mathscr{f}} = \left[\begin{array}{c} \underline{f} \\ \\ \underline{g} \\ \hline \underline{\mathscr{f}}_{a_1} \\ \hline \underline{\mathscr{f}}_{a_2} \\ \hline \underline{\mathscr{f}}_{\delta_1} + \underline{C}_{t_1 r}\underline{f}_{r_1} \\ \underline{\mathscr{f}}_{\alpha_1} + \underline{C}_{t_1 r}\underline{g}_{r_1} \\ \underline{\mathscr{f}}_{i_1} \\ \hline \underline{\mathscr{f}}_{\delta_2} + \underline{C}_{t_2 r}\underline{f}_{r_2} \\ \underline{\mathscr{f}}_{\alpha_2} + \underline{C}_{t_2 r}\underline{g}_{r_2} \\ \underline{\mathscr{f}}_{i_2} \\ \hline \underline{\mathscr{f}}_{r_1} \\ \hline \underline{\mathscr{f}}_{r_2} \end{array}\right] \begin{array}{l} \\ V \\ \\ \hline E_{a_1} \\ \hline E_{a_2} \\ \hline \\ E_{t_1} + E_{r_1} \\ \\ \hline \\ E_{t_2} + E_{r_2} \\ \\ \hline E_{r_1} \\ \hline E_{r_2} \end{array}$$

Vehicle $V = R_b + \sum_{n=1}^{2} (E_{a_n} + E_{t_n} + E_{r_n})$

$$\underline{f} = \underline{f}_b + \sum_{n=1}^{2} (\underline{C}_{ba_n} \underline{f}_{a_n} + \underline{C}_{bt_n} \underline{f}_{t_n} + \underline{C}_{br_n} \underline{f}_{r_n}) \tag{5.20}$$

and the total torque $\underline{g}$, about O_b, defined by

$$\begin{aligned} \underline{g} = \underline{g}_b + \sum_{n=1}^{2} [& (\underline{C}_{ba_n} \underline{g}_{a_n} + \underline{r}_{ba_n}^{\times} \underline{C}_{ba_n} \underline{f}_{a_n}) \\ & + (\underline{C}_{bt_n} \underline{g}_{t_n} + \underline{r}_{bt_n}^{\times} \underline{C}_{bt_n} \underline{f}_{t_n}) \\ & + (\underline{C}_{br_n} \underline{g}_{r_n} + \underline{r}_{br_n}^{\times} \underline{C}_{br_n} \underline{f}_{r_n})] \end{aligned} \tag{5.21}$$

Now that the kinetic energy (Section 3), potential energy (Section 4), and virtual work (this section) have been fully formulated, it is a straightforward task to write out the equations that govern the motion of the DEMSAT spacecraft structure.

6. MOTION EQUATIONS

The developments of the preceeding sections have been aimed toward the objective of obtaining motion equations and, more specifically, toward the use of Hamilton's principle in deriving those equations. The point has now arrived where all the required preliminaries have been completed.

Hamilton's principle (in its 'extended' form) is

$$\delta \int_{t_1}^{t_2} L \, dt = \int_{t_1}^{t_2} \delta W \, dt \tag{6.1}$$

where the Lagrangian L is defined to be

$$L \triangleq T - V \tag{6.2}$$

In the present case, $L = L(\underline{q}, \dot{\underline{q}})$. Now, the left side of (6.1) can be shown using the calculus of variations to be

$$\int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{q}}} \right) - \frac{\partial L}{\partial \underline{q}} \right]^T \delta\underline{q}\, dt$$

and the right side is

$$\int_{t_1}^{t_2} \underline{f}^T \delta\underline{q}\, dt$$

as witness (5.18). It follows from the fundamental lemma of variational calculus that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{q}}} \right) - \frac{\partial L}{\partial \underline{q}} = \underline{f} \tag{6.3}$$

These are Lagrange's equations of motion for DEMSAT.

We now call upon (3.52) and (4.6) to indicate that

$$T = \tfrac{1}{2}\dot{\underline{q}}^T \underline{M}\dot{\underline{q}} \quad ; \qquad V = \tfrac{1}{2}\underline{q}^T \underline{K}\underline{q} \tag{6.4}$$

whence, from (6.2) and (6.3),

$$\underline{M}\ddot{\underline{q}} + \underline{K}\underline{q} = \underline{f} \tag{6.5}$$

This represents, finally, the equations of motion for the present system. The mass matrix $\underline{M}$ is tabulated in Tables 3.2 and 3.3; the stiffness matrix $\underline{K}$ is recorded in Table 4.1; and the generalized forces (the elements of $\underline{f}$) are listed in Table 5.1.

7. SUBSTRUCTURAL MODELS IN MODAL COORDINATES

It frequently happens that the substructural model for a particular elastic body E is given, not in terms of 'physical coordinates,' but in terms of 'modal coordinates.' These modal coordinates are associated with the natural vibration mode shapes for the structure in question. *We shall assume that these vibration modes appertain to the structure when it is constrained not to translate at some point* O, *nor to rotate about* O. As an example of this assumption, and with reference to Fig. 2.1, we shall assume that the vibration modes for E_{r_2} are those modes that correspond to E_{r_2}

being constrained at O_{r2}, both translationally and rotationally.

All the foregoing analysis in this report remains valid if some of the coordinates in the formulation are modal coordinates.

To be more specific about this modal interpretation, let us consider what is probably a very likely situation for DEMSAT:

(i) the reflectors are specified using modal information;

(ii) the solar arrays are specified using modal information;

(iii) the towers are modeled using direct physical coordinates.

Then, as also described in Section 5 of Ref. 1, we can represent these assumptions by the following notational adaptations.

For the solar arrays, the following replacements are made (n = 1,2)

$$\underline{q}_a(t) \rightarrow \underline{\eta}_a(t) \quad ; \quad \underline{\eta}_a \triangleq \underline{E}_a^{-1} \underline{q}_a$$

$$\underline{\wp}_a(t) \rightarrow \underline{\gamma}_a(t) \quad ; \quad \underline{\gamma}_a \triangleq \underline{E}_a^{T} \underline{\wp}_a$$

$$\underline{M}_{aa} \rightarrow \underline{1} \quad ; \quad \text{unit matrix of appropriate dimension}$$

$$\underline{K}_{aa} \rightarrow \underline{\Omega}_a^2 \quad ; \quad \underline{\Omega}_a \triangleq \text{diag}\{\omega_{a1}, \omega_{a2}, \ldots\}$$

$$\underline{P}_a \rightarrow \underline{P}_{a\eta} \quad ; \quad \underline{P}_{a\eta} \triangleq \underline{P}_a \underline{E}_a$$

$$\underline{H}_a \rightarrow \underline{H}_{a\eta} \quad ; \quad \underline{H}_{a\eta} \triangleq \underline{H}_a \underline{E}_a$$

The columns of $\underline{E}_a$ are the modal eigenvectors for the solar array $\mathcal{E}_a$ (constrained at and about O_a) and $\{\omega_{a1}, \omega_{a2}, \ldots\}$ are its (constrained) natural frequencies.

The identical procedure holds also for the reflectors, $\mathcal{E}_r$.

8. CONCLUDING REMARKS

In this report it is assumed that one is in possession of an adequate set of structural data for each of the DEMSAT substructures. This data may either be in 'physical' coordinates (i.e., mass and stiffness matrices known), or in 'modal' coordinates (frequencies and modal momenta known). In either case, the formulation presented can be used to construct an overall model for a DEMSAT-like spacecraft.

REFERENCE

Hughes, P. C., "Structural Dynamics Modeling Plan for Control System Design and Evaluation," Dynacon Report MSAT-3, December 1981.

APPENDIX A

A Comparison of Formulations

This report uses Hamilton's principle and Lagrange's equations to derive the motion equations for the 'DEMSAT-like' flexible spacecraft shown in Fig. 2.1. This generic spacecraft is similar in layout to the one analysed on Ref. 1 and reproduced here as Fig. 4.1. We shall call the spacecraft of Ref. 1 "OMSAT" because it was motivated by certain configurations suitable for an <u>o</u>perational <u>m</u>obile communications <u>sat</u>ellite.

In comparing OMSAT (Fig. A.1) to DEMSAT (Fig. 2.1), the following points are noted:

(i) OMSAT has one solar array; DEMSAT has two.

(ii) OMSAT has one tower-reflector assembly; DEMSAT has two.

(iii) OMSAT has gimbal angles at the reflector hub; DEMSAT has none.

Based on these observations, a congruence between OMSAT and DEMSAT can be established by taking the following steps:

(i) strip DEMSAT of E_{a_2} (then $E_{a_1} \rightarrow E_a$);

(ii) strip DEMSAT of E_{t_2} (then $E_{t_1} \rightarrow E_t$);

(iii) strip DEMSAT of E_{r_2} (then $E_{r_1} \rightarrow E_r$);

(iv) strip OMSAT of gimbals.

The mathematical consequences of these strippings are as follows:

(i) Eliminate the elements associated with E_{a_2}, E_{t_2} and E_{r_2} from $\underline{M}$, $\underline{K}$, $\underline{f}$ and $\underline{q}$, the latter being tabulated respectively in Table 3.2, Table 4.1, Table 5.1, and Eq.(3.41).

(ii) Eliminate the elements associated with the gimbal angles $\underline{\beta}$ from $\underline{M}$, $\underline{K}$, $\underline{f}$ and $\underline{q}$ in Ref. 1, the latter being tabulated respectively in Eq. (4.12), Eq. (4.13), Eq. (4.16) and Eq. (4.11) (*all these equation references are to Ref. 1*).

It will be seen that the two system of motion equations are now identical.

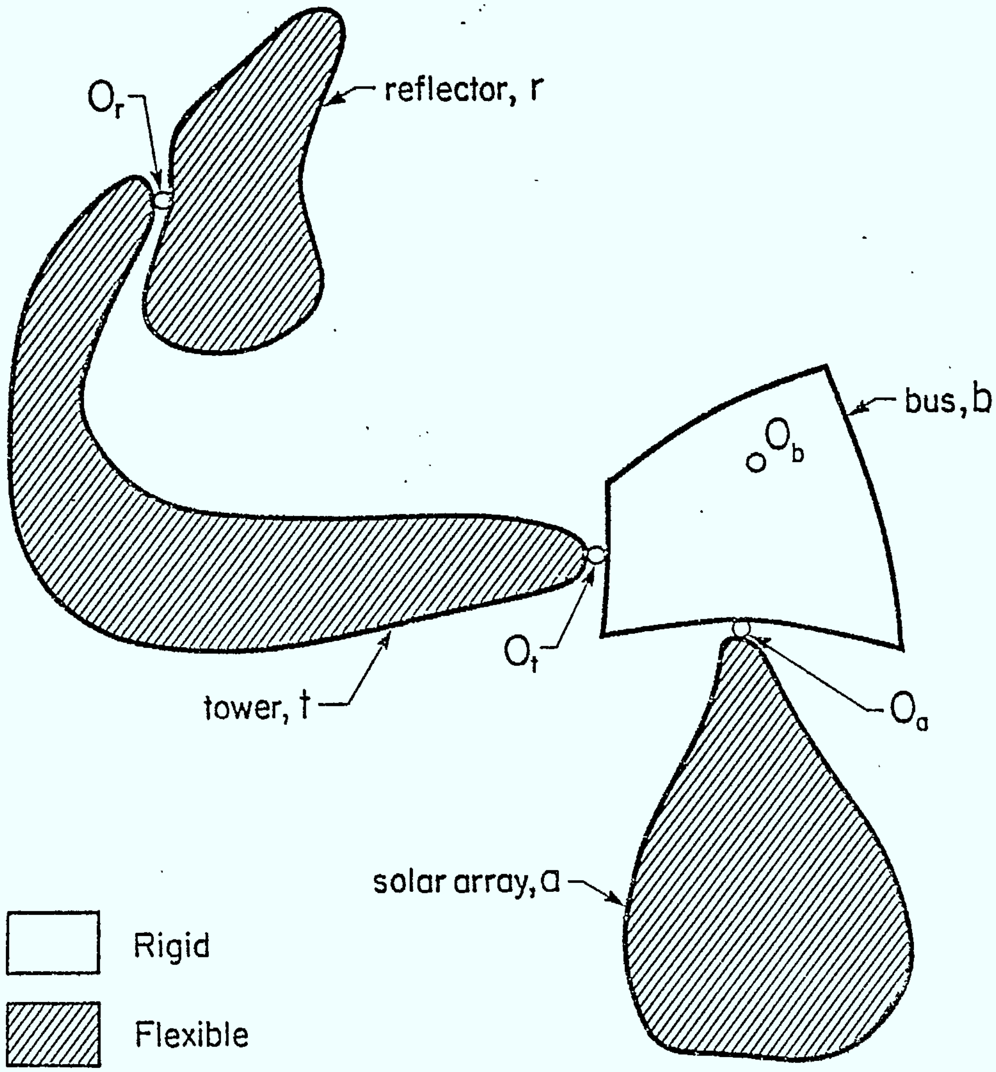

Fig. A.1: Morphology of "OMSAT" Satellite Analyzed in Ref. 1. (cf. Fig. 2.1.)

Since one of the two was derived using vectorial mechanics, and the other using Lagrange's equations, this comparison provides a major check on the correctness of both procedures. It should be noted in particular that in the present report no 'interbody' forces and torques were introduced. Therefore, these interbody interactions did not subsequently have to be eliminated. (This is a well-known advantage of the Lagrangian-Hamiltonian procedure.)