ACCELERATION FEEDBACK
IN THE CONTROL OF
FLEXIBLE STRUCTURES

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SUMMARY

The use of accelerometers as sensing elements for the control of large flexible space structures is explored. It is shown that, in addition to ordinary measurement error, the forces and torques on the vehicle produce additional direct errors in accelerometer interpretation. The portion of this additional error due to known forces and torques can be eliminated by feedforward within the control system. Moreover, because the interpretation of accelerometer outputs in terms of displacement and displacement-rate requires the structural math model itself, a still further source of control system error arises: parameter errors in the structural dynamics model used by the control system designer. This is examined in detail in this report for four types of control system design: simple (static) state feedback, simple (static) measurement feedback, state feedback based on a state vector estimated by a full-order state estimator, and state feedback based on a state vector estimated by a reduced-order (in fact, minimal order) state estimator. With these considerations in mind, and the theory in this report for guidance, additional numerical and simulation work can show quantitatively what the benefits and limitations of accelerometers are in specific situations.

PREFACE

Acknowledgments

The author gratefully acknowledges the encouragement and helpful advice of A. H. Reynaud of the Communications Research Centre, who acted as the Scientific Authority for this contract. Thanks are also due Mrs. J. Hughes for typing this report.

Proprietary Rights

Dynacon Enterprises Ltd. does not claim "proprietary rights" to the material in this report. Indeed, the hope is that the analyses, results, ideas and opinions in this report will be useful to others. In this event, a reference to this report would be appreciated.

Units and Spelling

This report uses S.I. units and North American spelling.

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1. INTRODUCTION

The measurement of acceleration is one of the most natural measurements — and one of the most useful — that can be made in the field of dynamics and control. There are (at least) two possible ways to process an acceleration measurement:

- integrate with respect to time, thus producing a measurement of velocity
- use a set of motion equations to interpret the acceleration measurement as an indirect measurement of position and velocity.

It is primarily the second of these interpretations that will concern us in this report.

It should be noted that all the kinematical quantities referred to above are *absolute* quantities; that is, the displacement, velocity, and acceleration are measured with respect to an *inertial* reference frame. The designation "absolute" is to distinguish these measurements from *relative* displacement, velocity, and acceleration, in which the motion one point of the system is measured with respect to another (not inertially fixed) point of the system.

1.1 Integration of Accelerometer Signal

In the first of the above two interpretations, when acceleration is integrated to produce velocity, the main point to note is that initial errors and spurious signals are also integrated. And, just as high-frequency noise is anathema to a differentiation process, so, too, low-frequency errors (in the limit, 'DC' errors) are the bane of an integration process. Therefore, whenever integration of accelerometer output is used to infer velocity, periodic up-dating (correction) of the reference velocity is required.

This argument is made symbollically in Fig. 1.1. Let the (ideal) accelerometer outputs be $\underline{a}(t)$. If the coordinates whose accelerations are being measured are denoted temporarily by $\underline{q}(t)$, then

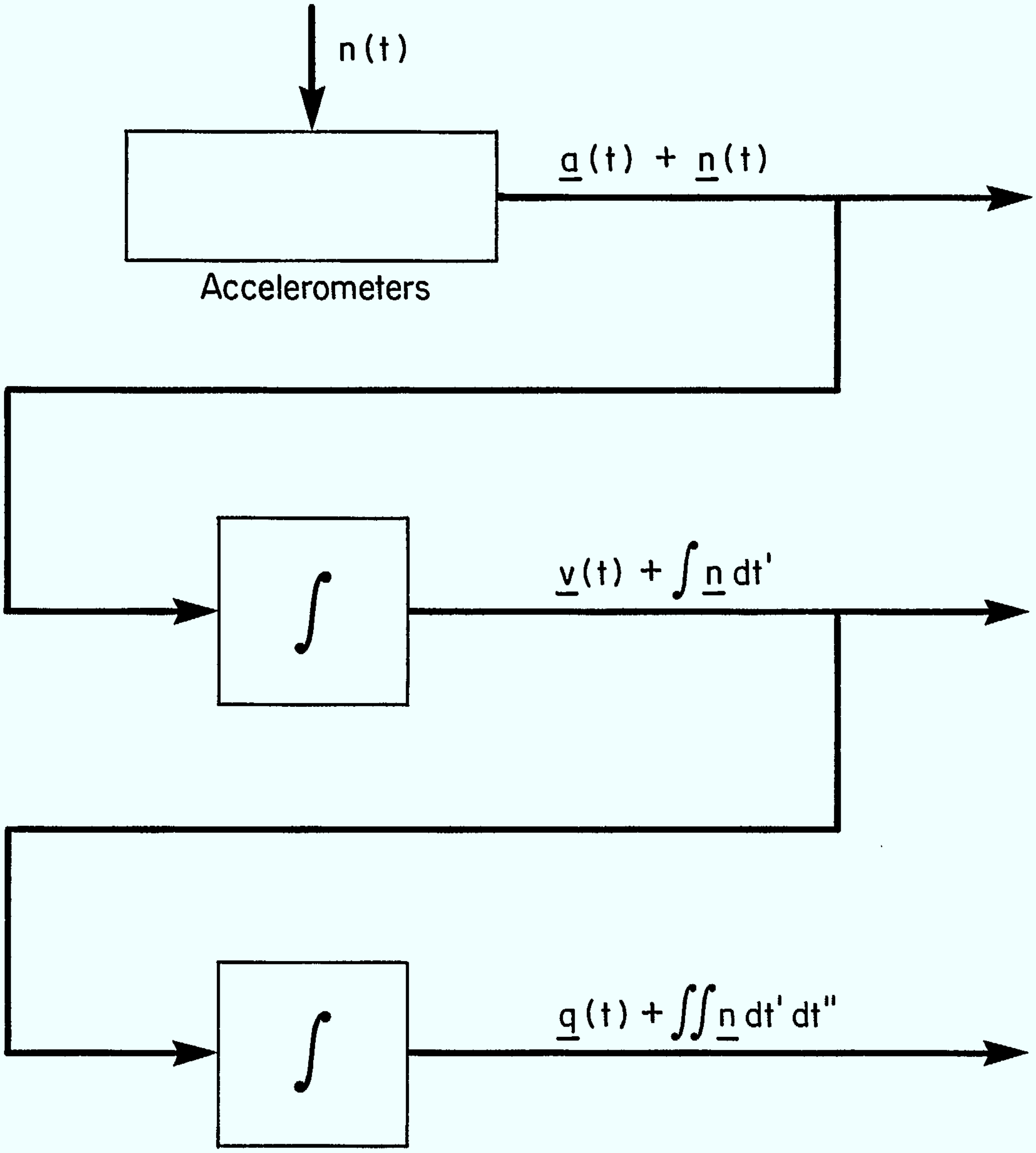


Fig. 1.1: Integration of Accelerometer Signals to Get Velocity and Position

$$\underline{a}(t) = \dot{\underline{v}}(t) = \ddot{\underline{q}}(t) \tag{1.1}$$

where the velocities $\underline{v}(t) = \dot{\underline{q}}(t)$. Thus, by direct integration of the accelerometer measurements, one can in principle produce the velocities $\dot{\underline{q}}(t)$, and the displacements, $\underline{q}(t)$, both of which are 'state variables' in the dynamical modeling of the system. (Note that acceleration itself is *not* a state variable.)

The problem of course, is that an accelerometer does not measure acceleration exactly. We denote the error (or 'noise') by $\underline{n}(t)$. Thus, as shown in Fig. 1.1, the outputs of the accelerometers are contained in the output vector $\underline{a}(t) + \underline{n}(t)$. Let $\underline{n}(t)$ be subdivided into two terms

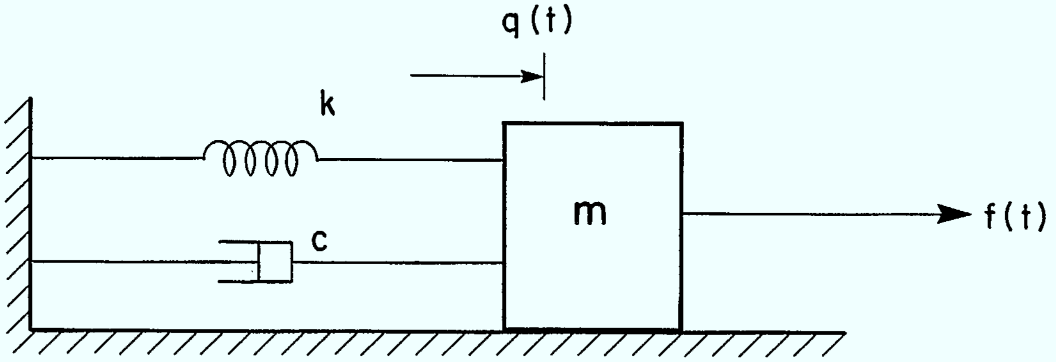
$$\underline{n}(t) = \underline{n}_{av} + \underline{n}_{\Delta}(t) \tag{1.2}$$

where the long-term integrated effect of $\underline{n}_{\Delta}(t)$ vanishes. Then the output of the first stage of integration in Fig. 1.1, after a long period of time is approximately $\dot{\underline{q}}(t) + \underline{n}_{av}t$. Clearly there is no limit to the contamination introduced by the second term.

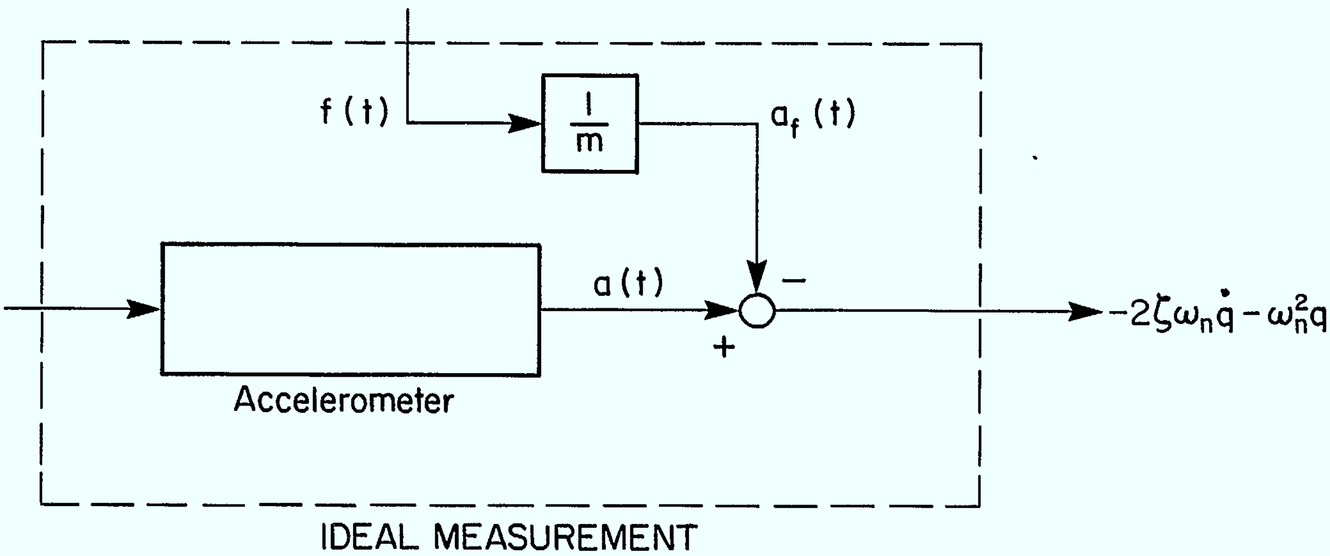
Even worse is the problem that arises if one attempts to integrate once again to produce a measurement of displacement by integrating a second time. The output of the second stage in Fig. 1.1 after a long time is $\underline{q}(t) + \frac{1}{2}\underline{n}_{av}t^2$, so that the integrated error builds up even faster than for $\dot{\underline{q}}$. Thus the use of accelerometers in this fashion to infer position and velocity requires some periodic correction from other instruments. We shall not pursue this subject further here.

1.2 Interpretation via Motion Equation

This report is concerned primarily with acceleration measurement as an indirect measurement of displacement and displacement rate. To illustrate this basic idea, consider Fig. 1.2, which shows the accelerometer output $\ddot{q}$ as measured for one-degree-of-freedom, mass-spring-dashpot system whose displacement is $q(t)$. The parameters are mass, m; damper constant, c; and spring constant, k. The external force on the



(a) Simplest System



(b) Interpretation Using Feedforward

Fig. 1.2: Use of Acceleration Measurement (Ideal Case)

mass is f(t). For present purposes we choose to write the well-known equation of motion

$$m\ddot{q} + c\dot{q} + kq = f(t) \tag{1.3}$$

as

$$a(t) \triangleq \ddot{q}(t) = a_f(t) - 2\zeta\omega_n\dot{q} - \omega_n^2 q \tag{1.4}$$

where

$$\omega_n^2 \triangleq k/m \quad ; \qquad 2\zeta\omega_n \triangleq c/m \tag{1.5}$$

as is conventional, and

$$a_f(t) \triangleq f(t)/m \tag{1.6}$$

Note that a_f has the dimensions of acceleration. Then, if $a_f(t)$ is *fed forward* as shown in Fig. 1.2b, the net output after the summing junction is a simple linear combination of position q(t), and rate $\dot{q}(t)$. This interpretation will become essential in the remainder of this report, which deals with multivariable control using a 'state-space' formulation.

Before leaving the simple example of Fig. 1.2, two idealizations should be noted. It is assumed in Fig. 1.2b that the excitation force f(t) is known. It is further assumed that the system parameters ζ and ω_n are accurately known. In reality, neither of these assumptions is precisely valid. With regard to the force, let

$$f(t) = f_{\checkmark}(t) + f_{?}(t) \tag{1.7}$$

where $f_{\checkmark}(t)$ is the known portion and $f_{?}(t)$ represents the unknown influences. Similarly,

$$a_{\checkmark}(t) \triangleq f_{\checkmark}(t)/m \quad ; \qquad a_{?}(t) \triangleq f_{?}(t)/m \tag{1.8}$$

Furthermore, we denote by $\hat{\zeta}$ and $\hat{\omega}_n$ the *assumed* values for damping factor and natural frequency (the *actual* values are still denoted by ζ and ω_n). Then the real situation is more like that shown in Fig. 1.3.

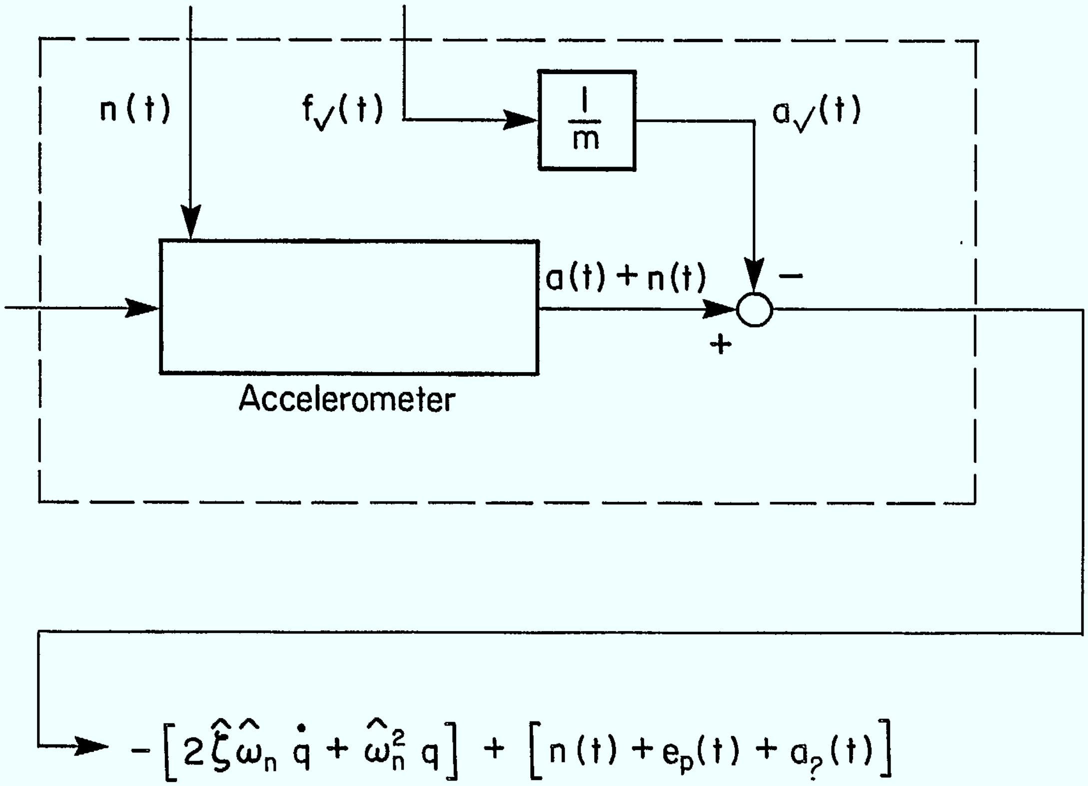


Fig. 1.3: Sources of Error in Actual Implementation

Now, the measurement actually made (after the feedforward correction has been made) is

$$\text{actual measurement} = -2\zeta\omega_n\dot{q} - \omega_n^2 q + n(t) + a_?(t) \tag{1.9}$$

Or, in terms of the measurement assumed, namely,

$$\text{assumed measurement} = -2\hat{\zeta}\hat{\omega}_n\dot{q} - \hat{\omega}_n^2 q + \ldots \tag{1.10}$$

the actual measurement is

$$\begin{matrix}\text{actual}\\ \text{measurement}\end{matrix} = -2\hat{\zeta}\hat{\omega}_n\dot{q} - \hat{\omega}_n^2 q + e_p(t) + n(t) + a_?(t) \tag{1.11}$$

where $e_p(t)$ is the 'error' associated with parameter errors. The latter is given by

$$e_p(t) \doteq -2(\hat{\zeta}\Delta\omega_n + \hat{\omega}_n\Delta\zeta)\dot{q} - 2(\hat{\omega}_n\Delta\omega)q \tag{1.12a}$$

$$\doteq -2(\zeta\Delta\omega_n + \omega_n\Delta\zeta)\dot{q} - 2(\omega_n\Delta\omega)q \tag{1.12b}$$

with

$$\Delta\zeta \triangleq \zeta - \hat{\zeta}$$

$$\Delta\omega_n \triangleq \omega_n - \hat{\omega}_n \tag{1.13}$$

The indications of approximateness in (1.12) reflect the assumption that $(\Delta\zeta)/\zeta$ and $(\Delta\omega_n)/\omega_n$ are small compared to unity: the expressions in (1.12) are correct to first order in these small quantities. (Parenthetically, one might question whether the current state of the art of damping modeling is sufficiently advanced to make realistic the assumption that $(\Delta\zeta)/\zeta$ is confined to small values.) In any case, the idea behind (1.12) — the idea that interpreting accelerometer data is clouded by uncertainties in the dynamical model of the structure — is a principal theme in this report.

1.3 Overview of Report

It is already clear that the use of accelerometer measurements in a state-space context poses special problems of correction and interpretation. When using the structural dynamics equations as a means of transformation from acceleration to position and velocity, special attention must be paid to the following factors:

- all known forces on the structure must be fed forward;
- unknown forces must be reduced to a minimum;
- the structural model must be known as accurately as possible.

In addition, it is almost trite to state that the acceleration measurement itself should be as accurate as possible.

The implications of these factors for all the control strategies that have been proposed in the literature for flexible space structures is a task so large that a complete discussion cannot be aspired to in this report. However, a major step in this direction can be taken by examining the implications of accelerometer feedback for four of the control strategies that are best known and widely used:

- simple state feedback
- simple output feedback
- full-order state estimation
- reduced-order state estimation.

[Note: What is herein called 'simple' feedback is often called 'static' feedback; and what is herein called a 'state estimator' is often called an 'observer.'] Each of the next four sections of this report deals with one of these strategies in turn. In each case, basic ideas are first reviewed and then parameter error effects are introduced into the discussion.

2. SIMPLE STATE FEEDBACK

Throughout this and the following sections we shall assume that the spacecraft can be represented as a linear time-invariant system of the standard form:

$$\dot{\underline{x}} = \underline{Ax} + \underline{Bu} \tag{2.1}$$

$$\underline{y} = \underline{Cx} \tag{2.2}$$

$$\underline{z} = \underline{Mx} \tag{2.3}$$

Here, $\underline{x}(t)$ is the state vector, $\underline{y}(t)$ is the output vector of regulated variables (or the 'regulated vector,' for short) and $\underline{z}(t)$ is the output vector of measured variables (the 'measured vector'). These two types of output vector may may contain one or more variables in common. They may even (in rare instances) be identical vectors. However, because one cannot always be guaranteed a direct measurement of a variable one wishes to regulate, it is a helpful generalization of the theory to make a distinction between the regulated vector, $\underline{y}$, and the measured vector, $\underline{z}$. Note that only variables appearing in $\underline{z}$ are actually available for feedback control.

2.1 The Basic Strategy

'Simple state feedback' is a concept that represents an idealistic limiting case: all the state variables are assumed available for feedback. An equivalent assumption is that there are as many independent measurements as there are state variables, so that $\underline{M}$ is a $n \times n$ nonsingular matrix. Then the state vector is immediately available from

$$\underline{x}(t) = \underline{M}^{-1}\underline{z}(t) \tag{2.4}$$

One is then in a position to 'feed back' this known state vector, and n arbitrary linear combinations of the state variables are allowed:

$$\underline{u}(t) = \underline{r}(t) - \underline{Kx}(t) \tag{2.5}$$

The coefficients k_{ij} are 'gains' and $\underline{K}$ is called the 'gain matrix.' The negative sign indicates negative feedback, and $\underline{r}(t)$ is a reference input, usually constant.

From (2.1) and (2.5) then,

$$\underline{x} = (\underline{A} - \underline{BK})\underline{x} + \underline{Br} \tag{2.6}$$

showing that the closed-loop system matrix is $\underline{A} - \underline{BK}$. We leave aside the question — the very important question — of how best to choose $\underline{K}$ to guarantee satisfactory closed-loop control characteristics. We wish to focus on the aspects of greatest relevance to the issue of acceleration feedback.

The implementation of simple state feedback is shown in Fig. 2.1. Note that the operations of measurement and state calculation have been separated. This separation is physically real — the measurements are made by sensors, while the state vector calculation (via $\underline{M}^{-1}$) is carried out via other hardware (or software).

2.2 Effect of Parameter Errors

Looking forward to the implementation of acceleration feedback, one might enquire what the implications might be of parameter errors, especially in the measurements. To aid in developing an answer to this question, Fig. 2.1 is modified as shown in Fig. 2.2. Although the actual measurement matrix is $\underline{M}$, the control system designer assumes it to be $\underline{\hat{M}}$, and so the state is calculated to be

$$\underline{\hat{x}} = \underline{\hat{M}}^{-1}\underline{z} \tag{2.7}$$

instead of $\underline{x}$. The feedback law is implemented as

$$\underline{u}(t) = \underline{r}(t) - \underline{K\hat{x}}(t) \tag{2.8}$$

so that the closed-loop system becomes

$$\underline{\dot{x}} = (\underline{A} - \underline{BK\hat{M}}^{-1}\underline{M})\underline{x} + \underline{Br} \tag{2.9}$$

which should be compared to the ideal result, expressed in (2.6).

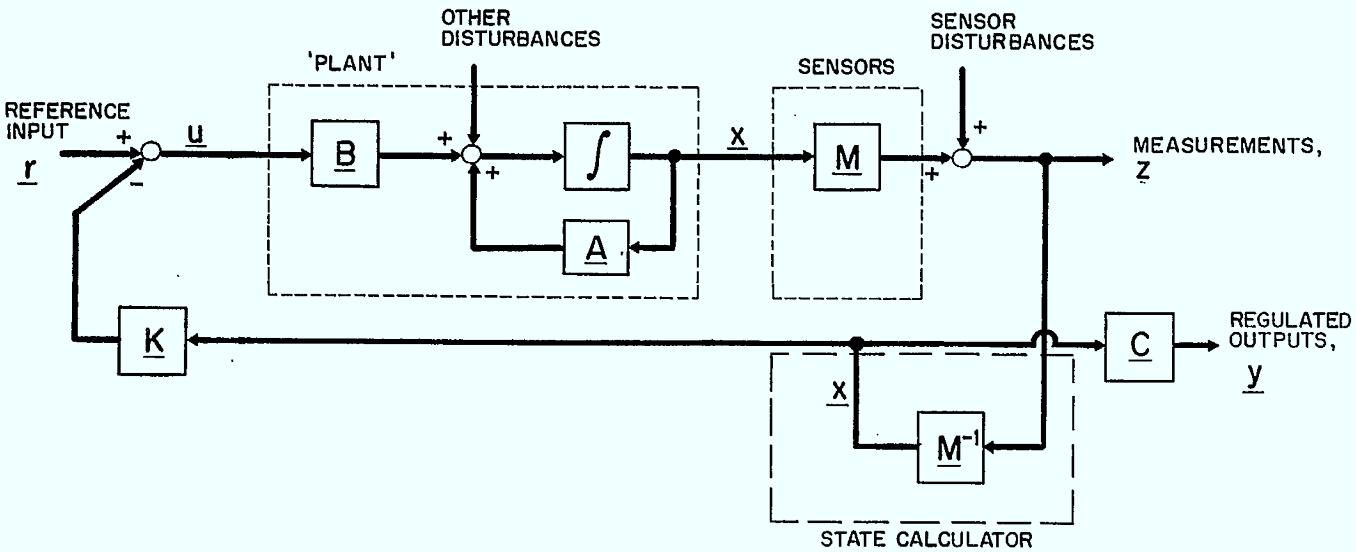


Fig. 2.1: Control System Design Based on Simple State Feedback

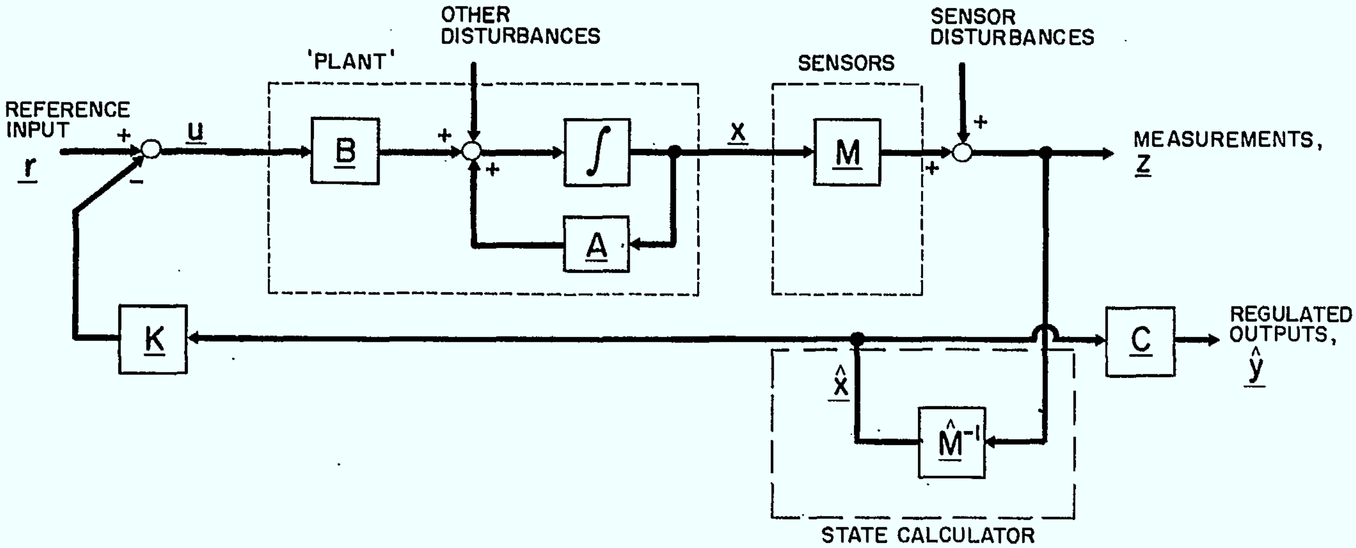


Fig. 2.2: Simple State Feedback — Actual and Assumed System Parameters

To isolate the pseudo-disturbance attributable to parameter error, in analogy with the simple example in Fig. 1.3 and Eqs. (1.11) and (1.12), we write

$$\underline{M} = \hat{\underline{M}} + \Delta\underline{M} \tag{2.10}$$

The errors $\Delta\underline{M}$ in the measurement matrix $\underline{M}$ will be expressed in terms of structural dynamics parameters in Section 6. Then

$$\dot{\underline{x}} = (\underline{A} - \underline{BK})\underline{x} + \underline{Br} - \underline{BK\hat{M}}^{-1}\underline{e}_p \tag{2.11}$$

where $\underline{e}_p$ is the error vector in the measurements arising from parameter errors:

$$\underline{e}_p(t) \triangleq (\Delta\underline{M})\underline{x}(t) \tag{2.12}$$

In other words, the measurements are

$$\underline{z} = \underline{\hat{M}x} + \underline{e}_p \tag{2.13}$$

At present, we are assuming the measurements themselves to be exact. The only error is in their *interpretation*, as used in the state calculator (Fig. 2.2). This explains the absence of a 'noise' term on the right-hand side of (2.13).

3. SIMPLE MEASUREMENT FEEDBACK

To measure all the state variables, directly or indirectly, is not usually possible. Usually a smaller number, m, of measurements is made, m < n, where n is the dimension of the state. That is,

$$\dot{\underline{x}} = \underline{Ax} + \underline{Bu} \tag{3.1}$$

$$\underline{z} = \underline{Mx} \tag{3.2}$$

and the measurement matrix $\underline{M}$ is m x n. Only the measurements $\underline{z}$ are available for feedback control.

3.1 The Basic Strategy

In "simple measurement feedback," the feedback law consists simply in assigning, as control variables, linear combinations of the measurements. Thus

$$\underline{u}(t) = \underline{r}(t) - \underline{K}_m \underline{z}(t) \tag{3.3}$$

where $\underline{K}_m$ is an r x m gain matrix (there are r control variables). The subscript 'm' is added to distinguish the r x $\underset{\uparrow}{m}$ measurement feedback matrix $\underline{K}_m$ in (3.3) from the r x $\underset{\uparrow}{n}$ state feedback matrix $\underline{K}$ in (2.5). As usual, $\underline{r}(t)$ in (3.3) is a reference input.

From (3.1) - (3.3), the closed-loop system is, ideally,

$$\dot{\underline{x}} = (\underline{A} - \underline{B}\underline{K}_m\underline{M})\underline{x} + \underline{B}\underline{r} \tag{3.4}$$

showing that the closed-loop system matrix is $\underline{A} - \underline{B}\underline{K}_m\underline{M}$. As with state feedback, considered earlier, the only remaining design question — What should $\underline{K}_m$ be? — is not answered here. The implementation of simple measurement feedback is shown in block diagram form in Fig. 3.1.

3.2 Effect of Parameter Errors

In reflecting upon the above equations and upon Fig. 3.1, one can see that parameter errors (in this case, errors in $\underline{M}$) do not have an impact on simple measurement feedback in the same manner as they do for simple state feedback. No calculation on $\underline{z}$ is made in Fig. 3.1 that corresponds to the $\underline{M}^{-1}$ calculation in Fig. 2.1. The only effect of an error $\Delta\underline{M}$ is a more subtle one, namely, the gain matrix chosen, $\underline{K}_m$, will be based on assumed values for $\underline{M}$ (and on $\underline{A}$ and $\underline{B}$ also) and so will not be as ideal a gain matrix for the actual values of $\underline{M}$ and $\underline{A}$ and $\underline{B}$. (In fact, this remark applies to state feedback also — Section 2 — and also to the control laws to be considered in Sections 4 and 5.) However, beyond remarking that this effect exists, no more will be said of a slightly off-design gain matrix in this report, because this report does not assign specific gain matrices.

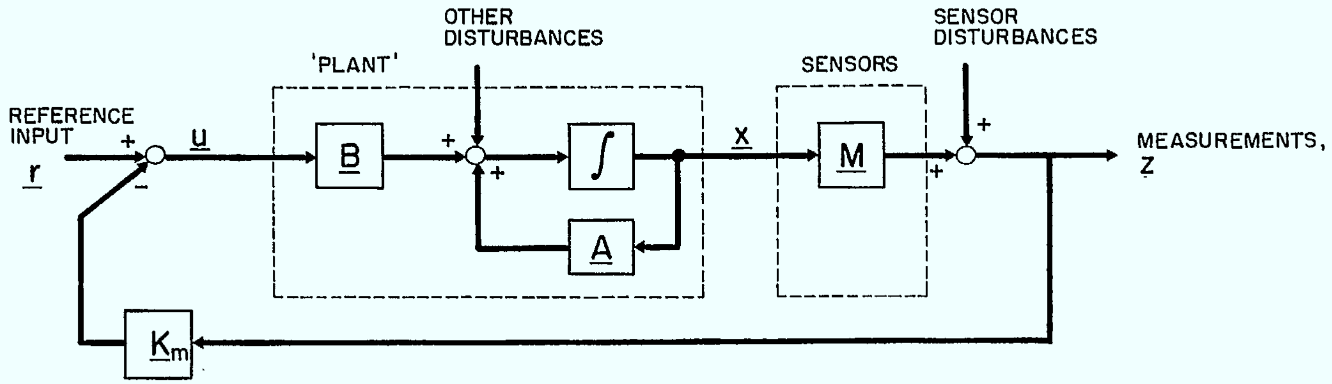


Fig. 3.1: Control System Design Based on Simple Measurement Feedback

4. FULL-ORDER STATE ESTIMATION

It has been remarked in the last two sections that a continuous knowledge of the state is desirable, but rarely possible. It *is* possible, however, to *estimate* the state of the 'plant' (i.e., the spacecraft dynamics) by making inferences from

- the measurements, and
- a system model.

Then, provided the plant is *observable*, a good estimate of the state can be calculated. The good news is that simple state feedback can then be used, with the 'estimated state' used in place of the (unavailable) 'state'; the bad news is the increased reliance on a system math model — a model that will inevitably contain parameter errors.

4.1 The Basic Strategy

The 'actual' (i.e., physical) system is represented by

$$\dot{\underline{x}} = \underline{Ax} + \underline{Bu} \tag{4.1}$$

as usual, and the measurements are

$$\underline{z} = \underline{Mx} \tag{4.2}$$

Let the *estimate* of $\underline{x}$ be denoted by $\hat{\underline{x}}$. At issue is the question of how to calculate $\hat{\underline{x}}$.

It is only common sense that $\hat{\underline{x}}$ should have the property that if, at some instant $t = t_1$, we know $\underline{x}(t_1)$, then for $t > t_1$ one could ideally produce $\hat{\underline{x}}(t)$ by integrating the differential system

$$\dot{\hat{\underline{x}}} = \underline{A\hat{x}} + \underline{Bu} \qquad (t > t_1) \tag{4.3}$$

with the initial condition $\hat{\underline{x}}(t_1) = \underline{x}(t_1)$. Then, ideally, $\hat{\underline{x}}(t) \equiv \underline{x}(t)$, for $t > t_1$.

Two characteristics are already clear from the somewhat overly-idealistic (4.3):

- the same controls that are fed to the actual system must also be fed to the state estimator
- to implement this state estimator requires that n integrations be performed in real time.

These characteristics will be shown (below) to persist even when (4.3) is modified to account for the assumptions behind it.

The difficulty with (4.3) as a state estimator is, of course, that one does not know $\underline{x}$ at $t = t_1$; that is part of the problem. A further difficulty is that even if one did know $\underline{x}$ at $t = t_1$ and integrated according to (4.3), the inevitable inaccuracies and disturbances would make their presence felt, even though the state estimator (4.3), proceeding open-loop as it does, would not take these inaccuracies and disturbances into account in any way. What is needed, evidently, is some sort of feedback within the estimator itself. This feedback would inform the estimator of the difference between its estimated state and the actual state (irrespective of whether this difference is due to a wrong initial condition or to subsequent inaccuracies and disturbances). Thus we wish to add to (4.3) a feedback term of the form

$$-\underline{K}_e(\hat{\underline{x}} - \underline{x})$$

Unfortunately, this is not possible because $\underline{x}$ is not known (if $\underline{x}$ were known, the estimator would not be needed!). The closest one can come is to compare, not the estimated state $\hat{\underline{x}}$ with the actual state $\underline{x}$, but the 'estimated measurement' $\underline{M\hat{x}}$ with the actual measurement $\underline{z}$. On this basis, we add a feedback term to (4.3), and the state estimator finally arrived at is as follows:

$$\dot{\hat{\underline{x}}} = \underline{A\hat{x}} + \underline{Bu} - \underline{K}_e(\underline{M\hat{x}} - \underline{z}) \tag{4.4}$$

The implementation of this estimator is shown in Fig. 4.1.

Also shown in Fig. 4.1 is the final use made of the estimated state, $\hat{\underline{x}}(t)$: it is treated as though it were the actual state, and 'simple' state feedback is applied:

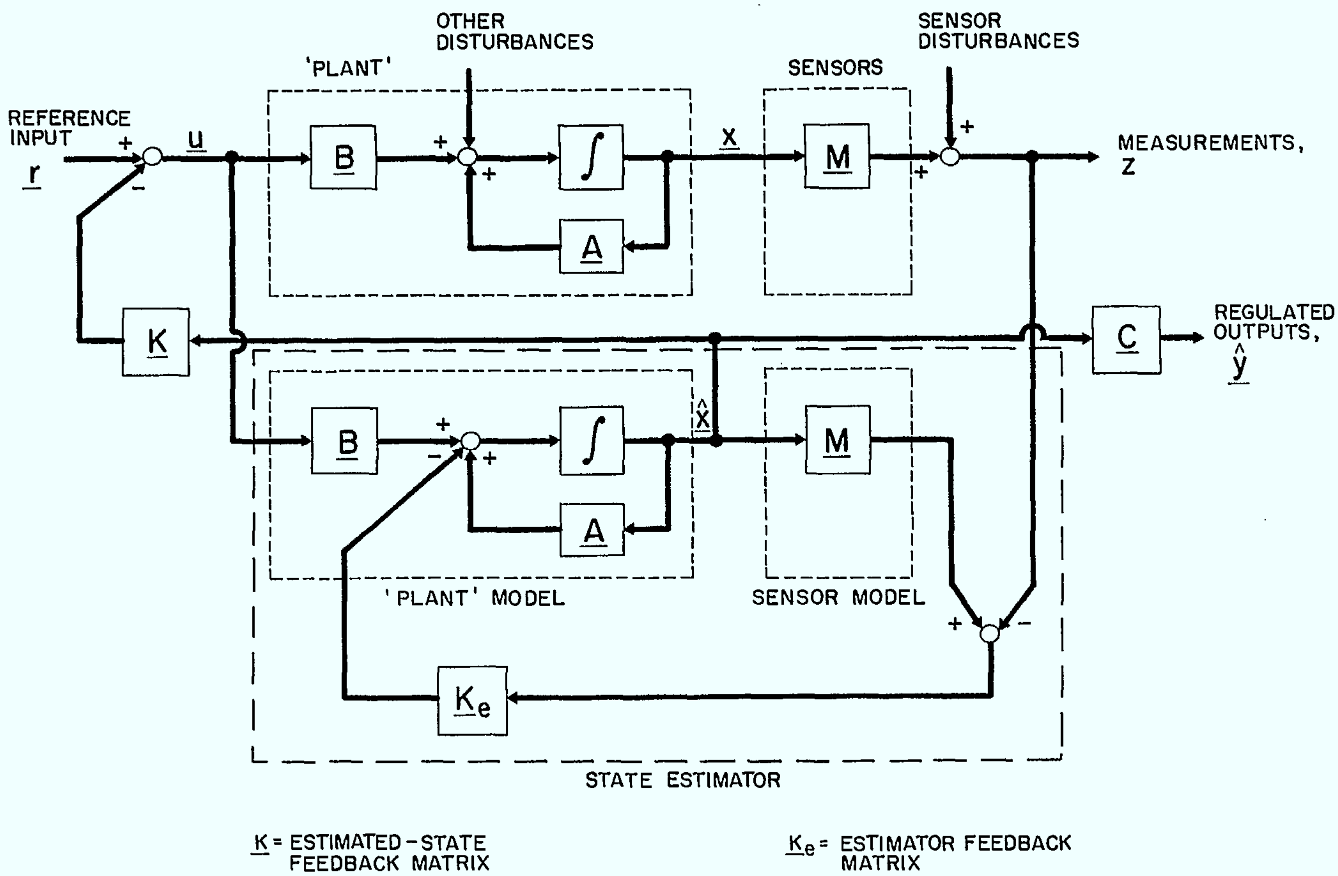


Fig. 4.1: Control System Design Based on Feedback of State Estimated by a Full-Order Estimator

$$\underline{u}(t) = \underline{r}(t) - \underline{K}\hat{\underline{x}}(t) \tag{4.5}$$

This completes a specification of the estimator. Note that the order of the system has been doubled — for every state variable $x_i(t)$ in the original model, a new state variable $\hat{x}_i(t)$ is present in the estimator. For this reason, the estimator discussed in this section is called a 'full-order' estimator.

The error in the state estimate is

$$\underline{e}(t) \triangleq \underline{x}(t) - \hat{\underline{x}}(t) \tag{4.6}$$

From (4.1,2,4,5),

$$\dot{\underline{e}} = (\underline{A} - \underline{K}_e\underline{M})\underline{e} \tag{4.7}$$

which shows that the transient characteristics of the error can be designed at will by a suitable choice of the estimator gain matrix $\underline{K}_e$. (This last statement assumes that the pair $\{\underline{M},\underline{A}\}$ is an 'observable pair', or that, equivalently, $\{\underline{A}^T,\underline{M}^T\}$ is a 'controllable pair.') In terms of $\underline{e}$, the closed-loop plant state $\underline{x}(t)$ evolves according to

$$\dot{\underline{x}} = (\underline{A} - \underline{BK})\underline{x} + \underline{BKe} + \underline{Br} \tag{4.8}$$

The transient characteristics of the closed-loop system matrix $\underline{A} - \underline{BK}$ can be varied arbitrarily (provided $\{A,B\}$ is a controllable pair) by an appropriate choice of the feedback gain matrix $\underline{K}$. As will be seen in Section 6, the pairs $\{\underline{A}^T,\underline{M}^T\}$ and $\{\underline{A},\underline{B}\}$ will virtually always be controllable in the application to flexible spacecraft.

4.2 Effect of Parameter Errors

The system matrices $\{\underline{A},\underline{B},\underline{M}\}$ are not precisely known in practice:

$$\begin{aligned} \underline{A} &= \hat{\underline{A}} + \Delta\underline{A} \\ \underline{B} &= \hat{\underline{B}} + \Delta\underline{B} \\ \underline{M} &= \hat{\underline{M}} + \Delta\underline{M} \end{aligned} \tag{4.9}$$

where $\hat{\underline{A}}$, $\hat{\underline{B}}$, and $\hat{\underline{M}}$ are the assumed values. Taking these parameter errors into account, Fig. 4.1 becomes transformed to look like Fig. 4.2. The 'real' plant is still represented by $\underline{A}$ and $\underline{B}$, and the 'real' sensors are still represented by $\underline{M}$, but in the state estimator the 'assumed' values for these matrices are used.

The system equations that correspond to the block diagram of Fig. 4.2 are as follows:

$$\begin{aligned} \dot{\underline{x}} &= \underline{Ax} + \underline{Bu} \\ \underline{z} &= \underline{Mx} \\ \dot{\hat{\underline{x}}} &= \hat{\underline{A}}\hat{\underline{x}} + \hat{\underline{B}}\underline{u} - \underline{K}_e(\hat{\underline{M}}\hat{\underline{x}} - \underline{z}) \\ \underline{u} &= \underline{r} - \underline{K}\hat{\underline{x}} \end{aligned} \tag{4.10}$$

Note that $\hat{\underline{x}}$ is now an *estimate* of the state $\underline{x}$ not only because it is the output of a state estimator, but also because the parameters used in the estimator are themselves estimated.

To investigate this combination of errors quantitatively, we define

$$\underline{e} \triangleq \underline{x} - \hat{\underline{x}}$$

as before, and re-write the system equations (4.10) as follows:

$$\dot{\underline{x}} = (\underline{A} - \underline{BK})\underline{x} + \underline{BKe} + \underline{Br} \tag{4.11}$$

$$\begin{aligned} \dot{\underline{e}} = {} & [\hat{\underline{A}} - \underline{K}_e\hat{\underline{M}} + (\Delta\underline{B})\underline{K}]\underline{e} \\ & + [(\Delta\underline{A}) - (\Delta\underline{B})\underline{K} - \underline{K}_e(\Delta\underline{M})]\underline{x} + (\Delta\underline{B})\underline{r} \end{aligned} \tag{4.12}$$

The plant equation (4.11) is unchanged; however, the estimator error has several new causes, as shown in (4.12). The system matrix is changed slightly, from $\hat{\underline{A}} - \underline{K}_e\hat{\underline{M}}$ (as assumed) to $\hat{\underline{A}} - \underline{K}_e\hat{\underline{M}} + (\Delta\underline{B})\underline{K}$. While not welcome, this small change in the 'error system matrix' should not cause any problems, perhaps a slight change in error decay rate at most, unless the design is extraordinarily sensitive (which it shouldn't be), or unless the errors $\Delta\underline{B}$ are very large, or unless a very-high-gain control system design has been selected. More worrisome, perhaps, are the other terms in (4.12), which show that the

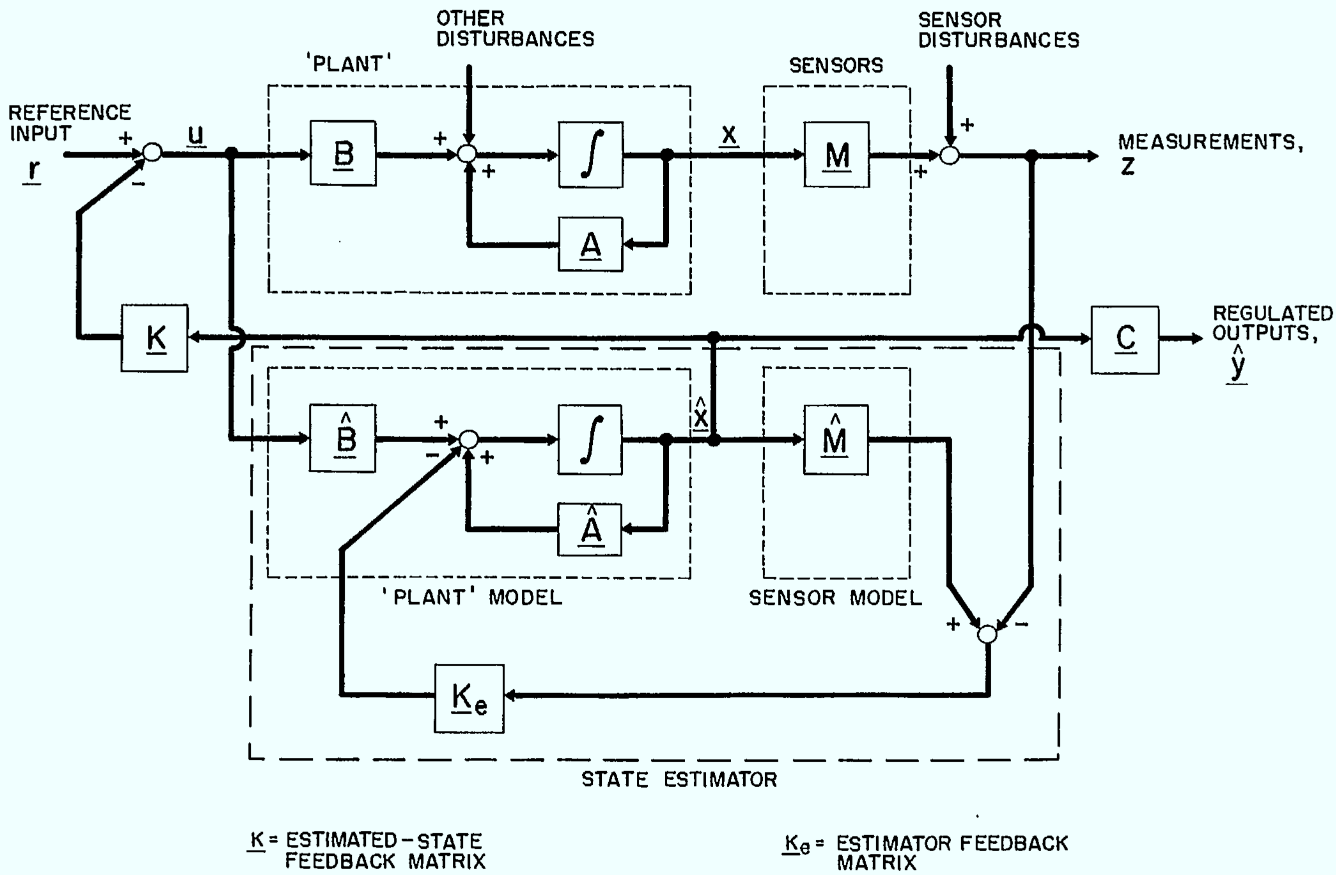


Fig. 4.2: Feedback of State Estimated by a Full-Order Estimator — Actual and Assumed System Parameters

error $\underline{e}$ is prevented from asymptotically vanishing by the persistent disturbance from the terms

$$[(\Delta\underline{A}) - (\Delta\underline{B})\underline{K} - \underline{K}_e(\Delta\underline{M})]\underline{x}$$

and

$$(\Delta\underline{B})\underline{r}$$

The former depends on parameter errors in $\underline{A}$, $\underline{B}$ and $\underline{M}$, while the latter depends only on errors in $\underline{B}$. The estimator error $\underline{e}$ will still vanish, of course, if the reference command $\underline{r} \equiv \underline{0}$ and $\underline{x} \to \underline{0}$ asymptotically. In fact, (4.11) and (4.12) can be written in assembled form as

$$\begin{bmatrix} \underline{x} \\ \underline{e} \end{bmatrix}^{\cdot} = \begin{bmatrix} \underline{A} - \underline{BK} & \underline{BK} \\ (\Delta\underline{A}) - (\Delta\underline{B})\underline{K} - \underline{K}_e(\Delta\underline{M}) & \hat{\underline{A}} - \underline{K}_e\hat{\underline{M}} + (\Delta\underline{B})\underline{K} \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{e} \end{bmatrix} + \begin{bmatrix} \underline{B} \\ \Delta\underline{B} \end{bmatrix} \underline{r} \qquad (4.13)$$

The eigenvalues of the square 2n x 2n composite system matrix in (4.13) will in general be only marginally different from the ideal composite system matrix

$$\begin{bmatrix} \hat{\underline{A}} - \hat{\underline{B}}\underline{K} & \hat{\underline{B}}\underline{K} \\ \underline{0} & \hat{\underline{A}} - \underline{K}_e\hat{\underline{M}} \end{bmatrix}$$

and so it could be argued that if the (ideal) state estimator is well designed, the effects of parameter errors will be small. That may well be, but the fact remains that, in (4.12), the estimator error is driven by a new type of term that is, according to (4.7), ideally absent. The estimator does not, as ideally assumed, become asymptotically exact regardless of what adventures the state is forced to undergo. The estimator is in fact compelled to participate in those adventures through the (weak) influence of parameter errors. In this connection it should be noted that the plant disturbances (without which a control system would be unnecessary) have not been symbolically included in the above discussion.

5. REDUCED-ORDER STATE ESTIMATION

It is possible to identify a control strategy that is, in a sense, mid-way between the rather spartan feedback scheme of Fig. 2.1 or 3.1 and the somewhat luxurious one of Fig. 4.1. In the former, measurements are just combined algebraically to form the control input variables; and, in the latter, the controller has as many integrators (or state variables) associated with it as has the original plant. It is not necessary, however, that a state estimator be of 'full' order, i.e., of order equal to the plant math model. As a special limiting case of this assertion, consider the situation discussed in Section 2, wherein n measurements were made. It was pointed out in that discussion that the state did not have to be estimated at all: it could be directly calculated, continuously and immediately, by the straightforward algebraic operation equivalent to the inversion of an $n \times n$ matrix.

In general, the number of measurements, m, will be fewer than the number of state variables, n. Nevertheless, it is not necessary (as will be shown below) to introduce n new state variables to be associated with the estimator, as done for the full-order state estimator. It is in fact necessary to introduce only $(n - m)$ new state variables in the estimator design, thus producing a 'minimal-order' estimator. Since we do not intend to discuss estimators whose orders lie between the extremes n and $n - m$, we can refer to an estimator of order $n - m$ unambiguously as a 'reduced-order state estimator.'

It is apparent that a considerable simplification in controller design can be derived by using a reduced-order estimator instead of a full-order estimator. The actual saving depends, of course, on the actual values of n and m. If a controller design is based on 6 rigid modes and 30 elastic modes ($n = 72$) and only 8 sensors (measurements) are used ($m = 8$), then the full-order estimator can be reduced in order from 72 order to 64 — only an 11% saving. On the other hand, if a controller design is based on 6 rigid modes and 10 elastic modes (n = 32) and 20 sensors are used, the full-order estimator can be reduced in order from 32 to 12 — a reduction in complexity by a factor of two-thirds. It is at all events quite clear that the idea of a reduced-order estimator is one worthy of further investigation. This is true in spite of the fact that although the system order is reduced, the algebra required to discuss the idea is increased.

5.1 The Basic Strategy

The idea of a reduced-order estimator harks back to an earlier one — state feedback (Section 2 and Fig. 2.1). It will be recalled that state feedback requires that as many measurements be made as there are state variables. The reason that an estimator is required is that the measurements are generally fewer than the number of state variables. We shall, however, approach the derivation of the reduced-order state estimator by postulating $n - m$ *pseudo*-measurements to replace the missing measurements.

The spacecraft dynamics is still represented by

$$\underline{\dot{x}} = \underline{Ax} + \underline{Bu} \tag{5.1}$$

and the (real) measurements

$$\underline{z} = \underline{Mx} \tag{5.2}$$

are augmented (in the control analyst's imagination) by the pseudo-measurements

$$\underline{z}_\Delta = \underline{M}_\Delta\underline{x} \tag{5.3}$$

When the m measurements $\underline{z}$ are combined with the $n - m$ pseudo-measurements $\underline{z}_\Delta$, we have

$$\begin{bmatrix} \underline{z} \\ \underline{z}_\Delta \end{bmatrix} = \begin{bmatrix} \underline{M} \\ \underline{M}_\Delta \end{bmatrix} \underline{x} \tag{5.4}$$

and the overall measurement matrix is now n x n. This allows solution for the state $\underline{x}$:

$$\underline{x} = \begin{bmatrix} \underline{M} \\ \underline{M}_\Delta \end{bmatrix}^{-1} \begin{bmatrix} \underline{z} \\ \underline{z}_\Delta \end{bmatrix} \tag{5.5}$$

We leave open the question of how best to choose $\underline{M}_\Delta$, except to say that it must be chosen such that the inverse in (5.5) exists. (And engineering judgment would further suggest that $\underline{M}_\Delta$ should be such that the overall measurement matrix in (5.5) is not even close to being singular.)

It will be convenient in the ensuing developments to have a notation for the inverse of the overall measurement matrix. Thus, let

$$[\underline{N} \quad \underline{N}_\Delta] \triangleq \begin{bmatrix} \underline{M} \\ \underline{M}_\Delta \end{bmatrix}^{-1} \tag{5.6}$$

It follows that

$$\begin{aligned} \underline{MN} &= \underline{1} && m \times m \\ \underline{M}_\Delta\underline{N}_\Delta &= \underline{1} && (n - m) \times (n - m) \\ \underline{MN}_\Delta &= \underline{0} && m \times (n - m) \\ \underline{M}_\Delta\underline{N} &= \underline{0} && (n - m) \times m \end{aligned} \tag{5.7}$$

and that

$$\underline{NM} + \underline{N}_\Delta\underline{M}_\Delta = \underline{1} \qquad n \times n \tag{5.8}$$

With these new symbols, (5.5) can be written directly as

$$\underline{x}(t) = \underline{Nz}(t) + \underline{N}_\Delta\underline{z}_\Delta(t) \tag{5.9}$$

The idea, to repeat, is that $\underline{z}$ is available, and, if $\underline{z}_\Delta$ *were* available simple state feedback could be used. To face reality, $\underline{z}_\Delta$ can only be *estimated*. In this way, the estimation problem is reduced from $\underline{x}$-estimation (order n) to $\underline{z}_\Delta$-estimation (order n - m). State estimation then proceeds according to

$$\underline{\hat{x}}(t) = \underline{Nz}(t) + \underline{N}_\Delta \underline{\hat{z}}_\Delta(t) \tag{5.10}$$

and one is led to consider appropriate means for estimating the missing measurements, $\underline{z}_\Delta$.

To begin, we derive a differential equation for $\underline{z}_\Delta$:

$$\begin{aligned}\underline{\dot{z}}_\Delta &= \underline{M}_\Delta \underline{\dot{x}} \\ &= \underline{M}_\Delta(\underline{Ax} + \underline{Bu}) \\ &= \underline{M}_\Delta \underline{A}(\underline{Nz} + \underline{N}_\Delta \underline{z}_\Delta) + \underline{M}_\Delta \underline{Bu}\end{aligned}$$

That is,

$$\underline{\dot{z}}_\Delta = (\underline{M}_\Delta \underline{AN}_\Delta)\underline{z}_\Delta + (\underline{M}_\Delta \underline{AN})\underline{z} + (\underline{M}_\Delta \underline{B})\underline{u} \tag{5.11}$$

Therefore, the basic construction of the differential equation for the estimate $\underline{\hat{z}}_\Delta$ must follow the pattern

$$\underline{\dot{\hat{z}}}_\Delta = (\underline{M}_\Delta \underline{AN}_\Delta)\underline{\hat{z}}_\Delta + (\underline{M}_\Delta \underline{AN})\underline{z} + (\underline{M}_\Delta \underline{B})\underline{u} + \cdots \tag{5.12}$$

This construction is, of course, incomplete. It is in the same primitive stage as was (4.3) for the full-order estimator. We have, thus far, only an open-loop estimator. Once perfect, it remains perfect; but it lacks information fed back on its unavoidable errors, just as, for the full-state estimator, (4.4) made amends for the inadequate (4.3).

The analogy between (5.12) and (4.3) might suggest that estimator feedback could be carried out in the form

$$\begin{aligned}\underline{\dot{\hat{z}}}_\Delta = (\underline{M}_\Delta \underline{AN}_\Delta)\underline{\hat{z}}_\Delta &+ (\underline{M}_\Delta \underline{AN})\underline{z} + (\underline{M}_\Delta \underline{B})\underline{u} \\ &- \underline{K}_e(\underline{M\hat{x}} - \underline{z}) \qquad \text{(tentative)}\end{aligned} \tag{5.13}$$

with $\underline{x}$ given by (5.10). But this approach does not work. Simple calculation from (5.10) and (5.7) shows that

$$\underline{M\hat{x}} - \underline{z} = \underline{MNz} + \underline{MN}_\Delta \underline{\hat{z}}_\Delta - \underline{z} = \underline{0} \tag{5.14}$$

In other words, the critical feedback term in (5.13) is always zero. When the state is estimated *algebraically* from *current* measurements, as in (5.10), rather than *dynamically*, from a *history* of measurements, the estimate of what the measurement should be, $\underline{M\hat{x}}$, will be merely the actual measurement, $\underline{z}$.

In place of the feedback term $\sim(\underline{M\hat{x}} - \underline{z})$ in (5.13), which, as we have seen, has only illusory benefits, we use instead a term that is related to the time derivative of the failed term, i.e., a term $\sim(\underline{M\dot{x}} - \underline{\dot{z}})$. Now, from the fact that

$$
\begin{aligned}
\underline{M\dot{x}} - \underline{\dot{z}} &= \underline{M}(\underline{Ax} + \underline{Bu}) - \underline{\dot{z}} \\
&= \underline{MA}(\underline{Nz} + \underline{N_\Delta z_\Delta}) + \underline{MBu} - \underline{\dot{z}} \qquad (5.15)
\end{aligned}
$$

an estimate of the missing measurements (the pseudo-measurements) can be constructed from (5.12) and (5.15):

$$
\begin{aligned}
\underline{\dot{\hat{z}}_\Delta} = {} & (\underline{M_\Delta A N_\Delta})\underline{\hat{z}_\Delta} + (\underline{M_\Delta AN})\underline{z} + (\underline{M_\Delta B})\underline{u} \\
& - \underline{\dot{K}_e}[(\underline{MAN_\Delta})\underline{\hat{z}_\Delta} + (\underline{MAN})\underline{z} + (\underline{MB})\underline{u} - \underline{\dot{z}}] \qquad (5.16)
\end{aligned}
$$

The symbol $\underline{\dot{K}_e}$ does not imply the time derivative of some matrix $\underline{K_e}$. Instead, $\underline{\dot{K}_e}$ is a constant gain matrix, and the notation is merely a reminder that the variables for which the gains (elements of $\underline{\dot{K}_e}$) serve as coefficients are the time derivatives of the elements of $(\underline{Mx} - \underline{z})$, or estimates of these elements. In this connection, note that the (unavailable) pseudo-measurements $\underline{z_\Delta}$ in (5.15) have been replaced by their (available) estimates in (5.16).

To reduce the estimator equation (5.16) to its bare essentials, we define

$$
\begin{aligned}
\underline{A_e} &\triangleq \underline{\Gamma AN_\Delta} & \underline{K'_e} &\triangleq \underline{\Gamma AN} \\
\underline{B_e} &\triangleq \underline{\Gamma B} & \underline{\Gamma} &\triangleq \underline{M_\Delta} - \underline{\dot{K}_e M}
\end{aligned}
\qquad (5.17)
$$

after which (5.16) condenses to

$$\dot{\hat{\underline{z}}}_\Delta = \underline{A}_e\hat{\underline{z}}_\Delta + \underline{B}_e\underline{u} + \underline{K}'_e\underline{z} + \dot{\underline{K}}_e\dot{\underline{z}} \tag{5.18}$$

All the quantities needed on the right-hand side of this estimator are physically available. It does look as though the measurements have to be differentiated — an undesirable operation. This differentiation can be avoided, however, by introducing $\underline{x}_e$, defined by

$$\underline{x}_e \triangleq \hat{\underline{z}}_\Delta - \dot{\underline{K}}_e\underline{z} \tag{5.19}$$

Thus the new state variables associated with estimator (n - m of them) are not $\hat{\underline{z}}_\Delta$, but $\underline{x}_e$. In terms of $\underline{x}_e$, the estimator equation (5.18) becomes

$$\dot{\underline{x}}_e = \underline{A}_e\underline{x}_e + \underline{B}_e\underline{u} + \underline{K}_e\underline{z} \tag{5.20}$$

where

$$\underline{K}_e \triangleq \underline{K}'_e + \underline{A}_e\dot{\underline{K}}_e \tag{5.21}$$

is the final 'estimator gain matrix' needed. The implementation of this state estimator is shown in Fig. 5.1. The output of the estimator is, as planned, $\hat{\underline{x}}$, where, from (5.10) and (5.19),

$$\hat{\underline{x}} = \underline{C}_e\underline{x}_e + \underline{D}_e\underline{z} \tag{5.22}$$

and the definitions

$$\begin{aligned} \underline{C}_e &\triangleq \underline{N}_\Delta \\ \underline{D}_e &\triangleq \underline{N} + \underline{N}_\Delta\dot{\underline{K}}_e \end{aligned} \tag{5.23}$$

have been introduced. Now that $\hat{\underline{x}}$ is available, 'simple' state feedback (Section 2) can be emulated:

$$\underline{u}(t) = \underline{r}(t) - \underline{K}\hat{\underline{x}}(t) \tag{5.24}$$

To conclude this discussion of the basic strategy of a reduced-order estimator, we derive a differential equation which governs how the

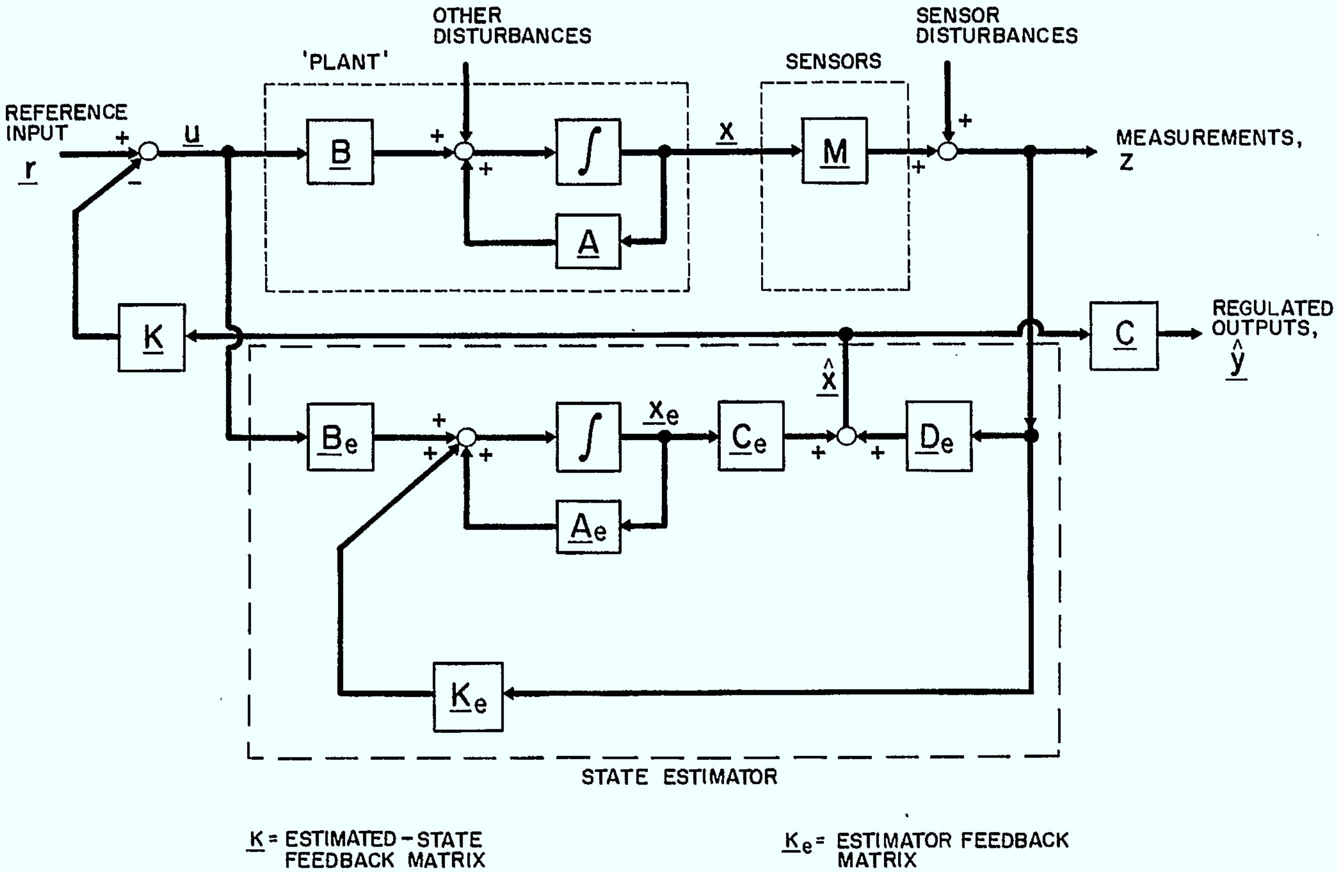


Fig. 5.1: Control System Design Based on Feedback of State Estimated by a Reduced-Order Estimator

estimator error evolves with time. The aim is to find an equation which does for the reduced-order estimator what (4.7) does for the full-order estimator. For the full-order estimator, we defined the error to be $\underline{e} \triangleq \underline{x} - \hat{\underline{x}}$. However, in the present case (the reduced-order estimator) we have seen that it is really $\underline{z}_\Delta$ that is being estimated; see, for example, (5.9) and (5.10). Therefore we define the relevant error to be

$$\underline{e}_z(t) \triangleq \underline{z}_\Delta(t) - \hat{\underline{z}}_\Delta(t) \tag{5.25}$$

Then, from (5.11) and (5.16),

$$\begin{aligned} \dot{\underline{e}}_z &= \dot{\underline{z}}_\Delta - \dot{\hat{\underline{z}}}_\Delta \\ &= (\underline{M}_\Delta \underline{A} \underline{N}_\Delta)\underline{e}_z + \underline{K}_e[(\underline{MAN}_\Delta)\hat{\underline{z}}_\Delta + (\underline{MAN})\underline{z} + (\underline{MB})\underline{u} - \dot{\underline{z}}] \end{aligned}$$

But the quantity in square brackets is, from (5.10),

$$\begin{aligned} [\cdot] &= \underline{MA\hat{x}} + \underline{MBu} - \dot{\underline{z}} \\ &= \underline{MAx} - \underline{MAe} + \underline{MBu} - \dot{\underline{z}} \\ &= \underline{M}(\underline{Ax} + \underline{Bu}) - \dot{\underline{z}} - \underline{MAe} \\ &= (\underline{M\dot{x}} - \dot{\underline{z}}) - \underline{MAe} = -\underline{MAe} \end{aligned}$$

Furthermore,

$$\begin{aligned} \underline{e} &= \underline{x} - \hat{\underline{x}} \\ &= (\underline{Nz} + \underline{N}_\Delta \underline{z}_\Delta) - (\underline{Nz} + \underline{N}_\Delta \hat{\underline{z}}_\Delta) \\ &= \underline{N}_\Delta \underline{e}_z \end{aligned} \tag{5.26}$$

So, combining the above calculations,

$$\dot{\underline{e}}_z = \underline{A}_e \underline{e}_z \tag{5.27}$$

This equation corresponds to (4.7) for full-order estimators. In fact, another form of (5.27) can be written using (5.26):

$$\underline{\dot{e}} = \underline{N}_{\Delta}\underline{\Gamma A e} \tag{5.28}$$

However, it makes more sense to integrate the n - m equations (5.27) and then apply (5.26), than to integrate the n equations (5.28).

The state equation for the plant can be written

$$\begin{aligned} \underline{\dot{x}} &= \underline{Ax} + \underline{Bu} \\ &= \underline{Ax} + \underline{B}(\underline{r} - \underline{K\hat{x}}) \\ &= \underline{Ax} + \underline{Br} - \underline{BK}(\underline{x} - \underline{e}) \\ &= (\underline{A} - \underline{BK})\underline{x} + \underline{Br} + \underline{BKN}_{\Delta}\underline{e}_z \end{aligned} \tag{5.29}$$

so that the overall system equations are

$$\begin{bmatrix} \underline{x} \\ \underline{e}_z \end{bmatrix}^{\bullet} = \begin{bmatrix} \underline{A} - \underline{BK} & \underline{BKN}_{\Delta} \\ \underline{0} & \underline{A}_e \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{e}_z \end{bmatrix} + \begin{bmatrix} \underline{B} \\ \underline{0} \end{bmatrix} \underline{r} \tag{5.30}$$

This system is equivalent to 2n - m scalar differential equations.

5.2 Effect of Parameter Errors

As in Section 4, the parameter errors in $\underline{A}$, $\underline{B}$ and $\underline{M}$ are now introduced,

$$\begin{aligned} \underline{A} &= \underline{\hat{A}} + \Delta\underline{A} \\ \underline{B} &= \underline{\hat{B}} + \Delta\underline{B} \\ \underline{M} &= \underline{\hat{M}} + \Delta\underline{M} \end{aligned} \tag{5.31}$$

and the consequences for the closed-loop control system derived. The first result to be presented is the error in $\underline{N}$ and $\underline{N}_{\Delta}$ due to the error in $\underline{M}$. (The definition (5.6) should be recalled.) In general, if a matrix $\underline{\hat{Q}}$ has an inverse, $\underline{\hat{R}}$, and $\underline{\hat{Q}}$ is changed slightly to $\underline{\hat{Q}} + \Delta\underline{Q}$, then the inverse is changed slightly to $\underline{\hat{R}} + \Delta\underline{R}$, where

$$\Delta\underline{R} = -\underline{\hat{R}}(\Delta\underline{Q})\underline{\hat{R}} \tag{5.32}$$

[The proof is straightforward: set

$$(\hat{\underline{Q}} + \Delta\underline{Q})(\hat{\underline{R}} + \Delta\underline{R}) = \underline{1}$$

which shows that, to first order,

$$\hat{\underline{Q}}\hat{\underline{R}} + \hat{\underline{Q}}(\Delta\underline{R}) + (\Delta\underline{Q})\hat{\underline{R}} = \underline{1}$$

and (5.32) follows from imposing $\hat{\underline{Q}}\hat{\underline{R}} = \underline{1}$.] Now, in the case at hand, we make the correspondence

$$\underline{Q} = \begin{bmatrix} \underline{M} \\ \\ \underline{M}_\Delta \end{bmatrix} \; ; \qquad \underline{R} = [\underline{N} \quad \underline{N}_\Delta] \tag{5.33}$$

and we note that $\Delta\underline{M}_\Delta = \underline{O}$ because $\underline{M}_\Delta$ occurs only in the control algorithm. It is set by the control designer and does not have a physical dual. With this in mind, the general formula (5.32) produces

$$\begin{aligned} \Delta\underline{N} &= -\hat{\underline{N}}(\Delta\underline{M})\hat{\underline{N}} \\ \Delta\underline{N}_\Delta &= -\hat{\underline{N}}(\Delta\underline{M})\hat{\underline{N}}_\Delta \end{aligned} \tag{5.34}$$

for the error in $\underline{N}$ and $\underline{N}_\Delta$ caused by the parameter errors $\Delta\underline{M}$.

Now, to proceed to the next step — finding the extra terms in the system equations caused by parameter errors. It is perhaps best to summarize the system equations in their elemental form (equation numbers on the left margin are for reference):

(5.1) $\qquad \dot{\underline{x}} = \underline{Ax} + \underline{Bu} \qquad$ (5.35a)

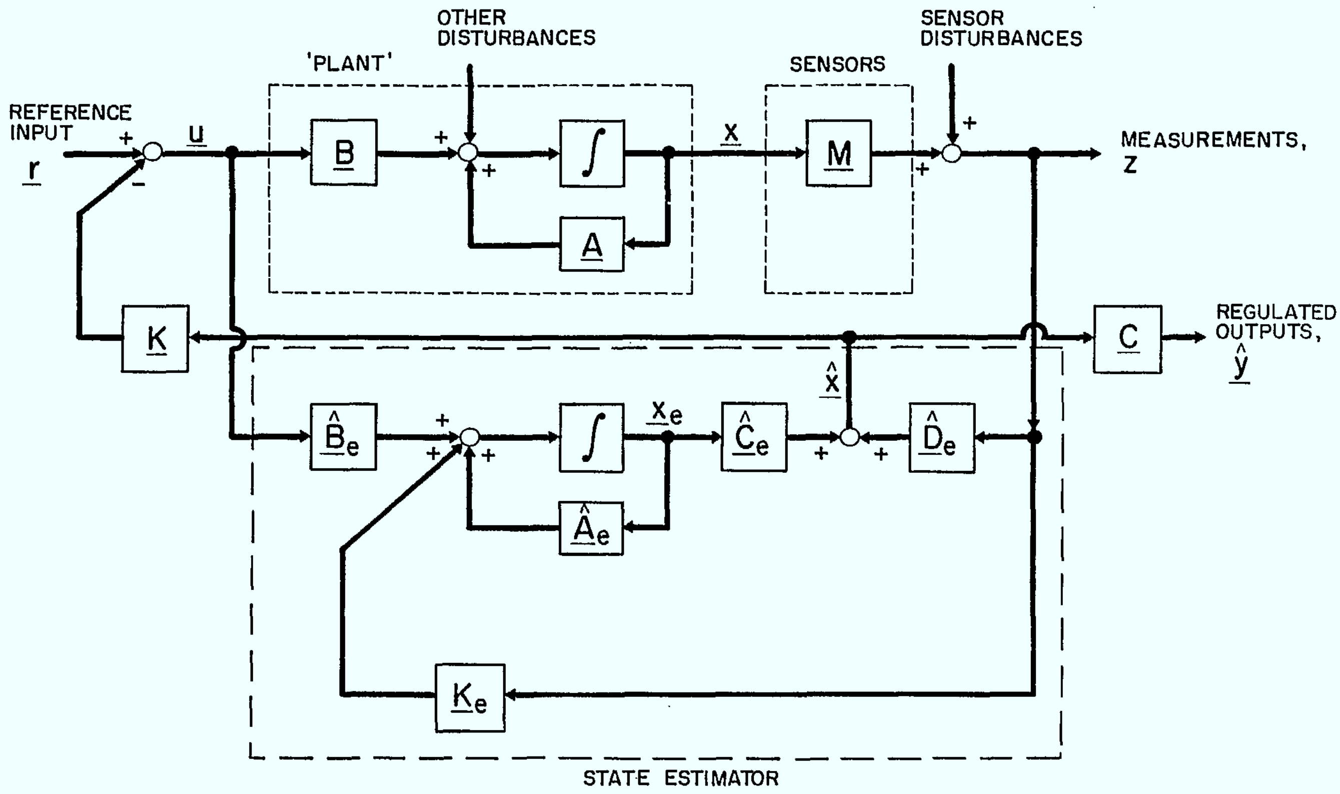
(5.2) $\qquad \underline{z} = \underline{Mx} \qquad$ (5.35b)

(5.20) $\qquad \dot{\underline{x}}_e = \hat{\underline{A}}_e\underline{x}_e + \hat{B}_e\underline{u} + \hat{\underline{K}}_e\underline{z} \qquad$ (5.35c)

(5.22) $\qquad \hat{\underline{x}} = \hat{\underline{C}}_e\underline{x}_e + \hat{\underline{D}}_e\underline{z} \qquad$ (5.35d)

(5.24) $\qquad \underline{u} = \underline{r} - \underline{K}\hat{\underline{x}} \qquad$ (5.35e)

where (see Fig. 5.2)



K = ESTIMATED-STATE FEEDBACK MATRIX

Ke = ESTIMATOR FEEDBACK MATRIX

Fig. 5.2: Feedback of State Estimated by a Reduced-Order Estimator — Actual and Assumed System Parameters

(5.17)	$\underline{\hat{A}}_e \triangleq \underline{\hat{\Gamma}\hat{A}\hat{N}}_\Delta$		(5.36a)
(5.17)	$\underline{\hat{B}}_e \triangleq \underline{\hat{\Gamma}\hat{B}}$		(5.36b)
(5.23)	$\underline{\hat{C}}_e \triangleq \underline{\hat{N}}_\Delta$		(5.36c)
(5.23)	$\underline{\hat{D}}_e \triangleq \underline{\hat{N}} + \underline{\hat{N}}_\Delta \underline{\dot{K}}_e$		(5.36d)
(5.17)	$\underline{\hat{K}}'_e \triangleq \underline{\hat{\Gamma}\hat{A}\hat{N}}$		(5.36e)
(5.21)	$\underline{\hat{K}}_e \triangleq \underline{\hat{K}}'_e + \underline{A}_e \underline{\dot{K}}_e \equiv \underline{\hat{\Gamma}\hat{A}\hat{D}}_e$		(5.36f)
(5.17)	$\underline{\hat{\Gamma}} \triangleq \underline{\hat{M}}_\Delta - \underline{\dot{K}}_e \underline{\hat{M}}$		(5.36g)

Our aim is to derive two differential equations, one for the 'plant state' $\underline{x}(t)$, and the other for the 'estimator error' $\underline{e}_z(t)$. These differential equations should also have their right sides expressed in $\underline{x}$ and $\underline{e}_z$ to form a system analogous to (4.13) for the full-order estimator.

We begin with the plant equation:

$$
\begin{aligned}
\underline{\dot{x}} &= \underline{Ax} + \underline{Bu} \\
&= \underline{Ax} + \underline{Br} - \underline{BK\hat{x}} \\
&= \underline{Ax} - \underline{BK}(\underline{x} - \underline{e}) + \underline{Br} \\
&= (A - \underline{BK})\underline{x} + \underline{BKe} + \underline{Br}
\end{aligned}
$$

It remains to express $\underline{e}$ in terms of $\underline{e}_z$. Before parameter errors were introduced, we had (5.26): $\underline{e} = \underline{N}_\Delta \underline{e}_z$. Now, with parameter errors, we must proceed carefully. It is true that

$$
\begin{aligned}
\underline{z} &= \underline{Mx} = (\underline{\hat{M}} + \Delta\underline{M})\underline{x} \\
\underline{z}_\Delta &= \underline{\hat{M}}_\Delta \underline{x}
\end{aligned}
\tag{5.37}
$$

$(\underline{\hat{M}}_\Delta = \underline{M}_\Delta)$. Then

$$
\underline{x} = (\underline{\hat{N}} + \Delta\underline{N})\underline{z} + (\underline{\hat{N}}_\Delta + \Delta\underline{N}_\Delta)\underline{z}_\Delta \tag{5.38}
$$

where $\Delta\underline{N}$ and $\Delta\underline{N}_\Delta$ were derived above to be as given in (5.34). On the other hand, $\underline{\hat{x}}$ exists only in the controller, so

$$\underline{\hat{x}} = \underline{\hat{N}z} + \underline{\hat{N}_\Delta \hat{z}_\Delta} \qquad (5.39)$$

Then

$$\begin{aligned} \underline{e} &= \underline{x} - \underline{\hat{x}} \\ &= (\Delta\underline{N})\underline{z} + \underline{\hat{N}_\Delta e_z} + (\Delta\underline{N_\Delta})\underline{z_\Delta} \\ &= \underline{\hat{N}_\Delta e_z} + (\Delta\underline{N})\underline{\hat{M}x} + (\Delta\underline{N_\Delta})\underline{\hat{M}_\Delta x} \end{aligned}$$

to first order in parameter errors. Using (5.34) and the orthonormality condition (5.8) gives

$$\underline{e} = \underline{\hat{N}_\Delta e_z} - \underline{\hat{N}}(\Delta\underline{M})\underline{x} \qquad (5.40)$$

Finally, then, the plant equation is

$$\underline{\dot{x}} = [\underline{A} - \underline{BK} - \underline{BKN}(\Delta\underline{M})]\underline{x} + \underline{BK\hat{N}_\Delta e_z} + \underline{Br} \qquad (5.41)$$

This is the first of the two system equations sought.

The second system equation desired is for $\underline{e_z}$. Starting with

$$\underline{\dot{e}_z} = \underline{\dot{z}_\Delta} - \underline{\dot{\hat{z}}_\Delta}$$

and using $\underline{\hat{M}_\Delta \dot{x}}$ for $\underline{\dot{z}_\Delta}$ and (5.16) for $\underline{\dot{\hat{z}}_\Delta}$, it can be shown after some effort that

$$\begin{aligned} \underline{\dot{e}_z} = {} & [\underline{\hat{\Gamma}}(\Delta\underline{A}) + (\Delta\underline{\Gamma})\underline{\hat{A}} - (\Delta\underline{B_e})\underline{K} - \underline{\hat{K}_e'}(\Delta\underline{M})]\underline{x} \\ & + [\underline{\hat{A}_e} + (\Delta\underline{B_e})\underline{K\hat{N}_\Delta}]\underline{e_z} + (\Delta\underline{B_e})\underline{r} \end{aligned} \qquad (5.42)$$

where

$$\begin{aligned} \Delta\underline{B_e} &\triangleq (\Delta\underline{\Gamma})\underline{B} + \underline{\Gamma}(\Delta\underline{B}) \\ \Delta\underline{\Gamma} &\triangleq -\underline{\dot{K}_e}(\Delta\underline{M}) \end{aligned} \qquad (5.43)$$

as is consistent with (5.17).

The overall closed-loop system is, therefore,

$$\begin{bmatrix} \underline{x} \\ \underline{e}_z \end{bmatrix}^{\bullet} = \begin{bmatrix} \underline{A} - \underline{BK} - \underline{BK\hat{N}}(\Delta\underline{M}) & \underline{BK\hat{N}}_\Delta \\ \hat{\underline{\Gamma}}(\Delta\underline{A}) + (\Delta\underline{\Gamma})\hat{\underline{A}} - (\Delta\underline{B}_e)\underline{K} - \hat{\underline{K}}'_e(\Delta\underline{M}) & \hat{\underline{A}}_e + (\Delta\underline{B}_e)\underline{K\hat{N}}_\Delta \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{e}_z \end{bmatrix}$$

$$+ \begin{bmatrix} B \\ \Delta\underline{B}_e \end{bmatrix} \underline{r} \qquad (5.44)$$

when parameter errors are included.

6. ACCELERATION FEEDBACK IN THE CONTROL OF FLEXIBLE SPACECRAFT

Four control strategies have been described in the preceding four sections. These have ranged from 'simple' measurement feedback to relatively sophisticated state estimators. Considerable attention was given to the untoward effects of errors in the assumed values of the system parameters; this attention was motivated by the realization that the interpretation of accelerometer feedback relies heavily on the structural math model being used. In this section, the parameter errors discussed thus far in the context of a general, linear, state-square formulation are expressed in terms of the errors in a spacecraft structural dynamics model. A second theme of this section is the entry into the system equations of the unknown forces and torques on the structure. With accelerometer feedback, these forces and torques not only disturb the structure, they disturb the control system itself that is attempting to control the sturcture.

6.1 Structural Model as State-Space Model

We begin by assuming that a structural model of the spacecraft of the following form is available:

$$\underline{\mathcal{M}}\ddot{\underline{q}} + \underline{\mathcal{D}}\dot{\underline{q}} + \underline{\mathcal{K}}\underline{q} = \underline{f}(t) \qquad (6.1)$$

After a standard vibration mode analysis, in which natural frequencies

ω_α and mode shapes $\underline{e}_\alpha$ are calculated, the structural model can be re-cast in modal form

$$\ddot{\underline{\eta}} + \underline{\tilde{\mathcal{D}}}\dot{\underline{\eta}} + \underline{\Omega}^2\underline{\eta} = \underline{\gamma}(t) \tag{6.2}$$

where $\underline{\eta}$ contains the modal coordinates and

$$\underline{\Omega} \triangleq \text{diag}\{\omega_1,\omega_2, \ldots\} \tag{6.3}$$

$$\underline{\tilde{\mathcal{D}}} \triangleq \underline{E}^T\underline{\mathcal{D}}\underline{E} \tag{6.4}$$

$$\underline{\gamma} \triangleq \underline{E}^T\underline{f} \tag{6.5}$$

$$\underline{E} \triangleq [\underline{e}_1 \quad \underline{e}_2 \quad \ldots] \tag{6.6}$$

The physical coordinates $\underline{q}(t)$ in the original model (6.1) are available from $\underline{\eta}$:

$$\underline{q} = \underline{E}\underline{\eta} \tag{6.7}$$

It is also true that $\underline{E}^T\underline{M}\underline{E} = \underline{1}$ and $\underline{E}^T\underline{K}\underline{E} = \underline{\Omega}^2$.

Equations (6.2 - 6.7) are quite standard, but inadequate in detail for our purposes because of the following two facts:

- For flexible vehicles, there are several 'rigid' modes. These have a natural frequency of zero, and piecewise-linear mode shapes. Moreover, there is no doubt (error) in these natual frequencies or mode shapes.
- Usually the number of physical coordinates (the dimension of $\underline{q}$) greatly exceeds the number of degrees of freedom wanted in the structural model. This leads to 'modal truncation' or, more generally, to 'mode selection.' After this process the dimension of $\underline{\eta}$ is much less than the dimension of $\underline{q}$.

To take these considerations into account, the modal matrix $\underline{E}$ is partitioned into three parts thus:

$$\underline{E} = [\underline{E}_r \quad \underline{E}_e \quad \underline{E}_t] \tag{6.8}$$

The subscripts $(\ )_r$, $(\ )_e$, and $(\ )_t$ mean 'rigid,' 'elastic' (retained), and elastic ('truncated'). The associated modal coordinates are $\underline{\eta}_r$, $\underline{\eta}_e$, $\underline{\eta}_t$, respectively, so that (6.7) becomes

$$\underline{q} = \underline{E}_r\underline{\eta}_r + \underline{E}_e\underline{\eta}_e \quad [+ \underline{E}_t\underline{\eta}_t] \tag{6.9}$$

The term in square brackets is, by definition, dropped.

The 'natural frequencies' associated with the 'rigid' modes are zero:

$$\underline{E}_r^T\underline{K}\underline{E}_r = \underline{O}$$

$$\underline{E}_r^T\underline{K}\underline{E}_e = \underline{O}$$

The remaining natural frequencies are collected in $\underline{\Omega}_e$:

$$\begin{aligned} \underline{E}_e^T\underline{K}\underline{E}_e &= \underline{\Omega}_e^2 \\ \underline{\Omega}_e &= \text{diag}\{\omega_1, \omega_2, \ldots\} \end{aligned} \tag{6.10}$$

Note that the frequencies have been re-ordered so that ω_1 is the first *nonzero* (i.e., elastic) natural frequency.

The relevant damping matrices are

$$\begin{aligned} \underline{E}_r^T\underline{\mathcal{D}}\underline{E}_r &= \underline{O} \\ \underline{E}_r^T\underline{\mathcal{D}}\underline{E}_e &= \underline{O} \\ \underline{\mathcal{D}}_e &\triangleq \underline{E}_e^T\underline{\mathcal{D}}\underline{E}_e \end{aligned} \tag{6.11}$$

Then the modal dynamic equations are, from (6.2),

$$\ddot{\underline{\eta}}_r = \underline{\gamma}_r(t) \tag{6.12}$$

$$\ddot{\underline{\eta}}_e + \underline{\mathcal{D}}_e\dot{\underline{\eta}}_e + \underline{\Omega}_e^2\underline{\eta}_e = \underline{\gamma}_e(t) - \underline{\mathcal{D}}_{et}\dot{\underline{\eta}}_t \tag{6.13}$$

with $\underline{\mathcal{D}}_{et} = \underline{E}_e^T\underline{\mathcal{D}}\underline{E}_t$. The term corresponding to truncated modes has been shifted to the right side of (6.13) to show that it now plays the role of an (unknown) disturbance.

The inputs to the modal equations, $\underline{\Upsilon}_r$ and $\underline{\Upsilon}_e$, consist of three types of inputs:

- known disturbances,
- control inputs (to compensate for the known disturbances),
- unknown disturbances.

To reflect this, let

$$\begin{aligned} \underline{\Upsilon}_r(t) &= \underline{\Upsilon}_{dr}(t) + \underline{\mathcal{B}_r u} \\ \underline{\Upsilon}_e(t) &= \underline{\Upsilon}_{de}(t) + \underline{\mathcal{B}_e u} \end{aligned} \tag{6.14}$$

For many spacecraft applications, the disturbances $\underline{\Upsilon}_{dr}$ and $\underline{\Upsilon}_{de}$ vary so slowly that they can be regarded as quasi-steady (e.g., they might be taken as constant for control system design). The control input variables are in $\underline{u}(t)$. Some contributions to $\underline{\Upsilon}_{dr}$ and $\underline{\Upsilon}_{de}$ may be known and calculable, others not. Also, for simplicity, we shall lump the term $-\underline{\mathcal{D}_{et}\dot{n}_t}$ in (6.13) in with $\underline{\Upsilon}_{de}(t)$ and call it $\underline{\Upsilon}^*_{de}$ to remind us that there is a dynamic 'spillover' term in $\underline{\Upsilon}^*_{de}$. This brings us to the final form for structural model:

$$\begin{aligned} \underline{\ddot{n}}_r &= \underline{\mathcal{B}_r u} + \underline{\Upsilon}_{dr} \\ \underline{\ddot{n}}_e + \underline{\mathcal{D}_e \dot{n}_e} + \underline{\Omega^2_e n_e} &= \underline{\mathcal{B}_e u} + \underline{\Upsilon}^*_{de} \end{aligned} \tag{6.14}$$

This can be placed in first-order (state-space) form, and thus adapted to the results of the previous four sections, by defining the state vector to be

$$\underline{x} \triangleq \mathrm{col}\{\underline{n}_r, \underline{n}_e, \underline{\dot{n}}_r, \underline{\dot{n}}_e\} \tag{6.15}$$

Then the 'plant' model is

$$\underline{\dot{x}} = \underline{Ax} + \underline{Bu} + \underline{v} \tag{6.16}$$

with

$$\underline{A} \triangleq \begin{bmatrix} \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{1} \\ \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & -\underline{\Omega}_e^2 & \underline{0} & -\underline{\mathcal{D}}_e \end{bmatrix} \tag{6.17}$$

$$\underline{B} \triangleq \begin{bmatrix} \underline{0} \\ \underline{0} \\ \underline{\mathcal{B}}_r \\ \underline{\mathcal{B}}_e \end{bmatrix} \qquad \underline{v} \triangleq \begin{bmatrix} 0 \\ \underline{0} \\ \underline{\Upsilon}_{dr} \\ \underline{\Upsilon}_{de}^{*} \end{bmatrix} \tag{6.18}$$

To complete the state-space model, and to place us in a position to use the results of Sections 2 - 5, we need a formulation for the measurements, $\underline{z}$. In this report, we consider acceleration measurements exclusively, although there will in general be other types of sensors as well. The outputs from such additional sensors can simply be concatenated with the accelerometer outputs considered below. With accelerations measured, we have

$$\underline{z}(t) = \underline{S}\ddot{\underline{q}} + \underline{n}(t) \tag{6.19}$$

where $\underline{n}(t)$ is measurement error, and $\underline{S}$ is a selection matrix. If the coordinates whose accelerations are measured are all contained in the physical model (6.11), then $\underline{S}$ consists of 1's and 0's. If the output axis of an accelerometer is skewed to the physical coordinates in the model, then the corresponding row of the selection matrix $\underline{S}$ consists of direction cosines (some of which may, however, be 1 or 0).

Now to find the measurement matrix $\underline{M}$ used throughout this report, we must use the structural dynamics model in (6.19). Employing (6.9) and (6.14), one arrived at

$$\underline{z}(t) = \underline{Mx}(t) + \underline{z}_{\surd}(t) + \underline{z}_{?}(t) + \underline{n}(t) \tag{6.20}$$

where

$$\underline{M} = [\underline{O} \quad -\underline{S}\underline{E}_e\underline{\Omega}_e^2 \quad \underline{O} \quad -\underline{S}\underline{E}_e\underline{\mathcal{D}}_e] \tag{6.21}$$

$$\underline{z}_{\surd}(t) = \underline{S}(\underline{E}_r\underline{\mathcal{B}}_r + \underline{E}_e\underline{\mathcal{B}}_e)\underline{u} + \underline{S}(\underline{E}_r\underline{\gamma}_{dr\surd} + \underline{E}_e\underline{\gamma}^*_{de\surd}) \tag{6.22}$$

$$\underline{z}_?(t) = \underline{S}(\underline{E}_r\underline{\gamma}_{dr?} + \underline{E}_e\underline{\gamma}^*_{de?}) + \underline{S}\underline{E}_t\ddot{\underline{n}}_t \tag{6.23}$$

However, (6.20) does not include the possibility of *feedforward* (recall the simple example in Section 1.2). The known measurement disturbance $\underline{z}_{\surd}$ can be eliminated by feeding it forward to cause cancellation. The feedforward would enter at the point labeled "sensor disturbances" in Figs. 2.1, 2.2, 3.1, 4.1, 4.2, 5.1 and 5.2. With feedforward applied, the measurement equation becomes

$$\text{(with feedforward)} \qquad \underline{z} = \underline{M}\underline{x} + \underline{z}_? + \underline{n} \tag{6.24}$$

This completes the state-space model of the flexible space structure.

Note in particular that external disturbances to the structure also directly disturb the measurement as well. One of the favorite assumptions conventionally made — that plant disturbances and measurement disturbances are unrelated — is clearly invalid when accelerometers are used as sensors.

6.2 Model Errors

Having found what the system matrices $\underline{A}$, $\underline{B}$, and $\underline{M}$ are for a flexible space vehicle, it is not difficult to find expressions for the errors caused in $\underline{A}$, $\underline{B}$ and $\underline{M}$ due to structural modeling errors. From (6.17),

$$\Delta\underline{A} = \begin{bmatrix} \underline{O} & \underline{O} & \underline{O} & \underline{O} \\ \underline{O} & \underline{O} & \underline{O} & \underline{O} \\ \underline{O} & \underline{O} & \underline{O} & \underline{O} \\ \underline{O} & -2\hat{\underline{\Omega}}_e\Delta\underline{\Omega}_e & \underline{O} & -\Delta\underline{\mathcal{D}}_e \end{bmatrix} \tag{6.25}$$

and, from (6.18),

$$\Delta \underline{B} = \begin{bmatrix} \underline{O} \\ \underline{O} \\ \Delta \underline{\mathcal{B}}_r \\ \Delta \underline{\mathcal{B}}_e \end{bmatrix} \tag{6.26}$$

Finally, from (6.21),

$$\Delta \underline{M} = [\underline{O} \quad -\underline{S}(\Delta \underline{E}_e \underline{\hat{\Omega}}_e^2 + 2\underline{\hat{E}}_e \underline{\hat{\Omega}}_e \Delta \underline{\Omega}_e) \quad \underline{O} \quad -\underline{S}(\Delta \underline{E}_e \underline{\hat{\mathcal{D}}}_e + \underline{\hat{E}}_e \Delta \underline{\mathcal{D}}_e)] \tag{6.27}$$

This raises naturally the question of how to estimate modal errors such as $\Delta \underline{\Omega}_e$, $\Delta \underline{\mathcal{D}}_e$, etc. There are no hard and fast rules for doing this and one must be satisfied with error estimates that are "not unreasonable."

7. CONCLUDING REMARKS

A number of conclusions emerge from the preceding formulations:

- By using a mathematical model of the structural dynamics, accelerometer measurements can be interpreted as a measurement of (a linear combination of) displacement and displacement rate.
- This interpretation is prone to error if there are errors in the structural model. In other words, there can be "measurement disturbances" owing to errors in mode shapes, modal frequencies, damping factors, etc.
- Any forces or torques that disturb the flexible vehicle also directly disturb the interpretation of accelerometer measurements. This disturbance is over and above simple measurement error and the parameter error mentioned above.
- Any *known* force or torque, of which control forces and torques are an important example, can be fed forward in the control system design to cancel part of the disturbance error mentioned above.
- For control systems that include either a full-order state estimator or a reduced-order state estimator, the estimation error does

not approach zero asymptotically when parameter errors are taken into account. The estimator's error will be continually excited by a non-zero state and by external disturbances (e.g., unmodeled torques).

Accelerometers can form part of a successful control system design, but only if the factors mentioned above are handled properly.