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# THE USE AND PROPERTIES OF WINDOW FUNCTIONS IN SPECTRUM ANALYSIS

by

A.W.R. Gilchrist

DEPARTMENT OF COMMUNICATIONS
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# THE USE AND PROPERTIES OF WINDOW FUNCTIONS IN SPECTRUM ANALYSIS

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ABSTRACT

This report is a handbook on spectrum windows. The subject is introduced by a discussion of the fundamentals of spectrum analysis, which shows how the four kinds of window arise in the practical estimation of spectra. The modern direct method of computation using sampled data and the DFT is emphasized, and it is shown that the effects of discrete analysis on the computed spectrum can be described completely by the appropriate spectrum window. Exact closed-form expressions are derived for the spectrum windows corresponding to most of the functions that have been proposed for use as data windows; the cases of continuous and discrete analysis are treated separately for each function. Some of these results are not available elsewhere. The characteristics of the spectrum windows are also presented in graphical form.

## 1. INTRODUCTION

The standard work on power-spectrum analysis (Ref. 1), which was published in 1959, deals almost exclusively with the indirect method, in which the power spectrum is obtained through the autocovariance. The autocovariance is computed from the signal values. The Fourier transform of a modified version of the autocovariance gives the estimate of the power spectrum. In recent years the indirect method has been almost completely superseded by the direct method of analysis, in which the Fourier transformation is performed upon the (modified) signal amplitudes themselves. The squared modulus of the transform gives the estimate of the power spectrum. Each of these methods involves modification, in the one case of the autocovariance, and in the other of the signal. The nature and effect of these modifications is the subject of this report.

The change from the indirect to the direct technique has come about as a consequence of the introduction of the Fast Fourier Transform (FFT) (Ref. 2), an algorithm for computing the Discrete Fourier Transform (DFT) with great efficiency. As its name implies, the DFT is adapted to the use of sampled data and digital computation. The direct method in current practice is therefore a digital one, although either analog or digital computation could be used with either the direct or the indirect method. This report covers both methods of analysis and both types of computation, but its primary concern is with current practice: the direct method and digital computation.

We now introduce some of the nomenclature of the subject, and describe in general terms the modifications carried out on the signal or on the autocovariance. If we denote either function by $z(t)$, and its modified version by $z_{mod}(t)$, the modification can be described by a function $g(t)$ such that

$$z_{mod}(t) = g(t) \cdot z(t).$$

This modification is called "windowing" $z(t)$, and $g(t)$ is the "window" function. The names arise from the analogy in which we consider t as a spatial coordinate, and $z(t)$ as the distribution of intensity of a light source; then, if $g(t)$ represents the transparency of a *window* through which the source is viewed, $z_{mod}(t)$ gives the observed distribution of intensity.

In the case of the indirect method, $z(t)$ is the autocovariance, and $g(t)$ is called the autocovariance window or the lag window. Let the Fourier transforms of $g(t)$, $z(t)$, and $z_{mod}(t)$ be denoted $G(f)$, $Z(f)$, and $Z_{mod}(f)$. According to the outline of the method given above, $Z_{mod}(f)$ is the estimate of the power spectrum; from the properties of Fourier transforms we can write

$$Z_{mod}(f) = G(f) * Z(f)$$

where * denotes convolution. $G(f)$ is called the power-spectrum window corresponding to the lag window $g(t)$. It is obviously important to know how the choice of a particular lag window affects the spectral estimates, and conversely to be able to select a lag window that leads to estimates having desirable properties. These matters are dealt with in subsequent sections.

When the direct method is used, $z(t)$ represents the signal, and $g(t)$ is referred to as the data window. $Z(f)$ is the Fourier transform of the signal; it is also called the complex amplitude spectrum of the signal. $G(f)$ is the amplitude-spectrum window. It is shown in Section 2 that the use of a data window imposes a window $H(f)$ on the power spectrum, and the relation between $H(f)$ and $g(t)$ is derived. Thus three windows are involved in power-spectrum analysis by the direct method, but they are not applied separately, or in cascade; if $g(t)$ is regarded as basic, $G(f)$ represents the effect of $g(t)$ in the transform plane, and $H(f)$ represents the effect of $g(t)$ on the power spectrum. If the direct method is used to obtain the autocovariance, a fourth window comes into play: the lag window corresponding to the data window used. The relation of this lag window to $g(t)$ is also given in Section 2. Again, the lag window is not applied as a separate operation; it arises as a consequence of having applied $g(t)$.

In Sections 3 and 4 the functions G(f) and H(f) are derived for the various functions g(t) that have been proposed for use as data windows or lag windows. In each case the characteristics are determined for both continuous data (and analog computation), and discrete or sampled data (and digital computation).

## 2. FUNDAMENTALS OF SPECTRUM ANALYSIS

In this section the important concepts of power-spectrum analysis are reviewed. The first subsection introduces the notion of the true autocovariance and the true power spectrum of a signal. To relate it to current methods of analysis, the power spectrum is expressed directly in terms of the signal, rather than indirectly through the autocovariance. For observed signals that are random, and, at least conceptually, of infinite duration, the true power spectrum cannot be determined, since infinitely long records would be required. Approximations that can be measured are discussed in Section 2.2, which shows how the various kinds of windows arise in spectrum analysis. It is shown in Section 2.3 that the effect of using sampled data and the DFT can be described by a modification of the window functions that apply to continuous data. The properties of the spectrum windows corresponding to specific data windows are examined in subsequent sections, for both the continuous and discrete cases.

### 2.1 THE POWER SPECTRUM OF A SIGNAL

Suppose that the signal x(t) is real and infinitely long, and has a mean value of zero. In order to define the power spectrum of x(t) we first consider an associated function, the autocovariance $C(\tau)$, which is the average of the product of x(t) with a displaced version of itself:

$$C(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t) \cdot x(t + \tau)\, dt. \qquad \ldots\ldots (1)$$

Equation (1) implies a further restriction on the signal: since $C(\tau)$ is written as a function of $\tau$ alone, the function defined on the right-hand side must be independent of the choice of origin. This condition requires that the signal have a kind of temporal homogeneity that is known as (wide-sense) stationarity.

Equation (1) can be written in the alternative form:

$$C(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} y(t)\, y(t + \tau)\, dt\ , \qquad \ldots\ldots (2)$$

where

$$y(t) = W_0(t)x(t) \qquad \ldots\ldots (3)$$

and

$$W_0(t) = \begin{cases} 1\ , & 0 \le t \le T \\ 0\ , & \text{otherwise.} \end{cases} \qquad \ldots\ldots (4)$$

We now introduce the Fourier transform Y(f) of the modified signal y(t), for which the following pair of relations hold:

$$Y(f) = \int_{-\infty}^{\infty} y(t) \exp\{-2\pi ift\}\, dt, \qquad \ldots\ldots (5)$$

$$y(t) = \int_{-\infty}^{\infty} Y(f) \exp\{2\pi ift\}\, df\ . \qquad \ldots\ldots (6)$$

C(τ) can be expressed in terms of Y(f) as follows:

$$C(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} y(t) \int_{-\infty}^{\infty} Y(f) \exp\{2\pi if(t+\tau)\}\, df\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} Y(f) \exp\{2\pi if\tau\} \int_{-\infty}^{\infty} y(t) \exp\{2\pi ift\}\, dt\, df$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} Y(f)\, Y(-f) \exp\{2\pi if\tau\}\, df$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |Y(f)|^2 \exp\{2\pi if\tau\}\, df. \qquad \ldots\ldots (7)$$

Equation (7) is in the form of the relation between the autocovariance and the power spectrum S(f) given by the Wiener-Khinchin theorem (Ref. 3):

$$C(\tau) = \int_{-\infty}^{\infty} S(f) \exp\{2\pi if\tau\}\, df \qquad \ldots\ldots (8)$$

From Eqns. (7) and (8) we obtain the formal identity:

$$S(f) = \lim_{T\to\infty} \frac{1}{T} |Y(f)|^2 \qquad \ldots\ldots (9)$$

$$= \lim_{T\to\infty} \frac{1}{T} \left| \int_{-\infty}^{\infty} y(t) \exp\{-2\pi ift\}\, dt \right|^2 \qquad \ldots\ldots (10)$$

$$= \lim_{T\to\infty} \frac{1}{T} \left| \int_{0}^{T} x(t) \exp\{-2\pi ift\}\, dt \right|^2 \qquad \ldots\ldots (11)$$

Equation (11) defines the true power spectrum of the signal x(t) directly in terms of the signal values. Equations (8) and (11) are together equivalent to Eqn. (1), and give the true autocovariance of the signal. Note that neither S(f) nor C(τ) is determinable for an observed signal, since the definitions imply that the observation time must be infinite. Analysis of finite records can yield only approximations at best, even for stationary signals. The value of such analysis depends upon our knowledge of the nature and quality of the approximation imposed by the record length and, possibly, by the method of analysis. We now turn to a consideration of these matters.

## 2.2 PRACTICAL ESTIMATION OF THE POWER SPECTRUM

It would appear, at first glance, that the function

$$P(f) = \frac{1}{T'} \left| \int_0^{T'} x(t) \cdot \exp\{-2\pi i f t\}\, dt \right|^2 \qquad .....(12)$$

should provide a reasonable approximation to the power spectrum S(f), as defined in Eqn. (11), when the signal x(t) is stationary. This function is known as the periodogram. When x(t) is a random signal, P(f) provides spectrum estimates of notoriously poor statistical stability, however long the duration T' of the signal. The stability can be improved if the available section of the signal is divided into several subsections of length T and the average of the periodograms is used as the spectrum estimate. A more extensive class of estimates can be obtained if the values of x(t) on the interval [0,T] are given arbitrary weightings--i.e., if x(t) is multiplied by a data window $W_n(t)$. The function $W_n(t)$ is defined arbitrarily on the interval [0,T], and is zero outside this interval; the subscript n denotes the particular function chosen. We therefore take the following function as our approximation to the true power spectrum:

$$S_n(f) = \text{Ave}\, \frac{1}{T} \left| \int_{-\infty}^{\infty} W_n(t) \cdot x(t) \cdot \exp\{-2\pi i f t\}\, dt \right|^2 , \qquad .....(13)$$

where Ave denotes the averaging of several periodograms, as described above. It may be considered as a statistical average over a finite ensemble, or as a finite time average. If the average is over the periodograms computed from K contiguous segments of the signal, each of length T, and the first segment starts at $t = t_0$, the explicit form of Eqn. (13) is:

$$S_n(f) = \frac{1}{K} \sum_{k=0}^{K-1} \left[ \frac{1}{T} \left| \int_{-\infty}^{\infty} W_n\{t - (t_0 + kT)\} \cdot x(t) \cdot \exp\{-2\pi i f t\}\, dt \right|^2 \right].$$

The effects of the weighting function will be examined later.

It should be noted that Eqn. (13) differs in an essential way from Eqn. (10), even if the averaging is supposed to be over an infinite ensemble, and $W_n(t)$ is replaced by the unit weighting function $W_0(t)$. According to Eqn. (13) the signal is analyzed in sections of fixed length T, while Eqn. (10) represents analysis of unlimited sections of the signal.

The approximation $S_n(f)$ to the power spectrum corresponds to an approximation $C_n(\tau)$ to the autocovariance function:

$$C_n(\tau) = \text{Ave}\, \frac{1}{T} \int_{-\infty}^{\infty} W_n(t) \cdot x(t)\, W_n(t + \tau) \cdot x(t + \tau)\, dt. \qquad .....(14)$$

$C_n(\tau)$ is the Fourier transform of $S_n(f)$. If the operation denoted by Ave were over an infinite ensemble (or a record of infinite duration), the right-hand side of Eqn. (14) would give the expected value of the estimator $C_n(\tau)$, which is denoted by $E[C_n(\tau)]$. In this case Eqn. (14) may be written:

$$E[C_n(\tau)] = D_n(\tau) \quad C(\tau), \qquad .....(15)$$

where $C(\tau)$ is the true autocovariance function defined by Eqn. (1), and

$$D_n(\tau) = \frac{1}{T} \int_{-\infty}^{\infty} W_n(t) \cdot W_n(t + \tau)\, dt. \qquad .....(16)$$

$D_n(\tau)$ is the autocovariance or lag window corresponding to the data window $W_n(t)$.

The same assumption (that Ave denotes averaging over an infinite ensemble) applied to Eqn. (13) gives the expected value of $S_n(f)$. The result is:

$$E[S_n(f)] = \mathcal{V}_n(f) \star S(f), \qquad .....(17)$$

where $S(f)$ is the true power spectrum, $\mathcal{V}_n(f)$ is the Fourier transform of $D_n(\tau)$ and $\star$ denotes convolution. $\mathcal{V}_n(f)$ can be expressed as:

$$\mathcal{V}_n(f) = \frac{1}{T} |\mathcal{W}_n(f)|^2. \qquad .....(18)$$

$\mathcal{W}_n(f)$ is the amplitude-spectrum window corresponding to the data window $W_n(t)$-- i.e., the Fourier transform of $W_n(t)$. The function $\nu_n(f)$ is the power-spectrum window corresponding to the data window $W_n(t)$.

Equation (17) shows that the estimates of the power spectrum given by $S_n(f)$ have an expected value equal to the convolution of the true power spectrum with the window function $\mathcal{V}_n(f)$. The scatter of the estimates about the expected value is determined by the number of segments of the signal used to form the average specified in Eqn. (13). It is usual to take the distribution of $S_n(f)$ to be chi-squared, with 2K degrees of freedom, where K is the number of independent signal segments used. This assumption is exact when the distribution of the signal amplitudes is Gaussian. On this basis, the standard deviation of the estimate $S_n(f)$ is $(1/\sqrt{K})$ times its expected value, given by Eqn. (17). The scatter of the estimates can therefore be made arbitrarily small if a sufficient number of segments is used to form the average, provided, of course, that the signal remains stationary over the period covered by the analysis.

We have thus succeeded in describing, in general analytical form, the nature of the approximation involved in our estimate of the power spectrum. It remains to describe in specific terms the estimates obtained with a given data window, and to consider the inverse question: the design of the data window to obtain estimates having desirable properties. Some answers to these questions are provided in later sections.

It is worth pointing out that averaging is an essential part of the definitions of the estimators even when the signals are deterministic. The estimate of the power spectrum of such a signal, obtained in accordance with Eqn. (13) except for the omission of averaging, is not, in general, equal to the convolution of the true power spectrum with the window $V_n(f)$. This is illustrated in Appendix A.

## 2.3 THE INFLUENCE OF SAMPLING AND DISCRETE ANALYSIS

The Fourier transform of a continuous signal x(t) is defined by

$$X_0(f) = \int_{-\infty}^{\infty} x(t) \exp\{-2\pi i f t\}\, dt. \qquad .....(19)$$

In the previous sections we have been led to consider the integral between finite limits:

$$X(f) = \int_0^T x(t) \exp\{-2\pi i f t\}\, dt, \qquad .....(20)$$

and have seen how, for a stationary signal, the autocorrelation function and the power spectrum derived from Eqn. (20) can be related to the true functions that would be obtained from a signal of infinite duration. The 'true' transform (in the same sense) of a random signal is neither measurable nor meaningful, since X(f) is not a statistical average, but describes exactly the particular member of the ensemble (or section of the signal) from which it is derived. If $W_0(t)$ is a function having the value unity on the interval [0,T] and zero elsewhere, Eqn. (20) can be written in the alternative form:

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot W_0(t) \cdot \exp\{-2\pi i f t\}\, dt. \qquad .....(21)$$

When digital computation techniques are to be used the signal must be sampled. Suppose that N samples are taken, at $t = n\Delta t$, $n = 0, 1, \ldots, N-1$, where the inter-sample period $\Delta t$ is equal to T/N. The sampling theorem[4] shows that these samples represent the continuous signal exactly, provided that the signal has no component of frequency greater than $f_{max}$, and that the sampling rate satisfies the relation:

$$f_{max} \leq \frac{N}{2T} . \qquad .....(22)$$

The samples may be used to approximate X(f) as follows:

$$\tilde{X}(f) = \Delta t \sum_{n=0}^{N-1} x(n\Delta t) \exp\{-2\pi i f n \Delta t\} . \qquad .....(23)$$

Equation (23) defines $\tilde{X}(f)$ as a continuous function of f. Let $x_n$ denote $x(n\Delta t)$, and $\{x_n\}$ the sequence of sample values; similarly, let $\tilde{X}_k$ denote $\tilde{X}(f_k)$, where $f_k$ is a given frequency. For any N distinct frequencies $f_k$, Eqn. (23) yields N linear equations in the N variables $x_n$. If the rank of the matrix of coefficients (the exponentials) is N, the finite sequence $\{\tilde{X}_k\}$ uniquely determines the sequence $\{x_n\}$, and vice versa. It follows that these N values

$\tilde{X}_k$ completely determine the continuous function $\tilde{X}(f)$. It is shown below that this condition is satisfied if we choose:

$$f_k = k\,\Delta f \quad , \quad -\frac{N}{2} \le k \le \frac{(N-1)}{2} \qquad \text{.....(24)}$$

$$\Delta f = \frac{1}{T} \,. \qquad \text{.....(25)}$$

In terms of these discrete frequencies, Eqn. (23) becomes

$$\tilde{X}_k = \frac{T}{N} \sum_{n=0}^{N-1} x_n \exp\left\{-2\pi i k \frac{n}{N}\right\}. \qquad \text{.....(26)}$$

Equation (26) is the standard form for the Discrete Fourier Transform (DFT), except for the factor T which is usually dropped; this is equivalent to measuring time in units of T and frequency in units of 1/T. It should be noted that $\tilde{X}_k$, defined by Eqn. (26), is periodic, of period N.

The DFT can be written in a form analogous to Eqn. (21):

$$\tilde{X}_k = \int_{-\infty}^{\infty} x(t)\, W_0'(t) \exp\{-2\pi i f t\}\, dt, \quad f = \frac{k}{T} \,, \qquad \text{.....(27)}$$

where the sampled-data window $W_0'(t)$ is defined by:

$$W_0'(t) = W_0(t)\ \text{Comb}(t) \,, \qquad \text{.....(28)}$$

and

$$W_0(t) = \begin{cases} 1, & -\frac{\Delta t}{2} \le t \le T - \frac{\Delta t}{2} \\ 0, & \text{otherwise} \end{cases} \qquad \text{.....(29)}$$

$$\text{Comb}(t) = \Delta t \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t). \qquad \text{.....(30)}$$

Equation (27) exhibits the relationship between the DFT and the Fourier transform for continuous functions.

Although we have derived the DFT as an approximation to the ordinary Fourier transform, it may be considered as a transform in its own right. In particular, the inverse transform exists. Consider the function

$$y_n = \frac{1}{T} \sum_{k=0}^{N-1} \tilde{X}_k \exp\left\{2\pi i k \frac{n}{N}\right\} \qquad \text{.....(31)}$$

$$= \frac{1}{T} \sum_{k=0}^{N-1} \left[ \frac{T}{N} \sum_{\ell=0}^{N-1} x_\ell \exp\left\{-2\pi i k \frac{\ell}{N}\right\} \right] \exp\left\{2\pi i k \frac{n}{N}\right\}$$

$$= \frac{1}{N} \sum_{\ell=0}^{N-1} x_\ell \sum_{k=0}^{N-1} \exp\left\{-2\pi i k \frac{(\ell - n)}{N}\right\}$$

$$= \frac{1}{N} \sum_{\ell=0}^{N-1} x_\ell \, N \, \delta(\ell - n)$$

$$= x_n \, .$$

Hence

$$x_n = \frac{1}{T} \sum_{k=0}^{N-1} \tilde{X}_k \exp\left\{2\pi i k \frac{n}{N}\right\} . \qquad \ldots\ldots(32)$$

Equation (32) gives the inverse transform corresponding to the DFT defined by Eqn. (26). The existence of the inverse implies, and is implied by, the non-singularity of the matrix of coefficients of $\{x_n\}$, as was asserted above. Note that although Eqn. (32) yields the correct values of $x_n$ for $n = 0,1,\ldots, N-1$, the values of $x_n$ are not zero outside this range, but repeat cyclically with period N.

Equations (21) and (27) show that the effect of using discrete data in spectrum analysis is completely described by substituting the data window $W_0'(t)$ for the continuous-data window $W_0(t)$. It is evident that a similar description holds if sampling is applied to an arbitrarily weighted signal. Equation (18) of the previous section shows how the use of a data window affects the measured power spectrum; the effect is given in terms of $\mathcal{W}_n(f)$, the amplitude-spectrum window corresponding to $W_n(t)$; $W_n(t)$ and $\mathcal{W}_n(f)$ form a Fourier-transform pair. Since the data windows of Eqns. (21) and (27) are different, so are the corresponding amplitude-spectrum and power-spectrum windows. The Fourier-transform pairs considered in Section 3 are therefore treated separately for the cases of continuous data (and analog computation), and sampled data (and digital computation). It should be remembered, despite the obvious differences between the data windows, that Eqn. (27) was derived as an approximation to Eqn. (21); we therefore expect similarities between corresponding frequency-domain windows for continuous and sampled data. The extent to which this expectation is realized will be seen in Section 3.

## 3. SPECIFIC WINDOW PAIRS

In this section most of the functions that are commonly used as data windows or lag windows, as well as a few others, are defined, and their transforms--the corresponding complex amplitude-spectrum window or the power-spectrum window--are derived. In each case results are given for both continuous and discrete methods of analysis. Since there is no need (in this section) to distinguish between data windows and lag windows, or between amplitude-spectrum windows and power-spectrum windows, the functions of time are referred to as data windows, and their transforms as spectrum windows.

Data windows are denoted by $W_n(t)$ or $W_n'(t)$; the subscript is a serial number to identify the function concerned, and the dash indicates that sampling is used. The spectrum windows corresponding to these data windows are written with script capitals: $\mathcal{W}_n(f)$ or $\mathcal{W}_n'(f)$.

A second system of notation has been found useful, and is employed in parallel with the first. Data windows can often be described as the product of two or more simpler functions. For example, in Section 3.1 we have $W_0'(t)$ = Box(t) • Comb(t). These simpler functions recur as factors of several data-window functions. The product form can be shortened slightly by running together the names of the factor functions: Boxcomb(t) = Box(t) • Comb(t). In this example, Boxcomb(t) is simply an alternative designation for $W_0'(t)$. The transform of one of the factor functions or their products is denoted by the descriptive name, prefixed by F as part of the name. Thus Fboxcomb(f) is a descriptive synonym for $\mathcal{W}_0'(f)$.

## 3.1 THE RECTANGULAR DATA WINDOW

### 3.1.1 Continuous Data

The rectangular window for continuous data is defined by:

$$W_0(t) = \text{Box}(t) = \begin{cases} 1, & \tau_1 \leq t \leq T + \tau_1 \\ 0, & \text{otherwise.} \end{cases} \qquad .....(33)$$

When a function x(t) is multiplied by Box(t) the product function is zero everywhere except on the interval of length T starting at $t = \tau_1$, and on this interval it is identical to x(t). Box(t) is therefore the analytical representation of finite duration.

Let X(f), Fbox(f), and $\hat{X}(f)$ denote the transforms of x(t), Box(t), and {x(t) • Box(t)}, respectively. The relation between $\hat{X}(f)$ and X(f) is given by the theorem that the transform of a product is the convolution of the transforms:

$$\hat{X}(f) = X(f) \star \text{Fbox}(f)$$

$$= \int_{-\infty}^{\infty} X(f') \,\text{Fbox}(f - f')\, df'. \qquad .....(34)$$

According to Eqn. (34), $\hat{X}(f)$ is derived from X(f) by distributing each element of the latter over all frequencies, and superimposing these distorted elements. The elements are distributed according to the shape of Fbox(f), translated along the frequency axis so that it is centred on the element in question. If $\hat{X}(f)$ is to reproduce X(f) exactly, Fbox(f) must be a delta function, for then $\hat{X}(f) = X(f) \star \delta(f) = X(f)$. However this is impossible, since $\delta(f)$ is the transform of a unit function of infinite extent: in other words, we can determine the exact transform of x(t) only if we know the values of x(t) for all t. At best, therefore, $\hat{X}(f)$ can be only a reasonably close approximation to X(f). This will be so if Fbox(f) approximates a delta function--it should have a high, narrow peak at f = 0 and be as small as possible elsewhere. Let us see how close to this specification Fbox(f) actually is.

$$\text{Fbox}(f) = \int_{-\infty}^{\infty} \text{Box}(t) \exp\{-2\pi i f t\}\, dt \qquad \ldots\ldots(35)$$

$$= \int_{\tau_1}^{T+\tau_1} 1 \exp\{-2\pi i f t\}\, dt$$

$$W_0(f) = \text{Fbox}(f) = T\left\{\frac{\sin \pi f T}{\pi f T}\right\} \exp\{-i\pi f\ (T + 2\tau_1)\}. \qquad \ldots\ldots(36)$$

Equation (36) gives the spectrum window corresponding to the data window $W_0(t)$. These windows are illustrated in Figure 1. We recognize the exponential in Eqn. (36) as the frequency-domain representation of a translation in the time domain; this factor would vanish if the data window were centred on $t = 0$: i.e., if $\tau_1 = -T/2$. Fbox(f) has the shape of the curve $(\sin x)/x$. It does indeed have a peak at $f = 0$, which becomes higher and narrower as T increases; its height is T and its width 1/T (the 'width' used here is half the distance between the points, closest to $f = 0$, at which Fbox(f) becomes zero). So far it seems that Fbox(f) is a fairly good approximation to the ideal delta function. However Fbox(f) has many secondary peaks, or sidelobes; the function oscillates, passing through zero at frequency intervals of 1/T. The sidelobes are smaller than the main lobe, and decrease as 1/f, according to Eqn. (36) but they are still undesirably large. The largest ones, centred on $f = \pm 3/2T$, have a height about 21 per cent of that of the main lobe. While increasing the duration of the signal makes the main lobe higher and narrower, it does not change the relative heights of the sidelobes; the $(\sin x)/x$ pattern, however, is compressed along the frequency axis, so that the first sidelobe occurs closer to $f = 0$, and the height of the sidelobe at any fixed non-zero frequency does become smaller.

### 3.1.2 Sampled Data

The rectangular window for sampled data is defined by:

$$W_0'(t) = \text{Box}(t) \cdot \text{Comb}(t)\ , \qquad \ldots\ldots(37)$$

where Box(t) is defined by Eqn. (33), and

$$\text{Comb}(t) = \Delta t \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t). \qquad \ldots\ldots(38)$$

$\Delta t$ is the inter-sample period, T/N, and N is the number of samples. Comb(t) represents the sampling process: when a function x(t) is multiplied by Comb(t), and integrated with respect to t, the result is a time series representing the values of the function at the points $t = 0, \pm\Delta t, \pm 2\Delta t$, etc. The delta functions in Eqn. (38) are given the weights $\Delta t$ to make the integral of $x(t) \cdot \text{Comb}(t)$ approximate the integral of x(t). In our use of Comb(t) the integration occurs in the process of Fourier transformation, and we consider the product $x(t) \cdot \text{Comb}(t)$ as representing the sampled version of x(t). Multiplication of Comb(t) by Box(t) results in a finite sum of delta functions. In the notation described at the beginning of Section 3, the product function is denoted Boxcomb(t).

The spectrum window $W_0'(f)$, or Fboxcomb(f), corresponding to the data window $W_0'(t)$ can be found by transforming the finite sum of delta functions

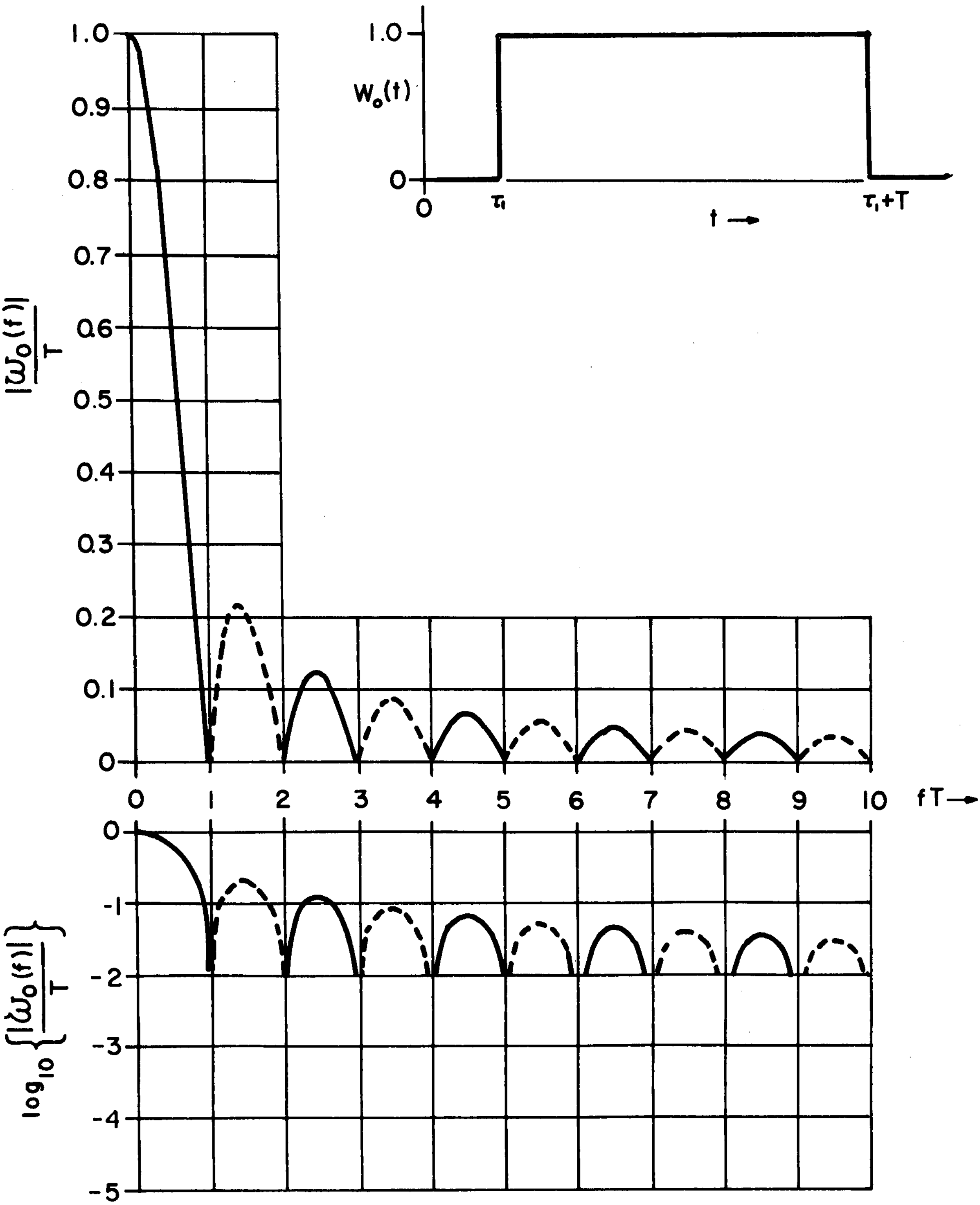


*Fig. 1. Spectrum window corresponding to rectangular data window.*

represented by Boxcomb(t). The alternative procedure of convolving Fbox(f) and Fcomb(f) is adopted here in order to exhibit separately the effects of finite duration and of sampling.

$$\begin{aligned} \text{Fboxcomb}(f) &= \text{Fbox}(f) * \text{Fcomb}(f) \\ &= \int_{-\infty}^{\infty} \text{Fbox}(f') \cdot \text{Fcomb}(f - f') \cdot df' \; . \end{aligned} \qquad .....(39)$$

Fbox(f) is given by Eqn. (36), and its effect has already been considered.

$$\text{Fcomb}(f) = \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{\Delta t}\right) . \qquad .....(40)$$

Fcomb(f) is the sum of an infinite number of delta functions, spaced at twice the Nyquist frequency, N/2T. The sampled function x(t) • Comb(t) has a spectrum given by the convolution with Fcomb(f) of the spectrum X(f) of the continuous function x(t). The effect of convolution with a comb of delta functions is very simple: each component of X(f), of frequency f, appears in the computed spectrum combined additively with all those components of X(f) whose frequencies are displaced from f by an integral multiple of twice the Nyquist frequency. This superposition of the components of the continuous spectrum is called aliassing. Note that aliassing implies that the computed spectrum is periodic; its period is twice the Nyquist frequency, and the period $-N/2T \leq f \leq N/2T$ is called the fundamental period, or the principal alias.

The aliassed spectrum computed from sampled data thus bears little relation, in general, to the true spectrum. However, if the signal is known to contain no frequencies greater than $f_0$, and the sampling rate is chosen to make the Nyquist frequency greater than or equal to $f_0$, no problem arises. The fundamental period of the aliassed spectrum is then the same as the true spectrum within this range, while the latter is known to be zero for greater frequencies. An alternative to adjusting the sampling rate is to filter the signal before sampling, to remove any components of frequencies greater than the fixed Nyquist frequency. In this case the fundamental period of the aliassed spectrum gives a correct analysis of the low-frequency components of the original signal.

Returning to the derivation of the spectrum window, we obtain from Eqns. (36), (39), and (40):

$$\text{Fboxcomb}(f) = T \sum_{k=-\infty}^{\infty} \left[ \frac{\sin \pi(f - \frac{k}{\Delta t})T}{\pi(f - \frac{k}{\Delta t})T} \right] \exp \; -i\pi(f - \frac{k}{\Delta t})(T + 2\tau_1) \; . \qquad .....(41)$$

This expression is reduced to closed form in Appendix B. The result is:

$$\text{Fboxcomb}(f) = \frac{T}{N} \left[ \frac{\sin \pi f T}{\sin \frac{\pi f T}{N}} \right] \exp \{-i\pi f(T + 2\tau_1)\}. \qquad .....(42)$$

It is shown in Appendix B that $\tau_1 = \tau_1' - \Delta t/2$, where $\tau_1'$ is the location of the first sample point. Fboxcomb(f), or $W_0'(f)$, is the spectrum window that results from the use of the rectangular data window with sampled data.

Fboxcomb(f) combines the features of Fbox(f) and Fcomb(f). Like Fcomb(f), Fboxcomb(f) is periodic, with period N/T. The spectrum of the sampled, time-limited function {x(t) • Boxcomb(t)} is therefore an aliassed version of the continuous spectrum. Each component of the spectrum is also spread in frequency, an effect similar to that of Fbox(f). This similarity is quantitative as well as qualitative, for it is easy to see that for small values of (fT/N) the expression on the right of Eqn. (42) reduces to that given for Fbox(f) in Eqn. (36). The approximation involved is quite good over about half of the frequency range (the fundamental period) of the spectrum, so that the main lobe and about half of the sidelobes--the largest ones--of Fboxcomb(f) almost coincide with those of Fbox(f). Fboxcomb(f) therefore varies as 1/f over the most important part of its range, but decreases less rapidly over the outer parts of the range.

There is one case in which this window behaves like a perfect filter: if the signal consists of one or several sine waves, of frequencies $f_i = m_i/T$, where $m_i$ is an integer less than or equal to N/2, then the computed spectrum is exact at $f = \pm f_i$, and is precisely zero at every other frequency (within the fundamental period) for which the DFT yields the spectrum. Since the latter frequencies are also integral multiples of 1/T, this conclusion follows from Eqn. (42). It will be noted that the sine waves specified are those whose periods divide the signal duration T exactly--in fact, $m_i$ times.

The frequency of any signal can be written $(m \pm \varepsilon)/T$, where m is an integer and $\varepsilon \leq \frac{1}{2}$. In the special cases just considered, $\varepsilon = 0$. As $\varepsilon$ increases two effects are observed: the value given by the DFT for f = m/T, which is the maximum of the spectrum, decreases; simultaneously the other values, which represent sidelobes, increase. This spreading of the spectrum is greatest when $\varepsilon = \frac{1}{2}$. Now it is shown in Section 2 that the spectrum of a sampled signal is a continuous function. The DFT gives only the values $\tilde{X}(k/T)$, where k is an integer, but this is a limitation imposed merely by the computational technique. It is possible to compute the spectrum for intermediate frequencies. One method for doing so is to translate the signal frequencies by an application of the shift theorem, and then transform the modified signal in the usual way. The frequency-shift is achieved by multiplying the $n^{th}$ signal sample by $\exp\{-2\pi i\varepsilon_0 n/N\}$; the transform of the modified sequence is $\tilde{X}((k + \varepsilon_0)/T)$. This method, used with $\varepsilon_0 = 0$ and $\frac{1}{2}$, yields 2N values of the spectrum spaced at intervals of 1/(2T). The four values $\varepsilon_0 = -1/4, 0, 1/4, 1/2$ give the spectrum for 4N frequencies at a spacing of 1/(4T). The transform for $\varepsilon_0 = 0$ is, of course, that of the unmodified signal. If the signal is a sinusoid of arbitrary but known frequency $(m + \varepsilon)/T$, the choice $\varepsilon_0 = \varepsilon$ makes the shifted frequency of the signal coincide with an analysis frequency in the positive half of the spectrum, but not simultaneously in the negative half. Provided m is large enough, the influence of the negative frequency component on the positive half of the spectrum may be neglected, and, for positive frequencies, the characteristics of the spectrum approximate those for the case $\varepsilon = 0$.

In most practical situations the frequency of the signal is not known beforehand, but the technique described above can be used to ensure that, whatever the frequency, it is arbitrarily close to one of the analysis frequencies. The reduction in the peak output indicated by Eqn. (42) for signals whose frequency is not an analysis frequency can therefore be avoided. In general, however, this method does not give any reduction in sidelobe levels, and the volume of computation is increased by a factor rather greater than the number of values of $\varepsilon_0$ employed. A similar result can be obtained, at a similar cost, by the method of the following section.

## 3.2 THE FRACTIONAL-WIDTH RECTANGULAR WINDOW

### 3.2.1 Continuous Data

The fractional-width rectangular data window is identical to the standard rectangular window (with the appropriate value of T) when the signal is continuous. The window warrants special consideration only when sampling techniques are used.

### 3.2.2 Sampled Data

The DFT is almost always computed nowadays by means of the efficient FFT algorithm, which relies on the factorization of N, the size of the data block. In most FFT programs, N is restricted to powers of two. However, when the number of data samples available is M, and M is not a power of two, the FFT algorithm can still be used. The procedure is to adjoin K zeros to the data sequence, so that N = M + K is a power of two; this augmented sequence is then transformed. The effect on the spectrum is obviously the same as if N, rather than M, data samples had been available, and had been transformed after modification by a special data window. This window, the fractional-width rectangular data window, is a function having the value unity for the first M samples, and zero for the remainder. The discrete frequencies at which the FFT gives the values of the transform remain spaced at $\Delta f = 1/T = 1/(N\,\Delta t)$, and the frequency coverage of the transform, which depends only on the sampling rate $(1/\Delta t)$, also remains the same as in the transformation of the full N data points with a full-width rectangular data window. What does not remain the same is the spectrum window; the change, although mathematically trivial, is important in practice.

The fractional-width rectangular data window is defined by:

$$W_1'(t) = \mathrm{Box}_1(t) \cdot \mathrm{Comb}(t), \qquad \ldots\ldots(43)$$

where

$$\mathrm{Box}_1(t) = \begin{cases} 1, & -\frac{\Delta t}{2} \le t \le \frac{MT}{N} - \frac{\Delta t}{2} \\ 0, & \text{otherwise}, \end{cases} \qquad \ldots\ldots(44)$$

and Comb(t) is given by Eqn. (38). $W_1'(t)$ 'transmits' M data samples, the first of which occurs at t = 0; the inter-sample period is $\Delta t$. The data window can be written:

$$W_1'(t) = \Delta t \sum_{n=0}^{M-1} \delta(t - n\Delta t), \qquad \ldots\ldots(45)$$

and the spectrum window is the transform of this:

$$\mathcal{W}_1'(f) = \int_{-\infty}^{\infty} \Delta t \sum_{n=0}^{M-1} \delta(t - n\Delta t) \exp\{-2\pi i f t\}\, dt, \qquad \ldots\ldots(46)$$

$$= \Delta t \sum_{n=0}^{M-1} \exp\{-2\pi i f\, n\Delta t\},$$

$$= \frac{T}{N}\left[\frac{\sin\left(\frac{\pi M f T}{N}\right)}{\sin\left(\frac{\pi f T}{N}\right)}\right] \exp\left\{-i\pi \frac{(M-1)fT}{N}\right\}. \qquad \ldots\ldots(47)$$

Equation (47) gives the spectrum window for this case. It may be compared with the spectrum window $\mathcal{W}_0'(f)$, to which it reduces when $M = N$ and $\tau_1 = -\Delta t/2$ (the appropriate value of $\tau_1$ when the first sample is taken at $t = 0$). It is periodic, with the same period $(N/T)$ as $\mathcal{W}_0'(f)$. If we replace the sine functions in the denominators of Eqns. (42) and (47) by their arguments, we obtain:

$$|\mathcal{W}_1'(f)| \simeq \frac{M}{N} \cdot \left|\mathcal{W}_0'\left(\frac{Mf}{N}\right)\right|, \qquad \ldots\ldots(48)$$

which shows that $\mathcal{W}_1'(f)$ is, approximately, a scaled and stretched version of the spectrum window corresponding to the full-width rectangular data window. The stretch factor is N/M. Thus, in particular, the main lobe is widened by this factor; its width, defined as half the interval between the first zeros of $\mathcal{W}_1'(f)$, is $(N/MT)$, or $(1/M\Delta t)$. As remarked earlier, the spacing of adjacent frequencies in the DFT remains unchanged when the fractional-width data window is used; this spacing, $(1/N\Delta t)$, is the same as the lobe width of $\mathcal{W}_0'(f)$; it is *not* the same as the lobe width of $\mathcal{W}_1'(f)$.

There is a close connection between the technique of 'filling up with zeros' and the interpolation technique discussed at the end of Section 3.1.2. Using the latter technique with two values of $\varepsilon$, $\varepsilon_1 = 0$ and $\varepsilon_2 = \frac{1}{2}$, we can compute two M-point DFT's: $\tilde{X}_1(k/T_1)$ and $\tilde{X}_1((k + \frac{1}{2})/T_1)$, where $k = 0,1,\ldots,M-1$, and $T_1 = M\Delta t$. The results can be interleaved to obtain a single spectrum, $\tilde{X}_1(k/T_2)$, where $k = 0,1,\ldots,(2M-1)$, and $T_2 = 2T_1$. The related 'filling up with zeros' procedure is to adjoin M zeros to the original sequence of M samples, and perform a single 2M-point DFT to obtain the spectrum $\tilde{X}_2(k/T_2)$, $k = 0,1,\ldots,(2M-1)$. The spectrum window for $\tilde{X}_1$ is given by Eqn. (42) with N replaced by M, and T by $T_1$. The spectrum window for $\tilde{X}_2$ is given by Eqn. (47) with N replaced by 2M, and T by $T_2 (= 2T_1)$. Making the substitutions, we find that the windows are identical. It follows that $\tilde{X}_1$ and $\tilde{X}_2$ are identical.

## 3.3 THE TRIANGULAR WINDOW

### 3.3.1 Continuous Data

The triangular data window, which applies a linear taper to the data, is defined as follows:

$$W_2(t) = \begin{cases} \frac{2}{T}(t - \tau_1), & \tau_1 \leq t \leq \tau_1 + \frac{T}{2} \\ -\frac{2}{T}(t - \tau_1 - T), & \tau_1 + \frac{T}{2} \leq t \leq \tau_1 + T \\ 0 & , \text{otherwise.} \end{cases} \qquad .....(49)$$

We shall consider, for simplicity, the triangular window centred on $t = 0$ ($\tau_1 = -T/2$), which is shown in Figure 2(a). Unlike the rectangular window, $W_2(t)$ is continuous everywhere; its first derivative, however, is discontinuous, as shown in Figure 2(b). The second derivative of $W_2(t)$ can be represented by delta functions in the way indicated in Figure 2(c). In general, if the $m^{th}$ derivative is discontinuous, the $(m + 1)^{th}$ derivative will require delta functions in its representation. There is an important general rule connecting the roll-off characteristic of the spectrum window with the order of the derivative of the data window in which discontinuities first appear. In order to determine this rule we shall compute the spectrum window corresponding to $W_2(t)$ in an indirect way, starting from the derivative involving delta functions. Let

$$V(t) = \frac{d}{dt}\{W_2(t)\}, \quad U(t) = \frac{d}{dt}\{V(t)\}. \qquad \begin{matrix} .....(50) \\ .....(51) \end{matrix}$$

Then

$$U(t) = \frac{2}{T}\left[\delta(t + \tfrac{T}{2}) - 2\delta(t) + \delta(t - \tfrac{T}{2})\right] . \qquad .....(52)$$

Let $\mathcal{U}(f)$ denote the transform of $U(t)$:

$$\mathcal{U}(f) = \frac{2}{T}\int_{-\infty}^{\infty}\left[\delta(t + \tfrac{T}{2}) - 2\delta(t) + \delta(t - \tfrac{T}{2})\right] \exp\{-2\pi ift\}\, dt ,$$

$$= -\frac{8}{T}\sin^2\left\{\frac{\pi fT}{2}\right\} . \qquad .....(53)$$

We can now write $U(t)$ as an inverse transform:

$$U(t) = -\frac{8}{T}\int_{\infty}^{\infty}\left[\sin^2\left\{\frac{\pi fT}{2}\right\}\right]\left[\exp\{2\pi ift\}\right] df . \qquad .....(54)$$

Note that the right side of Eqn. (54) is a function of t, which occurs only in the second bracket. The integral of this expression with respect to t is:

$$V(t) = -\frac{8}{T}\int_{-\infty}^{\infty}\left[\sin^2\left\{\frac{\pi fT}{2}\right\}\right]\left[\left\{\frac{1}{2\pi if}\right\} \exp\{2\pi ift\}\right] df. \qquad .....(55)$$

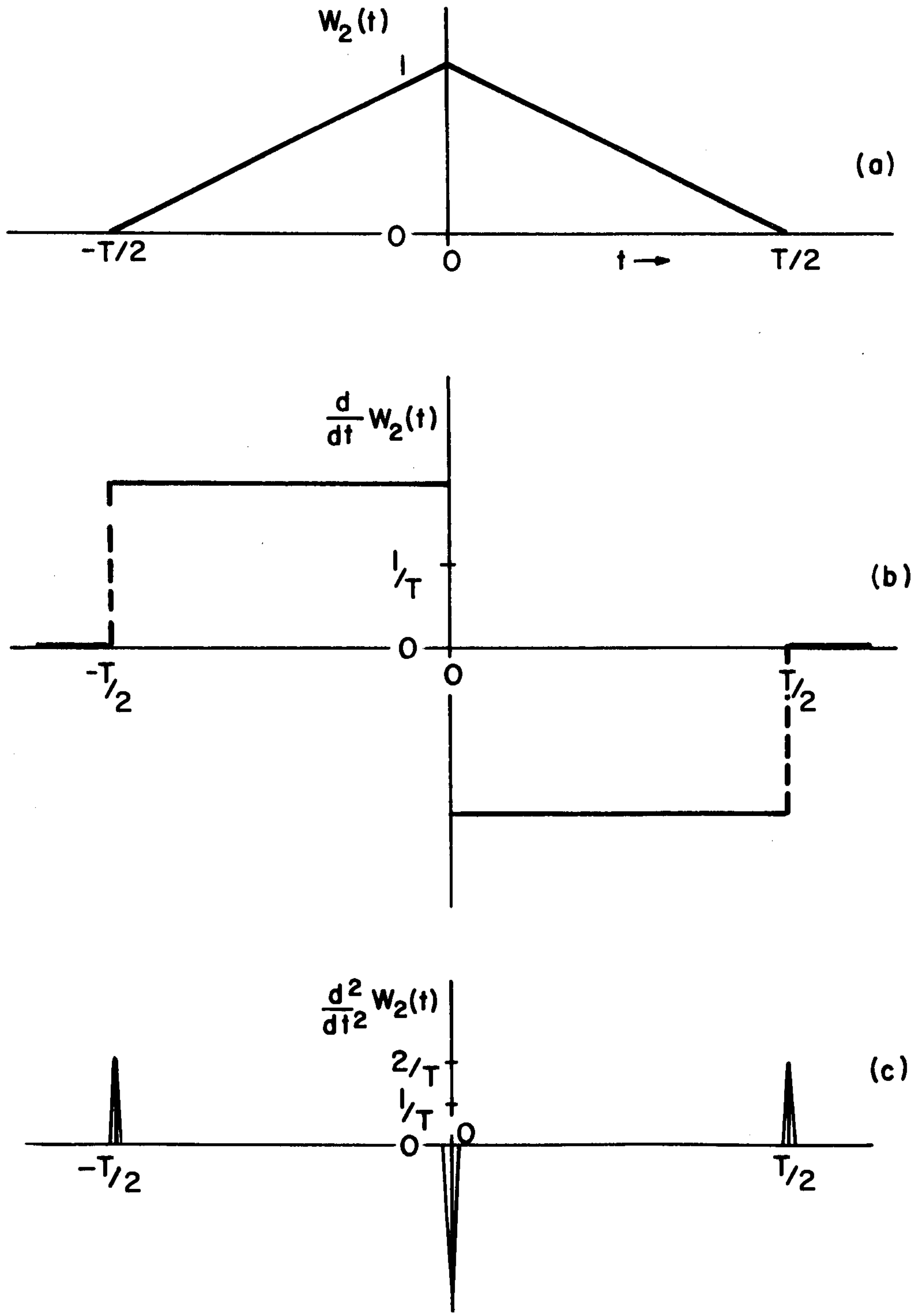


*Fig. 2. (a), (b) and (c). The triangular data window and its derivatives.*

Integrating a second time gives $W_2(t)$:

$$W_2(t) = -\frac{8}{T}\int_{-\infty}^{\infty}\left[\sin^2\left\{\frac{\pi f T}{2}\right\}\right]\left[\left\{\frac{1}{2\pi i f}\right\}^2 \exp\{2\pi i f t\}\right] df. \quad .....(56)$$

The transform of $W_2(t)$ can be written down by inspection of Eqn. (56):

$$\mathcal{W}_2(f) = \frac{T}{2}\left[\frac{\sin\left(\frac{\pi f T}{2}\right)}{\left(\frac{\pi f T}{2}\right)}\right]^2 . \quad .....(57)$$

This is the spectrum window corresponding to the triangular data window centred on $t = 0$. It will be observed that this function is non-negative. An application of the shift theorem yields the spectrum window for an arbitrarily located data window:

$$\mathcal{W}_2(f) = \frac{T}{2}\left[\frac{\sin\left(\frac{\pi f T}{2}\right)}{\left(\frac{\pi f T}{2}\right)}\right]^2 \exp\{-2\pi i f \tau_1\} . \quad .....(58)$$

A graph of $\mathcal{W}_2(f)$ is presented in Figure 2(d).

Equation (58) shows that the spectrum window has a $1/f^2$ roll-off. If we examine Eqns. (54), (55), and (56) the source of the $(1/f^2)$ factor is evident. The factor multiplying the kernel in Eqn. (54) is $\mathcal{U}(f)$. In Eqn. (55), as a result of an integration, it is $\{1/(2\pi i f)\}\mathcal{U}(f)$. In Eqn. (56), derived from Eqn. (54) by two integrations, the factor is $\{1/(2\pi i f)\}^2 \cdot \mathcal{U}(f)$, which is equal to $\mathcal{W}_2(f)$. Each integration of a derivative involving delta functions, or a lower order derivative, introduces a $(1/f)$ factor in the spectrum window. We are thus led to expect a roll-off as $f^{(-m+1)}$ if the $m^{th}$ order derivative of the data window is discontinuous. For the rectangular window $m = 0$, and the spectrum window exhibits a $(1/f)$ factor, in accordance with the rule. The cosine-bell window, to be considered later, is discontinuous in the second derivative, and its spectrum window has a $(1/f^3)$ factor, as we should have predicted. In qualitative terms the rule is: the smoother the data window, the faster the roll-off.

Turning now to other features of the spectrum window, we observe that the main lobe is wider than that of $\mathcal{W}_0(f)$. The first zeros occur at $f = \pm 2/T$, so that the lobe width, measured as half the separation of the first zeros, is twice that of $\mathcal{W}_0(f)$. However this is a crude criterion, and it overstates the relative lobe-widths in the present comparison. The widening of the main lobe that we have found in this case is an example of another general rule: the faster the roll-off, the wider the main lobe. The magnitude of $\mathcal{W}_2(f)$ is plotted in Figure 2(d), from which it can be seen that the height of the first sidelobe, which is the largest, is about 5 per cent of the height of the main lobe. The corresponding figure for $\mathcal{W}_0(f)$ is 21 per cent.

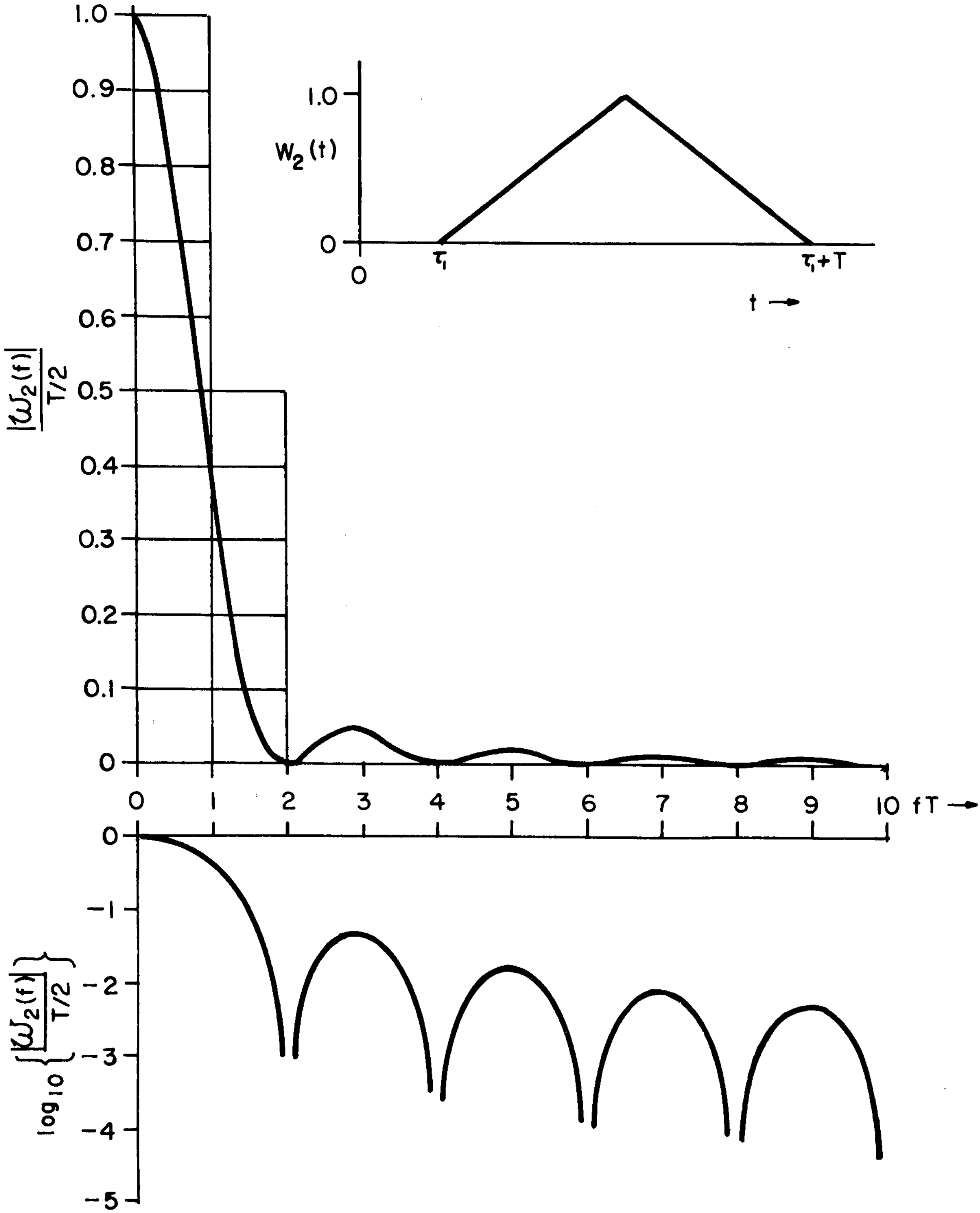


*Fig. 2. (d). Spectrum window corresponding to triangular data window.*

### 3.3.2 Sampled Data

The effect of sampling on the spectrum window will be computed for the triangular data window of span T, centred on t = T/2. The continuous data window, $W_2(t)$, is given by Eqn. (49) with $\tau_1 = 0$; the alternative name Tri(t) is given to this function. The window for sampled data is:

$$\text{Tricomb}(t) = \text{Tri}(t) \cdot \text{Comb}(t) \; . \qquad \ldots..(59)$$

Since Tri(t) is zero outside the range $0 \leq t \leq T$, Eqn. (59) represents a finite comb of delta functions, weighted according to the values of Tri(t) within this range. The spectrum window $W_2'(f)$ is Ftricomb(f), the transform of Tricomb(t):

$$\text{Ftricomb}(f) = \int_{-\infty}^{\infty} \left[ \frac{2\Delta t}{T} \left\{ \sum_{n=0}^{\frac{N}{2}-1} t - \sum_{n=\frac{N}{2}}^{N-1} (t-T) \right\} \delta(t - n\Delta t) \right] \exp\{-2\pi i f t\}\, dt, \qquad \ldots..(60)$$

$$= \frac{2T}{N^2} \left[ \sum_{n=0}^{\frac{N}{2}-1} n \exp\{-2\pi i f\, n\Delta t\} - \sum_{n=\frac{N}{2}}^{N-1} (n-N) \exp\{-2\pi i f\, n\Delta t\} \right] . \qquad \ldots..(61)$$

The summations in Eqn. (61) are of the forms:

$$\sum_{n=K}^{K+L-1} x^n = \left\{ x^K \left(1 - x^L\right) \right\} / (1 - x) \; , \qquad \ldots..(62)$$

$$\sum_{n=K}^{K+L-1} n x^n = x^K \left\{ K - (K-1)x - (K+L)x^L + (K+L-1)x^{L+1} \right\} / (1-x)^2 . \qquad \ldots..(63)$$

Making use of these results, we obtain:

$$W_2'(f) = \text{Ftricomb}(f) = \frac{2T}{N^2} \left[ \frac{\sin\left(\frac{\pi f T}{2}\right)}{\sin\left(\frac{\pi f T}{N}\right)} \right]^2 \exp(-i\pi f T). \qquad \ldots..(64)$$

Equation (64) gives the spectrum window corresponding to the triangular window for sampled data. The changes, as compared with the case of continuous data, are similar to those noted for the rectangular window. $W_2'(f)$ is periodic with period N/T. It reduces to $W_2(f)$ for small values of (fT/N), and, to a good approximation, $W_2(f)$ and $W_2'(f)$ are identical over a large part of the fundamental period of the latter. Within this range, therefore, $W_2'(f)$ exhibits a $1/f^2$ roll-off, but it falls off more slowly as (fT/N) approaches 1/2.

## 3.4 THE COSINE-BELL (HANNING) WINDOW

### 3.4.1 Continuous Data

The cosine-bell window, which is associated with the name of the Austrian physicist van Hann (to use it is to 'hann' the data), is defined by:

$$W_3(t) = \begin{cases} \cos^2\left\{\frac{\pi}{T}\left(t - \tau_2 - \frac{T}{2}\right)\right\}, & \tau_2 \le t \le \tau_2 + T \\ 0, & \text{otherwise.} \end{cases} \qquad .....(65)$$

The function in the first part of this definition has the shape of $\cos^2(t)$, the period T, and ranges in value between zero and unity; $t = \tau_2$ is the location of one of its minima. This function, unrestricted as to its region of definition, is given the descriptive name Bell(t). Bell(t) may be written in the following alternative forms:

$$\text{Bell}(t) = \frac{1}{2}\left[1 - \cos\left\{\frac{2\pi}{T}(t - \tau_2)\right\}\right], \qquad .....(66)$$

$$= \left[\frac{1}{2} - \frac{1}{4}\exp\left\{-2\pi i\frac{\tau_2}{T}\right\}\exp\left\{2\pi i\frac{t}{T}\right\} - \frac{1}{4}\exp\left\{2\pi i\frac{\tau_2}{T}\right\}\exp\left\{-2\pi i\frac{t}{T}\right\}\right]. \qquad .....(67)$$

The transform of Bell(t) is:

$$\text{Fbell}(f) = \left[\frac{1}{2}\delta(f) - \frac{1}{4}\delta\left(f - \frac{1}{T}\right)\exp\left\{-2\pi i\frac{\tau_2}{T}\right\} - \frac{1}{4}\delta\left(f + \frac{1}{T}\right)\exp\left\{2\pi i\frac{\tau_2}{T}\right\}\right]. \qquad .....(68)$$

$W_3(t)$ can be expressed in terms of Bell(t) and Box(t), where Box(t) is as defined in Eqn. (33) with $\tau_1 = \tau_2$:

$$W_3(t) = \text{Bell}(t) \cdot \text{Box}(t). \qquad .....(69)$$

Its transform is:

$$\mathcal{W}_3(f) = \text{Fbell}(f) \star \text{Fbox}(f), \qquad .....(70)$$

$$= \text{Fbellbox}(f). \qquad .....(71)$$

Performing the convolution, we obtain:

$$\text{Fbellbox}(f) = \left[\frac{1}{2}\text{Fbox}(f) - \frac{1}{4}\text{Fbox}\left(f - \frac{1}{T}\right)\exp\left\{-2\pi i\frac{\tau_2}{T}\right\} - \frac{1}{4}\text{Fbox}\left(f + \frac{1}{T}\right)\exp\left\{2\pi i\frac{\tau_2}{T}\right\}\right]. \qquad .....(72)$$

Fbox(f) is given by Eqn. (36). Equation (72) can be reduced to the form:

$$\text{Fbellbox}(f) = \text{Fbox}(f) \cdot G(f;\alpha), \qquad .....(73)$$

where $G(f;\alpha) = \frac{1}{2} [\{x^2(1 - \cos\alpha) - 1\} + i\{x \sin\alpha\}]/(x^2 - 1)$ , .....(74)

and

$$x = fT , \quad \alpha = 2\pi(\tau_2 - \tau_1)/T.$$

As noted earlier, the equivalence of Eqns. (65) and (69) requires $\tau_1 = \tau_2$, or $\alpha = 0$, and with this simplification Fbellbox(f), or $\mathcal{W}_3(f)$, can be expressed:

$$\mathcal{W}_3(f) = \frac{T}{2} \left[\frac{\sin \pi fT}{\pi fT}\right] \left[\frac{1}{1 - (fT)^2}\right] \exp\{-i\pi f\ (T + 2\tau_2)\} . \quad .....(75)$$

$\mathcal{W}_3(f)$ is plotted in Figure 3. Equation (75) gives the spectrum window corresponding to the cosine-bell data window for the case where the data are continuous. $\tau_2$ is zero for the data window symmetrically located on the interval [0,T].

It can be seen from Eqn. (75) that $\mathcal{W}_3(f)$ decreases as $(1/f^3)$ for $fT \gg 1$. This characteristic, and its connection with the smoothness of $W_3(t)$, was noted in Section 3.3.1. The first zeros of $\mathcal{W}_3(f)$ occur at $t = \pm 2/T$, so that the main lobe is wider than that of $\mathcal{W}_0(f)$; it is also slightly wider than the main lobe of $\mathcal{W}_2(f)$, although the latter has the same first zeros. The height of the first, and largest, sidelobe is less than three per cent of that of the main lobe, and, as we have seen, the remaining sidelobes decrease rapidly. In comparison with the rectangular window, the cosine-bell data window therefore offers much better control of the mutual interference of well-separated spectrum components, at the expense of greater interference between closely-spaced components--those separated by less than the width of the main lobe.

### 3.4.2 Sampled Data

When sampling is used, the cosine-bell data window can be written:

$$W_3'(t) = \text{Bell}(t) \cdot \text{Box}(t) \cdot \text{Comb}(t) , \quad .....(76)$$

$$= \text{Bellboxcomb}(t) , \quad .....(77)$$

where the factor functions are as defined previously. The corresponding spectrum window is:

$$\mathcal{W}_3'(f) = \text{Fbellboxcomb}(f) . \quad .....(78)$$

Since we have already found Fbellbox(f), we could, in principle, compute $\mathcal{W}_3'(f)$ as:

$$\mathcal{W}_3'(f) = \text{Fbellbox}(f) \star \text{Fcomb}(f) ,$$

but it is easier to combine the factors of $W_3(t)$ in a different order, which gives

$$\mathcal{W}_3'(f) = \text{Fbell}(f) \star \text{Fboxcomb}(f) . \quad .....(79)$$

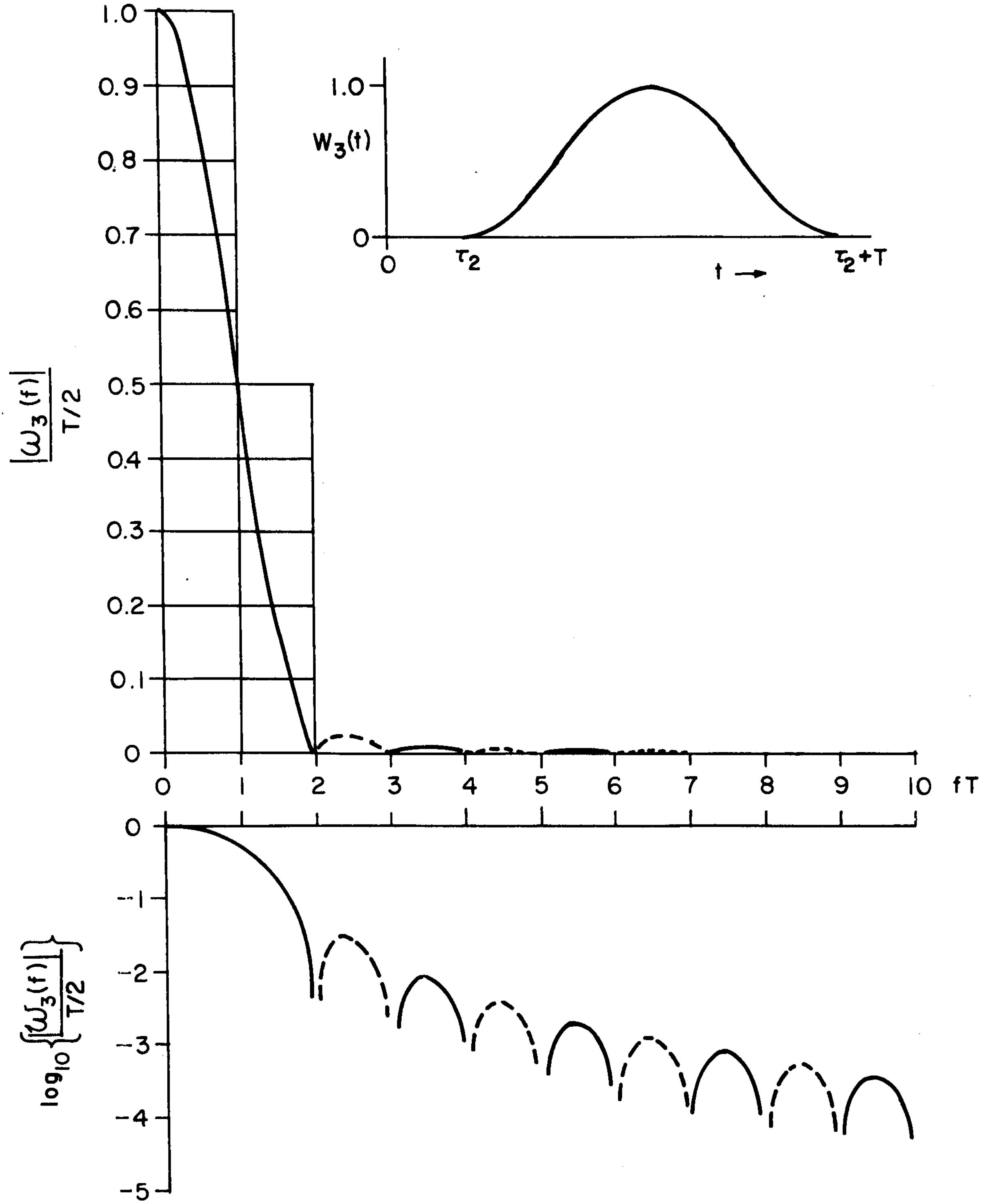


*Fig. 3. Spectrum window corresponding to cosine-bell data window.*

The two functions on the right of Eqn. (79) are given in Eqns. (68) and (42). The convolution is easy in this case, and the result is analogous to that of Eqn. (72) for continuous data:

$$W_3'(f) = \left[ \frac{1}{2}\,\mathrm{Fboxcomb}(f) - \frac{1}{4}\,\mathrm{Fboxcomb}(f - \tfrac{1}{T}) \cdot \exp\left\{-2\pi i \frac{\tau_2}{T}\right\} - \frac{1}{4}\,\mathrm{Fboxcomb}\,(f + \tfrac{1}{T}) \cdot \exp\left\{2\pi i \frac{\tau_2}{T}\right\} \right] . \qquad .....(80)$$

Equation (80), after some manipulation, yields the following expression:

$$W_3'(f) = \mathrm{Fboxcomb}(f) \cdot H(f;\, \alpha,\beta), \qquad .....(81)$$

where

$$H(f;\alpha,\beta) = \frac{[\{\cos\beta\sin^2 x(\cos\alpha-\cos\beta) + \sin^2\beta\cos^2 x\} - i\{\sin\alpha \cdot \sin\beta \cdot \sin x \cdot \cos x\}]}{2(\sin^2\beta-\sin^2 x)} \qquad .....(82)$$

and

$$x = \frac{\pi f T}{N}, \qquad \alpha = 2\pi(\tau_2-\tau_1)/T, \qquad \beta = \frac{\pi}{N} .$$

We now consider the values of $\tau_1$ and $\tau_2$. Let $\tau_1 = m\Delta t/2$, and $\tau_2 = n\Delta t/2$, where m and n are real numbers. To simplify the discussion, let the N sample points be located at $t = 0, \Delta t, \ldots, (N-1)\Delta t$; according to the observation following Eqn. (42), this choice makes $m = -1$. Two values of n are of interest, corresponding to two ways of applying the cosine-bell window:

(a) Bell(t) is symmetrical on the interval (0,T); this choice makes $n = 0$, $\alpha = \beta$.

(b) Bell(t) is symmetrical with respect to the N sample points; this gives $n = -1$, $\alpha = 0$.

In case (a) we have:

$$H(f;\beta,\beta) = \frac{\cos x \cdot \exp\{-ix\}}{2\left[1 - \left\{\dfrac{\sin x}{\sin\beta}\right\}^2\right]} , \qquad .....(83)$$

and, correspondingly,

$$W_3'(f) = \frac{T}{2N} \cdot \left[\frac{\sin \pi f T}{\tan\left(\dfrac{\pi f T}{N}\right)}\right] \left[1 - \left\{\frac{\sin\left(\dfrac{\pi f T}{N}\right)}{\sin\left(\dfrac{\pi}{N}\right)}\right\}^2\right]^{-1} \exp\{-i\pi f T\} . \qquad .....(84)$$

In case (b) we have:

$$H(f;0,\beta) = \frac{[\sin^2\beta-(1-\cos\beta)\sin^2 x]}{2[\sin^2\beta-\sin^2 x]} , \qquad .....(85)$$

and, correspondingly,

$$\mathcal{W}_3'(f) = \frac{T}{2N}\left[\frac{\sin \pi fT}{\sin\left(\frac{\pi fT}{N}\right)}\right]\left[1 - \left\{1-\cos\left(\frac{\pi}{N}\right)\right\}\left\{\frac{\sin\left(\frac{\pi fT}{N}\right)}{\sin\left(\frac{\pi}{N}\right)}\right\}^2\right]\left[1 - \left\{\frac{\sin\left(\frac{\pi fT}{N}\right)}{\sin\left(\frac{\pi}{N}\right)}\right\}^2\right]^{-1} \exp\left\{-\,i\pi\left(\frac{N-1}{N}\right) fT\right\} . \quad .....(86)$$

Equations (84) and (86) give the spectrum windows corresponding to the two ways of applying the cosine-bell window to sampled data.

When $(ft/N) << 1$, both expressions for $\mathcal{W}_3'(f)$ reduce to $\mathcal{W}_3(f)$, the spectrum window for continuous data, as given by Eqn. (75) with the appropriate values of $\tau_2$. Both functions are periodic, with period N/T, and have conjugate symmetry about $f = 0$. At large values of f (i.e., $f \to N/2T$) $\mathcal{W}_3'(f)$ falls off more rapidly than $\mathcal{W}_3(f)$ in case (a), while the opposite is true in case (b). This difference should normally be of little significance, since the values of both functions are very small in the region concerned.

A conclusion of some practical importance can be drawn from the form of the spectrum window given in Eqn. (80). The spectrum of a sampled signal is computed by means of the DFT, which yields the values of the spectrum at frequencies that are integral multiples of (1/T). When the rectangular data window is used, the value computed for the frequency $f = k/T$ is the convolution of the true spectrum with Fboxcomb(k/T). When the cosine-bell data window is used as prescribed for case (a), for which $\tau_2 = 0$, the value of the spectrum at $f = kT$ is the convolution of the true spectrum with $\mathcal{W}_3'(k/T)$, as given by Eqn. (80) with $\tau_2 = 0$. The equation shows that this is identical to the weighted sum of the values obtained at $f = (k-1)/T$, $k/T$, and $(k+1)/T$ with the rectangular data window; the weights are (-1/4), 1/2, -1/4. Hence the cosine-bell data window can be applied in the time domain, by modifying the signal samples according to the weights Bell $(n\Delta t)$, or in the frequency domain, by forming the indicated weighted sum of the spectrum values obtained from unmodified samples. The computational advantage appears to lie with the frequency-domain technique, since, because of the values of the weights, the weighted sums can be computed without multiplications.

## 3.5 THE HAMMING WINDOW

### 3.5.1 Continuous Data

The Hamming window is a modification of the cosine-bell window named after R.W. Hamming. It is defined by:

$$W_4(t) = \begin{cases} 0.54 - 0.46 \cos\left\{\frac{2\pi}{T}(t-\tau_2)\right\}, & \tau_2 \le t \le \tau_2 + T \\ 0, & \text{otherwise.} \end{cases} \quad .....(87)$$

Equation (87) can be written in the following alternative forms:

$$W_4(t) = \left[0.50\left\{1 - \cos\frac{2\pi}{T}(t-\tau_2)\right\} + 0.04\left\{1 + \cos\frac{2\pi}{T}(t-\tau_2)\right\}\right] \cdot \mathrm{Box}(t;\tau_2), \quad .....(87a)$$

$$= \left[0.50\left\{1 - \cos\frac{2\pi}{T}(t-\tau_2)\right\} + 0.04\left\{1 - \cos\frac{2\pi}{T}(t-(\tau_2 + \frac{T}{2}))\right\}\right] \cdot \mathrm{Box}\,(t;\tau_2). \quad .....(87b)$$

In these equations we exhibit explicitly the parameter of Box(t), which denotes the lower limit of the region in which Box(t) is non-zero. This parameter has previously been denoted by $\tau_1$. Equations (87a) and (87b) are equivalent to Eqn. (87) only if $\tau_1$ is equal to the parameter $\tau_2$ of the cosine functions. It can be seen by reference to Eqn. (87b) and Section 3.4.1 that $W_4(t)$ may be expressed in terms of $W_3(t)$, as follows:

$$W_4(t) = W_3(t;\ \tau_2;\ \tau_2) + 0.08\ W_3(t;\ \tau_2;\ \tau_2 + \frac{T}{2}). \quad .....(88)$$

The parameter lists for the functions on the right-hand side of this equation give, in order: the argument, the parameter of Box(t), and the parameter of Bell(t).

The transform of $W_3(t;\ \tau_2;\ \tau_2)$ is given by Eqn. (75). Since this equation applies only when the parameters are equal, it cannot be used to obtain the transform of $W_3(t;\ \tau_2;\ \tau_2 + T/2)$. The latter can be obtained from Eqns. (73) and (74), which give:

$$\mathcal{W}_3(f;\ \tau_2;\ \tau_2 + \frac{T}{2}) = \mathcal{W}_3(f;\ \tau_2;\ \tau_2) \cdot \left\{1 - 2(fT)^2\right\}. \quad .....(89)$$

Dropping the parameters, we can now write down the transform of $W_4(t)$:

$$\mathcal{W}_4(f) = \mathcal{W}_3(f) \cdot [1 + 0.08\ \{1 - 2(fT)^2\}]. \quad .....(90)$$

$$= 0.54T\left[\frac{\sin \pi fT}{\pi fT}\right]\left[\frac{1 - \frac{4}{27}(fT)^2}{1 - (fT)^2}\right] \exp\{-i\pi f(T + 2\tau_2)\}, \quad .....(91)$$

which, apart from the scale factor, differs from $\mathcal{W}_3(f)$ only in the numerator of the second bracket (compare Eqn. (75)). Graphs of $W_4(t)$ and $\mathcal{W}_4(f)$ are presented in Figure 4.

$\mathcal{W}_4(f)$ has the same zeros as $\mathcal{W}_3(f)$, plus an additional one given by $1 - (4/27)(fT)^2 = 0$: i.e., at $f = 2.598/T$. The first and second zeros of

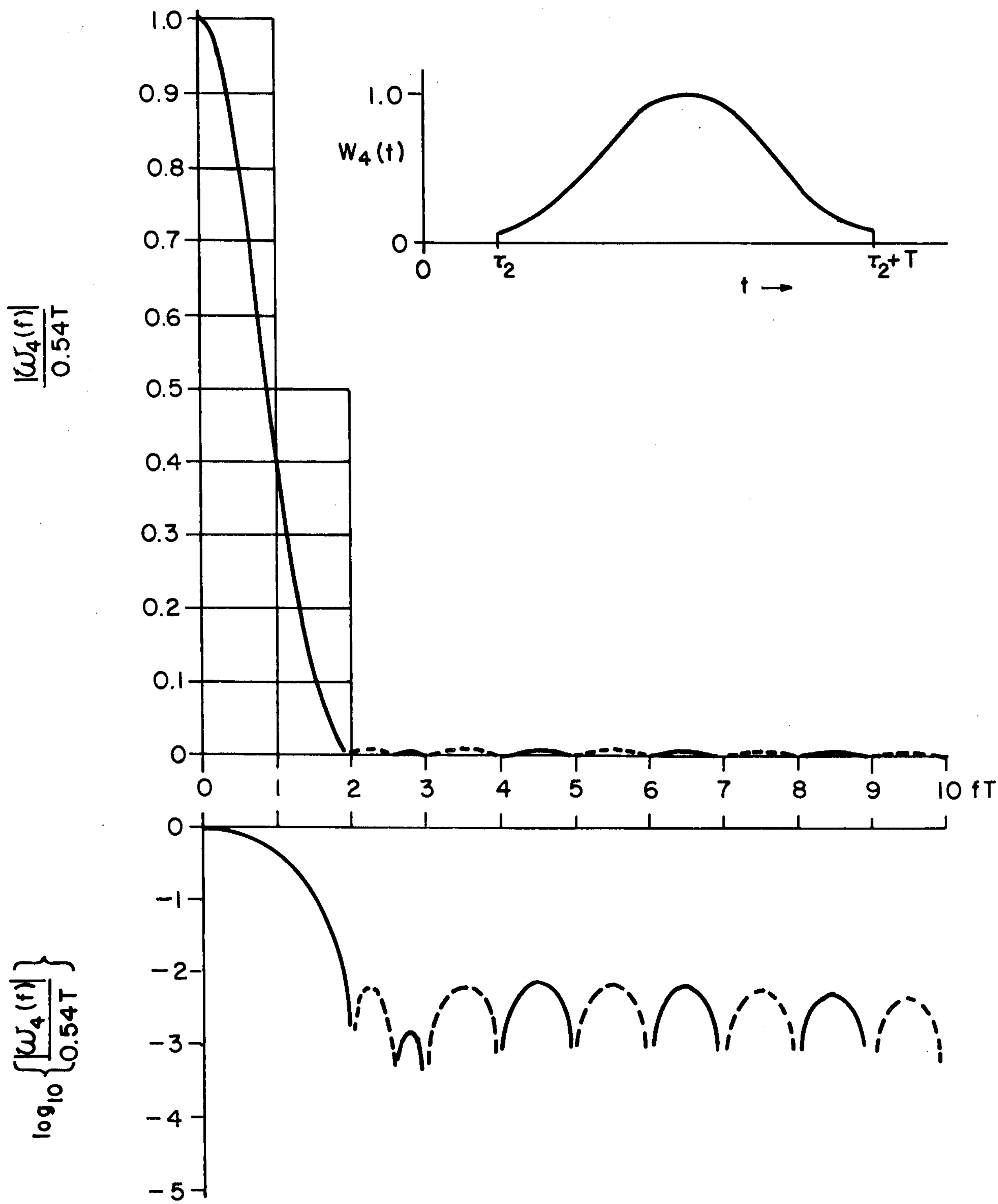


*Fig. 4. Spectrum window corresponding to Hamming data window.*

$\mathcal{W}_3(f)$ occur at f = 2/T and 3/T. The additional zero of $\mathcal{W}_4(f)$ occurs between these, and so exerts its principal effect in the region where the sidelobes of $\mathcal{W}_3(f)$ are largest. The effect is to reduce the height of the largest sidelobes to less than one per cent of the height of the main lobe. For comparison, the first sidelobe of $\mathcal{W}_3(f)$ has a height of about 2-1/2 per cent of that of the main lobe.

This improvement is not achieved without penalty: the sidelobes remote from the main lobe are larger than in $\mathcal{W}_3(f)$. While the latter window was shown to have a $1/f^3$ roll-off, Eqn. (91) shows that the Hamming window has a much smaller rate of decrease, which is proportional to 1/f. This difference is greater than one might, perhaps, have anticipated, since the Hamming and cosine-bell data windows are very much alike. However, the zero-order derivative of $W_4(t)$ - i.e., $W_4(t)$ itself--is discontinuous, and according to the rule given in Section 3.3.1 such functions lead to a 1/f roll-off. The roll-off characteristics of both windows are in accordance with the rule, and could have been predicted without the aid of Eqns. (75) and (91).

3.5.2 Sampled Data

The spectrum window for the case of discrete data is derived by the method of Section 3.4.2, which leads to a result similar to Eqn. (80):

$$\mathcal{W}_4'(f) = \Bigg[0.54\ \mathcal{W}_0'(f) - 0.23\ \mathcal{W}_0'(f - \tfrac{1}{T})\ \exp\left\{-2\pi i\ \frac{\tau_2}{T}\right\} - 0.23\ \mathcal{W}_0'(f + \tfrac{1}{T})\ \exp\left\{2\pi i\ \frac{\tau_2}{T}\right\}\Bigg] . \quad .....(92)$$

This result, incidentally, implies that 'hamming' can be applied in the frequency domain, exactly as described for the cosine-bell window in Section 3.4.2, but using the weights (-0.23, 0.54, -0.23) instead of (-0.25, 0.50, -0.25).

Equation (92) can be written

$$\mathcal{W}_4'(f) = \mathcal{W}_0'(f) \cdot K(f;\alpha,\beta) , \quad .....(93)$$

where

$$K(f;\alpha,\beta) = \frac{(K_1 - iK_2)}{(\sin^2\beta - \sin^2 x)} , \quad .....(94)$$

$$K_1 = 0.46 \cos\alpha \cos\beta \sin^2 x - 0.54 \cos^2\beta \sin^2 x + 0.54 \sin^2\beta \cos^2 x ,$$

$$K_2 = 0.46 \sin\alpha \sin\beta \sin x \cos x,$$

and $\alpha$, $\beta$, x are as defined in Section 3.4.2. When the data window is applied symmetrically with respect to the sample points, $\alpha$ is zero, and the spectrum window is:

$$\mathcal{W}_4'(f) = 0.54\,\frac{T}{N}\left[\frac{\sin \pi fT}{\sin\left(\frac{\pi fT}{N}\right)}\right]\left[\frac{1-\left\{1-\frac{23}{27}\cos\left(\frac{\pi}{N}\right)\right\}\left\{\frac{\sin\left(\frac{\pi fT}{N}\right)}{\sin\left(\frac{\pi}{N}\right)}\right\}^2}{1-\left\{\frac{\sin\left(\frac{\pi fT}{N}\right)}{\sin\left(\frac{\pi}{N}\right)}\right\}^2}\right]\exp\left\{-i\pi\left(\frac{N-1}{N}\right)fT\right\} \quad .....(95)$$

$\mathcal{W}_4'(f)$ reduces to $\mathcal{W}_4(f)$, with the appropriate time-shift exponential, when $(\pi fT/N) \ll 1$ and $(\pi/N) \ll 1$, which are valid approximations over the most important part of the period of $\mathcal{W}_4'(f)$--the region of the main lobe and the larger sidelobes. Equation (95) may also be compared with the corresponding result for the cosine-bell window (Eqn. (86)). Apart from the scale factor, the two expressions differ only in the numerical factor multiplying $\cos(\pi/N)$; it should be noted that $\cos(\pi/N)$ is very close to unity for the usual values of N.

## 3.6 GENERALIZED COSINE-BELL WINDOW

### 3.6.1 Continuous Data

A data window proposed by Bingham, Godfrey, and Tukey[5], can be described as a rectangular window, with short half-cosine bells added at each end. On the basis of the roll-off rule given in Section 3.3.1, this window would be expected to have the desirable sidelobe-suppression feature of the cosine-bell window, without the strong emphasis that the latter window places on the mid-range sample values (and their possible errors). These considerations apply however small we make the ratio of the span of the half-cosine bells to the total span of the window. However, as this ratio is made smaller, the window becomes more like the rectangular data window, and its corresponding spectrum window would be expected to resemble $\mathcal{W}_0(f)$ more and more closely. We shall see how these two apparently conflicting expectations are reconciled.

Reference 5 proposes a span ratio of 1/5. In the following definition the reciprocal of this ratio, m, is treated as a parameter that can assume any positive integral value.

$$W_5(t) = \begin{cases} 1 , & \frac{T}{2m} \le t \le \frac{(2m-1)T}{2m} \\ \frac{1}{2}\,[1 - \cos\,(2\pi mt/T)], & 0 \le t \le \frac{T}{2m},\ \frac{(2m-1)T}{2m} \le t \le T \\ 0 , & \text{otherwise.} \end{cases} \quad .....(96)$$

It is a straightforward matter to calculate $\mathcal{W}_5(f)$, the associated spectrum window, directly from this definition. An alternative method that makes use of previous results may also be used. Note that the cosine bell function in the definition above has the period T/m, so that there are m complete periods in the interval [0,T]. Since the cosine bell has the shape $\cos^2(t)$, we may exploit the fact that $\cos^2(t) + \sin^2(t) = 1$ to represent the constant part of $W_5(t)$ as the sum of two cosine bells:

$$W_5(t) = \frac{1}{2}\left[1 - \cos\frac{2\pi mt}{T}\right]\cdot \mathrm{Box}_1(t) + \frac{1}{2}\left[1 - \cos\left\{\frac{2\pi m}{T}\left(t - \frac{T}{2m}\right)\right\}\right]\cdot \mathrm{Box}_2(t), \quad \text{.....(97)}$$

where

$$\mathrm{Box}_1(t) = \begin{cases} 1\ , & 0 \le t \le T \\ 0\ , & \text{otherwise.} \end{cases} \quad \text{.....(98)}$$

and

$$\mathrm{Box}_2(t) = \begin{cases} 1\ , & \frac{T}{2m} \le t \le \frac{(2m-1)T}{2m} \\ 0\ , & \text{otherwise.} \end{cases} \quad \text{.....(99)}$$

The transforms of these functions can be written down at once from the results of Sections 3.1 and 3.4. The transform of Bell(t), modified to have the period T/m, is given by Eqn. (68) with the substitution of T/m for T. For the first cosine-bell function in Eqn. (97) $\tau_2 = 0$, and for the second $\tau_2 = T/2m$. The transform of $\mathrm{Box}_1(t)$ is given by Eqn. (36) with $\tau_1 = 0$, and the transform of $\mathrm{Box}_2(t)$ by the same equation with T replaced by (m-1)T/m, and $\tau_1 = T/2m$. Carrying out the algebra, we obtain the transform of the first term of $W_5(t)$:

$$\mathcal{W}_5^{(1)}(f) = \frac{T}{2}\left[\frac{\sin \pi fT}{\pi fT}\right]\left[\frac{1}{1 - \left(\frac{fT}{m}\right)^2}\right] \exp\{-i\pi fT\}\ , \quad \text{.....(100)}$$

and for the second term:

$$\mathcal{W}_5^{(2)}(f) = \frac{(m-1)T}{2m}\left[\frac{\sin \pi\left(\frac{m-1}{m}\right)fT}{\pi\left(\frac{m-1}{m}\right)fT}\right]\left[\frac{1}{1 - \left(\frac{fT}{m}\right)^2}\right] \exp\left\{-i\pi\left(\frac{m-1}{m} + \frac{1}{m}\right)fT\right\}\ ,$$

$$= \frac{T}{2}\left[\frac{\sin \pi\left(\frac{m-1}{m}\right)fT}{\pi fT}\right]\left[\frac{1}{1 - \left(\frac{fT}{m}\right)^2}\right] \exp\{-i\pi fT\}\ . \quad \text{.....(101)}$$

The sum of these terms is the spectrum window:

$$\mathcal{W}_5(f) = \left(\frac{2m-1}{2m}T\right)\left[\frac{\sin\left(\frac{2m-1}{2m}\right)\pi fT}{\left(\frac{2m-1}{2m}\right)\pi fT}\right]\left[\cos\left(\frac{\pi fT}{2m}\right)\right]\left[\frac{1}{1 - \left(\frac{fT}{m}\right)^2}\right] \exp\{-i\pi fT\}\ . \quad \text{.....(102)}$$

Graphs of this function for m = 2 and 4 are presented in Figures 5(a) and 5(b).

When m = 1, $\mathcal{W}_5(f)$ reduces to $\mathcal{W}_3(f)$, as given by Eqn. (75); $W_5(t)$ is then the standard cosine-bell. When $m = \infty$, $W_5(t)$ becomes $W_0(t)$, and $\mathcal{W}_5(f)$ reduces to $\mathcal{W}_0(f)$, as it should.

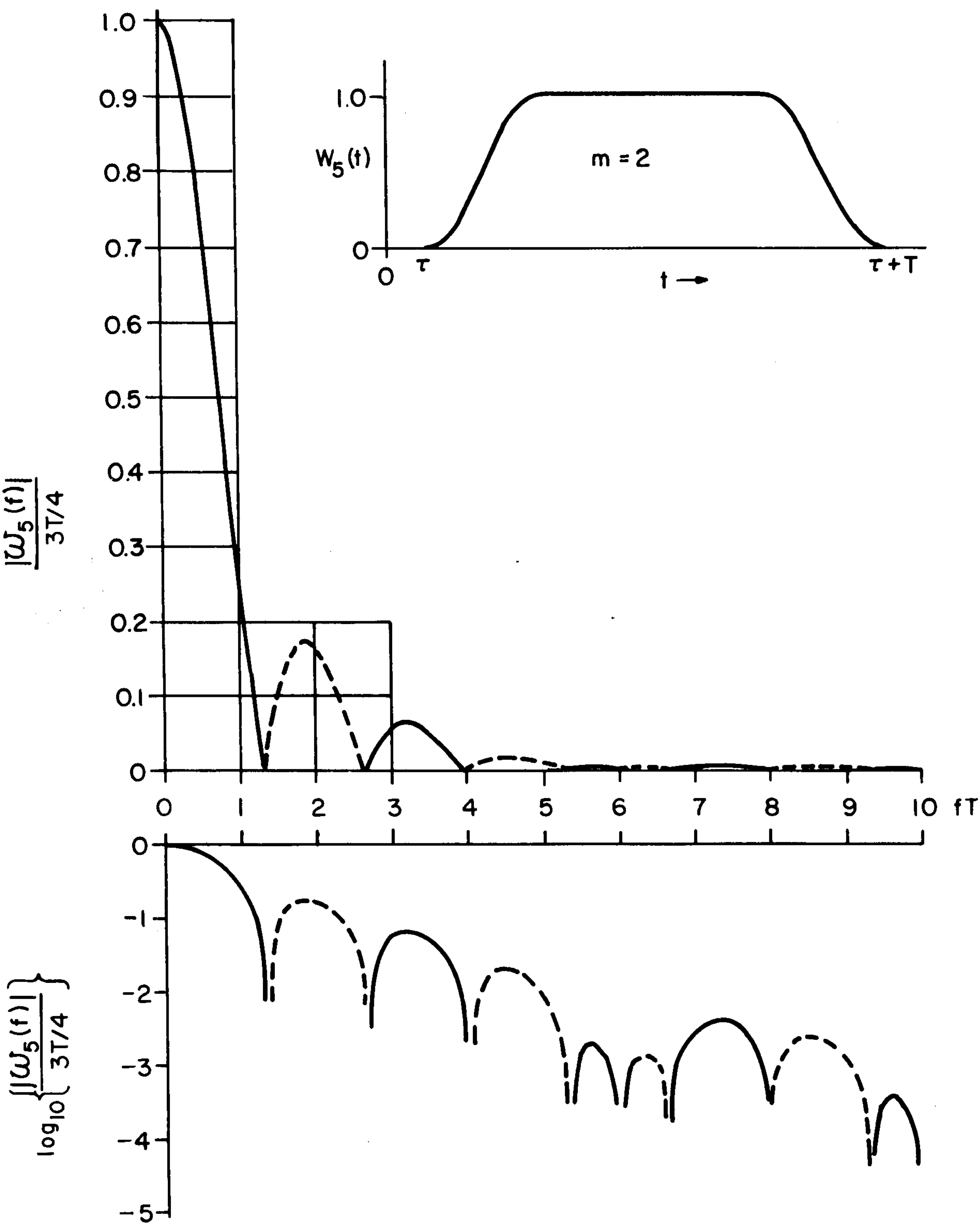


*Fig. 5.(a) Spectrum window corresponding to generalized cosine-bell data window (m = 2).*

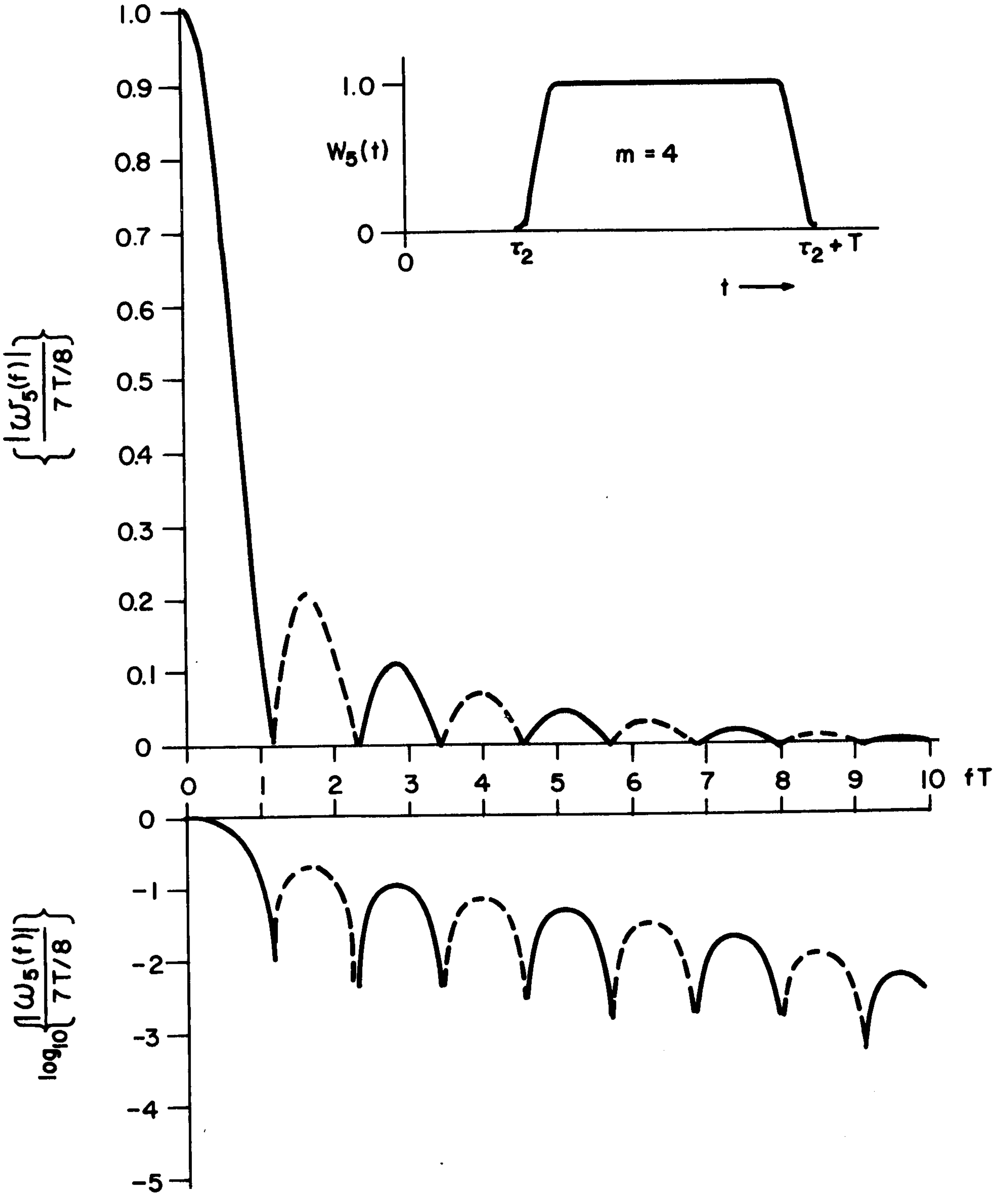


*Fig. 5.(b) Spectrum window corresponding to generalized cosine-bell data window (m = 4).*

$\mathcal{W}_5(f)$ has two sets of zeros, given by $f = (2r+1)m/T$, and $f = 2rm/(2m-1)T$, $r = 1,2,\ldots$; $r = 0$ does not correspond to a zero in either set. The first zeros are those at $f = \pm 2m/(2m-1)T$. The width of the main lobe therefore decreases rapidly with increase of m, from the value (specified as half the interval between the first zeros) of 2/T when m = 1, toward a limiting value of 1/T.

The rate of decrease of the size of the sidelobes is determined by the fourth bracket and the 1/f factor of the second bracket in Eqn. (102). The function $\{1 - (fT/m)^2\}$ decreases from unity at f = 0 to zero at f = m/T (which is also a zero of the third bracket), and becomes large and negative at higher frequencies. At frequencies less than m/T the window function therefore decreases essentially as 1/f, while at greater frequencies the asymptotic rate of decrease is $1/f^3$.

The characteristics noted represent the resolution of the apparent conflict discussed in the first paragraph of this section.

### 3.6.2 Sampled Data

The spectrum window for the case of discrete data, $\mathcal{W}'_5(f)$, is the convolution of $\mathcal{W}_5(f)$ with Fcomb(f). An expression for $\mathcal{W}'_5(f)$ based on this convolution is given in Reference 6, where the expression is left in the form of an infinite summation. To avoid the difficulties of this approach we proceed as follows:

$$W'_5(t) = \text{Bell}_1(t)\text{Boxcomb}_1(t) + \text{Bell}_2(t)\text{Boxcomb}_2(t) \;. \qquad \ldots\ldots(103)$$

The $\text{Bell}_i(t)$ functions are given by Eqn. (66) with T replaced by T/m; $\tau_2$ is zero for $\text{Bell}_1(t)$ and T/2m for $\text{Bell}_2(t)$. $\text{Boxcomb}_i(t)$ represents $\text{Box}_i(t)$. Comb(t); Comb(t) is given by Eqn. (38), and $\text{Box}_1(t)$ by Eqn. (33) with $\tau_1 = -\Delta t/2$; $\text{Box}_2(t)$ is defined as follows:

$$\text{Box}_2(t) = \begin{cases} 1 \;, & \left(\frac{T}{2m} - \frac{\Delta t}{2}\right) \le t \le \frac{(2m-1)T}{2m} - \frac{\Delta t}{2} \\ 0 \;, & \text{otherwise.} \end{cases} \qquad \ldots\ldots(104)$$

$\text{Box}_2(t)$ is the appropriate function, when the data are discrete, for selecting the region in which the continuous data window described by Eqn. (96) has the value unity. This statement is based on the argument presented in Appendix B, and involves the assumption that the point t = T/2m is a sample point. Our results will therefore be subject to the restriction that N/2m must be an integer; other assumptions would lead to other analytical expressions for the spectrum window. We may note, in passing, that an approximate analysis of this data window, employing a different technique, is presented in Reference 7; although not noted in the article, the analysis is subject to exactly the same restriction as that found in the present case.

The transforms of the functions on the right side of Eqn. (103) can be derived from previous results by making appropriate changes in the parameters. Each of the two terms yields an expression similar to that in Eqn. (80); these can be simplified and combined to give:

$$\mathcal{W}_5'(f) = \frac{T}{N}\left[\frac{\sin\left\{\left(\frac{2m-1}{2m}\right)\pi fT\right\}}{\tan\left(\frac{\pi fT}{N}\right)}\right]\left[\cos\left(\frac{\pi fT}{2m}\right)\right]\left[1-\left\{\frac{\sin\left(\frac{\pi fT}{N}\right)}{\sin\left(\frac{m\pi}{N}\right)}\right\}^2\right]^{-1}\exp\{-i\pi fT\}\ . \qquad \text{.....(105)}$$

This is the spectrum window corresponding to the sampled data window applied symmetrically on the interval (0,T), under the condition that N/2m is an integer.

It can easily be verified that $\mathcal{W}_5'(f)$, as given by Eqn. (105), reduces to $\mathcal{W}_5(f)$ for $(fT/N) \ll 1$. The data window $W_5'(t)$ becomes the standard cosine-bell window when m = 1, and, correspondingly, Eqn. (105) reduces to Eqn. (84). The largest possible value of m is N/2, which makes the time interval covered by the half cosine-bell equal to the inter-sample period. The data window function is then zero for the sample at t = 0, and unity (rectangular) for the remaining (N-1) samples. In this case Eqn. (105) becomes:

$$\mathcal{W}_5'(f) = \frac{T}{N}\left[\frac{\sin\left\{\frac{(N-1)\pi fT}{N}\right\}}{\sin\left(\frac{\pi fT}{N}\right)}\right]\exp\{-i\pi fT\}, \qquad m = N/2\ .$$

This is the spectrum window for the fractional-width rectangular data window that spans N-1 samples and is centred on t = T/2. Thus the expression of Eqn. (105) checks exactly with known results in these three limiting cases.

$\mathcal{W}_5'(f)$ is periodic, with period N/T (when N/2m is an integer), and it possesses conjugate symmetry about f = 0. We have already noted that the influence of sampling is small when $|fT/N| \ll 1$. The approximation is quite good over about half of the fundamental period, so that the characteristics of $\mathcal{W}_5'(f)$ in the region where its values are significant are substantially those of $\mathcal{W}_5(f)$. For larger values of $|fT/N|$ it can be shown that sampling increases the roll-off rate.

## 4. POWER-SPECTRUM WINDOWS

No distinction was made in Section 3 between transform pairs representing data windows and amplitude-spectrum windows, or lag windows and power-spectrum windows. The term 'data window' was used for either kind of time function, and 'spectrum window' for either frequency function. In this section the distinctions are once again observed.

According to Eqn. (18), the power-spectrum window $\mathcal{V}_n(f)$ corresponding to the data window $W_n(t)$ is $|\mathcal{W}_n(f)|^2/T$, where $\mathcal{W}_n(f)$ is the transform of $W_n(t)$.

The definition shows that $V_n(f)$ cannot be negative. The estimate of the power spectrum given by the direct method, as expressed in Eqn. (13), approximates the convolution of $V_n(f)$ with the true power spectrum of the signal. Since the true power spectrum is intrinsically non-negative, it is apparent that no negative power estimates can occur in the power spectrum computed by the direct method. This is not true, in general, of the results obtained by the indirect method. If $W_n(t)$ is used as a lag window, the power-spectrum window is $\mathcal{W}_n(f)$, and almost all of the frequency-domain functions derived in Section 3 exhibit negative values; the triangular window centred on t = 0 is an exception. The occurrence of negative lobes in $\mathcal{W}_n(f)$ does not necessarily mean that the computed power-spectrum will include negative estimates. Negative estimates are not likely to arise if the true spectrum is smooth and slowly varying, but they are common when the spectrum is sharply peaked, or when strong line components are present.

When the indirect method is used, the statistical stability of the computed power spectrum depends on the form of the lag window. Although the windows $W_n(t)$ of Section 3 could be used as lag windows, they would not give stable estimates. To achieve good stability, the computed power spectrum must be smoothed quite heavily. This could be done by forming weighted moving averages of the estimates; heavy smoothing corresponds to averaging over many adjacent estimates. Such averaging is most efficiently done in the lag domain, since long moving averages in the frequency domain correspond to windows of small span in the lag domain. The windows $W_n(t)$ of Section 3 are not short--with the exception of $W_1(t)$ they cover the entire span of the data. The treatment of $W_1(t)$ shows how the windows $W_n(t)$ can be modified to yield useful lag windows; for this application the windows must also, of course, be centred on t = 0. If the span of the lag window is reduced from T to T', the major features of the resulting power-spectrum window are well approximated by stretching $\mathcal{W}_n(f)$ along the frequency axis in the ratio (T/T'), as indicated in Section 3.2.2. The frequency interval between the first zeros of the stretched version of $\mathcal{W}_n(f)$, divided by (1/T), is a measure of the length of the equivalent moving average applied to the power spectrum. Hence for good stability (T/T') should be fairly large, of the order of 10. A thorough discussion of the stability of estimates of the power spectrum obtained by the indirect method is given in Reference 1.

When the direct method is used with a segmented signal, as outlined in Section 2.2, the stability of the power-spectrum estimates is achieved by averaging in time, rather than in frequency, and it is not a function of the window to provide stability.

The power-spectrum windows corresponding to the data windows of Section 3 are shown in Figures 6 to 11. The differences in frequency resolution (the width of the main lobe), in the size of the largest sidelobes, and in the rate of roll-off are apparent. The figures are drawn for the case of continuous data, since it was shown in Section 3 that, within the fundamental period, sampling has an appreciable effect only in the tails of the spectrum window. It should be remembered, however, that when sampling is used the power-spectrum window repeats periodically at frequencies beyond the Nyquist frequency.

## 5. ACKNOWLEDGEMENT

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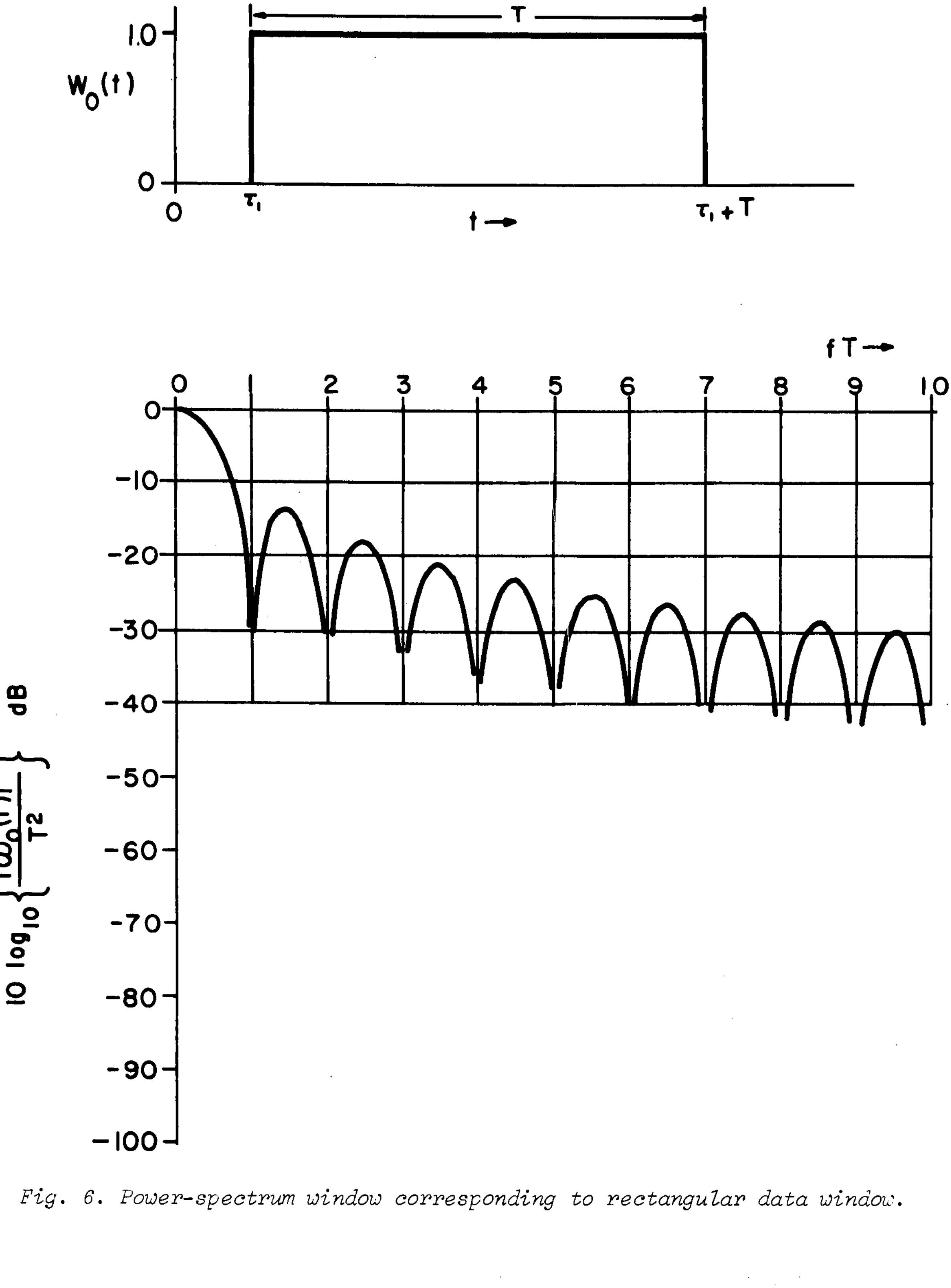


*Fig. 6. Power-spectrum window corresponding to rectangular data window.*

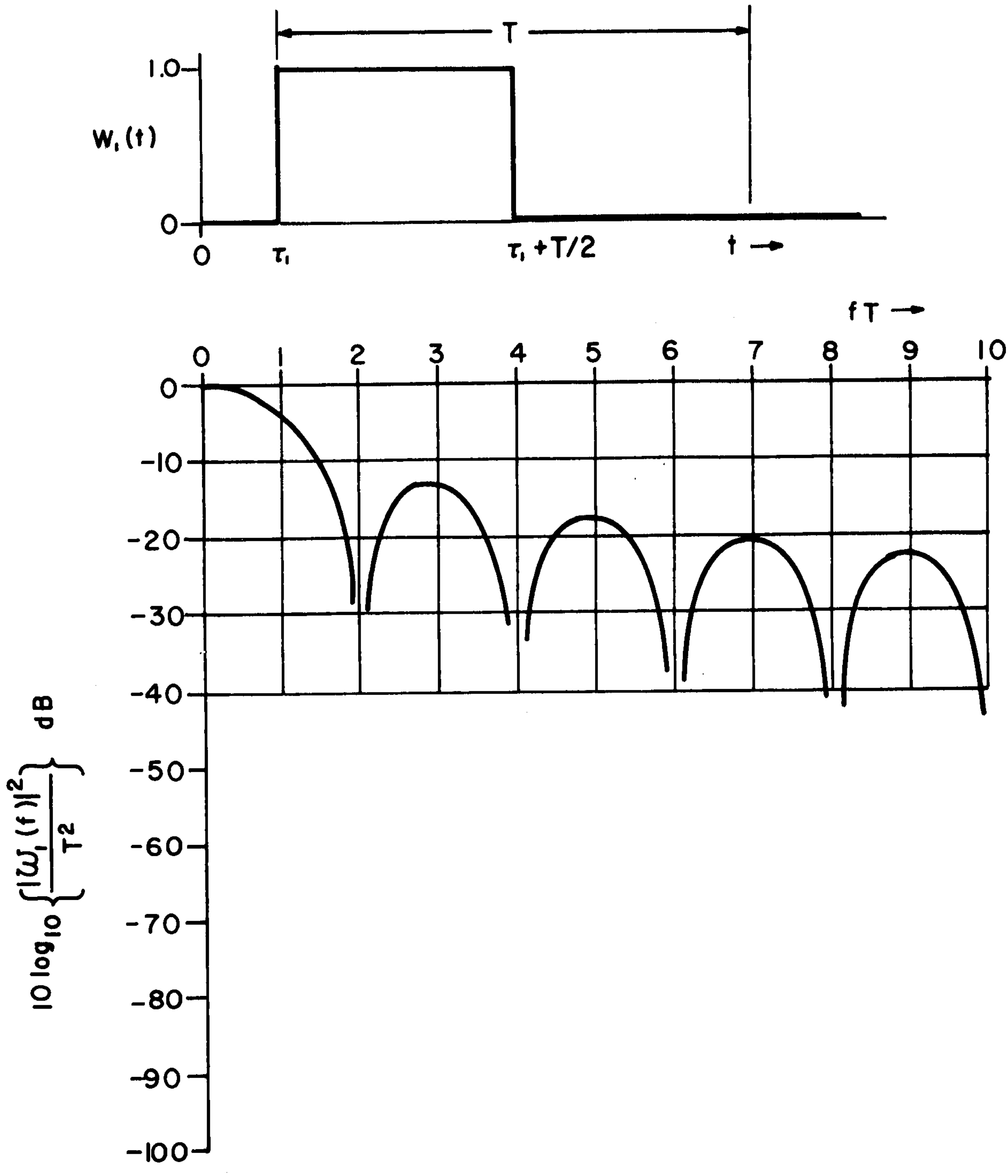


*Fig. 7. Power-spectrum window corresponding to half-width rectangular data window.*

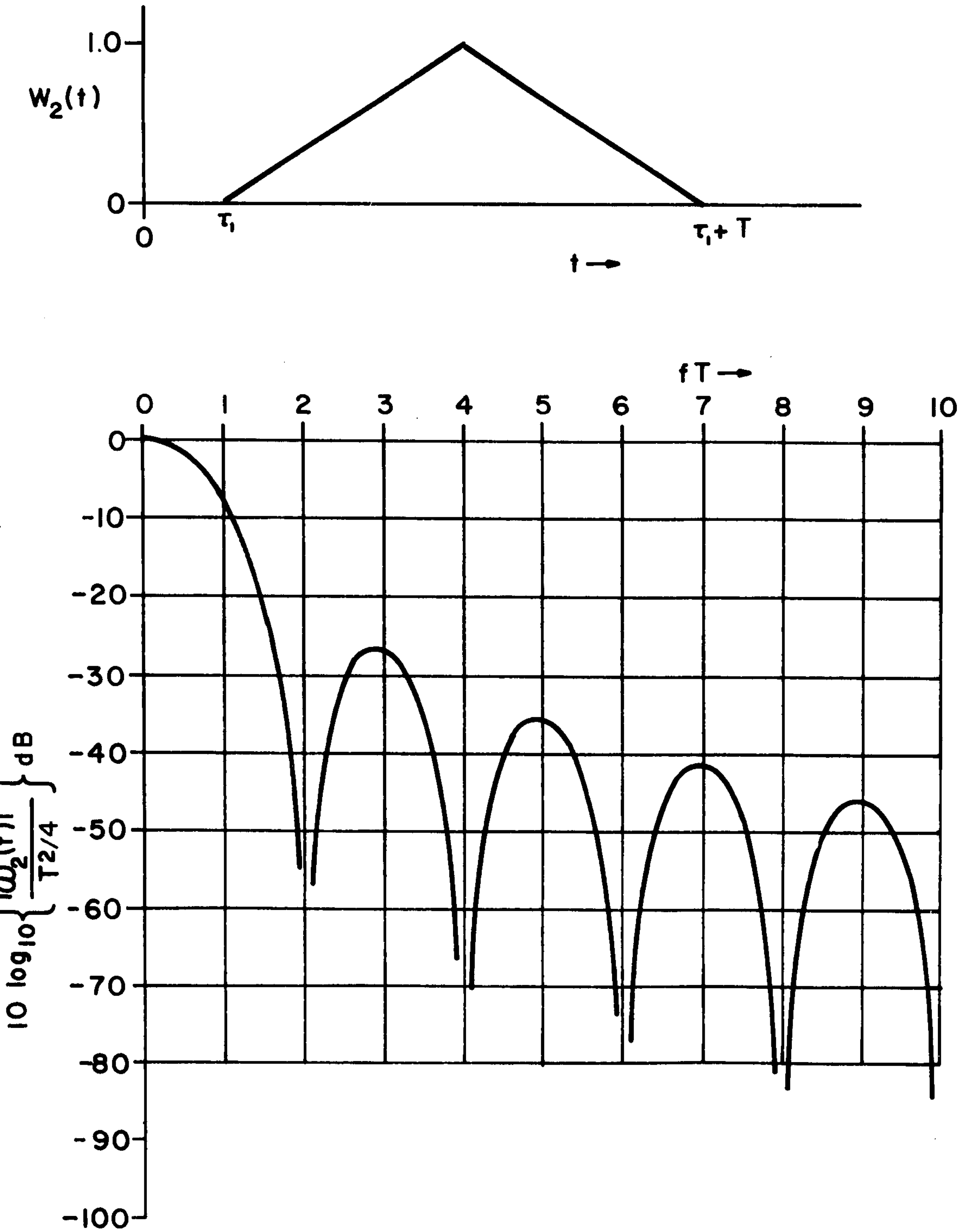


*Fig. 8. Power-spectrum window corresponding to triangular data window.*

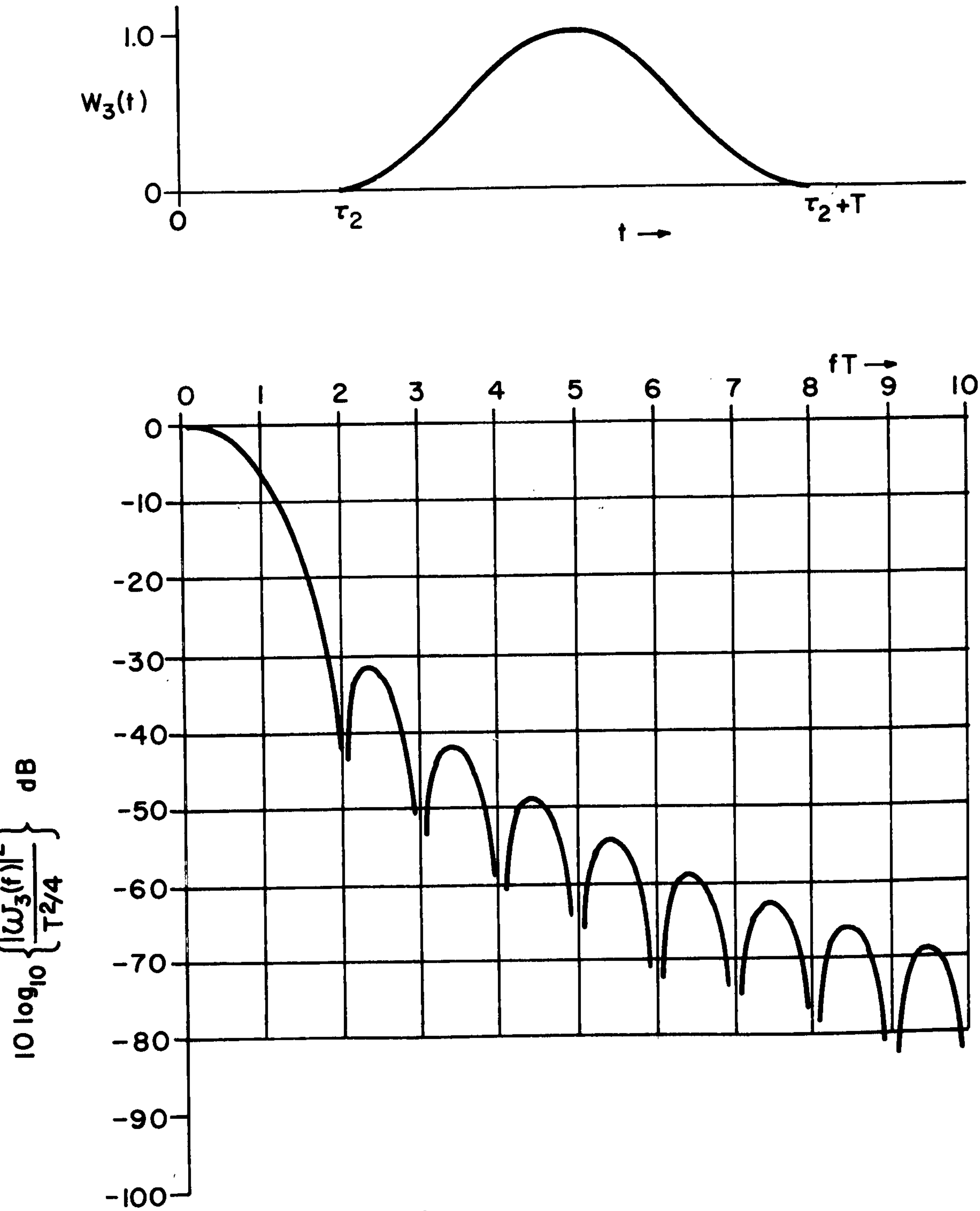


*Fig. 9. Power-spectrum window corresponding to cosine-bell data window.*

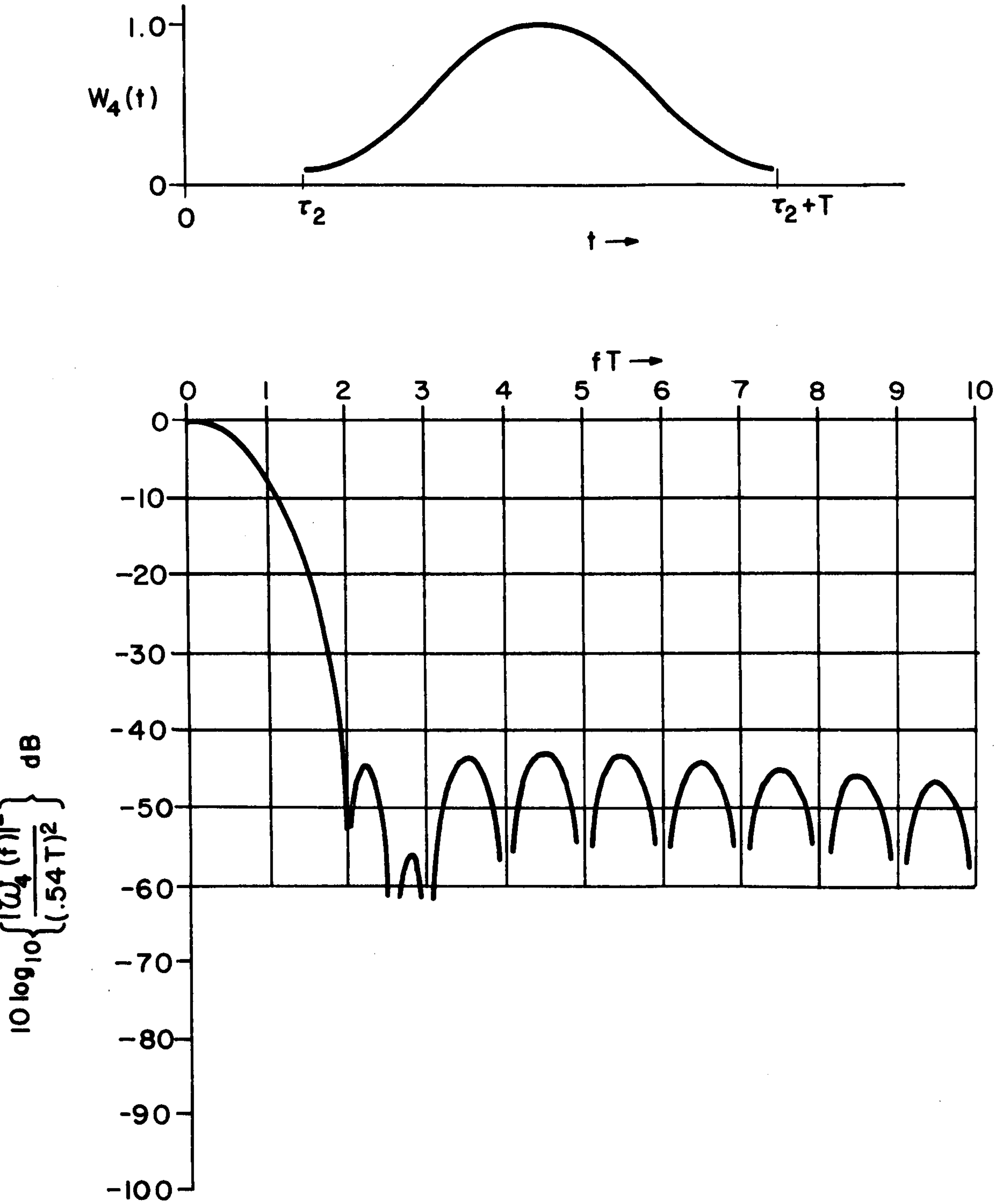


*Fig. 10. Power-spectrum window corresponding to Hamming data window.*

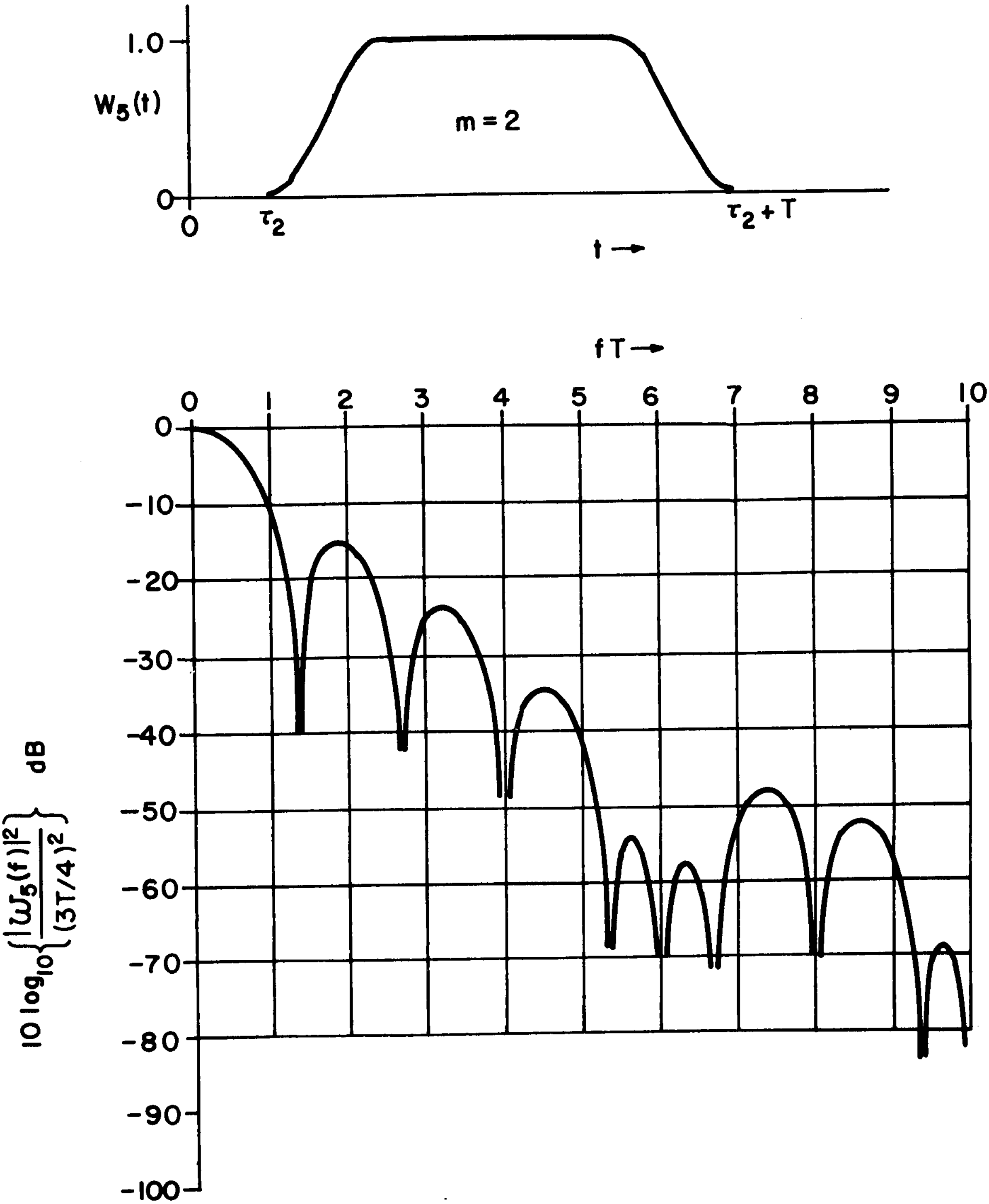


*Fig. 11.(a) Power-spectrum window corresponding to generalized cosine-bell data window (m = 2).*

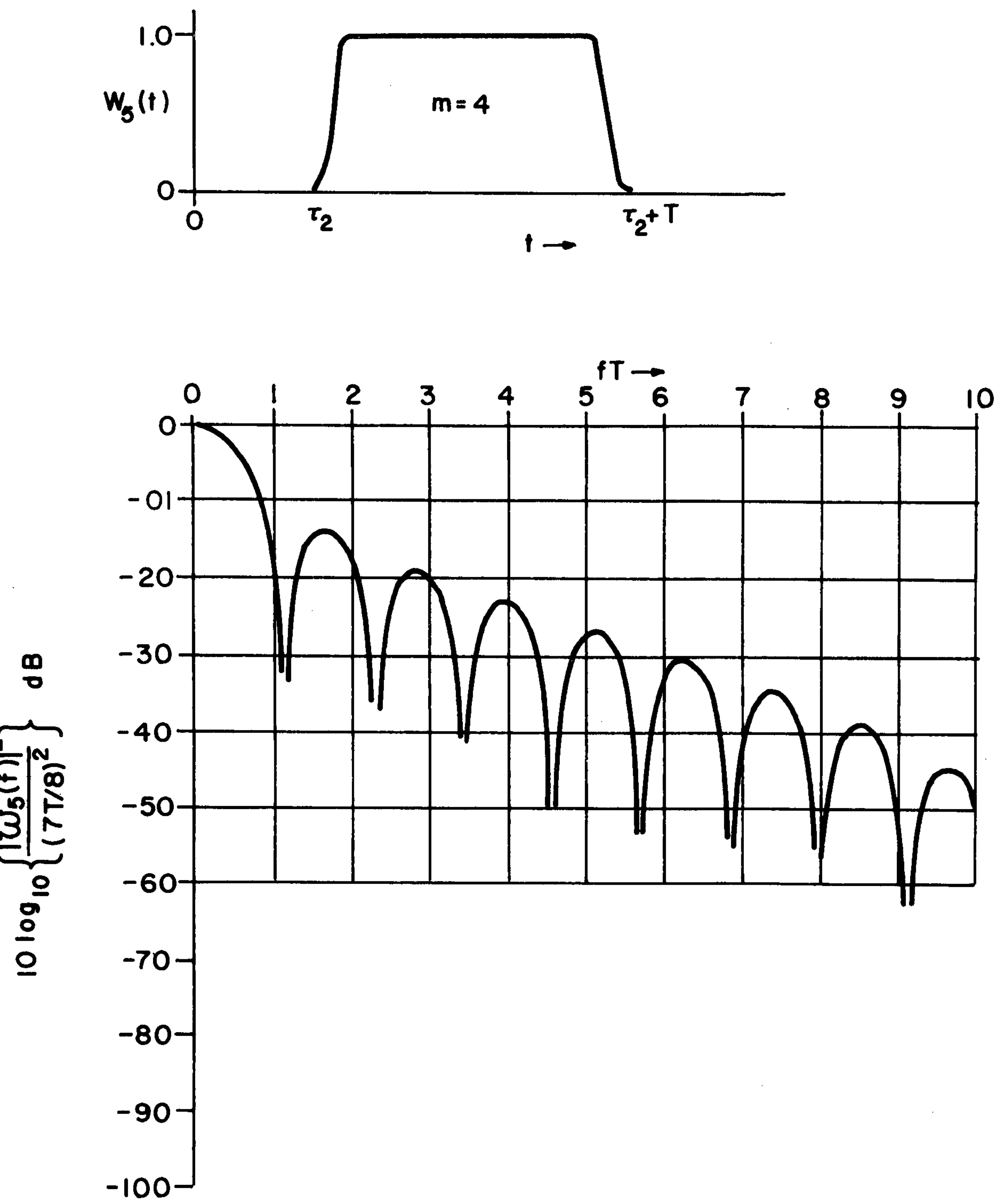


*Fig. 11.(b) Power-spectrum window corresponding to generalized cosine-bell data window (m = 4).*

# APPENDIX A

## THE ROLE OF AVERAGING IN THE DEFINITION OF THE POWER-SPECTRUM WINDOW

The estimator $S_n(f)$ for the power spectrum was defined in Section 2.2 as the average of a finite number of periodograms, and it was shown that, as this number is increased, $S_n(f)$ becomes arbitrarily close to the convolution of the true power spectrum with the power-spectrum window $V_n(f)$. It might be thought that averaging is required only to smooth out fluctuations due to the randomness of the signal, and consequently that the results of Section 2.2 hold, without averaging, when the signal is deterministic. This is not true, as is shown by the following example.

Suppose that the signal $x(t) = \cos(2\pi f_0 t + \alpha)$ is sampled at S samples/sec, and that N samples are analyzed by means of the DFT with a rectangular data window. In terms of the time index n and the frequency index $\sigma_0$, defined by $t = n/S$, $f_0 = S\sigma_0/N$ ($\sigma_0$ is not, in general, an integer), the signal can be written:

$$x(n\Delta t) = \cos\left(\frac{2\pi\sigma_0 n}{N} + \alpha\right) . \qquad .....(A1)$$

Using the amplitude-spectrum window (Eqn. (42)) corresponding to the rectangular data window, we can write the transform of the signal as follows:

$$X(\sigma) = \frac{T}{2N}\left[\frac{\sin \pi(\sigma_0-\sigma)}{\sin \pi \frac{(\sigma_0-\sigma)}{N}}\right] \exp(i\alpha) \cdot \exp\{i\pi(\sigma_0 - \sigma)(N-1)/N\}$$

$$+ \frac{T}{2N}\left[\frac{\sin \pi(\sigma_0+\sigma)}{\sin \pi \frac{(\sigma_0+\sigma)}{N}}\right] \exp(-i\alpha) \cdot \exp\{-i\pi(\sigma_0 + \sigma)(N-1)/N\} , \qquad .....(A2)$$

$$= z_1(\sigma) + z_2(\sigma) . \qquad .....(A3)$$

If we take the squared modulus of the transform and divide by T, the result is an estimate of the power spectrum identical to $S_0(f)$, as defined by Eqn. (13), except for the omission of the averaging operation. This estimate is:

$$S_{00}(\sigma) = \frac{1}{T} |z_1(\sigma) + z_2(\sigma)|^2 \ , \qquad \text{.....(A4)}$$

$$= \frac{1}{T} \{|z_1(\sigma)|^2 + |z_2(\sigma)|^2 + 2|z_1(\sigma)| \cdot |z_2(\sigma)| \cos(\phi_1 - \phi_2)\} \ , \qquad \text{.....(A5)}$$

where

$$(\phi_1 - \phi_2) = \left[\left\{\alpha + \pi\left(\frac{N-1}{N}\right)(\sigma_0 - \sigma)\right\} - \left\{-\alpha - \pi\left(\frac{N-1}{N}\right)(\sigma_0 + \sigma)\right\}\right] , \qquad \text{.....(A6)}$$

$$= 2\left\{\alpha + \pi\left(\frac{N-1}{N}\right)\sigma_0\right\} . \qquad \text{.....(A7)}$$

We now compute the right-hand side of Eqn. (17), for comparison with $S_{00}(\sigma)$. The power-spectrum window $V_0(\sigma)$ corresponding to the rectangular data window is:

$$V_0(\sigma) = \frac{T}{N^2} \left[\frac{\sin \pi\sigma}{\sin\left(\frac{\pi\sigma}{N}\right)}\right]^2 . \qquad \text{.....(A8)}$$

The true power spectrum of the signal is:

$$S(\sigma) = \frac{1}{4} \{\delta(\sigma - \sigma_0) + \delta(\sigma + \sigma_0)\} \ , \qquad \text{.....(A9)}$$

and the convolution of $S(\sigma)$ with $V_0(\sigma)$, which is denoted by $S_0(f)$, can be written:

$$S_0(\sigma) = \frac{1}{T} \{|z_1(\sigma)|^2 + |z_2(\sigma)|^2\} . \qquad \text{.....(A10)}$$

Comparing Eqns. (A5) and (A10), we see that $S_{00}(\sigma)$ is not the same as $S_0(\sigma)$ unless one or more of the factors of the cross term in $S_{00}(\sigma)$ is zero, which is so only in special cases. This demonstrates by counter-example that the concept of the power-spectrum window is not valid if averaging is not used.

Suppose the power spectrum of the same signal is estimated in accordance with Eqn. (13), by averaging over a large number of segments of the signal. If the length T of the segment is not an integral multiple of the period of the signal, the phase $\alpha$ will vary from segment to segment. The cross term in Eqn. (A5) will then average zero, because of the effect of the varying phase on the factor $\cos(\phi_1 - \phi_2)$. These phase variations do not occur if T is a multiple of the signal period, but then, as shown in Eqn. (A2), $z_1(\sigma)$ and $z_2(\sigma)$ are zero except for $\sigma = \pm\sigma_0$, and the cross term vanishes even without averaging. Thus Eqn. (17) correctly describes the power spectrum of this signal when averaging is used.

The need for averaging, demonstrated above for a sinusoidal signal, arises from the inequality (cf Eqns. (A5) and (A10)):

$$|\Sigma z_i|^2 \neq \Sigma |z_i|^2 \quad , \qquad \text{.....(A11)}$$

which holds, apart from special cases, for any set of components $z_i$--not just the pair forming a sinusoid.

# APPENDIX B

## A CLOSED-FORM EXPRESSION FOR Fboxcomb(f)

Equation (41) gives the spectrum window corresponding to the rectangular window for sampled data in the form:

$$\text{Fboxcomb}(f) = T \sum_{k=-\infty}^{\infty} \left[ \frac{\sin \pi \left(f - \frac{k}{\Delta t}\right) T}{\pi \left(f - \frac{k}{\Delta t}\right) T} \right] \exp \left\{ -i\pi \left(f - \frac{k}{\Delta t}\right) (T + 2\tau_1) \right\} . \qquad \text{.....(41)}$$

This expression can be simplified by the use of the following identities and definitions:

$$\Delta t = \frac{T}{N} ,$$

$$\sin \left\{ \pi \left(f - \frac{k}{\Delta t}\right) T \right\} = (-1)^{kN} \sin \pi f T ,$$

$$\exp \left\{ -i\pi \left(f - \frac{k}{\Delta t}\right) (T + 2\tau_1) \right\} = (-1)^{kN} \exp \{-i\pi f(T + 2\tau_1)\} \cdot \exp \{i2\pi N k\tau_1/T\},$$

$$\alpha = \frac{N}{fT} ,$$

$$\beta = 2\pi N \frac{\tau_1}{T} .$$

We obtain the result:

$$\text{Fboxcomb}(f) = \left[ T \frac{\sin \pi f T}{\pi f T} \exp \{-i\pi f(T + 2\tau_1)\} \right] \left[ \sum_{k=-\infty}^{\infty} \frac{1}{1-k\alpha} \exp (ik\beta) \right] . \qquad \text{.....(B1)}$$

The first bracketted factor in Eqn. (B1) is Fbox(f), as given by Eqn. (36), so that the second bracketted factor represents the effects of sampling.

We shall consider first the case where the N samples are taken at $t = 0, \Delta t, \ldots, (N-1)\Delta t$. It is necessary to distinguish between $\tau_1$, the starting point of the interval of length T from which the samples are taken, and the location of the first sample, which we shall denote by $\tau_1'$. If the sampled interval is written $-\varepsilon\Delta t \leq t \leq T-\varepsilon\Delta t$, where $0 < \varepsilon < 1$, then $\tau_1 = -\varepsilon\Delta t$, and $\tau_1' = 0$. The reason for introducing $\varepsilon$ is to ensure that the first sample at $t = 0$ is included in the interval, and that the $(N + 1)^{th}$ sample at $t = N\Delta t$ is not. The expression on the right-hand side of Eqn. (B1) depends on $\varepsilon$

through $\tau_1$ in the first bracket and $\beta$ in the second. In the final form of the equation this dependence must vanish, since obviously Boxcomb(t) is unaffected by changes in $\varepsilon$ within the specified limits.

For the case under consideration, Eqn. (B1) becomes:

$$\text{Fboxcomb}(f) = \left[ T \frac{\sin \pi fT}{\pi fT} \exp\left\{ -i\pi fT \left(1 - \frac{2\varepsilon}{N}\right)\right\}\right]\left[\sum_{k=-\infty}^{\infty} \frac{1}{1-k\alpha} \exp\{-2\pi ik\varepsilon\}\right]. \quad \text{.....(B2)}$$

It is shown later that

$$\sum_{k=-\infty}^{\infty} \frac{1}{1-k\alpha} \exp\{-2\pi ik\varepsilon\} = \left[\frac{\frac{\pi}{\alpha}}{\sin\left(\frac{\pi}{\alpha}\right)}\right] \exp\{i\pi(1-2\varepsilon)/\alpha\}, \quad \text{.....(B13)}$$

$$= \left[\frac{\frac{\pi fT}{N}}{\sin \frac{\pi fT}{N}}\right] \exp\{i\pi fT(1-2\varepsilon)/N\}, \quad \text{.....(B3)}$$

and hence:

$$\text{Fboxcomb}(f) = \frac{T}{N}\left[\frac{\sin \pi fT}{\sin\left(\frac{\pi fT}{N}\right)}\right] \exp\left\{-2\pi if\left(\frac{1}{2}\,\frac{N-1}{N}\,T\right)\right\}. \quad \text{.....(B4)}$$

Note that this expression for Fboxcomb(f) does not involve $\varepsilon$. The exponential factor in Eqn. (B4) has been written in a form that exhibits the time shift to which it corresponds: a shift to the right of $t_1 = \frac{1}{2}\,\frac{N-1}{N}\,T$. $t_1$ is the interval between the origin and the mid-point of the array of N sample points. The exponential would vanish if Fboxcomb(f) were modified to represent a shift of $(-t_1)$ in the time domain--i.e., if the sample points were symmetrically located with respect to the origin. In this case Eqn. (B4) becomes:

$$\text{Fboxcomb}(f) = \frac{T}{N}\left[\frac{\sin \pi fT}{\sin\left(\frac{\pi fT}{N}\right)}\right]. \quad \text{.....(B5)}$$

We can now deal with the general case, where the first sample point is at $t = \tau_1'$, by applying the appropriate time-shift exponential to the right-hand side of Eqn. (B5). The mid-point of the array of N sample points is at $t = (\tau_1' + \frac{1}{2}\,\frac{N-1}{N}\,T)$, and this is the magnitude of the shift required. Hence, in general:

$$\text{Fboxcomb}(f) = \frac{T}{N}\left[\frac{\sin \pi fT}{\sin\left(\frac{\pi fT}{N}\right)}\right] \exp\left\{-i\pi f\left[\left(\frac{N-1}{N}\right)T + 2\tau_1'\right]\right\}. \quad \text{.....(B6)}$$

Equation (B6) can also be written in the form:

$$\text{Fboxcomb}(f) = \frac{T}{N}\left[\frac{\sin \pi fT}{\sin\left(\frac{\pi fT}{N}\right)}\right] \exp\{-i\pi f(T + 2\tau_1)\}, \quad \text{.....(42)}$$

where $\tau_1 = \tau_1' - \Delta t/2$. Since $t = \tau_1'$ is the location of the first sample point, this form shows that the time interval of length T from which the N samples are taken must be regarded as starting $\Delta t/2$ to the left of the first sample point, and ending the same amount to the right of the last one. Thus, although $\varepsilon$ does not appear explicitly in the final equation, it is uniquely determined by the result; its value is $\varepsilon = 1/2$. Equation (42) is the desired equivalent, in closed form, of Eqn. (41).

## Summation of the Series

The even function f(x), defined on the interval $-\pi \leq x \leq \pi$ by the equation:

$$f(x) = \cos ax,$$

can be expanded in a Fourier cosine series, as follows:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

The coefficients are:

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos ax \cos nx \, dx \,,$$

$$= (-1)^n \frac{2}{\pi} \frac{a}{a^2 - n^2} \sin \pi a.$$

Hence,

$$\cos ax = \frac{2a \sin \pi a}{\pi} \left[ \frac{1}{2a^2} + \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{a^2 - n^2} \right] . \qquad \text{.....(B7)}$$

Equation (B7) is valid for $|x| \leq \pi$. Similarly, or by differentiating Eqn. (B7) term by term, we find:

$$a \sin ax = \frac{2a \sin \pi a}{\pi} \left[ \sum_{n=1}^{\infty} (-1)^n \frac{n \sin nx}{a^2 - n^2} \right] . \qquad \text{.....(B8)}$$

Substituting $(\pi - x)$ for x in Eqns. (B7) and (B8), we obtain:

$$\left(\frac{\pi a}{\sin \pi a}\right) \cos \{a(\pi - x)\} = \left[ 1 + 2 \sum_{n=1}^{\infty} \frac{a^2 \cos nx}{a^2 - n^2} \right] , \qquad \text{.....(B9)}$$

$$\left(\frac{\pi a}{\sin \pi a}\right) \sin \{a(\pi - x)\} = \left[ -2 \sum_{n=1}^{\infty} \frac{na \sin nx}{a^2 - n^2} \right] , \qquad \text{.....(B10)}$$

which can be combined to give:

$$\left(\frac{\pi a}{\sin \pi a}\right) \exp \{ia(\pi - x)\} = \left[ 1 + 2 \sum_{n=1}^{\infty} \frac{a^2 \cos nx - i\, na \sin nx}{a^2 - n^2} \right] , \qquad \text{.....(B11)}$$

$$= \sum_{n=-\infty}^{\infty} \frac{a}{a-n} \exp\{-inx\} \quad . \qquad .....(B12)$$

Equations (B9) to (B12) are valid for $0 \leq x \leq 2\pi$. With the substitutions $\alpha = 1/a$, and $2\pi\varepsilon = x$, Eqn. (B12) becomes:

$$\left\{ \frac{\frac{\pi}{\alpha}}{\sin\left(\frac{\pi}{\alpha}\right)} \right\} \exp\{i\pi(1-2\varepsilon)/\alpha\} = \sum_{k=-\infty}^{\infty} \frac{1}{1-k\alpha} \exp\{-2\pi ik\varepsilon\} \quad . \qquad .....(B13)$$

Equation (B13) gives the sum of the series in Eqn. (B2).