Communications Research Centre

A MICROWAVE ATTITUDE SENSING SYSTEM FOR SATELLITES - ANGULAR RELATIONSHIPS

by

Rolf Mamen

CRC REPORT NO. 1245

DEPARTMENT OF COMMUNICATIONS
MINISTÈRE DES COMMUNICATIONS

CANADA

OTTAWA OCTOBER 1973

COMMUNICATIONS RESEARCH CENTRE

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NOMENCLATURE

$\theta_1, \theta_2, \theta_3$ - Eulerian attitude error angles, roll, pitch and yaw, respectivel

ϕ_1, ϕ_2, ϕ_3 - Eulerian attitude angles with respect to the MASS set of axes

α, β, γ - the projected attitude angles measured by the MASS

ϕ - longitude of the ground station relative to the nominal or average subsatellite point

λ - latitude of the ground station

$\delta_1, \delta_2, \delta_3$ - Eulerian angles of the MASS coordinates with respect to spacecraft axes

$\delta'_1, \delta'_2, \delta'_3$ - Eulerian angles of the ground station antenna axes with respect to χ^E axes

χ^E - an earth centred set of axes

R_E - radius of the earth at the ground station

R_S - synchronous radius

R_{st} - radius at time t of an eccentric synchronous orbit

$\underline{r}_{SG}$ - a unit vector from the satellite to the ground station

ϕR - angle from ascending node to the perigee, in the orbit plane

e - eccentricity of orbit

f - true anomaly of spacecraft in its orbit

M - mean anomaly of spacecraft in its orbit

η' - orbital angle of satellite from ascending node, $\eta' = \phi R + f$

η - earth rate angle, $\eta = \phi R + M$

ω_E - spin rate of the earth

t - time measured from ascending node epoch.

NOTATION

$(\gamma)_i$ denotes the transformation corresponding to a rotation from an old set of coordinates χ^{OLD} to a new set χ^{NEW}, a rotation of magnitude $\gamma°$ and about axis i of the old set. The right-handed rule is the convention used to determine the sense of the rotation. Accordingly, an inertially fixed vector $\underline{V}_{OLD}$ in the old set of axes becomes $\underline{V}_{NEW}$ in the new representation, where $\underline{V}_{NEW} = (\gamma)_i \, \underline{V}_{OLD}$. The rotational transformations are defined in Appendix A.

Unless otherwise defined, a vector is a column vector in three dimensional Euclidean space. Accordingly, the rotational transformations are 3 x 3 matrices. The components of a vector $\underline{V}$ are indicated by subscripts: $\underline{V} = (V_1, V_2, V_3)^T$.

To abbreviate the trigonometric functions we employ the notation $C\theta$ for $\cos(\theta)$ and $S\theta$ for $\sin(\theta)$, for any angle represented by a Greek symbol such as θ. Since the angles δ_1, δ_2, δ_3 and δ'_1, δ'_2, δ'_3 appear frequently in the following, we use the further abbreviation of S_j and C_j to represent $\sin(\delta_j)$ and $\cos(\delta_j)$, respectively, for j = 1, 2 or 3. Similarly, S'_j and C'_j denote $\sin(\delta'_j)$ and $\cos(\delta'_j)$, respectively.

For further notational information, please refer to the Nomenclature.

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ABSTRACT

A Microwave Attitude Sensing Subsystem (MASS) may be used to measure the attitude of a satellite relative to an incident polarized RF signal from a ground station. The desired attitude of a spacecraft, however, may be specified with respect to a set of axes other than those of the MASS. This report describes the development of relations between the angles measured by the MASS and the spacecraft attitude specified by the conventional roll, pitch and yaw angles. In order to preserve generality, the ground station position is arbitrary and the satellite is assumed to be in an inclined eccentric synchronous orbit. Exact relations are developed for the roll, pitch and yaw angles in terms of the MASS measured angles, and conversely, for the MASS measured angles in terms of roll, pitch and yaw. Furthermore, the difficult-to-measure yaw angle is determined from roll and pitch plus any one of the MASS measured angles, for pseudo two-station yaw. The exact solutions are orbit dependent and lengthy, and various levels of approximations are suggested in order to simplify the solutions.

1. INTRODUCTION

The use of narrow-beam antennas aboard satellites requires accurate orientation of the satellites and even more precise measurement of the orientation.

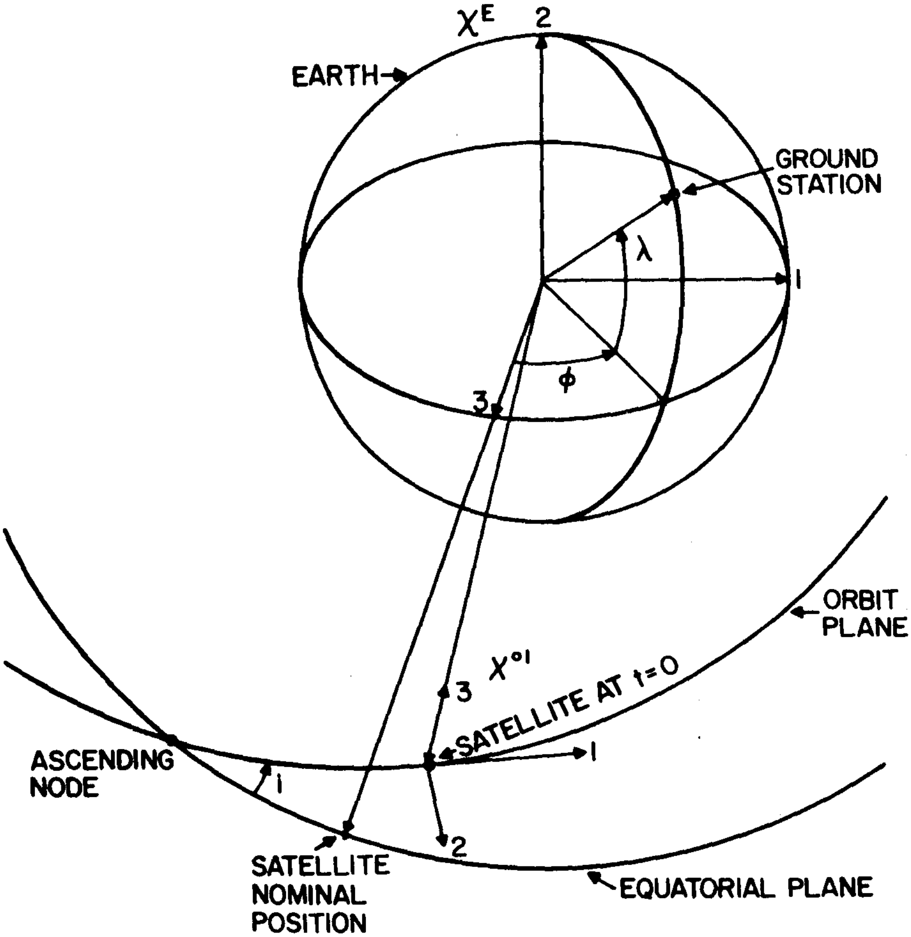

Fig. 1a. Pictorial of satellite and ground station.

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\chi^{E} \arrow[r, Rightarrow, "*(180^\circ)_1\ (\eta')_2\ (i)_3\ (-\eta)_2"] \arrow[d, Rightarrow, "\overset{*}{}(180^\circ)_1"'] & \chi^{01} \arrow[r, Rightarrow, "(\theta_1)_1\ (\theta_2)_2\ (\theta_3)_3"] & \chi^{SC} \arrow[d, Rightarrow, "(\delta_2)_2\ (\delta_1)_1"] \\
\chi^{0} \arrow[r, Rightarrow, "(\delta_2')_2\ (\delta_1')_1"] & \chi^{LS} \arrow[r, Rightarrow, "*\ (\phi_1)_1\ (\phi_2)_2\ (\phi_3)_3"] & \chi^{\overset{*}{RF}}
\end{tikzcd}
$$

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* Denotes additional parallel translation.

Fig. 1b. Rotational relations between coordinate sets χ.

Conventional sensors in the past have been basically one- or two-axis devices such as earth or sun sensors or star trackers, requiring a combination of two sensors in order to monitor the orientation of a satellite about the three conventional axes of roll, pitch and yaw.

One feature of the MASS proposed for the Communications Technology Satellite (CTS) is that the one device could measure three-axis information which can be transformed into roll, pitch and yaw angles. Two of the angles could be monitored on a principle similar to that of an interferometer, while the third, rotation about the antenna boresight, could be measured along polarimeter or polarization rotation lines. Since the proposed beacon would not be located at the subsatellite point, the measured angles would not correspond to the desired roll, pitch and yaw. However, the transformation may be effected if the beacon and satellite positions are both known. Furthermore, the cross coupling of the roll, pitch and yaw angles into the MASS measured angles enables yaw to be estimated from roll and pitch plus one of the MASS measured angles, in the so-called 'pseudo two-station yaw' calculation.

In this report the analyses are performed in full generality for any visible beacon location and satellite position, allowing the orbit to be both inclined and eccentric. After a definition of the pertinent coordinate systems and transformations, a solution of the desired roll, pitch and yaw error angles is found in terms of the MASS measured angles. For the purposes of error analysis and checking the results, the reciprocal solution of the MASS measured angles in terms of roll, pitch and yaw is also determined. A pair of rotational transformations required for these solutions is defined and found to be useful for directing both the beacon from the ground and the MASS antenna from the spacecraft. The dependence of these rotations on beacon and satellite position along with the dependence of the MASS measured angles on these rotations are shown to couple the orbital parameters into the MASS measured angles. Subsequently the same type of approach leads to a solution for yaw in terms of a MASS measured angle plus roll and pitch. Some other combinations of measurements are discussed and finally, future work is outlined.

2. COORDINATE SYSTEMS

Figure 1 displays pictorially the approximate relative position of the CTS and Ottawa, as well as the earth based coordinate system χ^E. The origin of χ^E is at the earth's centre; the #2 axis is northerly along the earth's spin axis, the #3 axis is directed toward the nominal or average subsatellite point on the equator and the #1 axis completes the right-handed system.

χ^{01}, originating at the spacecraft, is defined to have the #3 axis toward the earth's centre, the #2 axis along the southerly orbit normal and the #1 complementing the right-handed system (and approximately along the satellite velocity vector). χ^{01} is translated from the earth's centre and rotated by $(180°)_1\ (\eta')_2\ (i)_3\ (-\eta)_2$ from χ^E. χ^{SC}, originating at the spacecraft, has its axes along the physical axes of the spacecraft. The roll, pitch and yaw angles, to be defined in detail later, indicate the attitude of χ^{SC} with respect to χ^{01}. χ^0, centred at the beacon is derived from χ^E by translation and then the rotation $(180°)_1$. χ^{LS}, originating at the beacon, has the #3 axis directed away

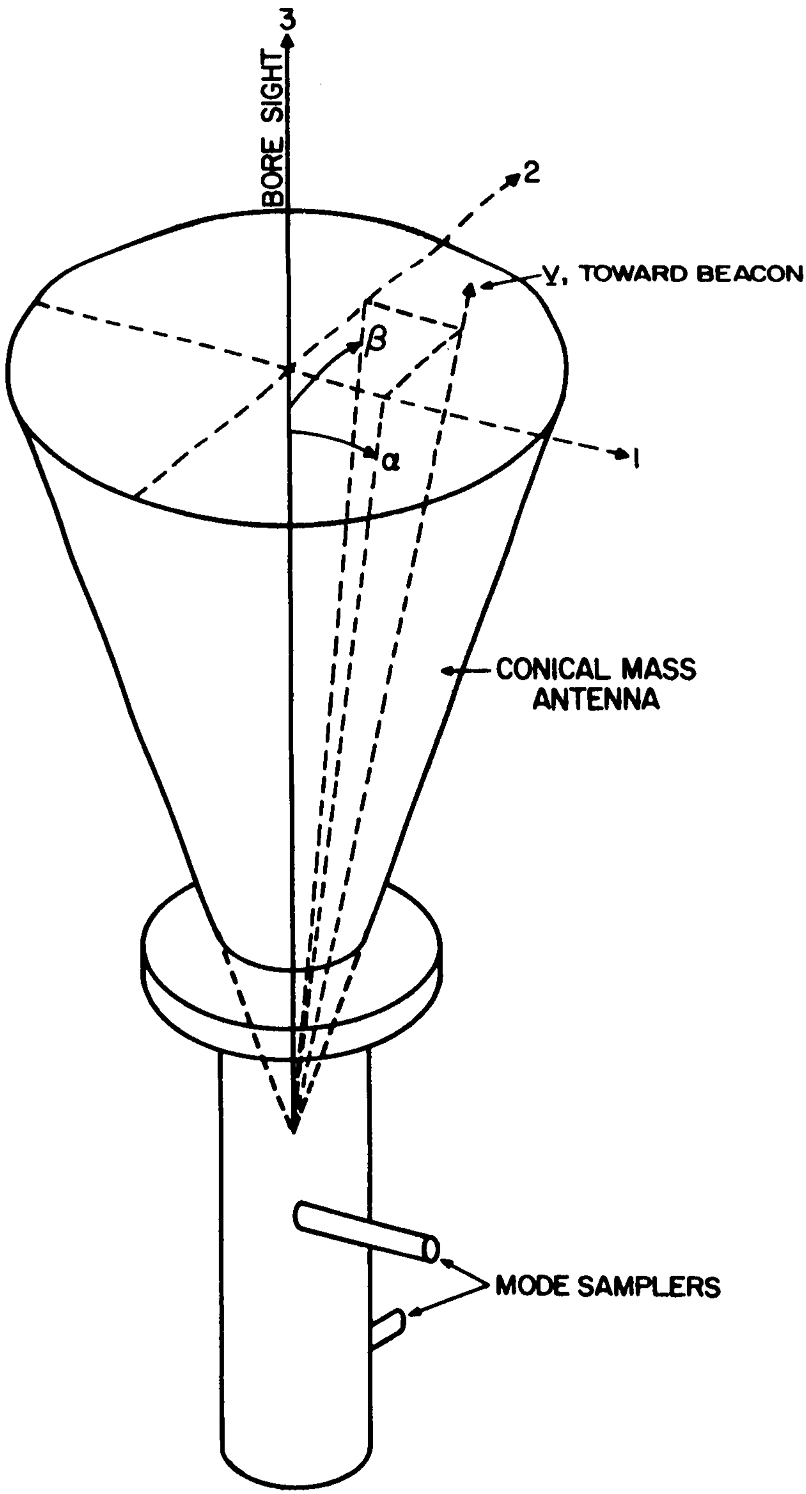

Fig. 2. MASS axes.

from the actual satellite position, and is derived from χ^0 by the rotation $(\delta_2')_2\ (\delta_1')_1$. χ^{RF}, originating at the satellite, is the set of coordinates forming the axes of the MASS and is defined to have the #3 axis directed at the beacon if the orbit inclination, eccentricity, station and attitude errors are all zero. χ^{RF} is derived from χ^{SC} by the rotation $(\delta_2)_2\ (\delta_1)_1$. The MASS measured angles indicate the attitude of χ^{RF} relative to χ^{LS}.

3. MASS-MEASURED ANGLES

An approximation of the physical appearance of the MASS antenna and the relative orientation of χ^{RF} is displayed in Figure 2. The #3 axis lies along the antenna boresight and the #1 and #2 axes are parallel to the mode samplers.

The 'interferometer' segment of the MASS measures the angles of incidence α and β of RF energy arriving at the 'horn' while the 'polarimeter' segment measures the effective rotation of the plane of polarization of the incident radiation. Thus if $\underline{V}$ denotes the direction of the beacon from the apex of the cone, observed in the χ^{RF} frame, α is the angle from the boresight to the projection of $\underline{V}$ onto the plane defined by axes #1 and #3, while β is the angle from boresight to the projection of $\underline{V}$ onto the plane of axes #2 and #3. Although the MASS electronics differ fundamentally from those of a pair of interferometers, the MASS measured angles α and β are identical to those that would be indicated by interferometers placed along the #1 and #2 axis, respectively.

The 'polarimeter' provides an indication of the rotation of a plane of polarization of the beacon signal. The rotation is measured in the plane defined by axes #1 and #2. The plane of polarization is defined by the vectors $\underline{V}$ and $\underline{W}$, $\underline{W} \perp \underline{V}$, and the polarimeter is assumed to measure the angle γ from the nominal position of $\underline{W}$ (at zero attitude and orbit errors) to the projection of the actual $\underline{W}$ into the plane of axes #1 and #2. We may arbitrarily set the nominal direction of $\underline{W}$ to $(0, 1, 0)^T$, and then, as shown in Figure 3, γ is formed by axis #2 and the component of $\underline{W} \perp$ axis #3.

Figure 3 shows that α, β and γ are in general not Eulerian angles[1] but projections. The Eulerian angles ϕ_3, ϕ_2 and ϕ_1 are required for the analysis to follow and are shown in Figure 4 as the rotations from χ^{LS} to χ^{RF}, about axes 3, 2 and 1 in that order. This sequence of rotations, $(\phi_1)_1\ (\phi_2)_2\ (\phi_3)_3$, was chosen to be consistent with the order of the roll, pitch and yaw errors defined in reference 2.

The relation between the measured angles (α, β, γ) and the desired rotational angles (ϕ_1, ϕ_2, ϕ_3) may be determined from the geometry of Figures 3 and 4. In χ^{LS}, $\underline{\tilde{V}} \equiv (0, 0, 1)^T$ and arbitrarily we defined $\underline{\tilde{W}} \equiv (0, 1, 0)^T$....(1)

The transformation from χ^{LS} to χ^{RF} is defined to be $(\phi_1)_1\ (\phi_2)_2\ (\phi_3)_3$, so that the ϕ's in the RF frame are analogous to the roll, pitch and yaw error angles in the SC frame. Thus for $\underline{\tilde{V}}$ and $\underline{\tilde{W}}$ in χ^{LS} transforming to $\underline{V}$ and $\underline{W}$ in χ^{RF}, respectively, we see $\underline{V} = (\phi_1)_1\ (\phi_2)_2\ (\phi_3)_3\ \underline{\tilde{V}}$ ....(2)

and $\underline{W} = (\phi_1)_1\ (\phi_2)_2\ (\phi_3)_3\ \underline{\tilde{W}}$. ....(3)

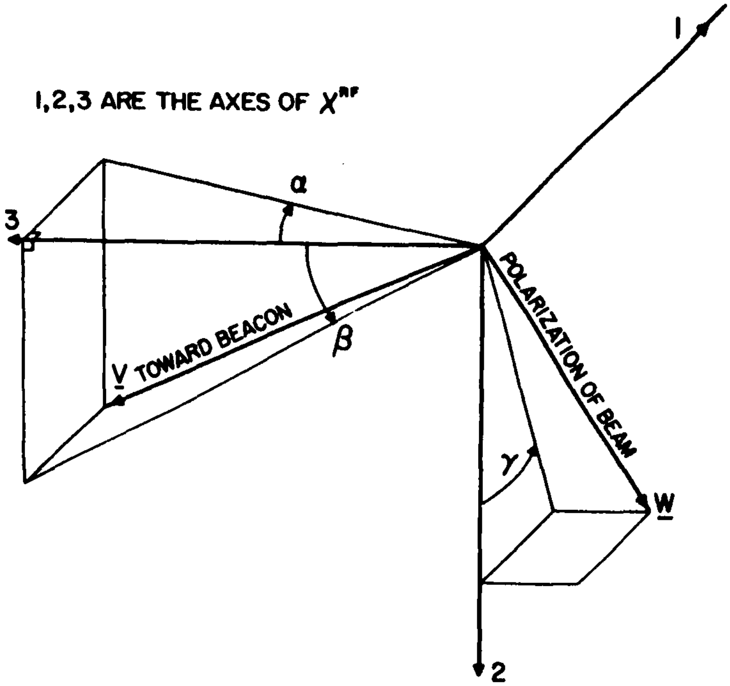

Fig. 3. MASS measured angles.

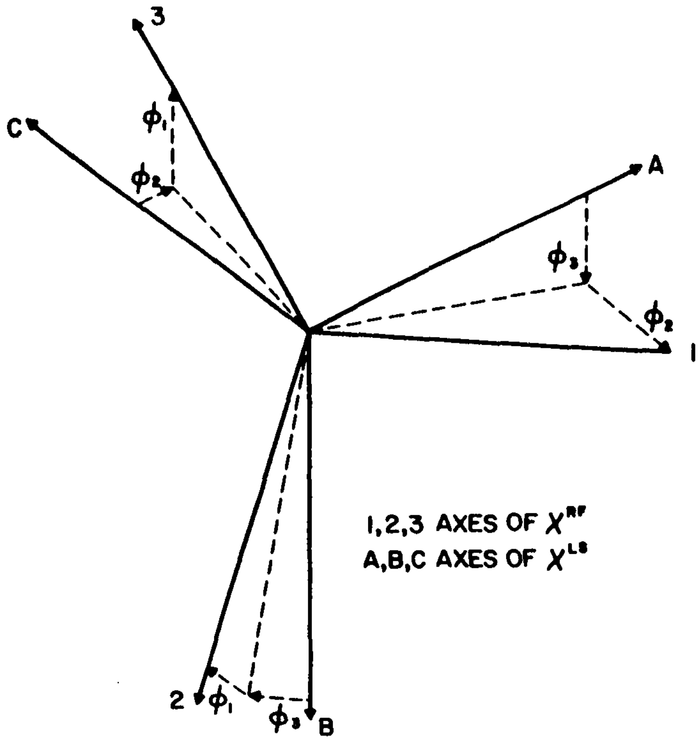

Fig. 4. Eulerian rotations $(\phi_1)_1\ (\phi_2)_2\ (\phi_3)_3$ from χ^{LS} to χ^{RF}.

From Figure 3 it is seen that:

$$\alpha = \tan^{-1} V_1/V_3 \qquad \text{....(4)}$$

$$\beta = \tan^{-1} -V_2/V_3 \qquad \text{....(5)}$$

$$\gamma = \tan^{-1} -W_1/W_2\ . \qquad \text{....(6)}$$

As mentioned earlier, the signs have been determined using the right hand rule.

Substituting in the numerical values of $\underline{\tilde{V}}$ and $\underline{\tilde{W}}$ and performing the operations indicated in (2) and (3) results in:

$$\underline{V} = \begin{bmatrix} -S\phi_2 \\ S\phi_1\ C\phi_2 \\ C\phi_1\ C\phi_2 \end{bmatrix} \text{ and } \underline{W} = \begin{bmatrix} C\phi_2\ S\phi_3 \\ S\phi_1\ S\phi_2\ S\phi_3 + C\phi_1\ C\phi_3 \\ C\phi_1\ S\phi_2\ S\phi_3 - S\phi_1\ C\phi_3 \end{bmatrix}. \qquad \text{....(7)}$$

Using these components, we find:

$$\tan\alpha = -\frac{S\phi_2}{C\phi_1\ C\phi_2} = -\frac{\tan\phi_2}{C\phi_1} \qquad \text{....(8)}$$

$$\tan\beta = -\frac{S\phi_1\ C\phi_2}{C\phi_1\ C\phi_2} = -\tan\phi_1 \qquad \text{....(9)}$$

$$\tan\gamma = -\frac{C\phi_2\ S\phi_3}{S\phi_1\ S\phi_2\ S\phi_3 + C\phi_1\ C\phi_3} \approx -\frac{C\phi_2}{C\phi_1}\tan\phi_3. \qquad \text{....(10)}$$

since for small ϕ's, $C\phi_1 C\phi_3 >> S\phi_1 S\phi_2 S\phi_3$.

For ϕ_1 and ϕ_2 small, α, β and γ are approximately the negatives of ϕ_2, ϕ_1 and ϕ_3, respectively. For ϕ_i less than 0.45°, $(C\phi_i)^{-1} < 1.0000308$, $C\phi_2/C\phi_1 < 1.0000308$, demonstrating the quality of the approximations.

The solution of ϕ_{1-3} in terms of α, β and γ is probably of greater interest in the analyses to follow, and is:

$$\phi_1 = -\beta \qquad \text{....(11)}$$

$$\begin{aligned} \phi_2 &= \tan^{-1}(-C\phi_1 \tan\alpha) \\ &= \tan^{-1}(-C\beta \tan\alpha) \end{aligned} \qquad \text{....(12)}$$

$$\begin{aligned} \phi_3 &= \tan^{-1}(-C\phi_1 \tan\alpha/C\phi_2) \\ &= \tan^{-1}\left(-C\beta \tan\alpha/C\left(\tan^{-1}(-C\beta \tan\alpha)\right)\right). \end{aligned} \qquad \text{....(13)}$$

ϕ_1, ϕ_2 and ϕ_3 are of the order of 0.5° to 3.0° normally and are restricted to magnitudes less than 90° in Equations (11) to (13).

4. THE TRANSFORMATION $(\delta_2)_2\ (\delta_1)_1$

The transformation $(\delta_2)_2\ (\delta_1)_1$ serves two functions; it relates χ^{RF} to χ^{SC} and nominally at least is the rotation from χ^0 to χ^{LS}. In the former capacity, $(\delta_2)_2\ (\delta_1)_1$ defines the correct orientation of the MASS antenna with respect to the spacecraft, such that for zero attitude and position errors, a beam $\underline{V}$ from the beacon arrives coaxially with the MASS antenna. In the latter capacity, $(\delta_2)_2\ (\delta_1)_1$ serves to direct the beacon antenna toward the nominal position of the spacecraft. In both cases the plane of polarization is taken into account, since the plane is defined by two axes of χ^{LS} and χ^{RF} is parallel for zero attitude and position errors.

Since the MASS antenna is assumed to be fixed rigidly to the spacecraft rather than gimballed, the $(\delta_2)_2\ (\delta_1)_1$ relating χ^{RF} to χ^{SC} is fixed, and those δ's are called 'the spacecraft δ's'. The ground station δ's, on the other hand, must vary as the position of the satellite changes, in order that the beacon beam reaches the spacecraft. These δ's will be denoted δ_1' and δ_2', the ground station δ's. δ_1 and δ_2 may be determined numerically as the rotations which cause axis #3 of χ^{RF} to be coincident with $\underline{V}$, a vector in the direction of the beacon from the spacecraft for zero position and attitude error. This vector may be defined first in χ^E as the unit vector $\underline{r}_{SG}$. The coordinates of the ground station are given by $\underline{R}_G$, where

$$\underline{R}_G = R_E\ (-\phi)_2\ (\lambda)_1\ (0,0,1)^T$$

$$= R_E\ (S\phi C\lambda,\ S\lambda,\ C\phi C\lambda)^T \qquad \text{....(14)}$$

where ϕ = relative longitude of the ground station with respect to the nominal or average subsatellite point,

λ = latitude of the ground station

R_E = radius of ground station (from the earth's centre).

Similarly, the nominal coordinates of a satellite are given by $\underline{R}_S$ where

$$\underline{R}_S = R_S\ (0,0,1)^T \qquad \text{....(15)}$$

where R_S = radius of satellite orbit, synchronous radius. Then a unit vector from the satellite to the ground station is parallel to

$$\underline{r}_{SG} = (\underline{R}_G - \underline{R}_S)/|\underline{R}_G - \underline{R}_S| \qquad \text{....(16)}$$

$$\underline{r}_{SG} = \begin{bmatrix} S\phi\ C\lambda \\ S\lambda \\ C\phi C\lambda - R_S/R_E \end{bmatrix} xk \qquad \text{....(17)}$$

where k is a normalising constant such that $|\underline{r}_{SG}| = 1$. For zero position errors, the transformation from χ^E to χ^{01} is $(180°)_1$, and for zero attitude errors, χ^{01} is parallel to χ^{SC}. χ^{RF} is derived from χ^{SC} by $(\delta_2)_2\ (\delta_1)_1$, so that in χ^{RF},

$$\underline{V} = (\delta_2)_2\ (\delta_1)_1\ (180°)_1\ \underline{r}_{SG} \qquad \text{....(18)}$$

or

$$\underline{V} = (180°)_1\ \underline{r}_{SG}\ . \qquad \text{....(19)}$$

Therefore,

$$\begin{bmatrix} S\delta_2 \\ -S\delta_1 C\delta_2 \\ C\delta_1 C\delta_2 \end{bmatrix} = \begin{bmatrix} S\phi C\lambda \\ -S\lambda \\ -C\phi C\lambda + R_S/R_E \end{bmatrix} xk \qquad \text{....(20)}$$

since for these conditions $\underline{V} = (0,0,1)^T$.

For the case of CTS at 114°W and a beacon at Ottawa, $\phi = 38.11028°$, $\lambda = 45.34889°$, $R_S/R_E = 6.62191$ and $\delta_1 = 6.685_7°$, $\delta_2 = 4.060_2°$. ....(21)

Obviously, for zero position error the δ''s are identical to the δ's just calculated. For a satellite off nominal position because of orbit eccentricity or inclination or both, the ground station δ''s are determined as follows. $\underline{R}_G$ remains the same, but $\underline{R}_S$ becomes

$$\underline{R}_S = R_{st}\ (\eta)_2\ (-i)_3\ (-\eta')_2\ (0,0,1)^T \qquad \text{....(22)}$$

where R_{st} = satellite orbit radius at the current time t

t = time measured from the epoch at which the satellite passes through the ascending node; t_p = time of perigee

η = earth rate angle, $\eta = \omega_E\ (t - t_p) + \phi R$, where ω_E = earth's rotational rate, ϕR = a constant

= angle through which χ^E has rotated since t = 0, plus offset at t = 0.

η' = orbit angle of satellite measured from the ascending node,

= $\omega_E t$ if orbit is circular

i = orbit inclination from the earth's equatorial plane.

The form of R_{st} and $\eta' - \eta$ are described in Appendix B. Expanding Eqn. (22), R_{st} becomes

$$\underline{R}_{st} = R_{st} \begin{bmatrix} Ci\ C\eta\ S\eta' - C\eta'\ S\eta \\ Si\ S\eta' \\ Ci\ S\eta\ S\eta' + C\eta\ C\eta' \end{bmatrix}. \qquad \ldots.(23)$$

Substituting this into Eqn. (16), we see

$$\underline{r}_{SG} = \begin{bmatrix} S\phi\ C\lambda - (Ci\ C\eta\ S\eta' - C\eta'\ S\eta)\ R_{st}/R_E \\ S\lambda - Si\ S\eta'\ R_{st}/R_E \\ C\phi\ C\lambda - (Ci\ S\eta\ S\eta' + C\eta\ C\eta')\ R_{st}/R_E. \end{bmatrix} x k_1 \qquad \ldots.(24)$$

In the frame χ^{LS}, in order to direct the third axis parallel to $\underline{r}_{SG}$, we see

$$\underline{\tilde{V}} = (0,0,1)^T = (\delta_2')_2\ (\delta_1')_1\ (180°)_1\ \underline{r}_{SG}\ , \qquad \ldots.(25)$$

so that

$$\begin{bmatrix} S\delta_2' \\ S\delta_1'\ C\delta_2' \\ -C\delta_1'\ C\delta_2' \end{bmatrix} = \begin{bmatrix} S\phi\ C\lambda - (Ci\ C\eta\ S\eta' - C\eta'\ S\eta)\ R_{st}/R_E \\ S\lambda - Si\ S\eta'\ R_{st}/R_E \\ C\phi\ C\lambda - (Ci\ S\eta\ S\eta' + C\eta\ C\eta')\ R_{st}/R_E \end{bmatrix} x k_1, \qquad \ldots.(26)$$

where k_1 is a normalising constant.

The dependence of δ_1' and δ_2' on orbit inclination and eccentricity are of interest both for beacon aiming and for the solution of attitude from ϕ_1 to ϕ_3. Taking the effects of inclination and eccentricity into account separately, Eqn. (26) produces variations of + 2.142° to - 2.161° from the nominal δ_1 and + 0.015° to - 0.021° from the nominal δ_2 for an inclination of 2.0°, and + 0.0126° to - 0.0126° from δ_1 and + 0.2156° to - 0.2157° from δ_2 for an eccentricity of ± 0.2°. The conclusions to be drawn from these figures are that the beacon must nod appreciably, predominantly up and down, and that it would be a poor approximation in an analysis to treat δ_1' and δ_2' as constants for inclined or eccentric orbits.

The importance of δ_1 and δ_2 becomes obvious when one realizes that any variation in δ_1 or δ_2 for a stationary spacecraft causes an almost identical and opposite variation in ϕ_1 and ϕ_2, respectively, which would be interpreted as a change in attitude. Changes in the satellite position in χ^E may be caused by orbit eccentricity and inclination, and can cause the beacon to drift from axis 3 of χ^{RF}, even though the roll, pitch and yaw remain zero. One may calculate the δ_1 and δ_2 to keep the beacon centred for any satellite position, and the difference of these 'desired' δ's from the nominal δ's shows up in δ_1 and δ_2 as apparent attitude change caused by position change.

To solve for the 'desired' on-board δ's, calculate $\underline{r}_{SG}$ for the particular satellite position, from Equation (24), and with $\underline{V}$ in χ^{RF}, $\theta_1 = \theta_2 = \theta_3 = 0$, determine δ_1, δ_2 from

$$\underline{V} = (0,0,1)^T = (\delta_2)_2\ (\delta_1)_1\ (180°)_1\ (\eta')_2\ (i)_3\ (-\eta)_2\ \underline{r}_{SG} \qquad \text{....(27)}$$

$$\begin{bmatrix} S\delta_2 \\ S\delta_1\ C\delta_2 \\ -C\delta_1\ C\delta_2 \end{bmatrix} = \begin{bmatrix} r_1\ (Ci\ C\eta\ C\eta' + S\eta\ S\eta') + r_2\ C\eta'\ Si + r_3\ (Ci\ C\eta'\ S\eta - C\eta\ S\eta') \\ -r_1\ Si\ C\eta \quad + r_2\ Ci \quad - r_3\ Si\ S\eta \\ r_1\ (Ci\ C\eta\ S\eta' - C\eta'\ S\eta) + r_2\ Si\ S\eta' + r_3\ (Ci\ S\eta\ S\eta' + C\eta\ C\eta') \end{bmatrix}$$

....(28)

where $\underline{r}_{SG} = (r_1, r_2, r_3)^T$.

Equation (27) is derived using the transformations $(180°)_1\ (\eta')_2\ (i)_3\ (-\eta)_2$ from χ^E to χ^{01}, $(\theta_1)_1\ (\theta_2)_2\ (\theta_3)_3$ from χ^{01} to χ^{SC} and finally $(\delta_2)_2\ (\delta_1)_1$ from χ^{SC} to χ^{RF}. An idea of the magnitude of the effect of satellite position on 'desired' on-board δ's is given by solving Equation (28) for various combinations of i, e, η and η'. Numerically, an inclination of 2.0° is seen to shift $\delta 1$ by + 0.142° to - 0.185° from the nominal 6.6857°, and δ_2 by + 0.230° to - 0.235° from the nominal 4.060°; an eccentricity causing ± 0.2° E-W drift varies δ_1 by + 0.0126° to - 0.0126° from nominal, and δ_2 by + 0.0170° to - 0.0170° from the nominal. While these variations may appear small, in magnitude as well as proportion, they are significant because ϕ_1 and ϕ_2 vary similarly and can disturb the roll and pitch angles by roughly the same amount. These variations reinforce the incentive to take the orbit parameter variation into account when solving for roll, pitch and yaw from the MASS measured angles, instead of assuming the transformation valid at the nominal satellite position will suffice. These solutions follow next, first the nominal and relatively simple transformation in order to 'get a feel' for the effects and then the exact solution, taking orbital parameter variations into account and making only one slight approximation.

5. BASIC MASS RELATIONS

Consider first the solution for θ_1, θ_2 and θ_3 from ϕ_1, ϕ_2 and ϕ_3, for a circular uninclined orbit. From Figure 1b it is apparent that the transformations from χ^E to χ^{RF} may be defined as either the 'upper' or the 'lower' path.

The translations may be ignored in this approach which concerns only vector differences or angles, neither of which is affected by parallel translation of coordinate frames. Accordingly we may equate the 'upper' and 'lower' transformation paths, which, for circular uninclined orbits, gives:

$$(\delta_2)_2\ (\delta_1)_1\ (\theta_1)_1\ (\theta_2)_2\ (\theta_3)_3\ (180°)_1 = (\phi_1)_1\ (\phi_2)_2\ (\phi_3)_3\ (\delta_2)_2\ (\delta_1)_1\ (180°)_1 \quad \text{....(29)}$$

Therefore,

$$(\theta_1)_1\ (\theta_2)_2\ (\theta_3)_3 = (-\delta_1)_1\ (-\delta_2)_2\ (\phi_1)_1\ (\phi_2)_2\ (\phi_3)_3\ (\delta_2)_2\ (\delta_1)_1 \quad \text{....(30)}$$

$$\begin{bmatrix} C\theta_2\ C\theta_3 & C\theta_2\ S\theta_3 & -S\theta_2 \\ S\theta_1\ S\theta_2\ C\theta_3 - C\theta_1\ S\theta_3 & C\theta_1\ C\theta_3 + S\theta_1\ S\theta_2\ S\theta_3 & S\theta_1\ C\theta_2 \\ C\theta_1\ S\theta_2\ C\theta_3 + S\theta_1\ S\theta_3 & C\theta_1\ S\theta_2\ S\theta_3 - S\theta_1\ C\theta_3 & C\theta_1\ C\theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} C\delta_2 & 0 & S\delta_2 \\ S\delta_1\ S\delta_2 & C\delta_1 & -S\delta_1\ C\delta_2 \\ -C\delta_1\ S\delta_2 & S\delta_1 & C\delta_1\ C\delta_2 \end{bmatrix} \cdot \begin{bmatrix} \text{(Same matrix as above but replacing } \theta \text{ with } \phi) \end{bmatrix} \cdot \begin{bmatrix} C\delta_2 & S\delta_1\ S\delta_2 & -C\delta_1\ S\delta_2 \\ 0 & C\delta_1 & S\delta_1 \\ S\delta_2 & -S\delta_1\ C\delta_2 & C\delta_1\ C\delta_2 \end{bmatrix} \quad \text{....(31)}$$

If one employs the approximation

$$C\theta_i \approx C\phi_i \approx 1,\ i = 1,\ldots,3$$

$$\text{and}\quad S\theta_i \approx \theta_i,\ i = 1,\ldots,3 \quad S\phi_i \approx \phi_i,\ i = 1,\ldots,3 \quad \text{....(32)}$$

where the angles are expressed in radians, and if one realizes $S\theta_1\ S\theta_2\ S\theta_3 << C\theta_1\ C\theta_3$, Equation (31) reduces to:

$$\begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ \theta_1\ \theta_2 - \theta_3 & 1 & \theta_1 \\ \theta_2 + \theta_1\ \theta_3 & \theta_2\ \theta_3 - \theta_1 & 1 \end{bmatrix} = \begin{bmatrix} C_2 & 0 & S_2 \\ S_1\ S_2 & C_1 & -S_1\ C_2 \\ -C_1\ S_2 & S_1 & C_1C_2 \end{bmatrix} \cdot$$

$$\begin{bmatrix} 1 & \phi_3 & -\phi_2 \\ \phi_1\ \phi_2 - \phi_3 & 1 & \phi_1 \\ \phi_2 + \phi_1\ \phi_3 & \phi_2\ \phi_3 - 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} C_2 & S_1\ S_2 & -C_1\ S_2 \\ 0 & C_1 & S_1 \\ S_2 & -S_1\ C_2 & C_1\ C_2 \end{bmatrix} \quad \text{....(33)}$$

where $C_i = C\delta_i$ and $S_i = S\delta_i$, $i = 1, 2$. From Eqn. (33) we may determine:

$$\theta_1 \approx C_2\ \phi_1 + S_2\ \phi_3 - C_1^2\ S_2\ \phi_1\ \phi_2 + C_1\ S_1\ C_2\ S_2\ \phi_1\ \phi_3 - S_1^2\ C_2\ \phi_2\ \phi_3 \quad \text{....(34a)}$$

$$\theta_2 \approx S_1\, S_2\, \phi_1 + C_1\, \phi_2 - S_1\, C_2\, \phi_3 + C_1\, S_2^2\, \phi_1\, \phi_3 - S_1\, S_2\, \phi_2\, \phi_3 \qquad \text{....(34b)}$$

$$\theta_3 \approx C_1\, S_2\, \phi_1 + S_1\, \phi_2 + C_1\, C_2\, \phi_3 + S_1\, S_2^2\, \phi_1\, \phi_3 + C_1\, S_2\, \phi_2\, \phi_3. \qquad \text{....(34c)}$$

Note that ϕ_i is a good rough indication of θ_i, since $C_j \approx 0.995$ and $S_j \approx 0.1$, $j = 1$ and 2. Furthermore an appreciable component of ϕ_3 couples into both θ_1 and θ_2. Since ϕ_3 may be an order of magnitude greater than ϕ_1 or ϕ_2, under normal operation this component is significant.

We next find the exact solution for $\underline{\theta}$ from $\underline{\phi}$ and orbital parameters. The dependence of δ_1 and δ_2 on these parameters has already been demonstrated, and $_3$ includes a sinusoidally varying factor of the inclination, so it is obviously necessary to use the exact expressions at this point in the solution. Subsequently simplifications will be considered and adopted if proven adequately accurate. In the following derivation δ_1 and δ_2 remain fixed at the nominal values, but δ_1' and δ_2' depend on the orbital parameters and must be determined from Equation (26). From the transformation equivalence of Figure 1b, we find:

$$(\phi_1)_1\ (\phi_2)_2\ (\phi_3)_3\ (\delta_2')_1\ (\delta_1')_1\ (180°)_1$$

$$= (\delta_2)_2\ \ (\delta_1)_1\ \ (\theta_1)_1\ (\theta_2)_2\ (\theta_3)_3\ (180°)_1\ (\eta')_2\ (i)_3\ (-\eta)_2 \qquad \text{....(35)}$$

so

$$(\theta_1)_1\ (\theta_2)_2\ (\theta_3)_3$$

$$= (-\delta_1)_1\ (-\delta_2)_2\ (\phi_1)_1\ (\phi_2)_2\ (\phi_3)_3\ (\delta_2')_2\ (\delta_1')_1\ (180°)_1\ (\eta)_2\ (-i)_3\ (-\eta')_2\ (180°)_1 \qquad \text{....(36)}$$

$$\begin{bmatrix} \sim & C\theta_2\, S\theta_3 & -S\theta_2 \\ \sim & \sim & S\theta_1\, C\theta_2 \\ \sim & \sim & \sim \end{bmatrix} = \begin{bmatrix} b7 & b8 & b9 \\ b4 & b5 & b6 \\ b1 & b2 & b3 \end{bmatrix} \cdot \begin{bmatrix} a7 & a8 & a9 \\ a4 & a5 & a6 \\ a1 & a2 & a3 \end{bmatrix} \qquad \text{....(37)}$$

In Eqn. 37, the symbol ~ is used to represent a matrix element which is not of interest here. On the r.h.s. of Eqn. 37, the first matrix represents the transformation $(-\delta_1)_1\ (-\delta_2)_2\ (\phi_1)_1\ (\phi_2)_2\ (\phi_3)_3$, while the second denotes $(\delta_2')_2\ (\delta_1')_1\ (180°)_1\ (\eta)_2\ (-i)_3\ (-\eta')_2\ (180°)_1$. The elements ai, bi, i = 1,...,9, are defined as follows:

$$b1 = -C_1\, S_2\, C\phi_2\, C\phi_3 + S_1\, (S\phi_1\, S\phi_2\, C\phi_3 - C\phi_1\, S\phi_3) + C_1\, C_2\, (C\phi_1\, S\phi_2\, C\phi_3 + S\phi_1\, S\phi_3)$$

$$b2 = -C_1\, S_2\, C\phi_2\, S\phi_3 + S_1\, C\phi_1\, C\phi_3 + C_1\, C_2\, (C\phi_1\, S\phi_2\, S\phi_3 - S\phi_1\, C\phi_3)$$

$$b3 = C_1\, S_2\, S\phi_2 + S_1\, S\phi_1\, C\phi_2 + C_1\, C_2\, C\phi_1\, C\phi_2$$

$$b4 = S_1\ S_2\ C\phi_2\ C\phi_3 + C_1\ (S\phi_1\ S\phi_2\ C\phi_3 - C\phi_1\ S\phi_3) - S_1\ C_2\ (C\phi_1\ S\phi_2\ C\phi_3 + S\phi_1\ S\phi_3)$$

$$b5 = S_1\ S_2\ C\phi_2\ S\phi_3 + C_1\ (C\phi_1\ C\phi_3) - S_1\ C_2\ (C\phi_1\ S\phi_2\ S\phi_3 - S\phi_1\ C\phi_3)$$

$$b6 = -S_1\ S_2\ S\phi_2 + C_1\ S\phi_1\ C\phi_2 - S_1\ C_2\ C\phi_1\ C\phi_2 \qquad \text{....(38a)}$$

$$b7 = C_2\ C\phi_2\ C\phi_3 + S_2\ (C\phi_1\ S\phi_2\ C\phi_3 + S\phi_1\ S\phi_3)$$

$$b8 = C_2\ C\phi_2\ S\phi_3 + S_2\ (C\phi_1\ S\phi_2\ S\phi_3 - S\phi_1\ C\phi_3)$$

$$b9 = -C_2\ S\phi_2 + S_2\ C\phi_1\ C\phi_2\ .$$

$$a1 = S_2'\ (Ci\ C\eta\ C\eta' + S\eta\ S\eta') + S_1'\ C_2'\ Si\ C\eta' - C_1'\ C_2'\ (Ci\ C\eta'\ S\eta - C\eta\ S\eta')$$

$$a2 = S_2'\ Si\ C\eta - S_1'\ C_2'\ Ci - C_1'\ C_2'\ Si\ S\eta$$

$$a3 = S_2'\ (S\eta\ C\eta' - Ci\ C\eta\ S\eta') - S_1'\ C_2'\ Si\ S\eta' + C_1'\ C_2'\ (Ci\ S\eta\ S\eta' + C\eta\ C\eta')$$

$$a4 = -C_1'\ Si\ C\eta' - S_1'\ (Ci\ C\eta'\ S\eta - C\eta\ S\eta')\ . \qquad \text{....(38b)}$$

$$a5 = C_1'\ Ci - S_1'\ Si\ S\eta$$

$$a6 = C_1'\ Si\ S\eta' + S_1'\ (Ci\ S\eta\ S\eta' + C\eta\ C\eta')$$

$$a7 = C_2'\ (Ci\ C\eta\ C\eta' + S\eta\ S\eta') - S_1'\ S_2'\ Si\ C\eta' + C_1'\ S_2'\ (Ci\ C\eta'\ S\eta - C\eta\ S\eta')$$

$$a8 = C_2'\ Si\ C\eta + S_1'\ S_2'\ Ci + C_1'\ S_2'\ Si\ S\eta$$

$$a9 = C_2'\ (C\eta'\ S\eta - Ci\ C\eta\ S\eta') + S_1'\ S_2'\ Si\ S\eta' - C_1'\ S_2'\ (Ci\ S\eta\ S\eta' + C\eta\ C\eta').$$

After isolating the three top right-hand corner elements of the matrices of Eqn. (37), we find for the exact solution:

$$\begin{aligned} -S\theta_2 = {} & a9\left[C_2\ C\phi_2\ C\phi_3 + S_2\ (C\phi_1\ S\phi_2\ C\phi_3 + S\phi_1\ S\phi_3)\right] \\ & + a6\left[C_2\ C\phi_2\ S\phi_3 + S_2\ (C\phi_1\ S\phi_2\ S\phi_3 - S\phi_1\ C\phi_3)\right] + a3\left[S_2\ C\phi_1\ C\phi_2 - C_2\ S\phi_2\right]. \end{aligned} \qquad \text{....(39)}$$

$$\begin{aligned} S\theta_1 = {} & (C\theta_2)^{-1}\Big\{a9\Big[S_1\ S_2\ C\phi_2\ C\phi_3 + C_1\ (S\phi_1\ S\phi_2\ C\phi_3 - C\phi_1\ S\phi_3) \\ & - S_1\ C_2\ (C\phi_1\ S\phi_2\ C\phi_3 + S\phi_1\ S\phi_3)\Big] + a6\Big[S_1\ S_2\ C\phi_2\ S\phi_3 + C_1\ C\phi_1\ C\phi_3 \\ & - S_1\ C_2\ (C\phi_1\ S\phi_2\ S\phi_3 - S\phi_1\ C\phi_3)\Big] + a3\Big[-S_1\ S_2\ S\phi_2 + C_1\ S\phi_1\ C\phi_2 \\ & - S_1\ C_2\ C\phi_1\ C\phi_2\Big]\Big\}. \end{aligned} \qquad \text{....(40)}$$

$$\begin{aligned} S\theta_3 = {} & (C\theta_2)^{-1}\Big\{a8\ \left[C_2\ C\phi_2\ C\phi_3 + S_2\ (C\phi_1\ S\phi_2\ C\phi_3 + S\phi_1\ S\phi_3)\right] \\ & + a5\ \left[C_2\ C\phi_2\ S\phi_3 + S_2\ (C\phi_1\ S\phi_2\ S\phi_3 - S\phi_1\ C\phi_3)\right] \\ & + a2\left[-C_2\ S\phi_2 + S_2\ C\phi_1\ C\phi_2\right]\Big\}. \end{aligned} \qquad \text{....(41)}$$

Equations (39) to (41) are obviously sufficiently complicated that simplifications would be at least desirable and probably imperative for implementation in an on-board processor. One method of analyzing the effect of candidate simplifications is to compute the difference between $\underline{\theta}$ calculated by (39) to (41) and the $\hat{\underline{\theta}}$ calculated using approximations, and then to determine the statistics of the difference for a large number of points in SP_1, the space of $(\underline{\phi}, i, e, \eta', f)$, where the latter four symbols denote inclination, eccentricity, orbital angle and true anomaly. This Monte Carlo type approach is useful, but requires random points in SP_1 that represent permitted and physically attainable combinations of attitude and position. The variables must be limited and one problem becomes apparent - that of specifying the combinations of $\underline{\phi}$ which can arise for a satellite whose attitude (in $\underline{\theta}$) and orbit (station, i and e) are limited a priori. To resolve this we may rewrite Eqn. (35) and solve for $\underline{\phi}$.

6. SOLUTION OF THE RF EULERIAN ANGLES ϕ_1, ϕ_2, ϕ_3

The transformation equivalence of Equation (35) is exact, and a reordering of the transformations produces:

$$(\phi_1)_1 (\phi_2)_2 (\phi_3)_3 = (\delta_2)_2 (\delta_1)_1 (\theta_1)_1 (\theta_2)_2 (\theta_3)_3 (180°)_1 (\eta')_2 (i)_3 (-\eta)_2 (180°)_1 \cdot (-\delta_1')_1 (-\delta_2')_2 \,. \qquad \ldots.(42)$$

Note the similarity of Equation (42) to Equation (36); they decompose to groups of matrices which are either of the same form as, or the transpose of, the corresponding group in the other equation, thereby reducing the number of algebraic steps to be followed. Ignoring the unneeded terms, Equation (42) expands to:

$$\begin{vmatrix} \sim & C\phi_2 S\phi_3 & -S\phi_2 \\ \sim & \sim & S\phi_1 C\phi_2 \\ \sim & \sim & \sim \end{vmatrix} = \begin{vmatrix} C_2 & S_1S_2 & -C_1S_2 \\ 0 & C_1 & S_1 \\ S_2 & -S_1C_2 & C_1C_2 \end{vmatrix} \cdot \begin{vmatrix} C\theta_2C\theta_3 & C\theta_2S\theta_3 & -S\theta_2 \\ S\theta_1S\theta_2C\theta_3-C\theta_1S\theta_3 & C\theta_1C\theta_3 & S\theta_1C\theta_2 \\ C\theta_1S\theta_2C\theta_3+S\theta_1S\theta_3 & C\theta_1S\theta_2S\theta_3-S\theta_1C\theta_3 & C\theta_1C\theta_2 \end{vmatrix}$$

$$\cdot \begin{vmatrix} a7 & a4 & a1 \\ a8 & a5 & a2 \\ a9 & a6 & a3 \end{vmatrix} \qquad \ldots.(43)$$

From Equation (43), the pertinent terms are found to be:

$$\begin{aligned} -S\phi_2 = {} & a1 \left[C_2C\theta_2C\theta_3 + S_1S_2(S\theta_1S\theta_2C\theta_3 - C\theta_1S\theta_3) - C_1S_2(C\theta_1S\theta_2C\theta_3 + S\theta_1S\theta_3)\right] \\ & + a2 \left[C_2C\theta_2S\theta_3 + S_1S_2C\theta_1C\theta_3 - C_1S_2(C\theta_1S\theta_2S\theta_3 - S\theta_1C\theta_3)\right] \\ & + a3 \left[-C_2S\theta_2 + S_1S_2S\theta_1C\theta_2 - C_1S_2C\theta_1C\theta_2\right]. \end{aligned} \qquad \ldots.(44)$$

$$\begin{aligned} S\phi_1 = (C\phi_2)^{-1} \Big\{ & a1 \left[C_1(S\theta_1S\theta_2C\theta_3 - C\theta_1S\theta_3) + S_1(C\theta_1S\theta_2C\theta_3 + S\theta_1S\theta_3)\right] \\ & + a2 \left[C_1C\theta_1C\theta_3 + S_1 (C\theta_1S\theta_2S\theta_3 - S\theta_1C\theta_3)\right] + a3 \left[C_1S\theta_1C\theta_2 + S_1C\theta_1C\theta_2\right]\Big\} \,. \end{aligned} \qquad \ldots.(45)$$

$$S\phi_3 = (C\phi_2)^{-1} \Big\{ a4 \left[C_2C\theta_2C\theta_3 + S_1S_2(S\theta_1S\theta_2C\theta_3 - C\theta_1S\theta_3) - C_1S_2(C\theta_1S\theta_2C\theta_3 + S\theta_1S\theta_3)\right]$$
$$+ a5 \left[C_2C\theta_2S\theta_3 + S_1S_2C\theta_1C\theta_3 - C_1S_2(C\theta_1S\theta_2S\theta_3 - S\theta_1C\theta_3)\right]$$
$$+ a6 \left[- C_2S\theta_2 + S_1S_2S\theta_1C\theta_2 - C_1S_2C\theta_1C\theta_2\right] \Big\} . \qquad \ldots.(46)$$

The factors a1 - a9 are again defined by Equation (38).

7. PSEUDO TWO-STATION YAW CALCULATION

In this approach, roll and pitch information from an earth sensor is used to replace angular information that would be available from a second MASS beacon, giving rise to the term 'pseudo two station' in the description. The offset of the MASS ground station from the subsatellite point introduces a coupling of the yaw error into the MASS Eulerian angles ϕ_1 and/or ϕ_2. If the offset is in latitude only, one of the two 'interferometer' angles ϕ_1 or ϕ_2 measures a portion of the yaw angle, while if the offset is in longitude, the remaining 'interferometer' angle is affected. If the offset is both in latitude and longitude, yaw is coupled into both the 'interferometer' angles ϕ_1 and ϕ_2. The effects of the coupling may appear detrimental in that all three MASS angles must be measured to permit the solution for any or all of roll, pitch and yaw, in the absence of information from a second sensor. On the other hand, this coupling may be used to advantage if ϕ_1 or ϕ_2 is measured and θ_1 and θ_2 are known, for the yaw may be determined from a combination of the three angles $(\phi_1, \theta_1, \theta_2)$ or $(\phi_2, \theta_1, \theta_2)$. Indeed, other combinations of three of $(\phi_1, \phi_2, \theta_1, \theta_2)$ contain the yaw information, but turn out to be less tractable algebraically. The use of ϕ_3 is not considered initially, as it would not be available in a simple MASS lacking the 'polarimeter' portion.

Several methods may be found to establish a relationship between θ_3 (yaw) and $(\theta_1, \theta_2, \phi_1, \phi_2)$. For example, a vector defined originally in χ^E may be transformed into χ^{RF} and the dependence of its final form on θ and ϕ may be employed to isolate an equation containing the desired terms which may be rearranged to define θ_3 in terms of the remaining angles. Alternatively, the transformation equivalence of Equation (35) may be manipulated as follows:

Equation (42) consists of nine 'elemental' equations, most of which involve combinations of two or more MASS angles. Fortunately the top right-hand element is simply $-S\phi_2$, the solution for which may be written as Equation (44) or alternatively:

$$-S\phi_2 = S\theta_3 \left[a1\ (-S_1S_2C\theta_1 - C_1S_2S\theta_1) + a2\ (C_2C\theta_2 - C_1S_2C\theta_1S\theta_2)\right]$$
$$+ C\theta_3 \left[a1\ (C_2C\theta_2 + S_1S_2S\theta_1S\theta_2 - C_1S_2C\theta_1S\theta_2) + a2\ (S_1S_2C\theta_1 + C_1S_2S\theta_1)\right]$$
$$+ a3 \left[-C_2S\theta_2 + S_1S_2S\theta_1C\theta_2 - C_1S_2C\theta_1C\theta_2\right] . \qquad \ldots.(47)$$

In Equation (47) the factors of $S\theta_3$ and $C\theta_3$ have been collected to facilitate the understanding of the dependence of θ_3 on ϕ_2, θ_1 and θ_2. As it is transcendental, Equation (47) must be solved iteratively for the numerical determination of θ_3 from ϕ_2, θ_1 and θ_2.

Similarly the relation between θ_3 and ϕ_1, θ_1 and θ_2 may be found from Equation (45), provided that Equation (44) or (47) is substituted in for ϕ_2. If a choice had to be made between Equation (47) and (45) for pseudo two-station yaw, the apparently simpler (47) would be preferred. Furthermore this choice would be reinforced if one considered the sensitivity of θ_3 to the accuracy of ϕ_1 or ϕ_2 plus θ_1 and θ_2, at least for the case of CTS with a ground station at Ottawa.

If one approximates $S\theta_i \approx \theta_i$, $C\theta_i \approx 1$, $S\phi_i \approx \phi_i$, $C\phi_i \approx 1$, the transformation equivalence of Equation (42) reduces to the following solutions:

$$\theta_3 = \frac{\phi_1 - (C_1\theta_1\theta_2 + S_1\theta_2)\ a1 - (C_1 - S_1\theta_1)\ a2 - (C_1\theta_1 + S_1)\ a3}{-\ C_1\ a1 + S_1\ \theta_1\ a1 + S_1\ \theta_2\ a2} \qquad \ldots.(48)$$

$$\theta_3 = \frac{\phi_2 + a1\ (C_2 + S_1S_2\theta_1\theta_2 - C_1S_2\theta_2) + a2\ (S_1S_2 + C_1S_2\theta_1)}{a1\ (S_1S_2 + C_1S_2\theta_1) + a2\ (C_1S_2\theta_2 - C_2)} + \frac{a3\ (-\ C_2\theta_2 + S_1S_2\theta_1 - C_1S_2)}{a1\ (S_1S_2 + C_1S_2\theta_1) + a2\ (C_1S_2\theta_2 - C_2)} \qquad \ldots.(49)$$

$$\theta_3 = \frac{-\ \phi_3 + \theta_1(C_1S_2\ a5 + S_1S_2\ a6 + S_1S_2\theta_2\ a4)}{S_1S_2\ a4 - C_2\ a5 + \theta_1\ C_1S_2\ a4 + \theta_2C_1S_2\ a5} + \frac{\theta_2\ (-\ C_1S_2\ a4 - C_2\ a6) + (C_2\ a4 + S_1S_2\ a5 - C_1S_2\ a6)}{S_1S_2\ a4 - C_2\ a5 + \theta_1C_1S_2\ a4 + \theta_2C_1S_2\ a5} \qquad \ldots.(50)$$

where the factors a1 - a6 are as defined in Equation (38).

The denominators of these equations (48) - (50) are approximately 0.116, 0.07 and -0.99 respectively, for a nominal orbit, so that measurement errors in the numerators are multiplied by the reciprocal of these numbers. On that basis, the order of preference becomes Equation (50), (49) and thirdly (48).

8. CONCLUDING REMARKS AND SUGGESTIONS FOR FURTHER WORK

The analysis contained in this report provides the basic relations necessary for a functional on-board MASS. Both exact and approximate solutions have been found, so that the error of any of the approximations may be predicted from pre-launch computations. Accordingly, the on-board processor need not be designed to handle the full and exact equations if the accuracy of the simplified ones is proven to be adequate. An alternate approach to minimizing the size and complexity of the on-board processor is to perform some of the computations on the ground and transmit the results to the spacecraft. Both approaches will probably be necessary, for the exact solutions of roll, pitch and yaw would require an unwieldy and impractical processor if performed on-board, and early efforts at simplifying the equations proved inaccurate.

Appendices A, B and C cover background material on rotational transformations and orbital parameters, and isolate the effects of the individual orbital parameters on the MASS measured angles. Appendix D describes some of the possible approximations to the equations, while in the other direction Appendix E extends the exact results to the case where $(\delta_2)_2$ $(\delta_1)_1$ and $(\delta_2')_2$ $(\delta_1')_1$ are followed by $(\delta_3)_3$, and $(\delta_3')_3$, respectively. The δ_3 and δ_3' may be either error angles in the antenna mountings, or deliberate and sizeable rotations of the antennas, or both. Thus a general or universally-applicable solution has been found, not only for roll, pitch and yaw from the MASS measured angles and vice-versa, but also for pseudo two-station yaw, requiring roll, pitch and any one of the MASS measured angles. The types of approximation suggested in Appendix D for the $\delta_3 = 0$ case are equally valid in the $\delta_3 \neq 0$ case, although the approximate solutions in the latter case will in general be more complex than in the former.

Appendix F concludes this report with a description of the relations between the δ_1 and δ_2 of this report and the conventional antenna aiming angles of azimuth and elevation.

The MASS analysis presented is a complete theoretical treatment of the topic, but a working MASS prototype would require further investigation, and several areas of research are suggested. The most immediate problem is the complexity of the exact solutions; a sensitivity analysis must be conducted to optimize the accuracy and effectiveness of a weight and power limited on-board processor utilizing approximations to the exact solutions.

A less immediate but theoretically interesting avenue of research is to examine the relations between spacecraft attitude, MASS measured angles ϕ_1 and ϕ_2 and an angle of incidence of sunlight on the spacecraft. It appears possible to solve for yaw during two sizeable portions of the orbit, using a simple two-axis MASS and an on-board sun sensor which could be required for other functions as well.

A variant to this problem is to employ an angle to another satellite of known position instead of a sun angle, thereby enabling yaw sensing uninterruptedly through the orbit.

A third future research topic is to manipulate the dependence of the MASS solutions on orbit parameters, in order to solve for the orbit parameters from the MASS measured angles and the spacecraft attitude.

A fourth area to examine is the correspondence between the MASS measured angles and attitude of the spacecraft specified in a way disregarding the conventional roll, pitch and yaw angles. Instead of roll, pitch and yaw, the restriction could be on more communications--significant parameters such as, for instance, deviations of an on-board antenna boresight from the intended target on earth and rotation of the spacecraft about the nominal boresight.

A fifth interesting topic is the feasibility of employing one MASS antenna and processor but two (widely separated) ground stations in a time shared mode to generate data from which both spacecraft attitude and position (and, by filtering, orbit) could be deduced.

Thus, although the basic theoretical problems of the MASS have been solved, much remains to be done for a practical implementation and further theoretically interesting topics have been suggested.

9. REFERENCES

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APPENDIX A

ROTATIONAL TRANSFORMATIONS

When a set of cartesian coordinates is rotated from one orientation in inertial space to another, the description of a fixed vector is transformed linearly in going from the old frame to the new. Assume that the rotation is about one of the axes of the old frame, and that the sense is defined according to the right-hand rule. For a rotation of $\theta°$ about axis i, a vector $\underline{W}$ in the new frame is derived from its description $\underline{V}$ in the old frame by the transformation $\underline{W} = (\theta)_i \underline{V}$ where the transformations $(\theta)_i$ are defined by:

$$(\theta)_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & S\theta \\ 0 & -S\theta & C\theta \end{bmatrix}, \quad (\theta)_2 = \begin{bmatrix} C\theta & 0 & -S\theta \\ 0 & 1 & 0 \\ S\theta & 0 & C\theta \end{bmatrix}, \quad (\theta)_3 = \begin{bmatrix} C\theta & S\theta & 0 \\ -S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \ldots.(1.1)$$

A rotation about other than a principal axis may be decomposed into two or more ordered rotations about principal axes, and the above transformations applied in the same sequence. The sequence is important for rotations (unlike vector additions or translations), since the transformations are not commutative. The transformations are orthogonal and the elements may be shown to be direction cosines.

Although the rotational transformation was explained for the case of a fixed vector and rotating coordinate frames, the same transformations are valid for fixed coordinates but rotating vector, the only difference being that now the sense of θ will be reversed from above, that is, the negative of the rotation from the old vector to the new, following the right-handed convention.

APPENDIX B

ORBITAL PARAMETERS

The position of the satellite affects δ_1' and δ_2' immediately and therefore the solution of $\underline{\theta}$ from $\underline{\phi}$ and vice versa, as well as two-station yaw. Figure 1a illustrates the nominal position of a synchronous satellite relative to a ground station such as Ottawa, and Figure B1 gives more details of the orbit in an inertial frame.

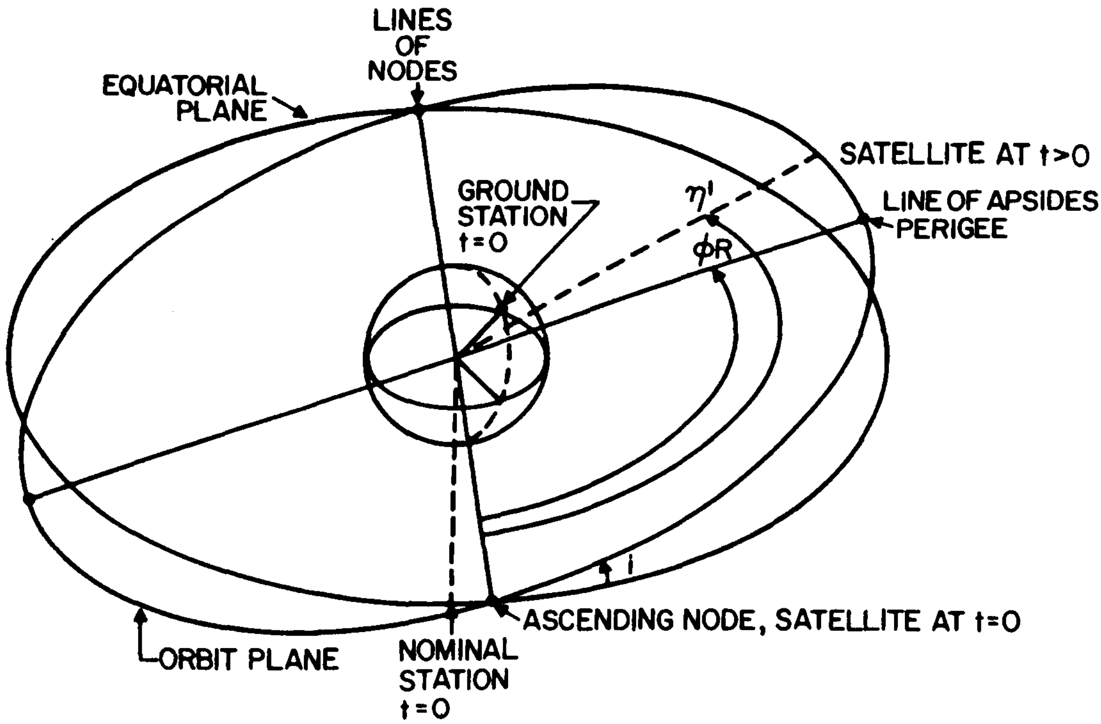

Fig. B1. Equatorial and orbit planes.

The orbit is assumed to be inclined with respect to the equatorial, eccentric and 'out of phase' by a steady state bias Ω in that the ascending node is shifted by Ω from the nominal longitude of the satellite, disregarding the effect of eccentricity. We define t to be zero at the instant the satellite passes through the ascending node.

η' is the orbital angle of the satellite, measured from the ascending node to satellite at any time t.

η is the rotation of χ^E in the same time plus a constant. $\eta = \omega_E t + \phi R - \omega_E t_p$, where ω_E is the earth's spin rate, t_p is the time of the perigee, and ϕR is defined below.

i denotes the inclination of the orbit plane with respect to the equatorial.

ϕR represents the angle from the ascending node line to the line of the apsides through the perigee. Figure B2 shows details of an eccentric orbit, with the earth located at the focus F2.

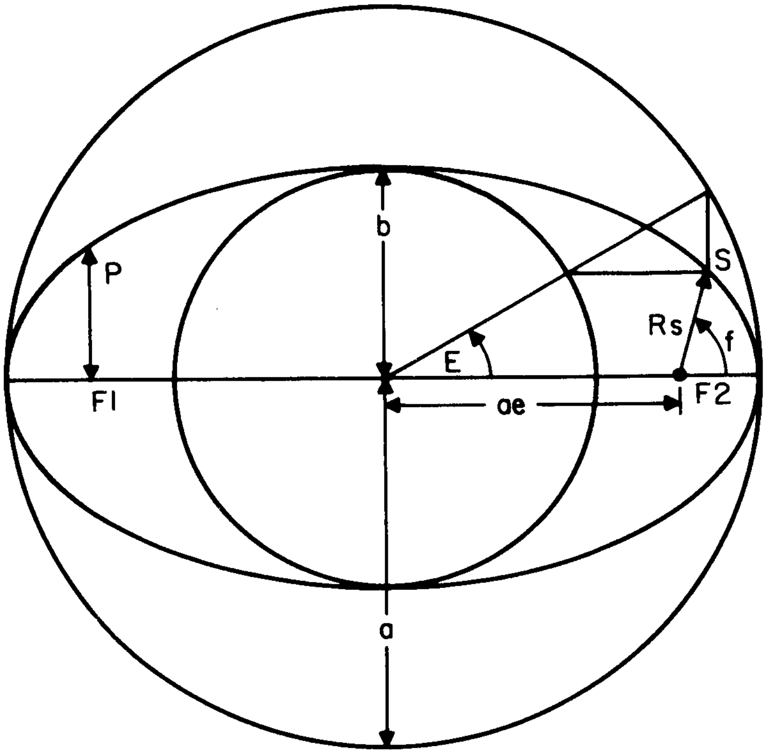

Fig. B2. Eccentric orbit parameters.

The letters a and b represent, respectively, the semi-major and semi-minor axes and the true anomaly f is measured at F2, counterclockwise from the perigee to the satellite $\underline{S}$. E is the eccentric anomaly and ae is half the distance from focus to focus, where e is the eccentricity of the orbit. In order to solve for δ_1' and δ_2' it is necessary to know $\underline{R}_s$ and η'. These may be determined if the orbital parameters and t are known as follows:

$$R_{st} = \frac{p}{1 + e \cos f} \quad \text{....(52)}$$

where p is the semi-latus rectum of the ellipse.

Now $$p = a\,(1 - e^2) \quad \text{....(53)}$$

and $$T = 2\pi\sqrt{\frac{a^3}{u}}\,, \quad \text{....(54)}$$

for a complete orbit of period T. u is a constant. For a constant period, 'a' does not vary and must be synchronous radius for a synchronous orbit. Accordingly, for $e << 1$, $p \approx a$, and synchronous radius $R_s = a$.

Therefore $$R_{st} = \frac{R_s (1 - e^2)}{1 + e \cos f} \approx R_s (1 - e \cos f), \text{ for } e << 1. \qquad \ldots.(55)$$

If M is the mean anomaly, $$M = \omega_E (t - t_p) \qquad \ldots.(56)$$

where t_p = time at perigee, then

$$f - M \approx 2e \sin f \text{ -(from Ref. 4)} \qquad \ldots.(57)$$

Now $$\eta' = \phi R + f \qquad \ldots.(58)$$

and $$\eta = \phi R + M \qquad \ldots.(59)$$

therefore $$\eta' = \eta + 2e \sin f \;. \qquad \ldots.(60)$$

On some satellites, such as CTS, the position of the satellite is restricted to an inclination of $\leq 2°$ and an apparent E - W drift of less than 0.2°. Accordingly,

$$|\Omega + 2e| \leq 0.2° \;. \qquad \ldots.(61)$$

In the equations developed earlier Ω does not appear explicitly; it is in fact incorporated in the actual ϕ or relative longitude of the ground station with respect to the subsatellite point.

APPENDIX C

EFFECTS OF ORBITAL PARAMETERS ON MASS

The three orbital parameters whose effects on the MASS are considered are:

(a) inclination

(b) eccentricity, and

(c) constant longitudinal error Ω of satellite station.

(a) Inclination Alone

The main effect of inclination on a synchronous orbit is to cause the satellite to trace a narrow and North-Southerly figure 8 with respect to an earth based observer. For an orbit inclination i, the northerly displacement of the satellite is given by $\sin^{-1}(\sin i \sin \omega t)$, and the lateral displacement by $\left(\omega t - \cos^{-1}\left(\frac{\cos \omega t}{\sqrt{1 - \sin^2 \omega t \sin^2 i}}\right)\right)$. Accordingly, a ground station antenna must nod, mainly up-down, to keep the satellite centered in its 'field of view'. In our case the main effect of a 2° inclination is that δ_2' varies little ($\approx \pm 0.02°$) while δ_1' changes by approximately ± the magnitude of i, and with a period of about 24 hours. δ_1' and δ_2' in turn affect the transformations between $\underline{\theta}$ and $\underline{\phi}$.

At the spacecraft the effects of the inclination are to cause the ground station to appear to move, implying different δ_1 and δ_2 are required to keep the ground station centered in the MASS antenna field of view for $\underline{\theta} = \underline{\phi}$. Since the δ_1 and δ_2 are fixed, ϕ_1 and ϕ_2 vary by approximately the values of $\pm 0.15°$ and $\pm 0.23°$ that δ_1 and δ_2, respectively, would vary if allowed to. There is a more marked effect on ϕ_3, which varies by close to the full amount of the inclination. Equation (46) reduces to

$$\phi_3 \approx - C_1' C_2 C\eta\, i - S_2 S\eta\, i \qquad \text{....(62)}$$

after applying several approximations.

(b) Eccentricity Alone

By considering the variation of orbit radius and η' with respect to η it may be seen that to an earth based observer the satellite appears to move around a banana shaped locus centered on the nominal satellite point. Accordingly, the ground station antenna must nod laterally by an angle slightly greater than the E-W drift measured at the earth centre, while the vertical nodding is considerably less, about 7 per cent of the δ_2 angle for the case of CTS and Ottawa and an allowed E-W drift of 0.2°.

At the spacecraft the ground station appears to rise and fall as the satellite radius varies, and appears to shift E-W as the satellite moves E-W from its nominal position. For the case of CTS, Ottawa and allowed E-W drift of 0.2°, δ_1 would ideally vary by ± 0.0126°, δ_2 by ± 0.0169°. Again, ϕ_1 and ϕ_2 vary instead, by close to the same amounts, and ϕ_3 is essentially unaffected.

(c) Station Error Ω Alone

δ_1' is basically unaffected while δ_2' varies by slightly more than Ω. For the case of CTS, Ottawa and Ω of 0.2°, δ_1 would vary by less than one per cent of Ω and δ_2 by less than 10 per cent of Ω. ϕ_3 remains essentially unaffected.

To sum up the main effects - δ_1' and δ_2' vary sufficiently with inclination, eccentricity and Ω that they cannot be safely assumed to be constants in order to reduce the complexity of the solutions, and ϕ_3 is biassed sinusoidally by almost the full value of the inclination. ϕ_1 and ϕ_2 are affected by all three, (a), (b) and (c), but for $i \leq 2°$, $e \leq .00174$ (rad), $\Omega \leq 0.2°$, the effects of inclination are bounded by about an order of magnitude above those of eccentricity or station error Ω. Note that if a substantial δ_3 ($\geq 15°$) is introduced, the above is no longer numerically exact.

APPENDIX D

APPROXIMATIONS AND SIMPLIFICATIONS

Since the exact solution of $\underline{\theta}$ from $\underline{\phi}$, $\underline{\phi}$ from $\underline{\theta}$ and two-station yaw are too complicated for a rapid interpretation of the various cross couplings, an attempt at simplifying approximations would be in order. Some of the possibilities are:

(a) replacing $C\theta_j$, $C\phi_j$ by 1, $S\theta_j$ by θ_j, $S\phi_j$ by ϕ_j, $j = 1,\ldots,3$;

(b) ignoring inclination, or eccentricity, or both (which means $\delta_1' = \delta_1$, $\delta_2' = \delta_2$, and $\eta' = \eta$);

(c) dropping second and higher order terms in θ_j and ϕ_j, $j = 1,\ldots,3$;

(d) replacing Ci by 1, Si by i (rads);

(e) combinations of the above.

The following equations result from applying approximation (d) above to the solution of $\underline{\theta}$ from $\underline{\phi}$ using the transformation equivalence:

$$(\phi_1)_1\ (\phi_2)_2\ (\phi_3)_3\ (\delta_2')_2\ (\delta_1')_1\ (180°)_1$$
$$= (\delta_2)_2\ (\delta_1)_1\ (\theta_1)_1\ (\theta_2)_2\ (\theta_3)_3\ (180°)_1\ (\eta')_2\ (i)_3\ (-\eta)_2 \quad \ldots.(63)$$

Approximations: $C\phi_j = 1$, $S\phi_j = \phi_j$, $j = 1,\ldots,3$.

Observing that

$$(\theta_1)_1\ (\theta_2)_2\ (\theta_3)_3 \approx \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ \theta_1\ \theta_2 - \theta_3 & 1 + \theta_1\ \theta_2\ \theta_3 & \theta_1 \\ \theta_2 + \theta_1\ \theta_3 & \theta_2\ \theta_3 - \theta_1 & 1 \end{bmatrix} \quad \ldots.(64)$$

for small θ_1, θ_2, θ_3,

the uncoupled solutions for θ_1, θ_2 and θ_3 become:

$$\begin{aligned} -\theta_1 = \Big\{ & \left[S_1\ S_2 + C_1\ (\phi_1\ \phi_2 - \phi_3) - C_2\ S_1\ (\phi_2 + \phi_1\ \phi_3)\right] \cdot \left[C_2'\ (Ci\ C\eta\ S\eta' - C\eta'\ S\eta) \right. \\ & \left. - S_1'\ S_2'\ Si\ S\eta' - C_1'\ S_2'\ (-\ Ci\ S\eta\ S\eta' - C\eta\ C\eta')\right] + \left[S_1\ S_2\ \phi_3 + C_1\ (1 + \phi_1\ \phi_2\ \phi_3) \right. \\ & \left. - C_2\ S_1\ (\phi_2\ \phi_3 - \phi_1)\right] \cdot \left[-C_1'\ Si\ S\eta' + S_1'\ (-\ Ci\ S\eta\ S\eta' - C\eta\ C\eta')\right] \\ & + \left[-S_1\ S_2\ \phi_2 + C_1\ \phi_1 - C_2\ S_1\right] \left[S_2'\ (Ci\ C\eta\ S\eta' - C\eta'\ S\eta) \right. \\ & \left. + C_2'\ S_1'\ Si\ S\eta' + C_1'\ C_2'\ (-Ci\ S\eta\ S\eta' - C\eta\ C\eta')\right] \Big\}. \end{aligned} \quad \ldots.(65)$$

$$\theta_2 = \Big\{ \left[C_2 + S_2 (\phi_2 + \phi_1 \phi_3)\right] \cdot \left[C_2' (Ci\ C\eta\ S\eta' - C\eta' S\eta) - S_1' S_2' Si\ S\eta' + C_1' S_2' (Ci\ S\eta\ S\eta' + C\eta\ C\eta')\right] + \left[C_2 \phi_3 + S_2 (\phi_2 \phi_3 - \phi_1)\right] \cdot \left[-C_1' Si\ S\eta' + S_1' (-Ci\ S\eta\ S\eta' - C\eta\ C\eta')\right] + \left[-C_2 \phi_2 + S_2\right] \cdot \left[S_2' (Ci\ C\eta\ S\eta' - C\eta' S\eta) + C_2' S_1' Si\ S\eta' + C_1' C_2' (-Ci\ S\eta\ S\eta' - C\eta\ C\eta')\right] \Big\} \quad \ldots.(66)$$

$$\theta_3 = \Big\{ \left[C_2 + S_2 (\phi_2 + \phi_1 \phi_3)\right] \cdot \left[C_2' Si\ C\eta + S_1' S_2' Ci + C_1' S_2' Si\ S\eta\right] + \left[C_2 \phi_3 + S_2 (\phi_2 \phi_3 - \phi_1)\right] \cdot \left[C_1' Ci - S_1' Si\ S\eta\right] + \left[-C_2 \phi_2 + S_2\right] \cdot \left[S_2' Si\ C\eta - C_2' S_1' Ci - C_1' C_2' Si\ S\eta\right] \Big\} \quad \ldots.(67)$$

Considerable simplifications result if δ_1' and δ_2' are taken to be δ_1 and δ_2, respectively, and if η' is treated as η. Under these assumptions and taking trigonometric cancellations into account, the solutions become:

$$\theta_1 = Si\ S\eta + \phi_1 \left[-C_1 S_2 (Ci\ C\eta\ S\eta - C\eta\ S\eta) + C_2 (Ci\ S^2\eta + C^2\eta)\right] + \phi_2 \left[S_1 (Ci\ C\eta\ S\eta - C\eta\ S\eta)\right] + \phi_3 \left[S_2 (Ci\ S^2\eta + C^2\eta) + C_1 C_2 (Ci\ C\eta\ S\eta - C\eta\ S\eta)\right] + \phi_1 \phi_2 \left[C_1 C_2 (C\eta\ S\eta - Ci\ C\eta\ S\eta) + C_1 S_1 S_2 Si\ S\eta - C_1^2 S_2 (Ci\ S^2\eta + C^2\eta)\right] + \phi_1 \phi_3 \left[S_1 C_2^2 (Ci\ C\eta\ S\eta - C\eta\ S\eta) - S_1^2 C_2 S_2 Si\ S\eta + C_1 C_2 S_1 S_2 (Ci\ S^2\eta + C^2\eta)\right] + \phi_2 \phi_3 \left[-C_1 S_1 C_2 Si\ S\eta + S_1^2 C_2 (-Ci\ S^2\eta - C^2\eta)\right] + \phi_1 \phi_2 \phi_3 \left[C_1^2 Si\ S\eta + C_1 S_1 (Ci\ S^2\eta + C^2\eta)\right] \quad \ldots.(68)$$

$$\theta_2 = \left[Ci\ C\eta\ S\eta - C\eta\ S\eta\right] + \phi_1 \left[S_2 C_1 Si\ S\eta + S_1 S_2 Ci\ S^2\eta + S_1 S_2 C^2\eta\right] + \phi_2 \left[C_1 Ci\ S^2\eta + C_1 C^2\eta - S_1 Si\ S\eta\right] + \phi_3 \left[-C_1 C_2 Si\ S\eta - S_1 C_2 Ci\ S^2\eta - S_1 C_2 C^2\eta\right] + \phi_1 \phi_3 \left[C_2 S_2 Ci\ C\eta\ S\eta - C_2 S_2 C\eta\ S\eta - S_1 S_2^2 Si\ S\eta + C_1 S_2^2 (Ci\ S^2\eta + C^2\eta)\right] + \phi_2 \phi_3 \left[-S_2 C_1 Si\ S\eta - S_1 S_2 Ci\ S^2\eta - S_1 S_2 C^2\eta\right] \quad \ldots.(69)$$

$$\theta_3 = Si\ C\eta + \phi_1 \left[-C_1 S_2 Ci + S_1 S_2 Si\ S\eta\right] + \phi_2 \left[S_1 Ci + C_1 Si\ S\eta\right] + \phi_3 \left[C_1 C_2 Ci - S_1 C_2 Si\ S\eta\right] + \phi_2 \phi_3 \left[C_1 S_2 Ci - S_1 S_2 Si\ S\eta\right] + \phi_1 \phi_3 \left[C_2 S_2 Si\ C\eta + S_1 S_2^2 Ci + C_1 S_2^2 Si\ S\eta\right] \quad \ldots.(70)$$

For further reduction in complexity (and accuracy) the second and third order ϕ-terms may be dropped.

In order to give an idea of the relative magnitudes of these terms, observe that: (1) η is a clock term and varies regularly from 0 to 2π rads with about a 24 hr period, (2) the inclination 'i' is to be limited to 2° so that

$$\left.\begin{array}{l} Ci = 0.999\ 390\ 827 \\ Si = 0.034\ 899\ 496 \end{array}\right|\ i = 2^\circ \qquad \text{....(71)}$$

(3) the nominal δ's for a ground station at Ottawa, using the NASA figures for earth and synchronous radii, become,

$$\begin{pmatrix} i = 0 \\ e = 0 \end{pmatrix} :$$

$$\begin{array}{lll} \delta_2 = 6.685\ 685\ 543^\circ & C_1 = 0.993\ 199\ 766\ 8 & S_1 = 0.116\ 422\ 605\ 2 \\ \delta_2 = 4.060\ 249\ 050^\circ & C_2 = 0.997\ 490\ 146\ 7 & S_2 = 0.070\ 805\ 417\ 76\ . \end{array} \qquad \text{....(72)}$$

If second and higher order terms in $\underline{\phi}$ are ignored and both i and e of the orbit are taken to be zero, the solutions for $\underline{\theta}$ reduce to:

$$\theta_1 = C_2\ \phi_1 + S_2\ \phi_3 \qquad \text{....(73)}$$

$$\theta_2 = S_1\ S_2\ \phi_1 + C_1\ \phi_2 - S_1\ C_2\ \phi_3 \qquad \text{....(74)}$$

$$\theta_3 = -C_1\ S_2\ \phi_1 + S_1\ \phi_2 + C_1\ C_2\ \phi_3\ . \qquad \text{....(75)}$$

These last equations demonstrate clearly the cross-coupling of the $\underline{\phi}$s in the $\underline{\theta}$ solutions. Accordingly, ϕ_1 serves as a good indication of θ_1 and so on, but for an accurate reading of θ_1, ϕ_1 and ϕ_3 must both be known. A similar statement may be made for θ_2 and θ_3. These statements are reinforced because θ_3 (and therefore ϕ_3) has approximately seven times the tolerance of θ_1 and θ_2.

APPENDIX E

EXACT SOLUTIONS FOR THE CASE OF AN ADDITIONAL TRANSFORMATION $(\delta_3)_3$

In the main body of the report there was the implicit assumption that the combined rotation $(\delta_2)_2\ (\delta_1)_1$ would specify the orientation of the MASS antenna with respect to the spacecraft axes. Even if no subsequent rotation $(\delta_3)_3$ about the antenna boresight is planned, there will, in practice, be some misalignment, or $\delta_3 \neq 0$. Furthermore, a large δ_3 (say, 45°), may be desirable, for example, to minimize the electrical interference of the spacecraft forward platform. Accordingly, a possibly non-zero δ_3 and δ_3' should be incorporated in the analysis for complete generality. In the following, δ_3 and δ_3' are assumed to be known, either having been calculated to satisfy some criterion or having been measured on a working model of the MASS.

A) Exact Solution of $\underline{\theta}$ from $\underline{\phi}$, Including δ_3, δ_3'

Following the example of the derivation in the main body of the report, we concentrate on the transformations between the various pertinent sets of coordinates. To distinguish from the δ_3 ignored derivation, we employ a superscript 2 for the coordinate sets χ^{LS} and χ^{RF}, the χ^{LS2} and χ^{RF2} now designating the axes of the line of sight and MASS R-F frames, respectively, following the δ_3' and δ_3 rotations. Thus the rotational sequences and pertinent frames become:

$$
\begin{tikzcd}
\underline{\chi}^{E} \arrow[rr, Rightarrow, "(180°)_1\ (\eta')_2\ (i)_3\ (-\eta)_2"] \arrow[d, Rightarrow, "(180°)_1"] & & \underline{\chi}^{01} \arrow[rr, Rightarrow, "(\theta_1)_1\ (\theta_2)_2\ (\theta_3)_3"] & & \underline{\chi}^{SC} \arrow[d, Rightarrow, "(\delta_3)_3\ (\delta_2)_2\ (\delta_1)_1"] \\
\underline{\chi}^{0} \arrow[rr, Rightarrow, "(\delta_3')_3\ (\delta_2')_2\ (\delta_1')_1"] & & \underline{\chi}^{LS2} \arrow[rr, Rightarrow, "(\phi_1)_1\ (\phi_2)_2\ (\phi_3)_3"] & & \underline{\chi}^{RF2}
\end{tikzcd}
$$

Equating the two transformation paths from χ^E to χ^{RF2}, we see

$$
(\delta_3)_3\ (\delta_2)_2\ (\delta_1)_1\ (\theta_1)_1\ (\theta_2)_2\ (\theta_3)_3\ (180°)_1\ (\eta')_2\ (i)_3\ (-\eta)_2
$$

$$
= (\phi_1)_1\ (\phi_2)_2\ (\phi_3)_3\ (\delta_3')_3\ (\delta_2')_2\ (\delta_1')_1\ (180°)_1\ . \qquad \ldots.(76)
$$

Therefore $(\theta_1)_1\ (\theta_2)_2\ (\theta_3)_3 = (-\delta_1)_1\ (-\delta_2)_2\ (-\delta_3)_3\ (\phi_1)_1\ (\phi_2)_2\ (\phi_3)_3$

$$\cdot\ (\delta_3')_3\ (\delta_2')_2\ (\delta_1')_1\ (180^\circ)_1\ (\eta)_2\ (-i)_3\ (-\eta')_2\ (180^\circ)_1 \qquad \ldots.(77)$$

The last seven transformations are identical to the last seven of the $\delta_3 = 0$ solution found in the main body of the report and are repeated here for convenience:

$$(\delta_2')_2\ (\delta_1')_1\ (180^\circ)_1\ (\eta)_2\ (-i)_3\ (-\eta')_2\ (180^\circ)_1 = A^\tau \qquad \ldots.(78)$$

where

$$A = \begin{bmatrix} a7 & a4 & a1 \\ a8 & a5 & a2 \\ a9 & a6 & a3 \end{bmatrix} \qquad \ldots.(79)$$

and where

$$a1 = S_2'\ (Ci\ C\eta\ C\eta' + S\eta\ S\eta') + S_1'\ C_2'\ Si\ C\eta' - C_1'\ C_2'\ (Ci\ C\eta'\ S\eta - C\eta\ S\eta')$$

$$a2 = S_2'\ Si\ C\eta - S_1'\ C_2'\ Ci - C_1'\ C_2'\ Si\ S\eta$$

$$a3 = S_2'\ (S\eta\ C\eta' - Ci\ C\eta\ S\eta') - S_1'\ C_2'\ Si\ S\eta' + C_1'\ C_2'\ (Ci\ S\eta\ S\eta' + C\eta\ C\eta')$$

$$a4 = -C_1'\ Si\ C\eta' - S_1'\ (Ci\ C\eta'\ S\eta - C\eta\ S\eta')$$

$$a5 = C_1'\ Ci - S_1'\ Si\ S\eta \qquad \ldots.(38b)$$

$$a6 = C_1'\ Si\ S\eta' + S_1'\ (Ci\ S\eta\ S\eta' + C\eta\ C\eta')$$

$$a7 = C_2'\ (Ci\ C\eta\ C\eta' + S\eta\ S\eta') - S_1'\ S_2'\ Si\ C\eta' + C_1'\ S_2'\ (Ci\ C\eta'\ S\eta - C\eta\ S\eta')$$

$$a8 = C_2'\ Si\ C\eta + S_1'\ S_2'\ Ci + C_1'\ S_2'\ Si\ S\eta$$

$$a9 = C_2'\ (C\eta'\ S\eta - Ci\ C\eta\ S\eta') + S_1'\ S_2'\ Si\ S\eta' - C_1'\ S_2'\ (Ci\ S\eta\ S\eta' + C\eta\ C\eta')\ .$$

Define $D(\delta) = (\delta_3)_3\ (\delta_2)_2\ (\delta_1)_1$ ....(80)

$$D(\delta) = \begin{bmatrix} d7 & d4 & d1 \\ d8 & d5 & d2 \\ d9 & d6 & d3 \end{bmatrix} \qquad \ldots.(81)$$

where the terms d1 to d9 are defined by

$$d1 = S_1\ S_3 - C_1\ S_2\ C_3$$

$$d2 = C_1\ S_2\ S_3 + S_1\ C_3$$

$$d3 = C_1\ C_2$$

$$d4 = S_1\ S_2\ C_3 + C_1\ S_3$$

$d5 = C_1\ C_3$

$d6 = -S_1\ C_2$

$d7 = C_2\ C_3$

$d8 = -C_2\ S_3$

$d9 = S_2$.

For example, $d9(\delta) = \mathrm{Sin}\ (\delta_2)$.

Also, $$(-\delta_1)_1\ (-\delta_2)_2\ (-\delta_3)_3 = D^\tau(\delta) \qquad \text{....(83a)}$$

and $$(\phi_1)_1\ (\phi_2)_2\ (\phi_3)_3 = D^\tau(-\phi)\ . \qquad \text{....(83b)}$$

Observing that $$(\phi_3)_3\ (\delta_3')_3 = (\phi_3 + \delta_3')_3\ , \qquad \text{....(84)}$$

Eqn. (77) may be written

$$D^\tau\ (-\theta) = D^\tau(\delta)\ D^\tau(-\phi')\ A^\tau \qquad \text{....(85)}$$

where ϕ_3' denotes $\phi_3 + \delta_3'$, while $\phi_1' = \phi_1$, $\phi_2' = \phi_2$.

Accordingly, the matrix multiplication may be performed to show:

$$\begin{aligned} d9(-\theta) = {} & d7(\delta)\ \left[d7(-\phi')a9 + d8(-\phi')a6 + d9(-\phi')a3\right] \\ & + d8(\delta)\ \left[d4(-\phi')a9 + d5(-\phi')a6 + d6(-\phi')a3\right] \\ & + d9(\delta)\ \left[d1(-\phi')a9 + d2(-\phi')a6 + d3(-\phi')a3\right] \end{aligned} \qquad \text{....(86)}$$

$$\begin{aligned} d6(-\theta) = {} & d4(\delta)\ \left[d7(-\phi')a9 + d8(-\phi')a6 + d9(-\phi')a3\right] \\ & + d5(\delta)\ \left[d4(-\phi')a9 + d5(-\phi')a6 + d6(-\phi')a3\right] \\ & + d6(\delta)\ \left[d1(-\phi')a9 + d2(-\phi')a6 + d3(-\phi')a3\right] \end{aligned} \qquad \text{....(87)}$$

$$\begin{aligned} d8(-\theta) = {} & d7(\delta)\ \left[d7(-\phi')a8 + d8(-\phi')a5 + d9(-\phi')a2\right] \\ & + d8(\delta)\ \left[d4(-\phi')a8 + d5(-\phi')a5 + d6(-\phi')a2\right] \\ & + d9(\delta)\ \left[d1(-\phi')a8 + d2(-\phi')a5 + d3(-\phi')a2\right]. \end{aligned} \qquad \text{....(88)}$$

These particular $d(-\theta)$ terms were singled out because of their simplicity; from Eqn. (82) we see that $d9(-\theta)$, $d6(-\theta)$ and $d8(-\theta)$ are, respectively, $-\mathrm{Sin}\ \theta_2$, $\mathrm{Sin}\ \theta_1\ \mathrm{Cos}\ \theta_2$ and $\mathrm{Cos}\ \theta_2\ \mathrm{Sin}\ \theta_3$, from which θ_2, θ_1, and θ_3 may be determined easily, and accurately for small θ's. Substituting Eqn. 82 into Eqns. 86, 87 and 88, we see

$$\begin{aligned} -S\theta_2 = {} & a9\ \Big[C_2\ C_3\ C\phi_2\ C\phi_3' - C_2\ S_3\ (S\phi_1\ S\phi_2\ C\phi_3' - C\phi_1\ S\phi_3') \\ & \qquad + S_2\ (S\phi_1\ S\phi_3' + C\phi_1\ S\phi_2\ C\phi_3')\Big] \\ & + a6\ \left[C_2\ C_3\ C\phi_2\ S\phi_3' - C_2\ S_3\ C\phi_1\ C\phi_3' + S_2\ (C\phi_1\ S\phi_2\ S\phi_3' - S\phi_1\ C\phi_3')\right] \\ & + a3\ \left[-C_2\ C_3\ S\phi_2 - C_2\ S_3\ S\phi_1\ C\phi_2 + S_2\ C\phi_1\ C\phi_2\right] \end{aligned} \qquad \text{....(89)}$$

$$S\theta_1 = (C\theta_2)^{-1} \Big\{ a9 \Big[(S_1\ S_2\ C_3 + C_1\ S_3)\ C\phi_2\ C\phi_3' + C_1\ C_3\ (S\phi_1\ S\phi_2\ C\phi_3' - C\phi_1\ S\phi_3') - S_1\ C_2\ (C\phi_1\ S\phi_2\ C\phi_3' + S\phi_1\ S\phi_3') \Big]$$
$$+ a6 \Big[(S_1\ S_2\ C_3 + C_1\ S_3)\ C\phi_2\ S\phi_3' + C_1\ C_3\ C\phi_1\ C\phi_3' - S_1\ C_2\ (C\phi_1\ S\phi_2\ S\phi_3' - S\phi_1\ C\phi_3') \Big]$$
$$+ a3 \Big[-(S_1\ S_2\ C_3 + C_1\ S_3)\ S\phi_2 + C_1\ C_3\ S\phi_1\ C\phi_2 - S_1\ C_2\ C\phi_1\ C\phi_2 \Big] \Big\} \qquad \ldots.(90)$$

$$S\theta_3 = (C\theta_2)^{-1} \Big\{ a8 \Big[C_2\ C_3\ C\phi_2\ C\phi_3' - C_2\ S_3\ (S\phi_1\ S\phi_2\ C\phi_3' - C\phi_1\ S\phi_3') + S_2\ (S\phi_1\ S\phi_3' + C\phi_1\ S\phi_2\ C\phi_3') \Big]$$
$$+ a5 \Big[C_2\ C_3\ C\phi_2\ S\phi_3' - C_2\ S_3\ C\phi_1\ C\phi_3' + S_2\ (C\phi_1\ S\phi_2\ S\phi_3' - S\phi_1\ C\phi_3') \Big]$$
$$+ a2 \Big[-C_2\ C_3\ S\phi_2 - C_2\ S_3\ S\phi_1\ C\phi_2 + S_2\ C\phi_1\ C\phi_2 \Big] \Big\}. \qquad \ldots.(91)$$

Equations (89) to (91) now specify the solutions of θ_2, θ_1 and θ_3 in terms of the MASS measured angles ϕ_1, ϕ_2, ϕ_3, the MASS antenna mounting rotations δ_1, δ_2, δ_3 and the satellite position dependent ground station antenna pointing angles δ_1', δ_2', δ_3', the last three of which are used in Eqn. (38b) which defines the parameters a1 to a9.

B) Exact Solution of $\underline{\phi}$ from $\underline{\theta}$, Including δ_3, δ_3'

Equation (85) may be manipulated to support:

$$D^T(-\phi') = D(\delta) D^T\ (-\theta)\ A\ .$$

Following the pattern of the previous section A), we find upon expansion that:

$$-S\phi_2 = a1 \Big[C_2\ C_3\ C\theta_2\ C\theta_3 + (S_1\ S_2\ C_3 + C_1\ S_3)\ (S\theta_1\ S\theta_2\ C\theta_3 - C\theta_1\ S\theta_3) + (S_1\ S_3 - C_1\ S_2\ C_3)\ (S\theta_1\ S\theta_3 + C\theta_1\ S\theta_2\ C\theta_3) \Big]$$
$$+ a2 \Big[C_2\ C_3\ C\theta_2\ S\theta_3 + (S_1\ S_2\ C_3 + C_1\ S_3)\ C\theta_1\ C\theta_3 + (S_1\ S_3 - C_1\ S_2\ C_3) \cdot (C\theta_1\ S\theta_2\ S\theta_3 - S\theta_1\ C\theta_3) \Big]$$
$$+ a3 \Big[-C_2\ C_3\ S\theta_2 + (S_1\ S_2\ C_3 + C_1\ S_3)\ S\theta_1\ C\theta_2 + (S_1\ S_3 - C_1\ S_2\ C_3)\ C\theta_1\ C\theta_2 \Big] \qquad \ldots.(92)$$

$$S\phi_1 = (C\phi_2)^{-1} \Big\{ a1 \Big[-C_2\ S_3\ C\theta_2\ C\theta_3 + C_1\ C_3\ (S\theta_1\ S\theta_2\ C\theta_3 - C\theta_1\ S\theta_3) + (C_1\ S_2\ S_3 + S_1\ C_3)\ (S\theta_1\ S\theta_3 + C\theta_1\ S\theta_2\ C\theta_3) \Big]$$
$$+ a2 \Big[-C_2\ S_3\ C\theta_2\ S\theta_3 + C_1\ C_3\ C\theta_1\ C\theta_3 + (C_1\ S_2\ S_3 + S_1\ C_3)\ (C\theta_1\ S\theta_2\ S\theta_3 - S\theta_1\ C\theta_3) \Big]$$

$$+ a3 \left[C_2\ S_3\ S\theta_2 + C_1\ C_3\ S\theta_1\ C\theta_2 + (C_1\ S_2\ S_3 + S_1\ C_3)\ C\theta_1\ C\theta_2 \right] \Bigg\} \quad \ldots (93)$$

$$S\phi_3' = (C\phi_2)^{-1} \Bigg\{ a4 \Big[C_2\ C_3\ C\theta_2\ C\theta_3 + (S_1\ S_2\ C_3 + C_1\ S_3)\ (S\theta_1\ S\theta_2\ C\theta_3 - C\theta_1\ S\theta_3) + (S_1\ S_3 - C_1\ S_2\ C_3)\ (S\theta_1\ S\theta_3 + C\theta_1\ S\theta_2\ C\theta_3) \Big]$$
$$+ a5 \Big[C_2\ C_3\ C\theta_2\ S\theta_3 + (S_1\ S_2\ C_3 + C_1\ S_3)\ C\theta_1\ C\theta_3 + (S_1\ S_3 - C_1\ S_2\ C_3) \cdot (C\theta_1\ S\theta_2\ S\theta_3 - S\theta_1\ C\theta_3) \Big]$$
$$+ a6 \Big[-C_2\ C_3\ S\theta_2 + (S_1\ S_2\ C_3 + C_1\ S_3)\ S\theta_1\ C\theta_2 + (S_1\ S_3 - C_1\ S_2\ C_3)\ C\theta_1\ C\theta_2 \Big] \Bigg\} \quad \ldots (94)$$

and $\phi_3' = \phi_3 + \delta_3'$.

Equations (92) to (94) (used with Eqn. (38b) again), define ϕ_1, ϕ_2, ϕ_3, in terms of θ_1, θ_2, θ_3, δ_1, δ_2, δ_3 and δ_1', δ_2', δ_3'. Accordingly, these equations may be used to calculate the angles the MASS should indicate for any specified combination of $\underline{\theta}$, $\underline{\delta}$ and $\underline{\delta}'$.

C) Exact Solution for Pseudo Two-Station Yaw, Including δ_3, δ_3'

Pseudo two-station yaw, for which the yaw angle is calculated from two normally earth-sensor measured angles (roll and pitch) and one angle from MASS. Theoretically, ϕ_1, ϕ_2 or ϕ_3 may be used, but ϕ_2 is the choice under normal conditions. ϕ_3 would produce the yaw estimate least sensitive to MASS angular measurement errors, but in the simplest MASS there would be no polarimeter or means of measuring ϕ_3. Of the two remaining MASS measured angles, ϕ_2 produces the least sensitivity to errors in ϕ of the final yaw estimate. Equation (92) contains terms in θ_1, θ_2, θ_3 and ϕ_2 but no terms in ϕ_1 nor ϕ_3. Accordingly, this equation may be manipulated to isolate the θ_3-dependent terms, as follows:

$$-S\phi_2 = C\theta_3 \Big[a1\ (C_2\ C_3\ C\theta_2 + (S_1\ S_2\ C_3 + C_1\ S_3)\ S\theta_1\ S\theta_2 + (S_1\ S_3 - C_1\ S_2\ C_3)\ C\theta_1\ S\theta_2) + a2\ ((S_1\ S_2\ C_3 + C_1\ S_3)\ C\theta_1 - (S_1\ S_3 - C_1\ S_2\ C_3)\ S\theta_1) \Big]$$
$$+ S\theta_3 \Big[a1\ (-(S_1\ S_2\ C_3 + C_1\ S_3)\ C\theta_1 + (S_1\ S_3 - C_1\ S_2\ C_3)\ S\theta_1) + a2\ (C_2\ C_3\ C\theta_2 + (S_1\ S_3 - C_1\ S_2\ C_3)\ C\theta_1\ S\theta_2) \Big]$$
$$+ a3 \Big[-C_2\ C_3 S\theta_2 + (S_1\ S_2\ C_3 + C_1\ S_3)\ S\theta_1\ C\theta_2 + (S_1\ S_3 - C_1\ S_2\ C_3)\ C\theta_1\ C\theta_2 \Big]. \quad \ldots (95)$$

This transcendental equation may be solved iteratively for θ_3; convergence may be expected to be rapid since θ_3 will normally be known beforehand to within a narrow tolerance. If one makes the approximation $C\theta_3 \approx 1$, $S\theta_3 \approx \theta_3$ in rads., Eqn. 95 ceases to be transcendental and may be written:

$$\begin{aligned}\theta_3 \quad & \Big\{ a1 \Big[(- S_1\ S_2\ C_3 + C_1\ S_3)\ C\theta_1 + (S_1\ S_3 - C_1\ S_2\ C_3)\ S\theta_1)\Big] \\ & + a2 \Big[C_2\ C_3\ C\theta_2 + (S_1\ S_3 - C_1\ S_2\ C_3)\ C\theta_1\ S\theta_2\Big]\Big\}^{-1} \Big\{ -S\theta_2 \\ & - \Big[a1 \Big[C_2\ C_3\ C\theta_2 + (S_1\ S_2\ C_3 + C_1\ S_3)\ S\theta_1\ S\theta_2 + (S_1\ S_3 - C_1\ S_2\ C_3)\ C\theta_1\ S\theta_2\Big] \\ & + a2 \Big[(S_1\ S_2\ C_3 + C_1\ S_3)\ C\theta_1 - (S_1\ S_3 - C_1\ S_2\ C_3)\ S\theta_1 \Big]\Big] \\ & - a3 \Big[-C_2\ C_3\ S\theta_2 + (S_1\ S_2\ C_3 + C_1\ S_3)\ S\theta_1\ C\theta_2 + (S_1\ S_3 - C_1\ S_2\ C_3)\ C\theta_1\ C\theta_2\Big]\Big\} \cdot\end{aligned}$$

....(96)

APPENDIX F

Relations Between Azimuth, Elevation and the Rotations $(\delta_2')_2\ (\delta_1')_1$

Although the ground station antenna pointing angles defined by $(\delta_2')_2\ (\delta_1')_1$ are correct, it is desirable to redefine the pointing angles in terms of the more conventional azimuth (az) and elevation (el) angles, since the majority of antenna drives are with respect to az and el. Accordingly, we develop the relations for az and el for arbitrarily located ground stations and satellite positions, the satellite assumed to be in an inclined eccentric synchronous orbit. The range, or satellite-ground station separation distance follows from the development, and these two steps are then used to demonstrate the relation between az and el and the $(\delta_2')_2\ (\delta_1')_1$ developed in the report.

The terms azimuth and elevation require clarification, for at least three different definitions of azimuth above appear in the literature[3], p. 231; [5], p. 44; [6], p. 22. For the following we define a plane P1 passing through the ground station and orthogonal to the vector $\underline{R}_G$ to the ground station from the centre of the earth, in the coordinate system χ^E. For a satellite situated at $\underline{R}_S$, then $\underline{R}_{GS} = \underline{R}_S - \underline{R}_G$ is the vector from the ground station to the satellite. If $\underline{R}'_{GS}$ is the projection of $\underline{R}_{GS}$ into the plane P1, we define the elevation of the satellite to be the angle subtended by $\underline{R}_{GS}$ and $\underline{R}'_{GS}$ at $\underline{R}_G$. Furthermore, if $\underline{R}_P$ denotes a vector parallel to the spin axis of the earth and passing through the ground station, and $\underline{R}'_P$ is its projection into P1, we define the azimuth of the satellite to be the angle from $\underline{R}'_P$ to $\underline{R}'_{GS}$, measured positive clockwise (i.e., to the East).

A ground station whose position is described by R_E, ϕ and λ (as shown on p. 2), is located at the tip of the vector $\underline{R}_G$,

$$\underline{R}_G = R_E(-\phi)_2\ (\lambda)_1\ (0,\ 0,\ 1)^T\ . \qquad \ldots.(97)$$

Similarly, in χ^E the position of the satellite is given by

$$\underline{R}_S = R_{ST}\ (\eta)_2\ (-i)_3\ \ (-\eta')_2\ (0,\ 0,\ 1)^T\ . \qquad \ldots.(98)$$

Accordingly, the vector from the ground station to the satellite is

$$\underline{R}_{GS} = \underline{R}_S - \underline{R}_G\ ,$$

the range being $|\underline{R}_{GS}|$.

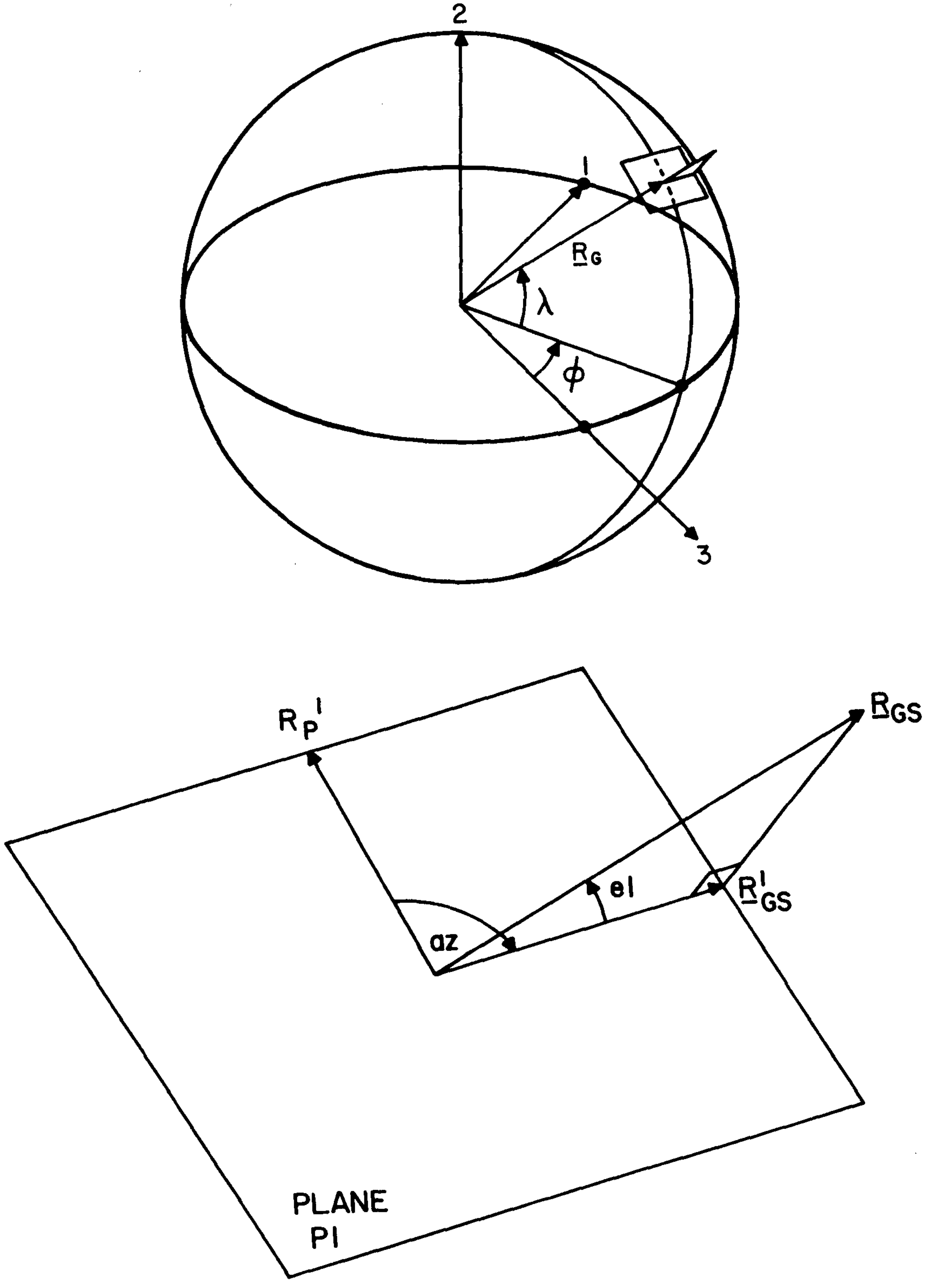

Fig. F1. Ground station antenna pointing angles.

That is,
$$\underline{R}_{GS} = \begin{bmatrix} R_{ST}\ (Ci\ C\eta\ S\eta' - C\eta'\ S\eta) - R_E S\phi C\lambda \\ R_{ST}\ Si\ S\eta' - R_E S\lambda \\ R_{ST}\ (Ci\ S\eta\ S\eta' + C\eta\ C\eta') - R_E C\phi C\lambda \end{bmatrix} . \qquad \ldots.(99)$$

Next we consider a set of coordinates χ^2 with origin at $\underline{R}_G$ and two axes in the plane P1 and the third axis parallel to $\underline{R}_G$, the set χ^2 being derived from χ^E by the rotations $(-\lambda)_1\ (\phi)_2$. Accordingly, axis #1 of χ^2 is the local E vector, axis #2 is the local N vector and axis #3 is along the local vertical. Thus the unit vector
$$\underline{r}_{GS} = \underline{R}_{GS}/|\underline{R}_{GS}| = (r_1,\ r_2,\ r_3)^T$$
in χ^E acquires the description in χ^2 of
$$\underline{r}^2_{GS} = (-\lambda)_1\ (\phi)_2\ \underline{r}_{GS} \qquad \ldots.(100)$$
or
$$\underline{r}^2_{GS} = \begin{bmatrix} C\phi r_1 - S\phi r_3 \\ C\lambda r_2 - S\lambda(S\phi r_1 + C\phi r_3) \\ S\lambda r_2 + C\lambda(S\phi r_1 + C\phi r_3) \end{bmatrix} . \qquad \ldots.(101)$$

Observing the definition of ax and el, we may write
$$(0,\ 1,\ 0)^T = (el)_1\ (-az)_3\ \underline{r}^2_{GS} \qquad \ldots.(102)$$

Therefore,
$$\underline{r}^2_{GS} = (az)_3\ (-el)_1\ (0,\ 1,\ 0)^T \qquad \ldots.(103)$$

Therefore,
$$\begin{bmatrix} S_{az}\ C_{el} \\ C_{az}\ C_{el} \\ S_{el} \end{bmatrix} = \begin{bmatrix} C\phi r_1 - S\phi r_3 \\ C\lambda r_2 - S\lambda(S\phi r_1 + C\phi r_3) \\ S\lambda r_2 + C\lambda(S\phi r_1 + C\phi r_3) \end{bmatrix} \qquad \ldots.(104)$$

Note that although elevation is bounded by 0 and 90°, the azimuth may range from 0 to 360°. Accordingly, care must be taken when inverting the sine of the azimuth that the correct quadrant is assigned to the result.

This derivation is easily adapted to other definitions of az and el. Relations to take the earth's oblateness into account have been developed elsewhere [5] p. 48, and [7], and will not be repeated here.

If the ground station antenna angles δ_1' and δ_2' have already been determined, az and el need not be calculated from the components of $\underline{r}_{GS}$ as in Eqn. (104), but may instead be found directly from δ_1' and δ_2'. $\underline{r}_{GS}$ was found in Eqn. (25) to be
$$\underline{r}_{GS} = (180°)\ (-\delta_1')_1\ (-\delta_2')_2\ (0,\ 0,\ 1)^T . \qquad \ldots.(105)$$

But $$\underline{r}_{GS} = - \underline{r}_{SG} \qquad \text{....(106)}$$

and from Eqn. (100) $$\underline{r}_{GS} = (-\phi)_2 \; (\lambda)_1 \; \underline{r}^2_{GS} \qquad \text{....(107)}$$

while from Eqn. (102), $$\underline{r}^2_{GS} = (az)_3 \; (-el)_1 \; (0,\ 1,\ 0)^T \; . \qquad \text{....(108)}$$

Therefore, $$\underline{r}_{GS} = (-\phi)_2 \; (\lambda)_1 \; (az)_3 \; (-el)_1 \; (0,\ 1,\ 0)^T \; . \qquad \text{....(109)}$$

From Eqns. (105), (106) and (109), we see

$$-(180°)_1 \; (-\delta'_1)_1 \; (-\delta'_2)_2 \; (0,\ 0,\ 1)^T = (-\phi)_2 \; (\lambda)_1 \; (az)_3 \; (-el)_1 \; (0,\ 1,\ 0)^T \; . \qquad \text{....(110)}$$

From Eqn. (110), we find

$$(az)_3 \; (-el)_1 \; (0,\ 1,\ 0)^T = -(-\lambda)_1 \; (\phi)_2 \; (180°)_1 \; (-\delta'_1)_1 \; (-\delta'_2)_2 \; (0,\ 0,\ 1)^T \qquad \text{....(111)}$$

Therefore $$\begin{bmatrix} S_{az} \; C_{el} \\ C_{az} \; C_{el} \\ S_{el} \end{bmatrix} = - \begin{bmatrix} C\phi S'_2 + S\phi C'_1 C'_2 \\ C\lambda S'_1 C'_2 - S\lambda(S\phi S'_2 - C\phi C'_1 C'_2) \\ S\lambda S'_1 C'_2 + C\lambda(S\phi S'_2 - C\phi C'_1 C'_2) \end{bmatrix} \; . \qquad \text{....(112)}$$

Thus az and el have been found in terms of δ'_1, δ'_2, ϕ and λ, and Eqn. (111) is easily manipulated to define δ'_1 and δ'_2 in terms of az, el, ϕ and λ.

CRC DOCUMENT CONTROL DATA

1. ORIGINATOR: Communications Research Centre, Department of Communications, Ottawa, Canada.

2 DOCUMENT NO: CRC Report No. 1245

3. DOCUMENT DATE: October, 1973

4. DOCUMENT TITLE: A Microwave Attitude Sensing System for Satellites - Angular Relationships

5. AUTHOR(S): Rolf Mamen

6. KEYWORDS (1) Satellite
(2) Attitude
(3) Sensing

7. SUBJECT CATEGORY (FIELD & GROUP: COSATI)

Space Technology

Spacecraft

22-01

8. ABSTRACT A Microwave Attitude Sensing Subsystem (MASS) may be used to measure the attitude of a satellite relative to an incident polarized RF signal from a ground station. The desired attitude of a spacecraft, however, may be specified with respect to a set of axes other than those of the MASS. This report describes the development of relations between the angles measured by the MASS and the spacecraft attitude specified by the conventional roll, pitch and yaw angles. In order to preserve generality, the ground station position is arbitrary and the satellite is assumed to be in an inclined eccentric synchronous orbit. Exact relations are developed for the roll, pitch and yaw angles in terms of the MASS measured angles, and conversely, for the MASS measured angles in terms of roll, pitch and yaw. Furthermore, the difficult-to-measure yaw angle is determined from roll and pitch plus any one of the MASS measured angles, for pseudo two-station yaw. The exact solutions are orbit dependent and lengthy, and various levels of approximations are suggested in order to simplify the solutions.

9. CITATION:

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