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CALCULATION OF EVANESCENT-WAVE GAIN IN THE TE_{0m} AND TM_{0m} MODES OF AN OPTICAL FIBRE

by

A. Watanabe, K.O. Hill and D. Mintz

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ABSTRACT

In this report we extend our previous perturbation-theory treatment of evanescent-wave gain in dielectric slabs to the case of a cylindrical dielectric waveguide, or optical fibre. The general formalism for treating evanescent-wave gain due to an active cladding material is developed, and detailed calculations of the gain coefficient ratio are presented for the TE_{0m} and TM_{0m} modes of the active optical fibre.

1. INTRODUCTION

Several types of waveguiding structures are known which can support the propagation of a finite number of bound electromagnetic modes. Two such structures are the dielectric slab and the optical fibre. In both structures the electromagnetic field associated with the bound modes extends beyond the central region; that part of the field which penetrates into the surrounding medium is termed evanescent. We have shown experimentally that when the surrounding medium is an active material with gain, then a signal propagating in a bound mode of the structure can exhibit gain[1]. In other work we have also observed laser action by evanescent-wave gain in an optical waveguide device[2,3,4].

In a previous paper we used first-order perturbation theory to treat the effects caused by a surrounding medium with gain on the bound modes of a dielectric slab[5], and derived formulae for the amplification of the bound modes by evanescent-wave interaction with the surrounding medium. In a subsequent work we treated evanescent-wave gain in a dielectric slab waveguide when the active material in the cladding layer was optically pumped by the evanescent field of the pump beam propagating in one of the bound modes of the structure[6]. In this paper we extend the earlier calculations on planar waveguides to active cylindrical dielectric waveguides, or active optical fibres.

2. THE BOUND MODES OF A PASSIVE OPTICAL FIBRE

Consider the case of a cylindrical waveguide with the z axis in the direction of propagation, core radius a, dielectric constant ε_1 and ε_2 of the core and cladding regions, respectively, and a relative permeability of one. The spatial variation of ε in the plane normal to the direction of propagation is shown in Figure 1. For a plane wave in the z direction the electric and magnetic fields can be written in cylindrical coordinates (ρ,ϕ,z) as $\underline{E}(\rho,\phi)\exp(ik_z z-i\omega t)$ and $\underline{B}(\rho,\phi)\exp(ik_z z-i\omega t)$, where the values of the angular frequency ω and the z-direction propagation constant k_z are both real and positive. The electric field satisfies the equation

$$\left(\nabla^2 + \frac{\varepsilon\omega^2}{c^2}\right)\underline{E} = 0,$$

$$\begin{aligned}\varepsilon &= \varepsilon_1 \text{ for } \rho \leq a \\ &= \varepsilon_2 \text{ for } \rho > a, \\ &\text{and } \varepsilon_2 < \varepsilon_1.\end{aligned} \qquad .....(1)$$

The magnetic field also satisfies the equation

$$\left(\nabla^2 + \frac{\varepsilon\omega^2}{c^2}\right)\underline{B} = 0. \qquad .....(2)$$

The Laplacian operator in cylindrical coordinates is given by

$$\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\,. \qquad .....(3)$$

Let $\hat{\rho}$, $\hat{\phi}$ and $\hat{z}$ be the unit vectors along the 3 axes of the cylindrical coordinate system. Then

$$\begin{aligned}\frac{\partial\hat{\rho}}{\partial\phi} &= \hat{\phi}, \\ \frac{\partial\hat{\phi}}{\partial\rho} &= -\hat{\rho},\end{aligned} \qquad .....(4)$$

and all other derivatives of the unit vectors are zero. Thus the equations for the derivatives of the individual components of the electric field vector $\underline{E} = (E_\rho,\ E_\phi,\ E_z)$ become

$$\frac{\partial^2}{\partial\phi^2} E_\rho\hat{\rho} = \frac{\partial^2 E_\rho}{\partial\phi^2}\,\hat{\rho} + 2\,\frac{\partial E_\rho}{\partial\phi}\,\hat{\phi} - E_\rho\hat{\rho}\,, \qquad .....(5)$$

and

$$\frac{\partial^2}{\partial\phi^2} E_\phi\hat{\phi} = \left(\frac{\partial^2 E_\phi}{\partial\phi^2} - E_\phi\right)\hat{\phi} - 2\,\frac{\partial E_\phi}{\partial\phi}\,\hat{\rho}\,. \qquad .....(6)$$

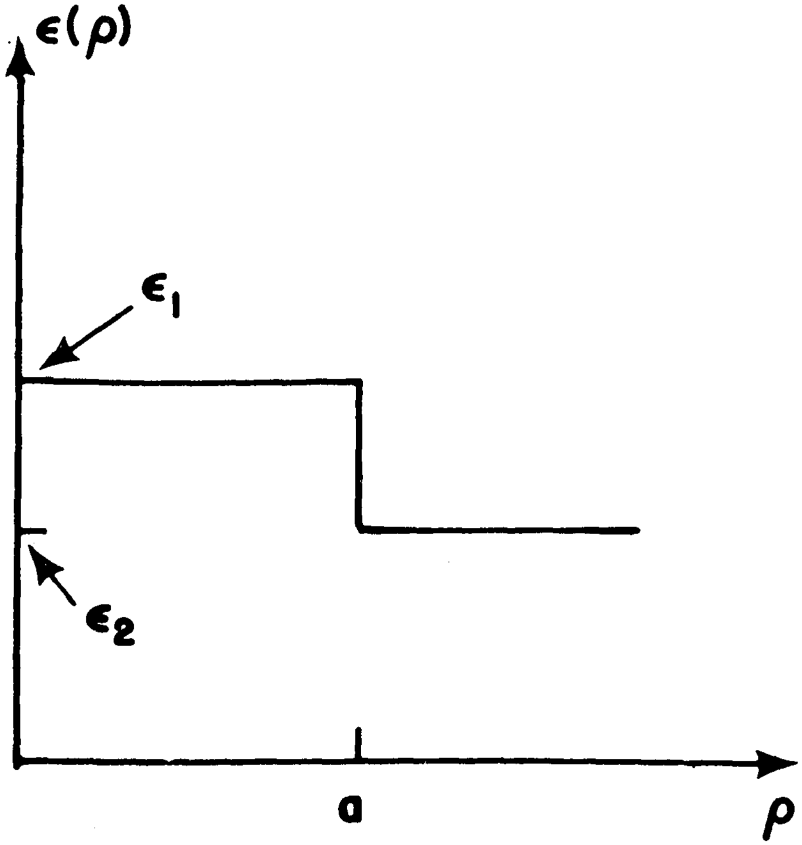

Fig. 1. The spatial variation of the dielectric constant $\varepsilon(\rho)$ for an optical fibre whose axis is in the z direction. ε_1 and ε_2 are the dielectric constants of the core and cladding regions, respectively. The diameter of the core is 2a.

With the use of equation (5) and (6), equation (1) can be written as

$$\frac{\partial^2 E_\rho}{\partial^2 \rho} + \frac{1}{\rho}\frac{\partial E_\rho}{\partial \rho} + \frac{1}{\rho^2}\left[\frac{\partial^2 E_\rho}{\partial \phi^2} - 2\frac{\partial E_\phi}{\partial \phi} - E_\rho\right] + \left(\frac{\varepsilon\omega^2}{c^2} - k_z^2\right) E_\rho = 0, \qquad \text{.....(7)}$$

$$\frac{\partial^2 E_\phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_\phi}{\partial \rho} + \frac{1}{\rho^2}\left[\frac{\partial^2 E_\phi}{\partial \phi^2} + 2\frac{\partial E_\rho}{\partial \phi} - E_\phi\right] + \left(\frac{\varepsilon\omega^2}{c^2} - k_z^2\right) E_\phi = 0, \qquad \text{.....(8)}$$

and

$$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \rho} + \frac{1}{\rho^2}\left[\frac{\partial^2 E_z}{\partial \phi^2}\right] + \left(\frac{\varepsilon\omega^2}{c^2} - k_z^2\right) E_z = 0, \qquad \text{.....(9)}$$

for the components along the ρ, ϕ and z axes, respectively. Equations (7) and (8) both contain E_ρ and E_ϕ and therefore are not easily solvable. On the other hand equation (9) contains only E_z and is readily solved. We assume a solution of the form

$$E_z = E_z(\rho)\ E_z(\phi)\ \exp(ik_z z - i\omega t). \qquad \text{.....(10)}$$

Equation (9) becomes

$$\frac{\rho^2}{E_z(\rho)}\left[\frac{\partial^2 E_z(\rho)}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_z(\rho)}{\partial \rho} + \left(\frac{\varepsilon\omega^2}{c^2} - k_z^2\right) E_z(\rho)\right] = -\frac{1}{E_z(\rho)}\left(\frac{\partial^2 E_z(\rho)}{\partial \phi^2}\right). \qquad \text{.....(11)}$$

In order to separate variables we let $+\lambda$ be the constant of separation, so that

$$+\lambda E_z(\phi) = -\frac{\partial^2 E_z(\phi)}{\partial \phi^2}. \qquad \text{.....(12)}$$

Because our space is symmetric in ϕ, the ϕ dependence should be symmetric in the interval $0 \leq \phi \leq 2\pi$. Thus the solution of equation (12) will have the form

$$E_z(\phi) = \cos(n\phi + \psi_n), \qquad \text{.....(13)}$$

where $\lambda = n^2$, n = 0,1,2,.... and ψ_n is the phase constant. We define propagation constants γ and β by

$$\gamma^2 = \frac{\varepsilon_1\omega^2}{c^2} - k_z^2, \qquad \text{.....(14)}$$

$$= k_z^2 - \frac{\varepsilon_2\omega^2}{c^2}. \qquad \text{.....(15)}$$

For γ and β real and positive we get propagation of energy inside the waveguide and no transverse flow of energy outside.

The differential equations for the ρ dependence of E_z become

$$\frac{\partial^2}{\partial\rho^2} E_z(\rho) + \frac{1}{\rho}\frac{\partial E_z(\rho)}{\partial\rho} + \left(\gamma^2 - \frac{n^2}{\rho^2}\right) E_z(\rho) = 0 \quad \text{for } \rho \leq a, \qquad \text{.....(16)}$$

and

$$\frac{\partial^2}{\partial\rho^2} E_z(\rho) + \frac{1}{\rho}\frac{\partial E_z(\rho)}{\partial\rho} - \left(\beta^2 + \frac{n^2}{\rho^2}\right) E_z(\rho) = 0 \quad \text{for } \rho > a. \qquad \text{.....(17)}$$

Equation (16) is a Bessel equation of order n and equation (17) is a modified Bessel equation of order n. The solutions of these equations are restricted by the requirement that they be finite at $\rho = 0$ and $\rho = \infty$ and that the field is localized to the vicinity of the waveguide core. The solutions have the form

$$E_z = \cos(n\phi + \psi_n)\exp(ik_z z - i\omega t)\begin{cases} A_n J_n(\gamma\rho) & \text{for } \rho \leq a \\ B_n K_n(\beta\rho) & \text{for } \rho > a, \end{cases} \qquad \text{.....(18)}$$

where A_n and B_n are constants, J_n is a Bessel function and K_n is a modified Bessel function of the second kind.

From a similar derivation for the magnetic field components it can be shown that B_z has the form

$$B_z = \cos(n\phi + \psi_n')\exp(ik_z z - i\omega t)\begin{cases} C_n J_n(\gamma\rho) & \text{for } \rho \leq a \\ D_n K_n(\beta\rho) & \text{for } \rho > a, \end{cases} \qquad \text{.....(19)}$$

where C_n and D_n are constants.

The components of the fields perpendicular to E_z and B_z can be read from the following equations[7]:

$$\left.\begin{aligned} (E_\rho, E_\phi, 0) &= \left(\frac{\varepsilon\omega^2}{c^2} - k_z^2\right)^{-1}\left[\nabla_t\left(\frac{\partial E_z}{\partial z}\right) - i\omega(\hat{z}\times\nabla_t B_z)\right] \\ (B_\rho, B_\phi, 0) &= \left(\frac{\varepsilon\omega^2}{c^2} - k_z^2\right)^{-1}\left[\nabla_t\left(\frac{\partial B_z}{\partial z}\right) + \frac{i\varepsilon\omega^2}{c^2}(\hat{z}\times\nabla_t E_z)\right], \end{aligned}\right\} \qquad \text{.....(20)}$$

where

$$\left.\begin{aligned}\nabla_t &\equiv \nabla - \hat{z}\frac{\partial}{\partial z}\\ \nabla &\equiv \left(\frac{\partial}{\partial \rho}, \frac{1}{\rho}\frac{\partial}{\partial \rho}, \frac{\partial}{\partial z}\right) .\end{aligned}\right\} \quad \ldots\ldots(21)$$

Thus the equations for the components of the field become[8]

$$\left.\begin{aligned}
E_\rho &= \left(\frac{\varepsilon\omega^2}{c^2} - k_z^2\right)^{-1}\left(ik_z\frac{\partial E_z}{\partial\rho} + \frac{i\omega}{\rho}\frac{\partial B_z}{\partial\phi}\right),\\
E_\phi &= \left(\frac{\varepsilon\omega^2}{c^2} - k_z^2\right)^{-1}\left(\frac{ik_z}{\rho}\frac{\partial E_z}{\partial\phi} - i\omega\frac{\partial B_z}{\partial\phi}\right),\\
B_\rho &= \left(\frac{\varepsilon\omega^2}{c^2} - k_z^2\right)^{-1}\left(ik_z\frac{\partial B_z}{\partial\rho} - i\frac{\varepsilon\omega}{c^2}\frac{1}{\rho}\frac{\partial E_z}{\partial\phi}\right),\\
B_\phi &= \left(\frac{\varepsilon\omega^2}{c^2} - k_z^2\right)^{-1}\left(ik_z\frac{1}{\rho}\frac{\partial B_z}{\partial\phi} + i\frac{\varepsilon\omega}{c^2}\frac{\partial E_z}{\partial\rho}\right).
\end{aligned}\right\} \quad \ldots\ldots(22)$$

The solutions for the field components are given by

$$\left.\begin{aligned}
E_z &= A_n J_n(\gamma\rho)\cos(n\phi + \psi_n)\exp(ik_z z - i\omega t),\\
E_\rho &= \left[A_n\frac{ik_z}{\gamma}J_n'(\gamma\rho)\cos(n\phi + \psi_n)\right.\\
&\qquad \left. - B_n\frac{ni\omega}{\gamma^2\rho}J_n(\gamma\rho)\sin(n\phi + \psi_n')\right]\exp(ik_z z - i\omega t),\\
E_\phi &= \left[-A_n\frac{ik_z n}{\gamma^2\rho}J_n(\gamma\rho)\sin(n\phi + \psi_n)\right.\\
&\qquad \left. - B_n\frac{i\omega}{\gamma}J_n'(\gamma\rho)\cos(n\phi + \psi_n')\right]\exp(ik_z z - i\omega t),\\
B_z &= B_n J_n(\gamma\rho)\cos(n\phi + \psi_n')\exp(ik_z z - i\omega t),\\
B_\rho &= \left[B_n\frac{ik_z}{\gamma}J_n'(\gamma\rho)\cos(n\phi + \psi_n')\right.\\
&\qquad \left. + A_n\frac{i\varepsilon_1\omega n}{c^2\gamma^3\rho}J_n(\gamma\rho)\sin(n\phi + \psi_n)\right]\exp(ik_z z - i\omega t),
\end{aligned}\right\} \quad \ldots\ldots(23)$$

$$B_\phi = \left[-B_n \frac{ik_z n}{\rho\gamma^2} J_n(\gamma\rho)\sin(n\phi + \psi_n') + A_n \frac{i\varepsilon_1\omega}{c^2\gamma} J_n'(\gamma\rho)\cos(n\phi + \psi_n)\right] \exp(ik_z z - i\omega t), \quad \ldots\ldots(23)$$

for $0 \leq \rho \leq a$ and

$$
\left.
\begin{aligned}
E_z &= C_n K_n(\beta\rho)\cos(n\phi + \psi_n)\exp(ik_z z - i\omega t),\\
E_\rho &= \left[-C_n \frac{ik_z}{\beta} K_n'(\beta\rho)\cos(n\phi + \psi_n) + D_n \frac{ni\omega}{\beta^2\rho} K_n(\beta\rho)\sin(n\phi + \psi_n')\right] \exp(ik_z z - i\omega t),\\
E_\phi &= \left[C_n \frac{ik_z n}{\beta^2\rho} K_n(\beta\rho)\sin(n\phi + \psi_n) + D_n \frac{i\omega}{\beta} K_n'(\beta\rho)\cos(n\phi + \psi_n')\right] \exp(ik_z z - i\omega t),\\
B_z &= D_n K_n(\beta\rho)\cos(n\phi + \psi_n')\exp(ik_z z - i\omega t),\\
B_\rho &= \left[D_n \frac{ik_z}{\beta} K_n'(\beta\rho)\cos(n\phi + \psi_n') - D_n \frac{i\varepsilon_2\omega n}{c^2\beta^2\rho} K_n(\beta\rho)\sin(n\phi + \psi_n)\right] \exp(ik_z z - i\omega t),
\end{aligned}
\right\} \ldots\ldots(24)
$$

$$B_\phi = \left[D_n \frac{ik_z n}{c^2\beta} K_n'(\beta\rho)\cos(n\phi + \psi_n)\right] \exp(ik_z z - i\omega t),$$

for $\rho > a$. We have written

$$
\left.
\begin{aligned}
J_n'(\gamma\rho) &= \frac{\partial}{\partial(\gamma\rho)} J_n(\gamma\rho)\\
K_n'(\beta\rho) &= \frac{\partial}{\partial(\beta\rho)} K_n(\beta\rho).
\end{aligned}
\right\} \quad \ldots\ldots(25)
$$

The boundary conditions at the core-cladding interface ($\rho = a$) require the continuity of the normal components of $\underline{B}$ and $\underline{D}$ and the tangential components of $\underline{E}$ and $\underline{H}$, where

$$\underline{B} = \mu_o \underline{H} \ , \qquad \text{.....(26)}$$

throughout space and

$$\left.\begin{aligned} \underline{D} &= \varepsilon \underline{H} \ , \\ \varepsilon &= \varepsilon_1 \text{ for } \rho \le a \\ &= \varepsilon_2 \text{ for } \rho > a. \end{aligned}\right\} \qquad \text{.....(27)}$$

From the continuity of the tangential components of $\underline{E}$ at the boundary and making use of equation (20) we obtain

$$\left.\begin{aligned} A_n J_n(\gamma a) &= C_n K_n(\beta a), \\ -A_n \frac{ik_z n}{\gamma^2 a} J_n(\gamma a) - B_n \frac{\omega}{\gamma} J_n'(\gamma a) &= C_n \frac{k_z n}{\beta^2 a} K_n(\beta a) + D_n \frac{\omega}{\beta} K_n'(\beta a), \end{aligned}\right\} \text{.....(28)}$$

and from the continuity of the tangential components of $\underline{B}$ we obtain

$$\left.\begin{aligned} B_n J_n(\gamma a) &= D_n K_n(\beta a), \\ -B_n \frac{ik_z n}{a\gamma^2} J_n(\beta a) + A_n \frac{i\varepsilon_1 \omega}{c^2 \gamma} J_n'(\gamma a) &= D_n \frac{k_z n}{a\beta^2} K_n(\beta a) - C_n \frac{\varepsilon_2 \omega}{c^2 \beta} K_n'(\beta a) \end{aligned}\right\}. \text{.....(29)}$$

3. UNPERTURBED SOLUTIONS FOR THE TE_{0m} AND TM_{0m} MODES

In an optical fibre of cylindrical geometry the low-order modes are the hybrid HE_{11} and HE_{21} modes and the TE_{01} and TM_{01} modes[8]. The electric and magnetic field components of the TE_{0m} and TM_{0m} modes have a relatively simple spatial distribution, since they have no ϕ dependence. In this case, equations (7) and (8) for E_ρ and E_ϕ are uncoupled and the equations become

$$\left.\begin{aligned} \frac{\partial^2 E_\rho}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_\rho}{\partial \rho} + \left(\frac{\varepsilon\omega^2}{c^2} - \frac{1}{\rho^2} - k_z^2\right) E_\rho &= 0, \\ \frac{\partial^2 E_\phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_\phi}{\partial \rho} + \left(\frac{\varepsilon\omega^2}{c^2} - \frac{1}{\rho^2} - k_z^2\right) E_\phi &= 0, \\ \frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \rho} + \left(\frac{\varepsilon\omega^2}{c^2} - k_z\right) E_z &= 0. \end{aligned}\right\} \qquad \text{.....(30)}$$

The components of $\underline{B}$ also satisfy equations of the form of equation (36). These are Bessel's equations or modified Bessel's equations for $(\varepsilon\omega^2/c^2 - k_z^2)$ positive or negative, respectively.

By the use of equations (23) and (24) and substituting for n, ψ_n' and ψ_n we can write down the solutions for the TE_{0m} and TM_{0m} modes. From the boundary conditions at p = a it can be shown[8] that

$$\left|\psi_n' - \psi_n\right| = \frac{\pi}{2} .$$

The solution for the TE_{0m} mode is obtained by taking n = 0, $\psi_n = \pi/2$ and $\psi_n' = 0$ make $E_z = 0$. Therefore for the TE_{0m} mode

$$\left.\begin{aligned}
E_\phi &= B_0 \frac{i\omega}{\gamma} J_1(\gamma\rho)\exp(ik_z z - i\omega t) \quad \text{for } \rho \le a,\\
&= -D_0 \frac{i\omega}{\beta} K_1(\beta\rho)\exp(ik_z z - i\omega t) \quad \text{for } \rho > a,\\
E_\rho &= E_z = 0\\
B_z &= B_0 J_0(\gamma\rho)\exp(ik_z z - i\omega t) \quad \text{for } \rho \le a,\\
&= D_0 K_0(\beta\rho)\exp(ik_z z - i\omega t) \quad \text{for } \rho > a,\\
B_\rho &= -B_0 \frac{ik_z}{\gamma} J_1(\gamma\rho)\exp(ik_z z - i\omega t) \quad \text{for } \rho \le a,\\
&= D_0 \frac{ik_z}{\beta} K_1(\beta\rho)\exp(ik_z z - i\omega t) \quad \text{for } \rho > a,\\
&\text{and}\\
B_\phi &= 0.
\end{aligned}\right\} \qquad \text{.....(31)}$$

The boundary conditions for the TE_{0m} mode are obtained from equations (28) and (29) to be

$$\left.\begin{aligned}
\frac{B_0}{\gamma} J_1(\gamma a) &= -\frac{D_0}{\beta} K_1(\beta a)\\
B_0 J_0(\gamma a) &= D_0 K_0(\beta a),
\end{aligned}\right\} \qquad \text{.....(32)}$$

where we have made use of the relationships

$$\left.\begin{aligned}
J_0'(\gamma\rho) &= -J_1(\gamma\rho)\\
K_0'(\beta\rho) &= -K_1(\beta\rho).
\end{aligned}\right\} \qquad \text{.....(33)}$$

By eliminating B_0 and D_0 from equation (32) we obtain the following transcendental equations for γ and β:

$$\left.\begin{aligned} \gamma^2 + \beta^2 &= (\varepsilon_1 - \varepsilon_2)\,\frac{\omega^2}{2} \\ \frac{J_1(\gamma a)}{\gamma J_0(\gamma a)} &+ \frac{K_1(\beta a)}{\beta K_0(\beta a)} = 0. \end{aligned}\right\} \qquad \ldots\ldots(34)$$

The solution of these transcendental equations leads to a discrete set of eigenvalues γ_m and β_m. The corresponding eigenfunctions are obtained substituting these values into equation (31). Now the TE_{01} is not the lowest-order mode and has a cut-off frequency. We examine the behaviour of the two terms $J_1(\gamma a)/\gamma J_0(\gamma a)$ and $K_1(\beta a)/\beta K_0(\beta a)$ of equation (34) with γa and βa, respectively. The maximum value of γa for a real value of β is $(\varepsilon_1 - \varepsilon_2)^{\frac{1}{2}}\,\omega a/c$, so that $K_1(\beta a)/\beta K_0(\beta a)$ is real only for $\gamma a < (\varepsilon_1 - \varepsilon_2)^{\frac{1}{2}}\,\omega a/c$. Also $-J_1(\gamma a)/\gamma J_0(\gamma a)$ is positive only for $\gamma a > 2.405$, which is the first zero of $J_0(\gamma a)$. Thus for an eigenvalue to exist, we must have

$$\omega > \frac{2.405c}{(\varepsilon_1 - \varepsilon_2)^{\frac{1}{2}} a}\,. \qquad \ldots\ldots(35)$$

A similar derivation can be carried out for the TM_{0m} mode by taking $n = 0$, $\psi_n = 0$, and $\psi_n' = \pi/2$ to make $B_z = 0$. Therefore for the TM_{0m} mode

$$\left.\begin{aligned} E_\rho &= -A_0\,\frac{ik_z}{\gamma}\,J_1(\gamma\rho)\,\exp(ik_z z - i\omega t) \text{ for } \rho \le a, \\ &= C_0\,\frac{ik_z}{\beta}\,K_1(\beta\rho)\exp(ik_z z - i\omega t) \text{ for } \rho > a, \\ E_z &= A_0 J_0(\gamma\rho)\exp(ik_z z - i\omega t) \text{ for } \rho \le a, \\ &= C_0 K_0(\beta\rho)\exp(ik_z z - i\omega t) \text{ for } \rho > a, \\ E_\phi &= 0 \\ B_\phi &= -A_0\,\frac{i\varepsilon_1\omega}{c^2\gamma}\,J_1(\gamma\rho)\exp(ik_z z - i\omega t) \text{ for } \rho \le a, \\ &= C_0\,\frac{i\varepsilon_2\omega}{c^2\beta}\,K_1(\beta\rho)\exp(ik_z z - i\omega t) \text{ for } \rho > a \\ B_\rho &= B_z = 0. \end{aligned}\right\} \qquad \ldots\ldots(36)$$

The boundary conditions lead to the following equations:

$$\left.\begin{aligned} A_0 J_0(\gamma a) &= C_0 K_0(\beta a) \\ -A_0 \frac{\varepsilon_1}{\gamma} J_1(\gamma a) &= C_0 \frac{\varepsilon_2}{\beta} K_1(\beta a). \end{aligned}\right\} \quad \text{.....(37)}$$

The transcendental equations are

$$\left.\begin{aligned} \gamma^2 + \beta^2 &= (\varepsilon_1 - \varepsilon_2) \frac{\omega^2}{c^2} \\ \frac{J_1(\gamma a)}{\gamma J_0(\gamma a)} + \frac{\varepsilon_2}{\varepsilon_1} \frac{K_1(\beta a)}{\beta K_0(\beta a)} &= 0 \end{aligned}\right\} \quad \text{.....(38)}$$

4. EVANESCENT-WAVE AMPLIFICATION

In order to account for the effects of a surrounding medium with gain on the bound modes of the optical fibre we follow the method that we developed previously for treating evanescent-wave gain in a dielectric slab[5]. We add a small imaginary component to the dielectric constant of the cladding region, so that

$$\varepsilon_2(\rho) = \varepsilon_2'(\rho) + i\, \varepsilon_2''(\rho). \quad \text{.....(39)}$$

The ρ variation of the real part of the dielectric constant is identical to that used for the passive waveguide case.

To solve for the effects of this small imaginary component on the modes of the fibre we use first-order perturbation theory. Perturbation theory[9] requires a complete set of orthogonal functions. The solutions of any wave equation form such a set of orthogonal functions[11]. A comparison of equation (9) with the Schrodinger wave equation shows that k_z^2 is the eigenvalue and $\varepsilon\omega^2/c^2$ plays the role of the potential. Thus the unperturbed solutions for the z components of fields as given in equations (18) and (19) are good eigenfunctions for use in calculating the values of the perturbed eigenvalues. The perturbed values of the eigenvalues of the n^{th} mode can be found[5] from

$$k'^2_{z;n} - k^2_{z;n} = \delta_n, \quad \text{.....(40)}$$

where

$$\delta_n = \langle\psi_n^*|\, i\varepsilon_2''(\rho)\, \frac{\omega^2}{c}\, |\psi_n\rangle, \quad \text{.....(41)}$$

where ψ_n is one of the unperturbed field components for the n^{th} mode. The z dependence of the amplitude of these components is given by

$$\exp[-i\, k'_{z;n} z] = \exp\left[-i\, k_{z;n} \left(1 + \frac{\delta_n}{2k_{z;n}}\right) z\right]. \quad \text{.....(42)}$$

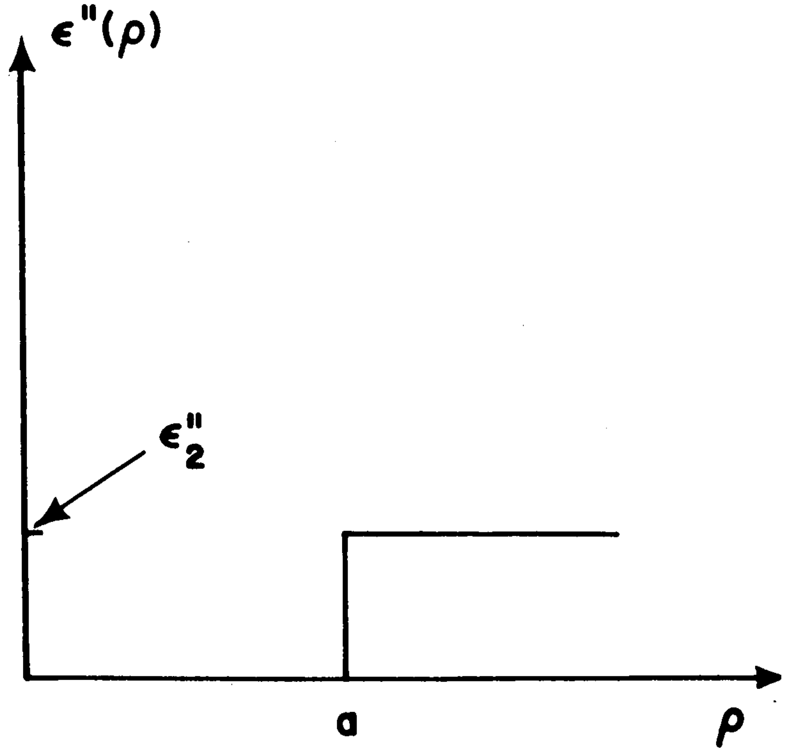

Fig. 2. The ρ *variation of the imaginary part of the dielectric constant of the active fibre with constant gain in the cladding region.*

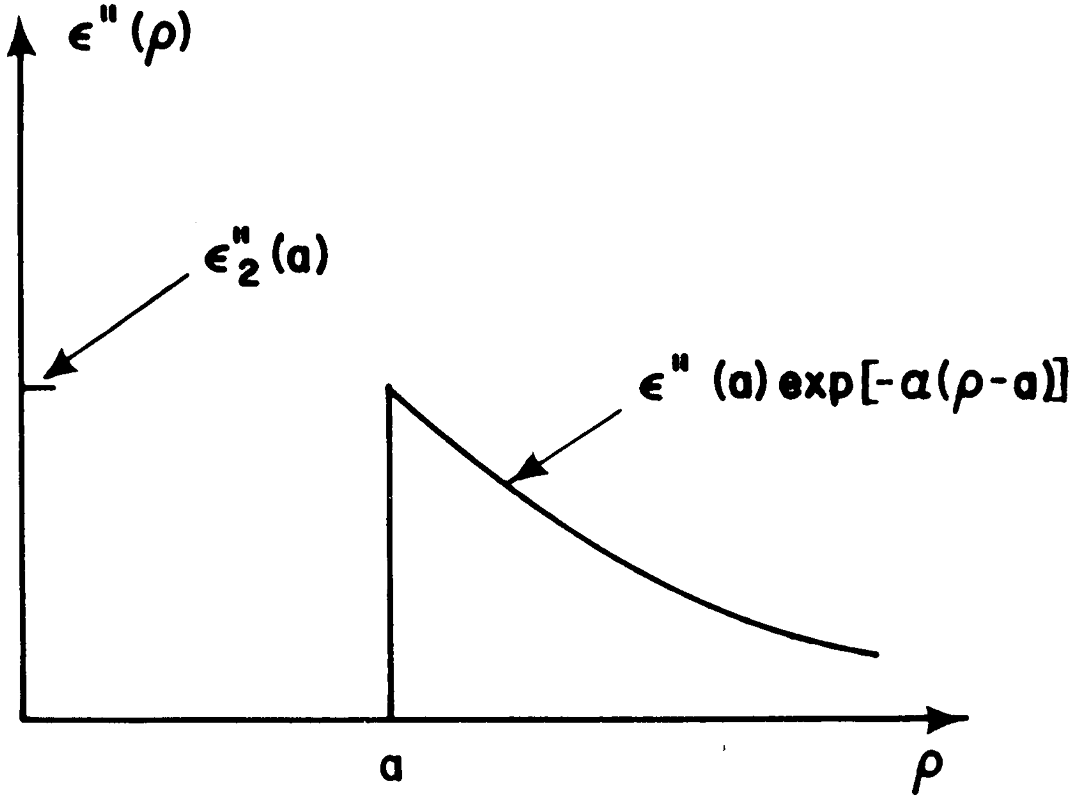

Fig. 3. The ρ *variation of the imaginary part of the dielectric constant of the evanescent-wave pumped active optical fibre.*

Thus as a result of the interaction of the evanescent waves with the medium, the n^{th} mode gains intensity as it propagates down the fibre in direct proportion to the quantity

$$\exp\left[-i\frac{\delta_n}{k_{z;n}}z\right]. \qquad .....(43)$$

The gain coefficient of the n^{th} mode is therefore

$$\alpha_n = \left[-i\frac{\delta_n}{k_{z;n}}\right]. \qquad .....(44)$$

We say that amplification takes place via evanescent-wave interaction, because the entire contribution to δ_n comes from the interaction of the evanescent field with the active medium surrounding the core of the fibre.

5. AMPLIFICATION IN THE TE_{0m} AND TM_{0m} MODES

In order to illustrate the effect of evanescent-wave amplification in optical fibres we treat in detail amplification in the TE_{0m} modes. In Section 3 we derived expressions for the eigenvalues and eigenfunctions for the unperturbed TE_{0m} modes. We now examine the perturbation caused by a cladding region with gain, following the method outlined in Section 4. The ρ variation of the real part of the dielectric constant is identical to that used in the passive case and is as shown in Figure 1. The ρ variation of the imaginary part of the dielectric constant is shown in Figure 2; it has a value of zero within the core and a constant value in the constant. Thus we are assuming a cladding with uniform gain.

For the case of evanescent-wave pumping, i.e., when the pump beam propagates in one of the bound modes of the optical fibre, the imaginary part of the dielectric constant will decrease exponentially with distance into the cladding, as shown in Figure 3. Also there will be a ϕ dependence in the field components if the pump mode is propagating in a mode other than the TE_{0m} or TM_{0m} modes. The effects arising in this case will be qualitatively similar to the case for uniform gain in the cladding, but will be somewhat more complex. We leave considerations of evanescent-wave pumping to a future work and limit our present considerations to the case of uniform gain in the cladding.

We make the assumption that the perturbation is small and that only first-order perturbations need to be considered. In almost all practical cases the gain is not large enough to make a significant perturbation in distances of the order of a wavelength[5]. Thus first-order perturbation theory is adequate for the treatment.

Since mode conversion will occur, the perturbed eigenfunctions will be given by a linear combination of the unperturbed eigenfunctions. Thus the gain in one of the unperturbed modes is somewhat lower than the total gain experienced by a wave propagating in that mode. We treat these mode-conversion processes as losses, as far as that particular mode is concerned, and calculate the net gain in one of the unperturbed TE_{0m} modes.

From an examination of equation (31) it can be seen that it is necessary to treat only one of the field components. The gain values for the other components follow immediately from equation (31). We carry through our treatment for E_ϕ, which has only a ρ dependence for the TE_{0m} modes. We derive an expression for E_ϕ in the normalized modal form[6] by assuming that

$$\int_o^\infty E_\phi^* E_\phi \, \rho d\rho = 1$$

$$= \int_o^a \left(-\frac{i\omega}{\gamma}\right) B_0^* J_1^*(\gamma\rho) \exp[-ik_z z + i\omega t] \left(\frac{i\omega}{\gamma}\right) B_0 J_1(\gamma\rho) \exp[ik_z z - i\omega t] \rho d\rho$$

$$+ \int_a^\infty \left(\frac{i\omega}{\beta}\right) D_0^* K_1^*(\beta\rho) \exp[ik_z z + i\omega t] \left(-\frac{i\omega}{\beta}\right) D_0 K_1(\beta\rho) \exp[ik_z z - i\omega t] \rho d\rho. \quad \text{.....(45)}$$

Making use of equation (32) to relate B_0 and D_0 and the values of γ and β from the solution of equation (38) we obtain

$$D_0^2 = \frac{2}{a^2\omega^2 \left(\frac{1}{\gamma^2} + \frac{1}{\beta^2}\right) \left[K_0^2(\beta a) + \frac{2K_0(\beta a) K_1(\beta a)}{\beta a}\right]}, \quad \text{.....(46)}$$

From equations (41) and (31)

$$\delta_m = \int_a^\infty \frac{\omega^2}{\beta_m^2} D_0^2 K_1^2(\beta_m \rho) \; i \frac{\varepsilon'' \omega^2}{c^2} \rho d\rho$$

$$= \frac{i\varepsilon'' \omega^2}{c^2} \frac{\omega^2}{\beta_m^2} D_0^2 \left\{ K_0^2(\beta_m a) + \frac{2K_1(\beta_m a) K_0(\beta_m a)}{\beta_m a} - K_1^2(\beta_m a) \right\} \frac{a^2}{2}. \quad \text{.....(47)}$$

Substitute for D_0^2 from equation (46) and for γ_m^2 from equation (38). Therefore

$$\delta_m = \frac{i\varepsilon'' \omega^2}{c^2} (1 - E_m)(1 - G_m), \quad \text{.....(48)}$$

where we have simplified the expression by writing

$$\left.\begin{aligned} E_m &= \frac{\beta_m^2}{(\varepsilon_1 - \varepsilon_2) \frac{\omega^2}{c^2}} \\ G_m &= \frac{\dfrac{K_1^2(\beta_m a)}{K_0^2(\beta_m a)}}{1 + \dfrac{2}{\beta_m a} \dfrac{K_1(\beta_m a)}{K_0(\beta_m a)}}. \end{aligned}\right\} \quad \text{.....(49)}$$

The value for the gain is found directly from equation (44), so that

$$\alpha_m = \frac{\frac{\varepsilon''\omega^2}{c^2}}{k_{z;m}} (1 - E_m)(1 - G_m).$$

On the other hand the gain coefficient for a wave propagating in bulk-material identical to the cladding material is given by

$$\alpha' = \frac{k_2 \varepsilon_2''}{\varepsilon_2} , \qquad \text{.....(50)}$$

where

$$k_2 = \frac{\omega}{c} \varepsilon_2^{\frac{1}{2}} .$$

Thus the gain-coefficient ratio (G.C.R.), which is the ratio of the gain coefficient in the cladding region of the optical fibre to the gain coefficient in the same material in bulk form, is given by

$$(\text{G.C.R.})_m = \frac{\alpha_m}{\alpha'} = \frac{k_2}{k_{z;m}} (1 - E_m)(1 - G_m) , \qquad \text{.....(51)}$$

for the TE_{0m} mode.

From a comparison of the equations (31) and (36) it can be seen that the gain-coefficient ratio is given by equation (51) for the TM_{0m} mode, as well.

6. NUMERICAL RESULTS

In order to see more clearly the behaviour of the gain-coefficient ratio as defined by equation (51), we calculate the gain-coefficient ratio as a function of mode number for some multimode fibres. These results are as shown in Figure 4 for three cases. The fibres have a core radius of 15 μm and a refractive index of 1.5. The refractive index of the cladding is lower than that of the core by Δn = 0.003, 0.01 and 0.03, to give fibres supporting 3, 5 and 9 modes, respectively, at a wavelength of 1 μm. We note that the higher order modes tend to have larger values of the gain-coefficient ratio.

From an examination of the form of the transcendental equations (38), it can be seen that the higher order modes have larger values of γa than do the lower-order modes. The maximum value of γa is given when $\beta a = 0$ by

$$(\gamma a)_{max} = (\varepsilon_1 - \varepsilon_2)^{\frac{1}{2}} \frac{\omega a}{c} . \qquad \text{.....(52)}$$

From equation (49), we see that $E_m = 0$ for $\beta a = 0$ and

$$\lim_{\beta a \to 0} G_m = 0 . \qquad \text{.....(53)}$$

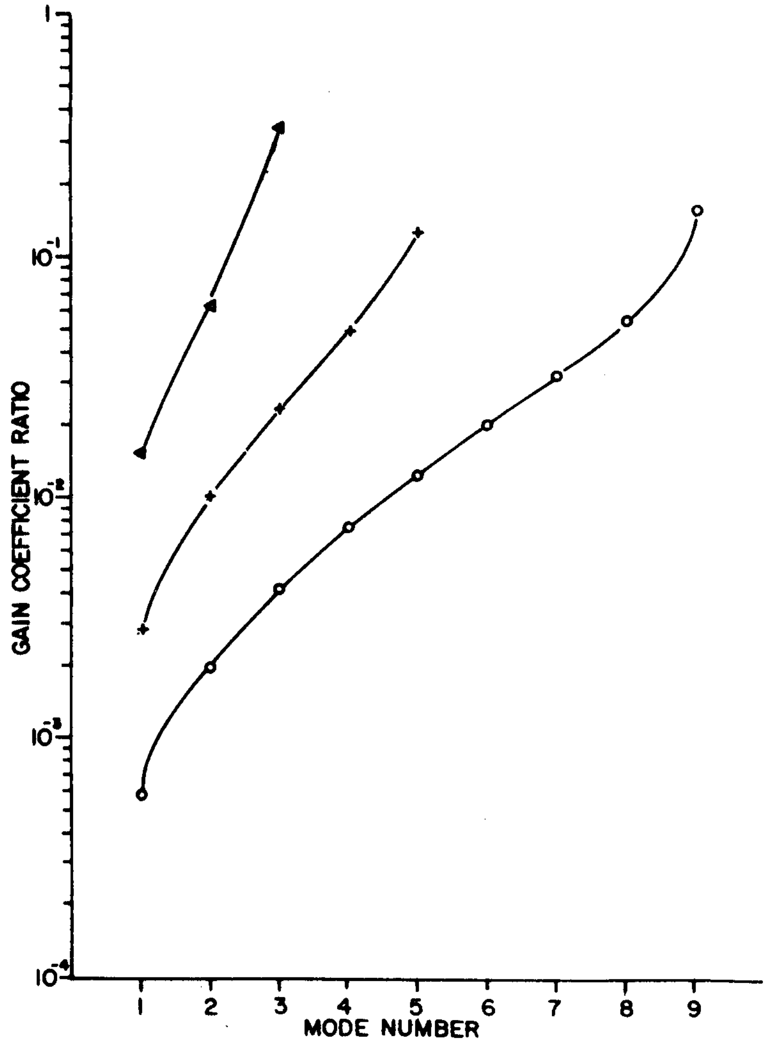

Fig. 4. The gain-coefficient ratio plotted against mode number m for the TE_{0m} modes of three active optical fibres at a wavelength of 1 μm. The fibres have a core radius of 15 μm and a core refractive index of 1.5. The refractive index of the cladding is lower than that of the core by Δn = 0.003, 0.01, and 0.03 for the three examples shown in the figure by the symbols ×, + and 0, respectively.

Thus

$$(\text{G.C.R.})_{m,\beta a=0} = \frac{k_2}{k_{z;m}}\,, \qquad \ldots\ldots(54)$$

which has a limiting value of unity. We see that the limiting value given by equation (54) is independent of a and of $(\varepsilon_1 - \varepsilon_2)$. Thus for any slab mode which is limitingly close to cut-off, the G.C.R. has a value of unity, since the evanescent field of the mode penetrates uniformly into the entire medium surrounding the slab. Figure 5 shows the variation of the G.C.R. against βa for the case of a fibre with a 15 μm diameter and a core refractive index of 1.50. Curves are shown for three cases, with a refractive-index difference of 0.003, 0.01 and 0.03 indicated by a, b and c, respectively. The value of the refractive-index difference affects only the factor $\varepsilon_1 - \varepsilon_2$ in the denominator of the expression for E_m as given in equation (49). It can be seen from Figure 5 the G.C.R. is not very sensitive to the value of $\varepsilon_1 - \varepsilon_2$ for small values of βa, i.e., for modes near cut-off. At cut-off ($\beta a = 0$) the G.C.R. has a value of unity for any fibre, independent of the refractive-index difference or the fibre diameter. Thus a value of the G.C.R. close to unity can be obtained for any active fibre by fine tuning the value of $\varepsilon_1 - \varepsilon_2$ to bring the highest-order propagating mode close to cut-off.

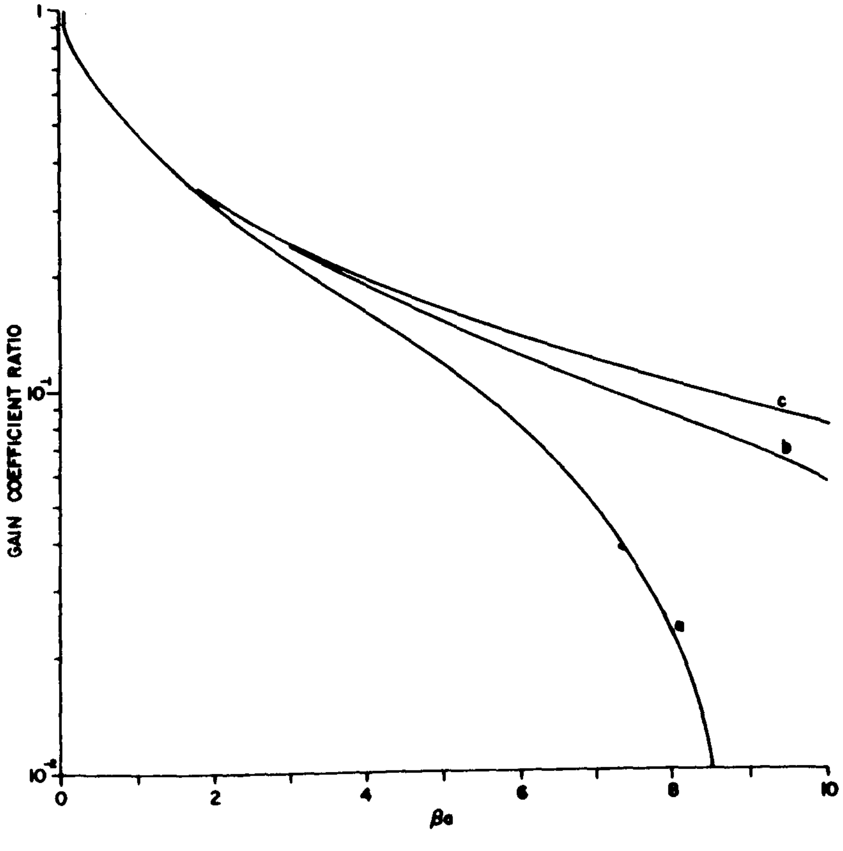

Fig. 5. The gain-coefficient ratio plotted against βa for the TE_{0m} modes of a fibre with a core radius of 15 μm and core refractive index of 1.5. The refractive index of the cladding is lower than that of the core by 0.003, 0.01 and 0.03 for the three curves marked a, b and c, respectively.

7. SUMMARY

We have used first-order perturbation theory to treat evanescent-wave gain due to an active cladding material on an optical fibre. Detailed calculations of the gain in the TE_{0m} and TM_{0m} modes of the fibre have been carried out and some numerical results have been presented for the gain-coefficient ratio. These results show that for modes close to cut-off the evanescent-wave gain is nearly equal to the gain attainable in the bulk material.

For the cases of the TE_{nm} and TM_{nm} modes for $m > 0$ and for the hybrid modes HE_{nm} and EH_{nm} there is a ϕ dependence and normalization must be carried out over both ρ and ϕ variables. Also equations (7) and (8) for the ρ and ϕ independence of the field components are coupled. Nevertheless the method we have outlined here for treating evanescent-wave gain in the cladding region of an optical fibre are still applicable. These cases are more complex to treat, and do not fall within the scope of this present work.

8. REFERENCES

1. Hill, K.O., Watanabe, A. and J.G. Chambers. *Amplification of bound modes via evanescent-wave interaction.* J. Opt. Soc. Am. 61, 1579, (1971).

2. Hill, K.O. and A. Watanabe. *A distributed-feedback side-coupled laser.* Opt. Comm. 5, 389, (1972).

3. Hill, K.O. and A. Watanabe. *Experimental and theoretical considerations of distributed-feedback side-coupled laser.* CRC Report No. 1235, October 1972.

4. Hill, K.O. and A. Watanabe. *A passive-core corrugated-waveguide laser.* Appl. Opt. 12, 430, (1973).

5. Hill, K.O., Watanabe, A. and J.G. Chambers. *Evanescent-wave interactions in an optical wave-guiding structure.* Appl. Opt. 11, 1952, (1972).

6. Watanabe, A., Hill, K.O. and R.I. MacDonald. *Amplification of light in an optical wave-guiding structure with evanescent-wave pumping.* Can. J. Phys. in press.

7. Jackson, J.D. *Classical electrodynamics.* (J. Wiley and Sons, Inc., New York, 1962) Chapter 8. The formulae given in this reference have been translated into rationalized MKS units by means of the table given on page 619 of this reference.

8. Snitzer, E. *Cylindrical dielectric waveguide modes.* J. Opt. Soc. Am. 51, 491, (1961).

9. Landau, L.D. and E.M. Lifshitz. *Quantum mechanics.* (Addison-Wesley Publishing Co., Inc., Reading, 1958), Chapter VI.

10. Pauling, L. and E.B. Wilson. *Introduction to quantum mechanics.* (McGraw-Hill Book Co., Inc., New York, 1958) p. 154.

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