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NATURAL MODES AND REAL MODAL VARIABLES FOR FLEXIBLE SPACECRAFT

by

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CAUTION

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TABLE OF CONTENTS

NOMENCLATURE

A	Matrix, Eq. (9) or (39). Positive definite and symmetric. Order (m x m) and rank m.
B	Sum of symmetric and skew symmetric D and G, respectively. Order and rank m.
C	Damping matrix (n x n) associated with one appendage. Symmetric, positive definite.
D	Damping matrix (m x m). Eq. (6) or (39). Symmetric.
E_2	$[0,1,0,0]^T$, (m x 1).
G_K	Gain Matrix, (m x m), Eq. (41).
G	Matrix (m x m) associated with stiffness, Eq. (6) or (39). Skew-symmetric.
g_k	Modal gain, real scalar, Eq. (35b and c), associated with pitch modes.
g_{2k}	Modal gain, real scalar, associated with G_K and Figure 4. Equals $E_2^T G_K \Lambda$.
H_K	Gain Matrix (m x m), Eq. (41).
h_o	Angular momentum stored in the momentum wheel.
h_k	Modal gain, real scalar, Eq. (35d). Associated with pitch modes.
h_{2k}	Modal gain, real scalar, associated with H_k and Figure 4. Equals $E_2^T H_K \Lambda$.
I_1, I_2, I_3, I_{13}	Moments and products of inertia of the undeformed spacecraft about the center of mass.
K	Stiffness matrix (n x n) of one appendage. Symmetric, positive definite.
L_1, L_2, L_3	Torque components about the center of mass of the spacecraft. Magnitude of the torque component in the roll-yaw plane.
L_o	Magnitude of the torque component in the roll-yaw plane.
M	Mass matrix (n x n), $\int \Phi\Phi^T dm$, from one appendage. Symmetric, positive definite.

m	Order of the eigenvalue problem. $m = 2n$ or $2n + 2$, respectively, in the two examples.
R	Matrix (m x m) defined in Eq. (6). Positive definite symmetric.
P_K	Real part of Π_K, column matrix (m x 1).
iQ_K	Imaginary part of Π_k, column matrix (m x 1).
S_1	Coefficient column matrix (n x 1), $\int y\Phi\Phi^T dm$ for one appendage.
S_2	Coefficient column matrix (n x 1), $\int x\Phi\Phi^T dm$ for one appendage.
U_K	Real part of X_K, column matrix (m x 1).
iV_K	Imaginary part of X_K, column matrix (m x 1).
$W(t)$	Deflection function, column matrix (n x 1).
W_K	The part of X_K which corresponds to deflection.
$w(x,t)$	Scalar deflection depicted in Figure 1.
X_K	Eigencolumn (m x 1), solution of Eq. (9).
Y_k^T	Eigenrow (1 x m), solution of Eq. (13).
$Z(t)$	Variable, column matrix (m x 1). Eq. (4) or (38).
$z(t)$	Modal variable, scalars, complex in general.
α	Angle between the roll axis and the roll-yaw component of torque.
$i\beta_k$	Imaginary part of λ_k.
ζ_k	Modal damping factor.
Λ	$[\cos\alpha, \sin\alpha, 0, 0]^T$, (m x 1).
$\theta(t)$	Attitude angle about pitch relative to a fixed line.
$\mathring{\theta}_k$	Part of eigenvector, Eq. (22).
λ_k	Eigenvalue, complex scalar, Eq. (11).
σ_k	Real part (scalar) of λ_k, Eq. (11).
T	Column matrix (m x 1), Eq. (6).

$\Phi(x)$	Column matrix (n x 1) of coordinate functions (assumed modes) of deflection. A complete set satisfying natural boundary conditions of the appendage. Depicted in Figure (1a) and (1b).
Π_k	Column matrix (m x 1), defined in Eq. (20).
ω_k	Modal natural frequency.
$\omega_1, \omega_2, \omega_3$	Angular velocity about roll, pitch, and yaw axes.
*	Indicates complex conjugate.
–	Indicates Laplace transform.

NATURAL MODES AND REAL MODAL VARIABLES FOR FLEXIBLE SPACECRAFT

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ABSTRACT

This report develops and illustrates a natural modal transformation theory which is applicable to flexible spacecraft with damping and gyroscopic forces. The theory is arranged into a form which is a generalization of the classical normal modes transformation theory. Modal differential equations are given in terms of real-valued scalars. Block diagrams in the time and Laplace transform domains demonstrate the feed-forward and second-order filter characteristics of the structure of the equations. Results for a single-axis flexible dynamics example are compared with earlier published results to show the correlation with the classical normal modes transformation theory.

1. INTRODUCTION

Flexible spacecraft which can be modelled with linear differential equations with time-invariant coefficients can be transformed to natural modal variables that correspond to natural modes of deformation. Natural modes in this context are eigenvectors of the homogeneous part of the linear differential equations, and the natural modal variables are ones for which the modal differential equations are uncoupled.

If the time derivatives of the linear equations are of second order only, or if certain proportionalities are present amongst the coefficients, then the applicable transformation theory is the 'classical normal modes' transformation, where the differential equations in terms of the natural modal variables (also called principal or normal coordinates) are uncoupled second-order real-valued linear oscillators, and the modal eigenvectors are real-valued column matrices [1,2]. The normal modes formulation is useful and popular because of the following features: (a) the variables and parameters of the modal differential equations are real-valued scalars, and the mode shapes are real-valued and physically-comprehensible; (b) the theory and computational procedures are well established for the mode shapes; (c) the modal equations are uncoupled second order linear differential equations which are easily integrated or analysed; (d) if the equations are arranged in a signal flow transfer function diagram (in the time or Laplace transform domains) they are found to have a feed-forward structure composed of a number of parallel 'second order bandpass filters' and this permits application of approximate techniques such as mode separation, modal truncation, and frequency domain analysis; (e) the modal frequencies and modal mass coefficients (which are the parameters which define the structural flexibility) are often related more or less proportionally to quantities which specify control, stabilization and structural integrity. These features enable an analyst to obtain a physical feeling for the flexible body dynamics directly from the formulation. The formulation has been found to be particularly useful for synthesizing or analyzing controllers via output feed-back techniques, and for helping to decipher complicated simulation or flight data. It also seems to be suited for synthesizing controllers by optimal and suboptimal methods, and for developing parameter identification methods. These observations are illustrated in, for example, the applications published in [3-7].

For situations where the linear model is not transformable using the classical normal modes theory, the analyst can use a more general, but more complicated, transformation mathematics. Ref. 8 gives an appropriate transformation theory in terms of complex-valued matrices and variables, and Refs. 2, 9, and 10 give relevant partial formulations which use modal vectors and complex variables. However in these works many of the above-noted attractive features associated with the normal modes transformation theory are absent or not evident.

This report demonstrates, by way of examples, that the general natural modes theory can be derived and arranged in a form which retains the attributes of the classical normal modes theory. Two well-known representative examples are worked, namely the attitude dynamics about pitch of a vehicle with a flexible appendage and linear damping, and the roll-yaw dynamics of the same vehicle with roll and yaw coupled by gyroscopic stiffness from a momentum wheel. The transformation theory is developed in such a way as to parallel the normal modes development insofar as possible. The modal equations are given in real-valued scalar form. Block diagrams are illustrated in both the time domain and the Laplace transform domain. For the pitch axis example, the results are compared with Ref. 4 to show the correlation with the classical normal modes transformation theory.

2. ATTITUDE DYNAMICS ABOUT PITCH

The satellite, depicted in Figure 1, consists of a central rigid body, two similar symmetric flexible appendages which deform in bending, and a momentum wheel aligned along the pitch axis. Because of symmetry, the pitch axis dynamics are independent of the roll and yaw dynamics, and are not influenced by the stored momentum. The model is*

* — *Symbols are defined in the Nomenclature and the figures.*

$$I_2\ddot{\theta} - 2S_2^T\ddot{W} = L_2 \tag{1a}$$

$$M\ddot{W} + C\dot{W} + KW - S_2\ddot{\theta} = 0 \tag{1b}$$

The variable W(t) arises from a discretization of the deformation in the form

$$w(x,t) = \Phi^T(x)\ W(t) \tag{2}$$

The Φ (x) are shape functions depicted in Figure 1(a). M, C, and K are of order nxn, and are non-diagonal, symmetric, and positive definite. The variable V is defined as $V = \dot{W}$. One may then form the equation

$$K\dot{W} - KV = 0 \tag{3}$$

Introduce the variable

$$Z = \begin{bmatrix} V \\ W \end{bmatrix} \tag{4}$$

Z is of order mx1 where m = 2n.

Equations (1), may be rewritten in terms of the variable Z:

$$I_2\ddot{\theta} - 2T^T\dot{Z} = L_2 \tag{5a}$$

$$R\dot{Z} + (D + G)Z - T\ddot{\theta} = 0 \tag{5b}$$

where

$$R = \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix} \quad T = \begin{bmatrix} S_2 \\ 0 \end{bmatrix} \quad D = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 0 & K \\ -K & 0 \end{bmatrix} \tag{6}$$

2.1 MODAL EIGENVALUES, MODAL EIGENCOLUMNS, AND MODAL EIGENROWS

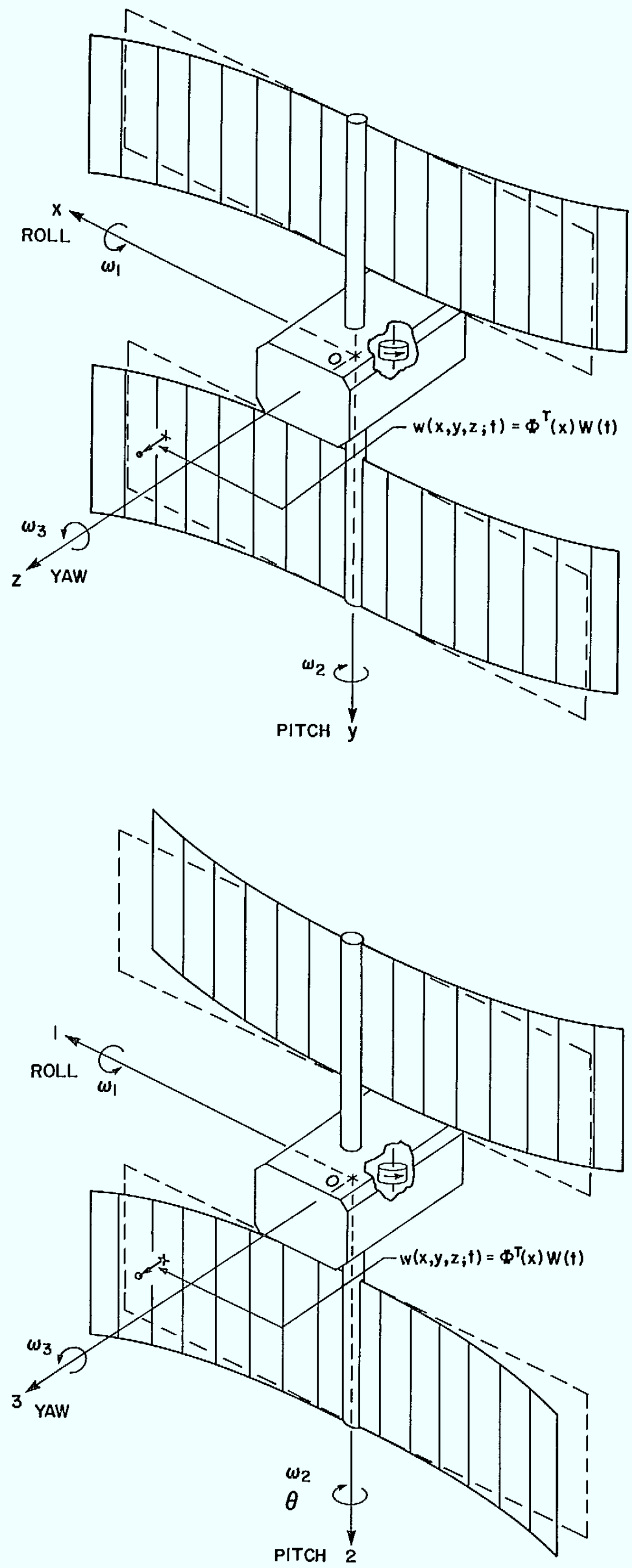
The coordinate functions for the modal transformation are the eigenvalues of the homogeneous part of Eq. 5. Substitution of

$$\dot{\theta} = \dot{\theta}_k e^{\lambda_k t} \text{ ,} \quad Z = X_k e^{\lambda_k t} \tag{7}$$

into the homogeneous part yields the eigenvalue problem in the form

$$I_2\dot{\theta}_k - 2T^T X_k = 0 \tag{8a}$$

$$(\lambda_k R + B)X_k - \lambda_k T\dot{\theta}_k = 0 \quad . \tag{8b}$$



(A)

(B)

Figure 1. Configuration of Spacecraft: (a) Deformations Coupled to Pitch; (b) Deformations Coupled to Roll and Yaw.

A solution of the eigenvalue problem directly from Eqs. (8) would entail solving for rigid body modes. To avoid them, $\dot{\theta}_k$ is next eliminated from Eqs. (8), to result in the eigenvalue problem in a more conventional form,

$$(\lambda_k A + B)X_k = 0 \tag{9}$$

where $B = D + G$, and $A = R - 2TT^T/I_2$. A is symmetric, positive-definite, and has rank and dimension equal to $m = 2n$. D and G are also of dimension m, and are respectively symmetric and skew-symmetric. B is thus composed of a symmetric plus a skew-symmetric matrix, and is of rank m. Equation (9) can be regarded as the standard (desired) form for the eigenvalue problem, and the arrangement of the equations into this form is a worthwhile step in the process of transformation to natural modes.

The eigenvalues, λ_k, are solutions of

$$\det(\lambda_k A + B) = 0 \quad . \tag{10}$$

Since A and B are of dimension m, Eq. (10) is a polynomial of degree m. From the theory of polynomial equations it can be shown that the equation yields m eigenvalues which are real or complex. Since the rank of A and B is also m, the eigenvalues are non-zero. (The methods of this section can also be extended to the case where A or B have rank less than m. Rank less than m implies the existence of λ's which are zero, and this physically represents 'rigid body' modes.) The complex eigenvalues occur in complex conjugate pairs

$$\lambda_p = -\sigma_p + i\beta_p \tag{11a}$$

$$\lambda_p = -\sigma_p - i\beta_p \tag{11b}$$

The eigenvalues can be assumed distinct since non-distinct roots can be assumed to be rendered distinct by a small change in physical parameters. From Eq. (7) it is evident that the components σ and β related to damping and frequency, respectively. λ's which are complex correspond to 'less than critically damped' modes, and real λ's correspond to 'greater than critically damped' modes. A change in physical parameters in Eq. (1a) (particularly an increase in damping) can cause β_p in Eq. (11) to change from a real quantity to an imaginary one, and thus change two complex roots to two real ones; thus some of the real λ's may also be in pairs.

The λ's of Eq. (11) that are complex can be converted to the conventional natural modal frequencies and modal damping ratios, ω and ζ, by the formula

$$\zeta_k = \sigma_k/(\beta_k^2 + \sigma_k^2)^{1/2}; \quad \omega_k^2 = \beta_k^2 + \sigma_k^2 \tag{12a}$$

The converse is

$$\sigma_k = \zeta_k\omega_k \; ; \; \beta_k^2 = \omega_k^2(1 - \zeta_k^2) \quad . \tag{12b}$$

Then, $|\lambda_k| = \omega_k$. $\zeta_k < 1$ for less than critical damping, and $\zeta_k = 1$ for critical damping.

X_k is referred to as a modal eigencolumn, and can be calculated by solving Eq. (9), given a particular λ. Also a matrix row corresponding to the particular λ can be calculated by solving

$$Y_r^T(\lambda_r A + B) = 0 \tag{13}$$

The matrix row, Y_r^T, is referred to as a modal eigenrow. If λ_r is complex, the corresponding X_r and Y_r can be expected to be complex. If λ_r is real, then X_r and Y_r will be real. Taking the transpose of Eq. (13), and recognizing that $A = A^T$, leads to

$$(\lambda_r A + B^T)Y_r = 0 \tag{14}$$

Comparing Eqs. (9) and (10) shows that X_r and Y_r are different since B is not equal to B^T. Premultiply Eq. (9) by Y_r^T:

$$Y_r^T(\lambda_k A + B)X_k = 0 \quad . \tag{15}$$

Postmultiply Eq. (13) by X_k:

$$Y_r^T(\lambda_r A + B)X_k = 0 \quad . \tag{16}$$

Substract Eq. (16) from Eq. (15):

$$(\lambda_k - \lambda_r)Y_r^T A X_k = 0 \quad . \tag{17}$$

Hence for distinct eigenvalues,

$$Y_r^T A X_k = 0 \ , \ Y_r^T B X_k = 0 \ , \quad r \neq k \tag{18}$$

Eqs. (18) are orthogonality properties. Also from Eqs. (9) and (14):

$$\lambda_r = - Y_r^T B X_r / (Y_r^T A X_r) = - X_r^T B^T Y_r / (X_r^T A Y_r) \quad . \tag{19a}$$

If X_r and Y_r correspond to λ_r, then the eigencolumn and eigenrow corresponding to λ_r^* are X_r^* and Y_r^*. It then follows that

$$\lambda_r^* = - Y_r^{*T} B X_r^* / (Y_r^{*T} A X_r^*) = - X_r^{*T} B_r^T Y_r^* / (X_r^{*T} A Y_r^*) \tag{19b}$$

If the quantity Π_k is defined as

$$\Pi_k = Y_k / (Y_k^T A X_k) \tag{20}$$

then the complex conjugate of Π_k can be shown to be

$$\Pi_k^* = Y_k^*/(Y_k^{*T} A X_k^*) \ . \tag{21}$$

The part of the eigenvector corresponding to pitch motion is obtained from Eq. (8a):

$$\dot{\theta}_k = 2T^T X_k / I_2 \tag{22a}$$

The structure of the eigencolumn may be noted to be

$$X_k = \begin{bmatrix} \lambda W_k \\ \\ W_k \end{bmatrix} . \tag{22b}$$

X_k may be written in terms of its real parts as

$$X_k = U_k + iV_k \tag{23a}$$

$$X_k^* = U_k - iV_k \ . \tag{23b}$$

2.2 TRANSFORMATION TO MODAL COORDINATES

The following transformation is defined

$$Z(t) = \sum_{k=1}^{m} \{X_k z_k(t) + X_k^* z_k^*(t)\} \tag{24a}$$

$$\mathring{\theta}(t) = \mathring{\Theta}(t) + \sum_{k=1}^{m} \{\mathring{\theta}_k z_k(t) + \mathring{\theta}_k^* z_k^*(t)\} \qquad . \tag{24b}$$

$z_k(t)$ is a scalar, and

$$z_k(t) = \xi_k(t) + i\eta_k(t) \tag{25a}$$

$$z_k^*(t) = \xi_k(t) - i\eta_k(t) \tag{25b}$$

where ξ_k and η_k are real-valued.

Equation (24) transforms the m + 1 real scalar variables contained in the set $\{\dot{\theta}(t), Z(t)\}$, to m + 1 new real scalar variables contained in $\{\mathring{\Theta}(t), z(t)\}$. The right-hand side is real-valued due to the pairing of complex and complex conjugate quantities.

The $\Theta(t)$ can be interpreted as the mean (or equivalent rigid body) motion, and $z_k(t)$ can be thought of as a superimposed modal vibration. $z_k(t)$ is a scalar modal coordinate variable which is understood to be complex if paired with a complex X and to be real (i.e. $\eta = 0$) if paired with a real-valued X (i.e. V = 0). For the real-valued modes, $X = X^* = U$ and $z(t) = z^*(t) = \xi(t)$. There are p real-valued z's and (m-p)/2 complex z's. There are (m-p)/2 η's and (m+p)/2 ξ's. In this example m is even (equals 2n) and this implies that p is also even. (In an example with a discrete damper, m would turn out odd, and p would also be odd.) The transformation may be expressed directly in terms of real-valued scalars by combining Eqs. (24) with Eqs. (22 - 23) and (25):

$$Z(t) = 2 \sum_{k=1}^{m} \{U_k \xi_k(t) - V_k \eta_k(t)\} \tag{26a}$$

$$\dot{\theta}(t) = \dot{\Theta}(t) + \frac{4T^T}{I_2} \sum_{k=1}^{m} \{U_k \xi_k(t) - V_k \eta_k(t)\} \tag{26b}$$

In Eq. (26) the summation applies to both real and complex λ's; for the λ's which are real, the V_k and η_k are understood to be equal to zero.

Substitution of Eqs. (24) into Eq. (5a), and use of Eqs. (22) results in

$$I_2 \ddot{\Theta} = L_2 \tag{27}$$

Substitution of Eqs. (24) into Eq. (5b) and use of Eqs. (22) results in

$$\Sigma (R - \frac{2TT^T}{I_2}) X_k \dot{z}_k + \Sigma D X_k z_k = T\ddot{\Theta} \tag{28}$$

Multiplication of this expression by Y_j^T and use of the Eqs. (18) and (20) results in, for both real and complex z's:

$$\dot{z}_j - \lambda_j z_j = \Pi_j^T \, T\ddot{\Theta} \tag{29}$$

Eqs. (27) and (29) are the transformed differential equations, where Eq. (27) describes the mean (or equivalent rigid) angular motion and Eqs. (29) are uncoupled complex-valued modal equations.

If Eq. (28) is multiplied by Y_j^{*T}, the same procedures lead to

$$z_j^* - \lambda_j^* z_j^* = \Pi_j^{*T} \, T\ddot{\Theta} \tag{30}$$

Eqs. (29) and (30) can be converted to the real variables $\xi(t)$ and $\eta(t)$ by substituting Eqs. (25) into them and then successively adding and subtracting the equations; for complex λ's one obtains

$$\dot{\xi}_k + \sigma_k \xi_k + \beta_k \eta_k = P_k^T \, T\ddot{\Theta} \tag{31a}$$

$$\dot{\eta}_k - \beta_k \xi_k + \sigma_k \eta_k = Q_k^T \, T\ddot{\Theta} \tag{31b}$$

and for real λ's,

$$\dot{\xi}_k + \sigma_k \xi_k = P_k^T \, T\ddot{\Theta} \tag{31c}$$

Eqs. (31) are the modal differential equations in terms of real variables. Thus for each complex eigenvalue there are two intercoupled first order modal differential equations, which are uncoupled from other modes. For real eigenvalues there is one first-order modal differential equation per mode. There are m(= 2n) λ's. p are real, and m-p are complex. Thus there are p modes corresponding to real λ's and (m-p)/2 modes corresponding to complex λ's, for a total of (m+p)/2 modes.

2.3 BLOCK DIAGRAM FORM FOR DEFLECTION

Eqs. (31), (27), (26) and (2) are readily arranged into the block diagram form shown in Figure 2. The figure illustrates the modal and feed-forward features.

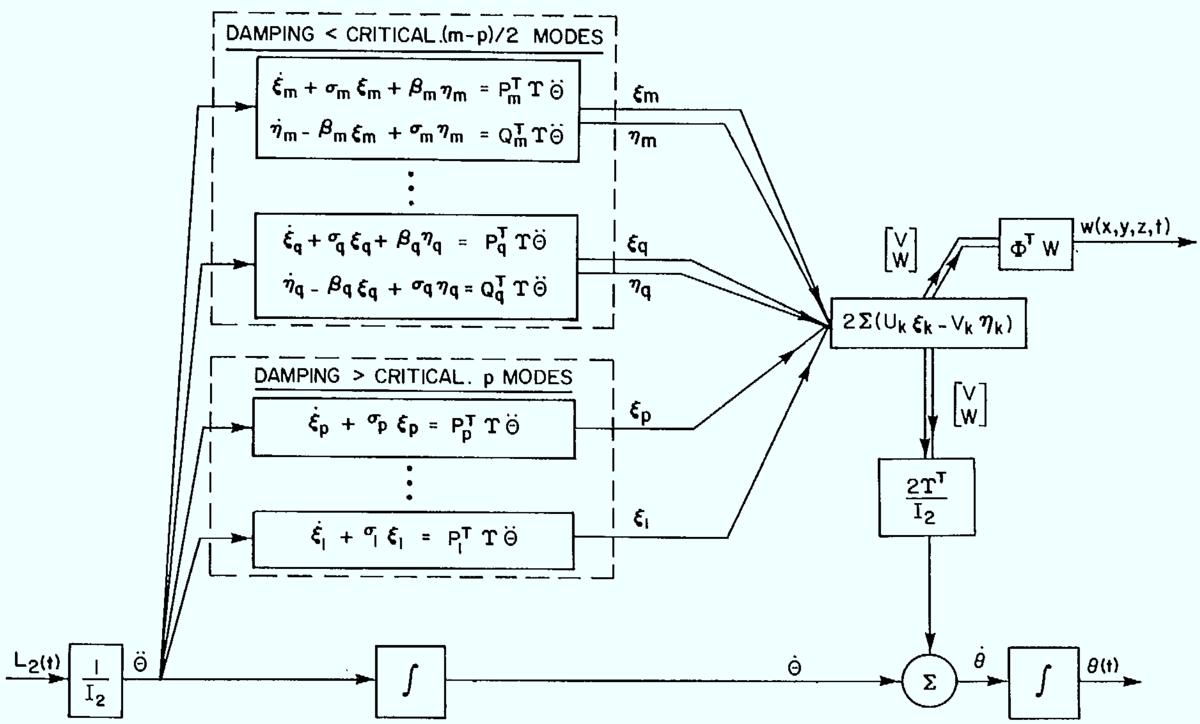


Figure 2. Transfer Function Between Input (L_2) and Output (θ and ω)

2.4 LAPLACE-TRANSFORMED EQUATIONS AND COMPARISON WITH CLASSICAL NORMAL MODES TRANSFORMATION

Taking Laplace transforms of Eqs. (31a and b) with zero initial states results in

$$(s + \sigma_k)\bar{\xi}_k(s) + \beta_k\bar{\eta}_k(s) = P_k^T T s^2 \bar{\Theta}(s) \tag{32a}$$

$$-\beta_k\bar{\xi}_k(s) + (s - \sigma_k)\bar{\eta}_k(s) = Q_k^T T s^2 \bar{\Theta}(s) \quad . \tag{32b}$$

Eqs. (32) may be solved for ξ_k and η_k:

$$\bar{\xi}_k(s) = \{(s + \sigma_k)P_k^T - \beta_k Q_k^T\} \frac{T s^2 \Theta(s)}{(s + \sigma_k)^2 + \beta_k^2} \tag{33a}$$

$$\bar{\eta}_k(s) = \{(s + \sigma_k)Q_k^T + \beta_k P_k^T\} \frac{T s^2 \Theta(s)}{(s + \sigma_k)^2 + \beta_k^2} \quad . \tag{33b}$$

Eqs. (33) apply for the $(2n - p)/2$ modes corresponding to complex λ's. For the other p modes, Eq. (31c) yields

$$\bar{\xi}_k(s) = \frac{P_k^T \; T s^2 \bar{\Theta}(s)}{s + \sigma_k} \tag{34}$$

Substitution of Eqs. (33) and (34) into the Laplace-transformed form of Eq. (26a), and conversion of σ, β to ω, ζ by Eqs. (12) results in

$$\bar{\theta}(s) = \bar{\Theta}(s) + \sum_{k=1}^{p} \frac{g_k}{s + \sigma_k} s\bar{\Theta}(s)$$

$$+ \sum_{k = p + 1}^{(2n+p)/2} \frac{g_k(s + \zeta_k\omega_k) - h_k\omega_k \sqrt{1 - \zeta_k^2}}{s^2 + 2\omega_k\zeta_k s + \omega_k^2} s\bar{\Theta}(s) \tag{35a}$$

where

$$g_k = \frac{4}{I_2} T^T U_k P_k T \quad \text{for } k = 1 \text{ to } p \tag{35b}$$

$$g_k = \frac{4}{I_2} T^T \{U_k P_k^T - V_k Q_k^T\} T \quad \text{for } k = (p+1) \text{ to } (2n+p)/2 \tag{35c}$$

$$h_k = \frac{4}{I_2} T^T \{U_k Q_k^T + V_k P_k^T\} T \quad . \tag{35d}$$

Equations (35) and (27) combine into the block diagram of Figure 3.

The counterpart of the above Eqs. (32 - 35) and Figure 3, for the classical normal modes theory with undamped modes and non-rigourously added modal damping is given by Eqs. (30 - 34) and Figure 5 of Ref. 4. In comparing the Laplace-Transform results with the normal modes transformation, one notes: (a) the work herein includes 'greater than critically damped' modes as well as the 'less than critically damped' modes of Ref. (4); (b) if there are no 'greater than critically damped' modes, then in both cases the number of modes is n; if there are p 'critically damped' modes, then the number of modes is (n+p)/2; (c) the numerators of the modes of Figure 3 contain two 'modal gains' g_k and h_k per mode, whereas the corresponding results of Ref. 4 contain only the g_k.

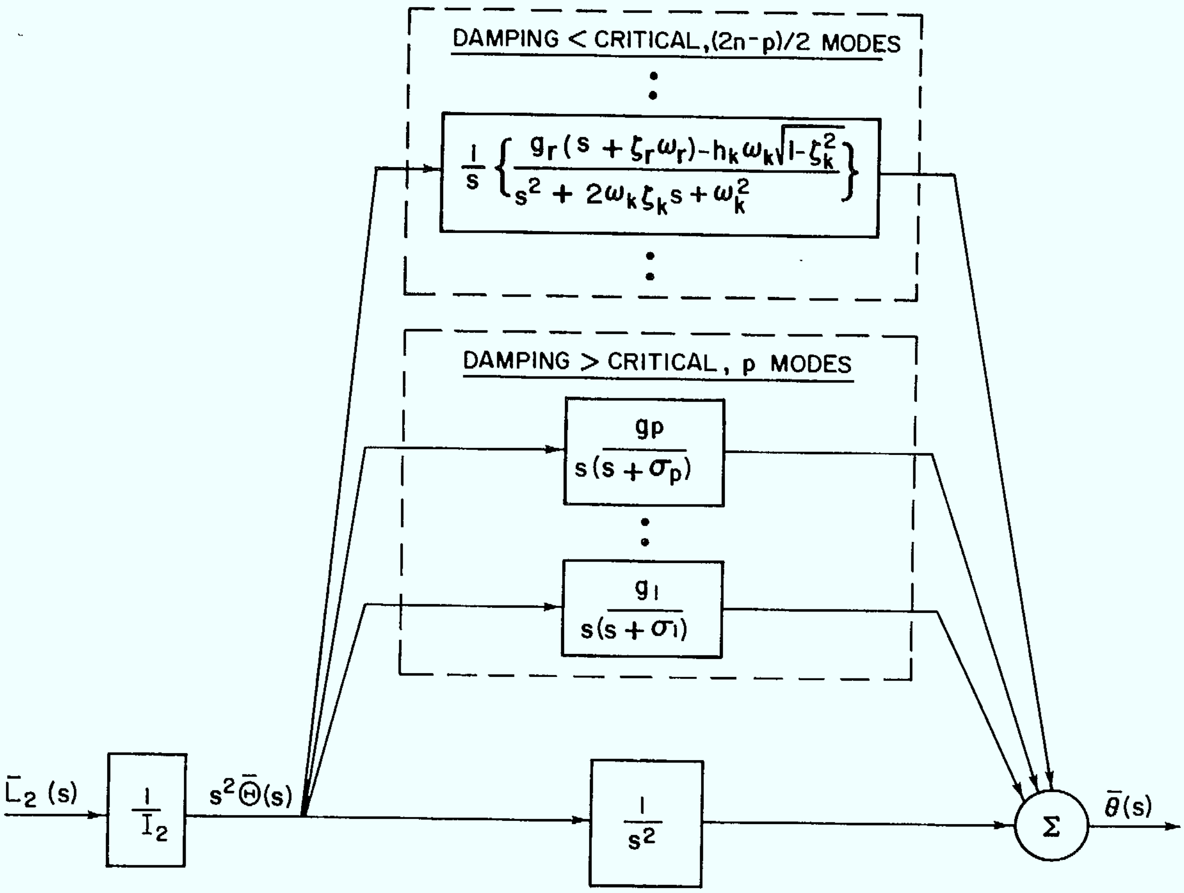


Figure 3. Transfer Function Between $\bar{L}_2(s)$ and $\bar{\theta}(s)$. All Coefficients and Variables are Real-Valued Scalars.

If damping is set equal to zero in the work herein, then $B = -B^T$, the eigenvalues are then purely imaginary, and $Y_k = \bar{X}_k$. Also the structure of the eigenvector, X_k, is $[i\omega_k W_k^T, W_k^T]^T$. The modal equations (31) reduce to

$$\ddot{\eta}_k + \omega_k^2 \eta_k = - \frac{S^T W_k}{2\omega_k W_k^T M W_k} \ddot{\theta} \tag{36}$$

and by comparing Eq. (36) with Eq. (28) of Ref. 4, it is noted that the η_k are proportional to and the first derivative of the normal modal coordinate, $\dot{q}_k$.

Eqs. (31) are the same as ones obtained in Ref. 7 by a non-rigourous induction. Ref. 7 also includes useful solutions for the complex differential equations (29 and 30).

3. ROLL AND YAW ATTITUDE DYNAMICS

The dynamics of roll and yaw are coupled by the stored momentum of the momentum wheel. The equations are

$$I_1\dot{\omega}_1 + I_{13}\dot{\omega}_3 + h_o\omega_3 + 2S_1^T\ddot{W} = L_o \cos\alpha \tag{37a}$$

$$I_3\dot{\omega}_3 + I_{13}\dot{\omega}_1 - h_o\omega_1 = L_o \sin\alpha \tag{37b}$$

$$M\ddot{W} + C\dot{W} + KW + S_1\dot{\omega}_1 = 0 \tag{37c}$$

The deformation of the array is discretized as per Eq. (2) with shape functions depicted in Figure 1(b).

Define the variable, Z, as

$$Z = \begin{bmatrix} \omega_1 \\ \omega_2 \\ V \\ W \end{bmatrix} \tag{38}$$

where $\dot{W} = V$. Then the equations may be arranged into the form

$$A\dot{Z} + BZ = F \tag{39}$$

where

$$A = \begin{bmatrix} I_1 & I_{13} & 2S_1^T & 0 \\ I_{13} & I_3 & 0 & 0 \\ 2S_1 & 0 & M & 0 \\ 0 & & 0 & K \end{bmatrix}, \quad F = \begin{bmatrix} L_o \cos\alpha \\ L_o \sin\alpha \\ 0 \\ \\ \end{bmatrix}$$

$$B = D + G,$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & h_o & 0 & 0 \\ -h_o & 0 & 0 & 0 \\ 0 & 0 & 0 & K \\ 0 & 0 & -K & 0 \end{bmatrix}$$

A is symmetric, positive definite, and has rank equal to dimension and equal to $m = 2n + 2$. D and G are symmetric and skew symmetric, respectively. B has rank m.

3.1 TRANSFORMATION TO MODAL COORDINATES

Upon substitution of $Z = X_k \exp(\lambda_k t)$, Eq. (39) leads directly to the eigenvalue problem in the desired standard form, Eq. (9). The derivation of the modes and frequencies follows the theory given in Eqs. (9 - 21). As there is no equivalent of θ_k (i.e. no free rigid body motion), the transformation is of the form of Eq. (24a) or (26a). The modal equations are the same as Eqs. (31), except that on the right hand side, F replaces $T\ddot{\Theta}$. The Laplace transform relation for Z may be found by substituting the equivalents of Eqs. (33) and (34) into Eq. (26a). One obtains

$$\begin{bmatrix} \bar{\omega}_1 \\ \bar{\omega}_3 \\ \bar{V} \\ \bar{W} \end{bmatrix} = \sum_{\text{cplx}} \frac{(s + \sigma_k)G_k - \omega_k H_k}{(s + \sigma_k)^2 + \omega_k^2} s^2 \bar{F} + \sum_{\text{real}} \frac{G_k}{s + \omega_k} s^2 \bar{F} \tag{40}$$

where G_k and H_k are matrices,

$$G_k = U_k P_k^T - V_k Q_k^T \quad , \quad H_k = U_k Q_k^T + V_k P_k^T \tag{41}$$

Eq. (40) can be converted into scalar Laplace block diagrams. As an example the relationship between $\bar{\omega}_3(s)$ and $\bar{L}_o(s)$ is shown in Figure 4. The feed-forward structure is evident. A similar block diagram is developed in Ref. 12 for this example by a different approximate method.

4. CONCLUSION AND DISCUSSION

This work illustrates transformations, differential equations and block diagrams for natural modal theory in terms of real scalar modal variables and modal column vectors. The theory is demonstrated to be a generalization of the classical normal modes transformation theory. Although the examples chosen to illustrate the principles are specific, it is evident that the theory as outlined in Eqs. (7 - 37) is applicable to a wide class of flexible spacecraft. Other examples, some of which involve gear drives and stepper motors, appear in Ref. 13.

This work offers a procedure that is not limited to light damping for calculating the structural modal damping factors of a spacecraft from data on its components or substructures. The procedure is more satisfactory than current non-rigorous ones which involve adding modal damping terms to modal equations that are derived with damping absent, or which involve ignoring non-diagonal terms of transformed damping matrices.

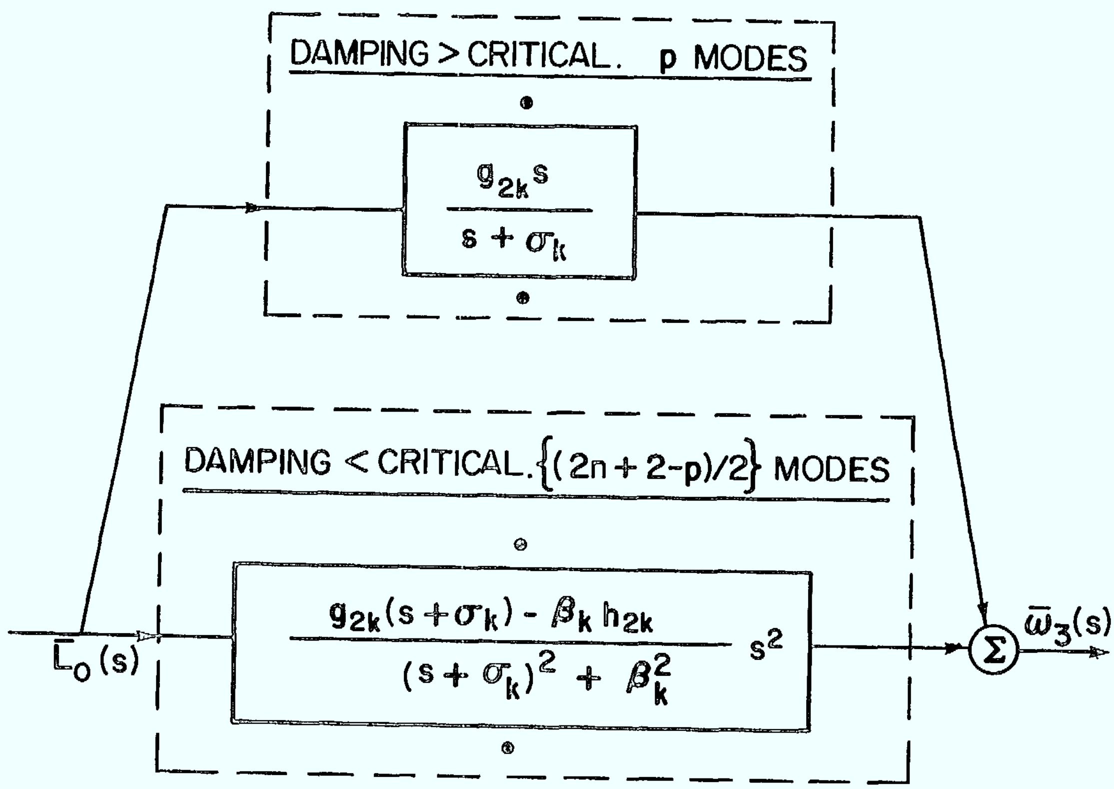


Figure 4. Transfer Function Between $\bar{L}_o(s)$ and $\bar{\omega}_3(s)$. All Coefficients and Variables are Real-Valued Scalars.

The equations of the eigenvalue problem can be arranged into several different forms. The choice of arrangement is a key decision. Herein, as a result of the use of $K\dot{W} - KV = 0$ (Eq. 4), the arrangement has resulted in the eigenvalue problem appearing with D and G as symmetric and skew symmetric matrices, respectively; the associated eigenvalue theory, which also appears in Ref. 10 and other sources, is quite general and accommodates both examples 1 and 2. A less general procedure, which can be applied to the first example but not the second, entails using $M\dot{W} - MV = 0$ in place of Eq. (4), and this arrangement leads to the form Eq. (9) where A and B are both symmetric. Although this procedure offers the advantage that the eigenvectors and orthogonalities are simpler [9] and that a corresponding well-developed technology for parameter estimation is available [14], it is less general relative to satellite applications because it cannot accommodate gyroscopic stiffness. Finally Ref. 11 presents theory for an arrangement where $\dot{W} - V = 0$ is used in place of Eq. (4), which is general but very complicated.

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8. **ABSTRACT:**

This report develops and illustrates a natural modal transformation theory which is applicable to flexible spacecraft with damping and gyroscopic forces. The theory is arranged into a form which is a generalization of the classical normal modes transformation theory. Modal differential equations are given in terms of real-valued scalars. Block diagrams in the time and Laplace transform domains demonstrate the feed-forward and second-order filter characteristics of the structure of the equations. Results for a single-axis flexible dynamics example are compared with earlier published results to show the correlation with the classical normal modes transformation theory.

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