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# A NATURAL MODES FORMULATION AND MODAL IDENTITIES FOR STRUCTURES WITH LINEAR VISCOUS DAMPING

by

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## A NATURAL MODES FORMULATION AND MODAL IDENTITIES FOR STRUCTURES WITH LINEAR VISCOUS DAMPING

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## ABSTRACT

A modal model is developed for an elastic structure with linear viscous damping. The transfer functions and normalizations that are of use in experimental modal parameter estimation are given special attention. Procedures for extraction of damped natural modes from experiment-derived residues are outlined. Mass-properties-related modal identities are obtained for the damped modes.

## 1.0 INTRODUCTION

There are several modal models that are in use or relevant to current R&D in modal parameter estimation by regression (curve fitting) and in substructure coupling. The following three are of specific interest.

(i) A first modal model corresponds to an undamped linear elastic structure. In this case, the modal equations in the time domain are uncoupled linear second order undamped vibratory equations. The modal parameters include natural frequencies and mode shape amplitude coefficients. The modal displacements of a particular mode are in phase with each other. The basic transformation is between physical displacements and modal displacements (a point transformation)[1].

(ii) A second modal model corresponds to a linear elastic structure with linear viscous damping proportional to a combination of the stiffness and mass properties. The modal equations in the time domain are uncoupled linear second order damped vibratory equations, and the modal parameters include modal natural frequencies, mode shape amplitude coefficients and modal damping factors. As in (i), the basic transformation from physical to modal variables is a point transformation, and the modal displacements of a particular mode shape are in phase with each other.[2,3]

(iii) A third modal model corresponds to a linear elastic structure with general linear viscous damping.[3,4] In this case, the modal differential equations in the time domain are paired first order differential equations. The modal parameters include modal natural frequencies, mode shape amplitude and phase coefficients, and modal damping factors. The basic transformation links physical displacements and velocities with modal displacements and velocities (contact transformation). The modal displacements of a particular mode shape are not in phase with each other.

The first and second models are subcases of the third.

There are additional modal models that apply in special and or limited situations. For example, the single-degree-of-freedom model with viscous or hysteretic damping [5,6], a model based on appropriated input forces [7,8], and models with centrifugal stiffening and gyroscopic stiffness [9,10].

In the direct modal parameter methods of parameter estimation, an analytical model of the output/input response is defined in terms of modal parameters (modal frequencies, damping factors, mode shape coefficients); the identified parameters are the numerical values of the modal parameters of the analytical modal model that yield the best curve fit of the analytical model of the response to a corresponding measured function. In this context, various techniques use the second, third, or single-degree-of-freedom viscous on hysteretic modal models. Associated modal orthogonality checks and adjustment of theoretical models to match measured results are currently done on underlying assumption of the first or second model.

Initial substructure coupling techniques were based on the first model. Recent substructure coupling work proposes use of the second or third models in order to extend earlier methods to include damping synthesis.

Until recently, the distinction between the models was not particularly important in most practical situations, because the accuracy of measurement-derived modal data was limited. However, with recent interactive computer processing of measurements and with parameter estimation techniques such as the complex exponentials methods, the phases of the modes are now being accurately resolved. For most real structures, the displacements are observed to be out of phase with each other. Thus the generality offered by the third model seems needed, relative to the first or second, for modal parameter estimation, modal completeness and or orthogonality checks and for substructure coupling.

Due to wide use over many years, the characteristics of the first model are well understood from both physical and mathematical viewpoints. Recently, new mass-properties-related modal identities have been developed that have significance to analytical and experimental verification of mode set completeness and modal truncation procedures[11]. The characteristics of the third modal model are not nearly as well developed and understood.

The intent of this report is twofold: first, to present a development of the transformation from physical variables to 'damped natural modal variables' for structures with linear viscous damping, in a manner that gives visibility into the structure of the equations and into aspects relevant to parameter determination applications; second, to derive modal identities similar to those of Ref. 11, for the damped natural modal model.

## 2.0 STRUCTURE IN TERMS OF PHYSICAL VARIABLES

Consider a flexible structure as is depicted schematically in Fig. 1. The structure is defined by $N$ points, relative to a coordinate system $(0xyz)$. In the context of modal parameter identification, where a finite number of points are excited and or instrumented, it is convenient to regard the number, $N$, as finite. The development to follow applies for the finite situation, and as well in the limit as $N$ tends to infinity in which case it becomes a continuum representation. A mass, $m^i$, is associated with each point. The stiffness and damping forces of the structure are assumed to be linear functions of the deformation variables, $(u^i, v^i, w^i)$. The structure is further considered to be restrained so that rigid body translations and rotations relative to $(0xyz)$ are not possible.

Define the deformation matrices, $U, V$ and $W$, of order $N \times 1$, as

$$U = \begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ \vdots \\ u^N \end{bmatrix} \qquad V = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \\ \vdots \\ v^N \end{bmatrix} \qquad W = \begin{bmatrix} w^1 \\ w^2 \\ w^3 \\ \vdots \\ w^N \end{bmatrix}, \tag{2-1}$$

and the corresponding position matrices of order $N \times 1$ as

$$X = \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ \vdots \\ x^N \end{bmatrix} \qquad Y = \begin{bmatrix} y^1 \\ y^2 \\ y^3 \\ \vdots \\ y^N \end{bmatrix} \qquad Z = \begin{bmatrix} z^1 \\ z^2 \\ z^3 \\ \vdots \\ z^N \end{bmatrix}. \tag{2-2}$$

The motion equations follow.

$$\begin{aligned} M^N\ddot{U} + C_{xx}\dot{U} + C_{xy}\dot{V} + C_{xz}\dot{W} + K_{xx}U + K_{xy}V + K_{xz}W &= f_x \\ M^N\ddot{V} + C_{xy}\dot{U} + C_{yy}\dot{V} + C_{yz}\dot{W} + K_{xy}U + K_{yy}V + K_{yz}W &= f_y \\ M^N\ddot{W} + C_{xz}\dot{U} + C_{yz}\dot{V} + C_{zz}\dot{W} + K_{xz}U + K_{yz}V + K_{zz}W &= f_z. \end{aligned} \tag{2-3}$$

In Eq. (2-3), $M^N$ is a diagonal matrix of dimension $N \times N$,

$$M^N = \begin{pmatrix} m^1 & 0 & \cdot & \cdot \\ 0 & m^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & m^N \end{pmatrix}, \tag{2-4}$$

$K_{xx}$, $K_{xy}$, etc. are $N \times N$ stiffness matrices, $C_{xx}$, $C_{xy}$, are damping matrices, and $f_x$, $f_y$, and $f_z$ are $N \times 1$ column matrices of external force components at the points of the structure. Define the $3N \times 1$ column matrix, $\mathbf{q}$, as

$$\mathbf{q} = \begin{bmatrix} U \\ V \\ W \end{bmatrix}. \tag{2-5}$$

Then Eqs. (2-3) may be written as

$$M\ddot{\mathbf{q}} + C\dot{\mathbf{q}} + K\mathbf{q} = \mathbf{f}, \tag{2-6a}$$

where

$$M = \begin{bmatrix} M^N & 0 & 0 \\ 0 & M^N & 0 \\ 0 & 0 & M^N \end{bmatrix}. \qquad C = \begin{bmatrix} C_{xx} & C_{xy} & C_{xz} \\ C_{xy} & C_{yy} & C_{yz} \\ C_{xz} & C_{yz} & C_{zz} \end{bmatrix};$$

$$K = \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{xy} & K_{yy} & K_{yz} \\ K_{xz} & K_{yz} & K_{zz} \end{bmatrix}; \qquad \mathbf{f} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}.$$

The order, $n$, of Eq. (2-6a) is $3N$. $M, C$ and $K$ are each of order $n \times n$, and are positive definite and symmetric. For this choice of deformation coordinates $M$ is also diagonal; there are other choices (such as finite element coordinates) where $M$ turns out non-diagonal. The initial conditions associated with Eq. (2-6a) are

$$\mathbf{q}(t_0) = \mathbf{q}_0; \qquad \dot{\mathbf{q}}(t_0) = \dot{\mathbf{q}}_0. \tag{2-6b}$$

Equations (2-6a) and (2-6b) model the structure in terms of physical variables.

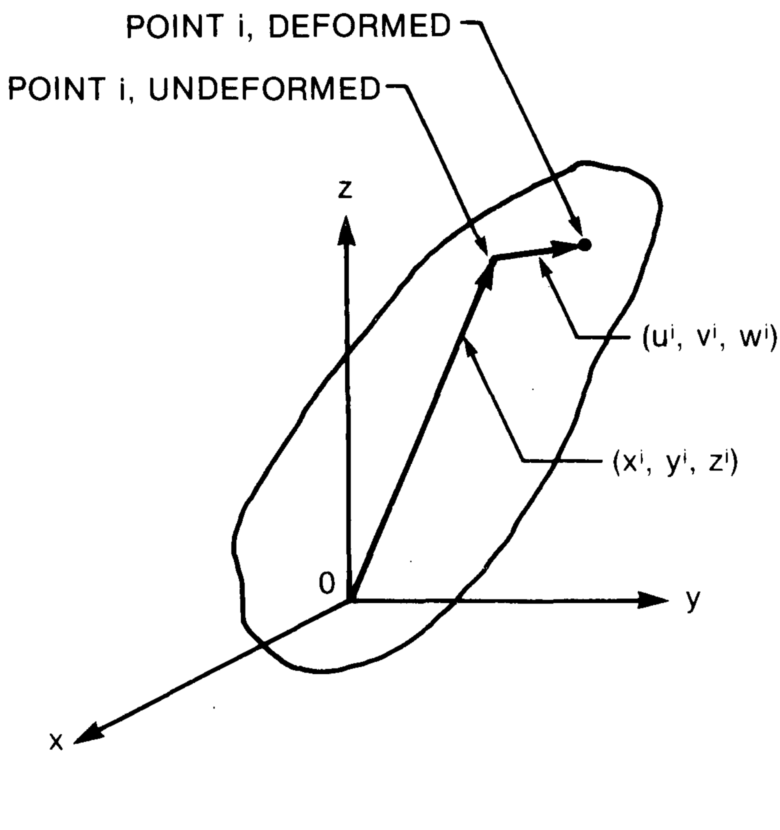


Figure 1. Coordinates of Structure

# 3.0 TRANSFORMATION TO DAMPED NATURAL MODAL VARIABLES

It is necessary at the outset to arrange the system's equations in a state variable form. The most familiar such form is due to Foss[3]. However, Foss's arrangement leads to parameter matrices that are not positive definite; as a consequence the desired modal identities (the 'damped mode' equivalents of Ref. 11) cannot be established directly and easily. An alternate arrangement introduced in Refs. 9 and 10 turns out to have the needed positive definite parameter matrix; as a consequence, the modal vectors constitute a basis in a vector space that has a Euclidean inner product and for which Bessel's and Parseval's equations can be established. Then the modal identities follow directly. Accordingly, this report develops the damped modes theory following the state variable arrangement of Refs. 9-10.

## 3.1 Structural Model in State Variable Form

The following equation is self evident:

$$K\dot{\mathbf{q}} - K\dot{\mathbf{q}} = 0 \tag{3-1}$$

Equation (3-1) may be combined with Eq. (2-6a), to obtain

$$\begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix} \frac{d}{dt}\begin{bmatrix} \dot{\mathbf{q}} \\ \mathbf{q} \end{bmatrix} + \begin{bmatrix} C & K \\ -K & 0 \end{bmatrix}\begin{bmatrix} \dot{\mathbf{q}} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}. \tag{3-2}$$

Equation (3-2) may be written in the first order state variable form,

$$A\dot{\mathbf{Q}} + B\mathbf{Q} = \mathbf{F}, \tag{3-3a}$$

where

$$\mathbf{Q} = \begin{bmatrix} \dot{\mathbf{q}} \\ \mathbf{q} \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix} \tag{3-4a}$$

$$A = \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}, \qquad B = \begin{bmatrix} C & K \\ -K & 0 \end{bmatrix}. \tag{3-4b}$$

$\mathbf{Q}$ and $\mathbf{F}$ are of order $2n \times 1$. $A$ is of order $2n \times 2n$ and rank $2n$, and is positive definite and symmetric. $B$ is order $2n \times 2n$, and is the sum of a symmetric part involving $C$ and a skew symmetric part involving $K$. The initial condition, Eq. (2-6b), becomes

$$\mathbf{Q}(t_0) = \mathbf{Q}_0. \tag{3-3b}$$

Equations (3-3) model the system in state variable form.

## 3.2 Eigenproblem Analysis

Substitution of

$$\mathbf{Q} = \boldsymbol{\Upsilon}_k e^{\lambda_k t} \tag{3-5}$$

into the homogeneous part of Eq. (3-3a) results in the eigenproblem

$$(\lambda_k A + B)\boldsymbol{\Upsilon}_k = 0. \tag{3-6}$$

The eigenvalues, $\lambda_k$, are solutions of

$$det(\lambda_k A + B) = 0. \tag{3-7}$$

Since $A$ and $B$ are of dimension $2n$, Eq. (3-7) is a polynomial of degree $2n$. From the theory of polynomial equations it can be shown that the equation yields $2n$ eigenvalues which are real or complex. Since the rank of $A$ is also $2n$, the eigenvalues are non-zero. (The methods of this section can also be extended to the case where $A$ has rank less than $2n$. Rank less than $2n$ implies the existence of $\lambda$'s which are zero, and this physically represents 'rigid body' modes.) The complex eigenvalues occur in complex conjugate pairs

$$\lambda_k = -\sigma_k + i\nu_k \tag{3-8a}$$

$$\lambda_k^* = -\sigma_k - i\nu_k. \tag{3-8b}$$

The eigenvalues can be assumed distinct since non-distinct roots can be rendered distinct by a small change in physical parameters. $\lambda$'s which are complex correspond to 'less than critically damped' modes, and real $\lambda$'s correspond to 'greater than critically damped' modes. A change in physical parameters in Eq. (3-7) (particularly an increase in damping) can cause $\nu_k$ to change from a real quantity to an imaginary one, and thus change two complex roots to two real ones; thus real $\lambda$'s are also in pairs. The $\lambda$'s of Eq. (3-8) that are complex can be converted to the conventional natural modal frequencies and modal damping ratios, $\omega$ and $\varsigma$, by the formula

$$\varsigma_k = \sigma_k/(\nu_k^2 + \sigma_k^2)^{1/2}; \qquad \omega_k^2 = \nu_k^2 + \sigma_k^2. \tag{3-9a}$$

The converse is

$$\sigma_k = \varsigma_k \omega_k; \qquad \nu_k^2 = \omega_k^2(1 - \varsigma_k^2). \tag{3-9b}$$

Then, $|\lambda_k| = \omega_k$ . $\varsigma_k < 1$ for less than critical damping, and $\varsigma_k = 1$ for critical damping.

A matrix column, $\Upsilon_k$, is calculated (and determinable to within a complex scalar constant) by solving Eq. (3-6) for a particular $\lambda_k$. Also a matrix row, $\Gamma_r^T$, can be calculated for a particular $\lambda_r$ from

$$\Gamma_r^T(\lambda_r A + B) = 0. \tag{3-10}$$

If $\lambda_k$ is complex, the corresponding column ($\Upsilon_k$) and row ($\Gamma_k$) can be expected to be complex. If $\lambda_r$ is real, then the $\Upsilon_k$ and $\Gamma_k$ would be real. Taking the transpose of Eq. (3-10) and recognizing that $A = A^T$ yields

$$(\lambda_r A + B^T)\Gamma_r = 0. \tag{3-11}$$

Since $B$ is not equal to $B^T$ Eqs. (3-6) and (3-11) are different, and thus $\Upsilon_k$ and $\Gamma_k$ are not equal.

Premultiply Eq. (3-6) by $\Gamma_r^T$:

$$\lambda_k \Gamma_r^T A \Upsilon_k + \Gamma_r^T B \Upsilon_k = 0. \tag{3-12}$$

Post-multiply Eq. (3-10) by $\Upsilon_k$:

$$\lambda_r \Gamma_r^T A \Upsilon_k + \Gamma_r^T B \Upsilon_k = 0. \tag{3-13}$$

Subtract Eq. (3-13) from Eq. (3-12) to obtain,

$$(\lambda_k - \lambda_r)\Gamma_r^T A \Upsilon_k = 0. \qquad (3-14)$$

Hence, for two non-equal eigenvalues $\lambda_r$ and $\lambda_k$,

$$\Gamma_r^T A \Upsilon_k = 0; \qquad \Gamma_r^T B \Upsilon_k = 0 \qquad r \neq k. \qquad (3-15)$$

It follows from Eq. (3-12), and from Eq. (3-11) after multiplication $\Upsilon_k^T$, that

$$\lambda_k = -\frac{\Gamma_k^T B \Upsilon_k}{\Gamma_k^T A \Upsilon_k} = -\frac{\Upsilon_k^T B^T \Gamma_k}{\Upsilon_k^T A \Gamma_k}. \qquad (3-16)$$

$\lambda_k$ and $\lambda_k^*$ are two distinct eigenvalues, and consequently $\Gamma_k^{*T} A \Upsilon_k$ and $\Gamma_k^{*T} B \Upsilon_k$ are equal to zero (i.e., $\lambda_r = \lambda_k^*$ in Eqs. (3-12) to (3-13)).

$\Upsilon_k$ and $\Gamma_k$ are thus right and left eigenvectors that are orthogonal relative to $A$.

Each $\Upsilon_k$ obtained by solving Eq. (3-6) is determinable only to within an arbitrary complex scalar constant (scale factor) and also may be viewed as having arbitary physical units.

The most convenient choice for the arbritary scale factor and units depends on the application; options will be discussed in Chapter 4. In this chapter (Chapter 3) all expressions are valid for any scalar factor and type of normalization.

To further demonstrate the structure of $\Upsilon_k$, consider the upper and lower internal columns, $\Upsilon_k^u$ and $\Upsilon_k^l$, each of order $n \times 1$:

$$\Upsilon_k = \begin{bmatrix} \Upsilon_k^u \\ \Upsilon_k^l \end{bmatrix}. \qquad (3-17)$$

Substitute Eq. (3-17) and the parameter values for **A** and **B** of Eq. (3-4b) into Eq. (3-6):

$$\begin{pmatrix} \lambda_k M & 0 \\ 0 & \lambda_k K \end{pmatrix} \begin{pmatrix} \Upsilon_k^u \\ \Upsilon_k^l \end{pmatrix} + \begin{pmatrix} C & K \\ -K & 0 \end{pmatrix} \begin{pmatrix} \Upsilon_k^u \\ \Upsilon_k^l \end{pmatrix} = 0. \qquad (3-18)$$

Eq. (3-18) reduces to

$$\lambda_k M \Upsilon_k^u + C \Upsilon_k^u + K \Upsilon_k^l = 0 \qquad (3-19a)$$

$$\lambda_k K \Upsilon_k^l - K \Upsilon_k^u = 0. \qquad (3-19b)$$

From the latter equation,

$$\Upsilon_k^u = \lambda_k \Upsilon_k^l. \qquad (3-20)$$

Denote the $n \times 1$ column matrix, $\Upsilon_k^l$ by $\Phi_k$ ($\Phi_k$ is, in general, complex). Then from Eqs. (3-20) and (3-17),

$$\Upsilon_k = \begin{bmatrix} \lambda_k \Phi_k \\ \Phi_k \end{bmatrix}. \qquad (3-21)$$

Eqs. (3-19) are further easily arranged to obtain

$$(\lambda_k^2 M + \lambda_k C + K)\Phi_k = 0. \qquad (3-22)$$

A similar parallel development where $\Gamma_k$ is partitioned into upper and lower parts and Eq. (3-11) is decomposed in a similar manner yields

$$\Gamma_k^u = -\lambda_k \Gamma_k^l \tag{3-23a}$$

$$(\lambda_k^2 M + \lambda_k C + K)\Gamma_k^l = 0. \tag{3-23b}$$

Thus $\Gamma_k^l$ equals $\Phi_k$, and

$$\Gamma_k = \begin{bmatrix} -\lambda_k \Phi_k \\ \Phi_k \end{bmatrix}. \tag{3-24}$$

### 3.3 Vector Space Corresponding to Damped Natural Modes

The quantities $(\Upsilon_k + \Upsilon_k^*)$ and $i(\Upsilon_k - \Upsilon_k^*)$, $k = 1$ to $n$, are $2n$ real-valued linearly-independent column matrices (each of order $2n$), and are thus a vector basis of a real vector space, having dimension $2n$. For any vector, $\mathbf{G}$ (i.e., real-valued column matrix) contained in the space, $\mathbf{G}^T A \mathbf{G} \geq 0$ since $A$ is positive definite. Thus $\mathbf{G}_1^T A \mathbf{G}_2$ defines a real inner product for two arbitrary vectors, $\mathbf{G}_1$ and $\mathbf{G}_2$, contained in the space. Thus the vector space is a real inner product space (a Euclidean Space, $E^{2n}$). The corresponding dual basis consists of $(\Gamma_k + \Gamma_k^*)$ and $i(\Gamma_k - \Gamma_k^*)$, $k = 1$ to n. In accordance with the Representation Theorem, a real-valued vector, $\mathbf{G}$, contained in $E^{2n}$, may be represented in terms of the $\Upsilon$ - basis by

$$\mathbf{G} = \sum_{k=1}^{n} a_k(\Upsilon_k + \Upsilon_k^*) + b_k i(\Upsilon_k - \Upsilon_k^*), \tag{3-25}$$

where $a_k$ and $b_k$ are real-valued scalars. The above expression rearranges to the form

$$\mathbf{G} = \sum_{k=1}^{n} (\alpha_k \Upsilon_k + \alpha_k^* \Upsilon_k^*) \tag{3-26}$$

where $\alpha_k = a_k + ib_k$. Thus one may regard the basis vectors as $\Upsilon_k$ and $\Upsilon_k^*$, if at the same time appropriate pairing of complex and complex conjugate quantities is maintained to ensure that the total expression is real-valued. Similarly $\mathbf{G}$ may be represented in terms of the dual basis as

$$\mathbf{G} = \sum_{k=1}^{n} (\beta_k \Gamma_k + \beta_k^* \Gamma_k^*). \tag{3-27}$$

It proves convenient to work directly with the complex $\Upsilon$'s and $\Gamma$'s and the representation in the form of Eq. (3-26) as opposed to the real-valued basis vectors and the form of Eq. (3-25).

### 3.4 System Model in Terms of Complex Modal Variables

The real-valued column matrix, $\mathbf{Q}(t)$, of Eq. (3-3), can be represented as

$$\mathbf{Q}(t) = \sum_{k=1}^{n} \left[\Upsilon_k \rho_k(t) + \Upsilon_k^* \rho_k^*(t)\right]. \tag{3-28}$$

In Eq. (3-28), $\rho_k(t)$ is a complex-valued scalar variable

$$\rho_k(t) = \xi_k(t) + i\eta_k(t). \tag{3-29}$$

Eq. (3-28) defines a one-to-one transformation from the $2n$ real scalar physical variables contained in $\mathbf{Q}$ (i.e. $u^s(t), v^s(t), w^s(t)$, etc. of Eq. (2-1)) to $n$ complex scalar modal variables, $\rho_k(t)$ (or equivalently, a transformation to the $2n$ real modal variables, $\xi_k(t)$ and $\eta_k(t)$). To transform the state variable differential equation, Eq. (3-3a), to the equivalent modal variable differential equation substitute Eq. (3-28) into Eq. (3-3a), premultiply by $\Gamma_r^T$, and use the orthogonality properties of Eqs. (3-15) and (3-16), to obtain

$$\dot{\rho}_k - \lambda_k \rho_k = \frac{\Gamma_k^T \mathbf{F}(t)}{\Gamma_k^T A \Upsilon_k}. \tag{3-30}$$

The denominator, $\Gamma_k^T A \Upsilon_k$, is

$$[-\lambda_k \mathbf{\Phi}_k^T, \mathbf{\Phi}_k^T] \begin{pmatrix} M & 0 \\ 0 & K \end{pmatrix} \begin{bmatrix} \lambda_k \mathbf{\Phi}_k \\ \mathbf{\Phi}_k \end{bmatrix},$$

when combined with Eqs. (3-4b), (3-21), and (3-24). The above reduces to

$$-\lambda_k^2 \mathbf{\Phi}_k^T M \mathbf{\Phi}_k + \mathbf{\Phi}_k^T K \mathbf{\Phi}_k.$$

Multiplication of Eq. (3-22) by $\mathbf{\Phi}_k^T$ and combination with the above gives the final result,

$$\Gamma_k^T A \Upsilon_k = -\lambda_k \mathbf{\Phi}_k^T (2\lambda_k M + C) \mathbf{\Phi}_k. \tag{3-31}$$

Likewise, for the numerator of Eq. (3-30)

$$\Gamma_k^T \mathbf{F} = -\lambda_k \mathbf{\Phi}_k^T \mathbf{f}. \tag{3-32}$$

Substitution of Eqs. (3-31) and (3-32) into Eq. (3-30) results in

$$\dot{\rho}_k(t) - \lambda_k \rho_k(t) = \frac{\mathbf{\Phi}_k^T \mathbf{f}(t)}{\mathbf{\Phi}_k^T (2\lambda_k M + C) \mathbf{\Phi}_k}. \tag{3-33a}$$

A similar procedure with $\Gamma_r^{*T}$ as the premultiplier leads to

$$\dot{\rho}_k^*(t) - \lambda_k^* \rho_k^*(t) = \frac{\mathbf{\Phi}_k^{*T} \mathbf{f}(t)}{\mathbf{\Phi}_k^{*T} (2\lambda_k^* M + C) \mathbf{\Phi}_k^*}. \tag{3-33b}$$

Equations (3-33) are the differential equations of the system in terms of complex modal variables. The corresponding initial conditions, $\rho(t_0)$, can be obtained by substitution of Eq (3-28) into Eq (3-3b), and inversion.

### 3.5 Transfer Matrix and Residues

For parameter identification, the transfer matrix, $H$, between $\mathbf{f}$ and $\mathbf{q}$ is needed in terms of modal parameters. Appropriate forms of $H$ are derived in this section.

The transformation between $\mathbf{q}$ and $\rho$ may be deduced from Eqs. (3-28), (3-4a), and (3-21) to be

$$\mathbf{q}(t) = \sum_{k=1}^{n} \{\mathbf{\Phi}_k \rho_k(t) + \mathbf{\Phi}_k^* \rho_k^*(t)\}. \qquad (3-34)$$

Equations (3-33) and (3-34) are next transformed by the Laplace Transform. In this context the two-side transform of a complex variable is implied with $\mathbf{Q}(t)$ taken to be zero at $t_0 = -\infty$. Then $\rho(-\infty)$ and $\rho^*(-\infty)$ are zero. The transformed modal equations, are then, from Eq. (3-34);

$$\bar{\rho}_k(s) = \frac{\mathbf{\Phi}_k^T \bar{\mathbf{f}}(s)}{\mathbf{\Phi}_k^T(2\lambda_k M + C)\mathbf{\Phi}_k} \cdot \frac{1}{(s - \lambda_k)}. \qquad (3-35a)$$

$$\bar{\rho}_k^*(s) = \frac{\mathbf{\Phi}^{*T} \bar{\mathbf{f}}(s)}{\mathbf{\Phi}_k^{*T}(2\lambda_k^* M + C)\mathbf{\Phi}_k^*} \cdot \frac{1}{(s - \lambda_k^*)}. \qquad (3-35b)$$

Substitution of Eqs. (3-35) into a transformed version of Eq. (3-34) yields

$$\bar{\mathbf{q}}(s) = \sum_{k=1}^{n} \left\{ \frac{\mathbf{\Phi}_k \mathbf{\Phi}_k^T}{\mathbf{\Phi}_k^T(2\lambda_k M + C)\mathbf{\Phi}_k} \cdot \frac{1}{(s - \lambda_k)} + \frac{\mathbf{\Phi}_k^* \mathbf{\Phi}_k^{*T}}{\mathbf{\Phi}_k^{*T}(2\lambda_k^* M + C)\mathbf{\Phi}_k^*} \cdot \frac{1}{(s - \lambda_k^*)} \right\} \bar{\mathbf{f}}(s). \qquad (3-36)$$

Define the residue matrices, $R^k$ and $R^{*k}$, as

$$R^k = \frac{\mathbf{\Phi}_k \mathbf{\Phi}_k^T}{\mathbf{\Phi}_k^T(2\lambda_k M + C)\mathbf{\Phi}_k}; \qquad R^{*k} = \frac{\mathbf{\Phi}_k^* \mathbf{\Phi}_k^{*T}}{\mathbf{\Phi}_k^{*T}(2\lambda_k^* M + C)\mathbf{\Phi}_k^*}, \qquad (3-37)$$

and the 'transfer matrix', $\bar{H}(s)$, as

$$\bar{H}(s) = \sum_{k=1}^{n} \left[ \frac{R^k}{s - \lambda_k} + \frac{R^{*k}}{s - \lambda_k^*} \right]. \qquad (3-38)$$

Then Eq. (3-36) may be written as

$$\bar{\mathbf{q}}(s) = \bar{H}(s)\bar{\mathbf{f}}(s) \qquad (3-39)$$

Equation (3-28) can be inverted to the time domain, to yield the unit impulse response function, $H(t)$:

$$H(t) = \sum_{k=1}^{n} (R^k e^{\lambda_k t} + R^{*k} e^{\lambda_k^* t}). \qquad (3-40)$$

$R^k$, $\bar{H}(s)$, and $H(t)$ are symmetric and of dimension $n \times n$.

### 3.6 Invariance of Residues to Mode Shape Scale Factor

As noted in Section 3.2, the mode shapes, $\mathbf{\Phi}_k$, are determinable only to within a multiplicative scalar complex constant. Suppose that a mode shape has been calculated, and has a value, $\mathbf{\Phi}_k'$. Suppose that is then rescaled to a new value, $\mathbf{\Phi}_k$, (by dividing each element of the $n \times 1$ column matrix by $d_k$) so that

$$\mathbf{\Phi}_k' = d_k \mathbf{\Phi}_k. \qquad (3-41)$$

Then, with reference to Eq. (3-37),

$$\frac{\Phi'_k \Phi'^T_k}{\Phi'^T_k (2\lambda_k M + C)\Phi'_k} = \frac{d_k^2 \Phi_k \Phi_k^T}{d_k^2 \{\Phi_k^T (2\lambda_k M + C)\Phi_k\}} = R^k.$$

Thus the residue expressions, and also expressions for $\bar{H}(s)$ and $H(t)$ given above are demonstrated to be invariant to the scale factor $d_k$.

### 3.7 System Model in Terms of Real Modal Variables

The system model can be expressed in terms of real-valued modal variables, $\xi_k(t)$ and $\eta_k(t)$, as an alternate to $\rho(t)$and $\rho^*(t)$ of Chapter 3.4. Substitution of Eq. (3-29) into (3-34) results in

$$\mathbf{q}(t) = 2\sum_{k=1}^{n}\{Re\Phi_k.\xi_k(t) - Im\Phi_k.\eta_k(t)\}. \qquad (3-42)$$

Substitution of Eqs. (3-29) and (3-8) into Eqs. (3-33a) and (3-33b), and successive addition and subtraction of the two equations leads to

$$\dot{\xi}_k + \sigma_k\xi_k + \nu_k\eta_k = \mathbf{S}_k^T\mathbf{f} \qquad (3-43a)$$

$$\dot{\eta}_k + \sigma_k\eta_k - \nu_k\xi_k = \mathbf{T}_k^T\mathbf{f}, \qquad (3-43b)$$

where $\mathbf{S}_k^T$ and $\mathbf{T}_k^T$ are the real and imaginary parts, respectively, of $\Phi_k^T/\{\Phi_k^T(2\lambda_k M + C)\Phi\}$. Equations (3-43) are two first-order modal differential equations that are the damped-natural-modes counterpart of the familiar single uncoupled second order modal equation

$$\ddot{p}_k + 2\varsigma_k\omega_k\dot{p}_k + \omega_k^2 p_k = \pi^T\mathbf{f}. \qquad (3-44)$$

of the proportional-damping theory. Of significance, however, is the fact that Eqs. (3-43) cannot be put into the form of Eq. (3-44), except for the special cases of proportional and zero damping. Because of this, the physical concepts of "modal mass", "modal damping", and "modal stiffness" are of limited use in the damped natural modes theory. The transformation of Eqs. (3-43) from variables $(\xi_k, \eta_k)$ to $(r_k, p_k)$ by

$$r_k(t) = -\frac{\sigma_k}{\nu_k}\xi_k)t) + \eta_k(t) \qquad (3-45a)$$

$$p_k(t) = \frac{\sigma_k^2 + \nu_k^2}{\nu_k}\xi_k(t), \qquad (3-45b)$$

together with appropriate substitutions and use of Eqs. (3-9) lead to:

$$\ddot{p}_k + 2\varsigma_k\omega_k\dot{p}_k + \omega_k^2 p_k = \omega_k^2\left(\frac{\varsigma_k\mathbf{S}_k^T}{\sqrt{1-\varsigma_k^2}} - \mathbf{T}_k\right)\mathbf{f} + \mathbf{S}_l^T\dot{\mathbf{f}}. \qquad (3-46)$$

The above equation is different in structure from Eq. (3-44) because of the term in $\dot{\mathbf{f}}$.

It was noted in Chapter 3.2 that

$$\Gamma_k^{*T} A \Upsilon_k = 0. \qquad (3-47)$$

Subsitution of Eqs. (3-4b), (3-21) and (3-24) into this expression and reduction leads to

$$\lambda_k^* \lambda_k = \frac{\Phi^{*T} K \Phi_k}{\Phi_k^{*T} M \Phi_k}. \tag{3 – 48}$$

$\lambda_k^* \lambda_k$ equals $\sigma_k^2 + \nu_k^2$, which in turn equals $\omega_k^2$. If a "modal mass" and a "modal stiffness" are defined by

$$m_k = \Phi_k^{*T} M \Phi_k \tag{3 – 49a}$$

$$k_k = \Phi_k^{*T} K \ \Phi_k, \tag{3 – 49b}$$

then,

$$\omega_k^2 = \frac{k_k}{m_k}. \tag{3 – 50}$$

Likewize, the relation

$$\Gamma_k^{*T} B \Upsilon_k = 0 \tag{3 – 51}$$

can be used to show that

$$\lambda_k + \lambda_k^* = -\frac{\Phi_k^{*T} C \Phi_k^T}{\Phi_k^{*T} M \Phi_k}. \tag{3 – 52}$$

From this expression follows a definition of "modal damping" as

$$c_k = \Phi_k^{*T} C \Phi_k, \tag{3 – 49c}$$

Since $\lambda_k + \lambda_k^*$ equals $-2\omega_k \varsigma_k$, Eq. (3-52) yields

$$2\omega_k \varsigma_k = \frac{c_k}{m_k}. \tag{3 – 53}$$

Equations (3-50) and (3-53) can be used to convert Equation (3-46) into a second order differential equation involving $m_k, c_k$, and $k_k$.

For the special case of zero damping, $C = 0$, $\sigma_k = 0$, $\nu_k = \omega_k$, and the $\Phi_k$ are real. Then

$$\mathbf{S}_k = 0; \qquad \mathbf{T}_k = \frac{-\Phi_k}{2\omega_k \Phi_k^T M \Phi_k}.$$

Substitution of this specialization into Eqs. (3-43), and elimination of $\eta_k$ in favour of $\xi_k$ leads to the familiar second order form

$$\ddot{\xi}_k + \omega_k^2 \xi_k = \frac{\Phi_k^T \mathbf{f}}{2\Phi_k^T M \Phi_k}. \tag{3 – 54}$$

Additional information on the real modal variables formulation is available in Ref 9.

# 4.0 EXTRACTION OF MODAL PARAMETERS FROM EXPERIMENTAL DATA

To obtain experimental estimates of the modal parameters ($\lambda_k$, $\Phi_k$, $R^k$, and others), a modal survey test is first done in which vibratory forces are input to the structure at one or more locations, and structural responses are measured at the same and additional locations. The basic experimental data, namely measurements of the input forces and the responses with time, are preprocessed with frequency or time domain methods to obtain an experiment-derived estimate of part of the transfer matrix, $H$, in the frequency or time domain. Estimates of the complex frequencies, $\lambda_k$, and the residues, $R^k$ are next deduced from the experiment-derived $H$ by a curve fit type of parameter estimation method. In this type of method, the experiment-derived $\lambda_k$ and $R^k$ are the numerical values of these parameters that result in a best curve fit of the analytical expression for $H$ of Eq. (3-38) or (3-40), to the corresponding measurement-derived $H$.

Estimates of the modal frequencies, $\omega_k$, and damping factors, $\zeta_k$, are directly calculated from the estimates of $\lambda_k$ (using Eqs. (3-9) for example). It is not as straightforward to deduce the mode shapes, $\Phi_k$, and appropriate normalization factors from the estimates of $R^k$ and $\lambda_k$. Ways of achieving this are given in Ref. 13 for the proportionally-damped ($\Phi$ real-valued) theory. This chapter extends the methods of Ref. 13 to apply for the damped natural modes theory given herein.

## 4.1 Normalization Constant

The normalization constant, $Q_k$, is consistent with Ref. 13 when defined as

$$Q_k = \frac{1}{\Phi_k^T(2\lambda_k M + C)\Phi_k}. \qquad (4-1)$$

$Q_k$ is a complex scalar, and its conjugate is further given by

$$Q_k^* = \frac{1}{\Phi_k^{*T}(2\lambda_k^* M + C)\Phi_k^*}. \qquad (4-2)$$

The numerical value and units of $Q_k$ thus depend on numerical value and units of the mode shape, or vice-versa. With this definition of normalization constant, the residues assume the form,

$$R^k = Q_k\Phi_k\Phi_k^T; \qquad R^{k*} = Q_k^*\Phi_k^*\Phi_k^{*T}, \qquad (4-3)$$

and $\bar{H}(s)$ of Eq. (3-38) takes the form

$$\bar{H}(s) = \sum_{k=1}^{n}\left\{\frac{Q_k\Phi_k\Phi_k^T}{s-\lambda} + \frac{Q_k^*\Phi_k^*\Phi_k^{*T}}{s-\lambda^*}\right\}. \qquad (4-4)$$

Three of several possible ways of assigning the arbitrary constant associated with each mode shape, and hence $Q_k$, follow.

(a) Choose the numerical scaling for each mode so that the scalar element of the mode shape at the main exciter's driving point is $1+i0$. As will be seen, this implies that $Q_k$ equals the scalar value of the residue associated with the driving point. This choice of normalization is compatible with current experimental modal analysis conventions.

If the $\Phi_k$ is further considered dimensionless, then the units of $\rho(t)$ and $Q_k$ are $m$ and $sec/kg$, respectively.

(b) Choose the scaling for each mode so that $\Phi_k^T(2\lambda_k M + C)\Phi_k$ equals $2\lambda_k$. This degenerates to $\Phi_k^T M \Phi_k = 1$ when damping is zero, and is thus consistent with the unity modal mass convention of finite element and experimental modal engineering practice. For this choice, $Q_k = 1/2\lambda_k$. In this case the units of $\Phi_k$ are $kg^{-1/2}$, and of $Q$ are sec.

(c) Choose the numerical scaling of each mode so that $\Gamma_k^T A \Upsilon_k$ equals $1 + i0$. This then implies that $Q_k = -\lambda_k$. This choice is the most convenient one for theoretical work because it simplifies algebra a great deal. Unfortunately, this normalization has no counterpart in the classical undamped or porportionally-damped modal theories, and consequently would not be compatible with current engineering practice and developed software.

### 4.2 Normalisation Procedure For Experiment-Derived Modes

We assume that experiment-derived estimates of the residue matrices, $\hat{R}^k$, are available from modal test procedures described in the foregoing (the symbol $\wedge$ denotes a numerical estimate (a complex number)).

The structure of $R^k = Q_k \Phi_k \Phi_k^T$ at the scalar level is,

$$\begin{bmatrix} R_{11} & \cdot & R_{d1} & \cdot & R_{1n} \\ R_{12} & \cdot & R_{d2} & \cdot & R_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ R_{1d} & \cdot & R_{dd} & \cdot & R_{dn} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ R_{1n} & \cdot & R_{dn} & \cdot & R_{nn} \end{bmatrix}^k = Q_k \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \cdot \\ \Phi_d \\ \cdot \\ \Phi_n \end{bmatrix}_k \left[ \Phi_1 \Phi_2 \cdot \Phi_d \cdot \Phi_n \right]_k \qquad (4-5)$$

$$= \begin{bmatrix} Q\Phi_1\Phi_1 & Q\Phi_2\Phi_1 & \cdot & Q\Phi_d\Phi_1 & \cdot & Q\Phi_n\Phi_1 \\ Q\Phi_1\Phi_2 & Q\Phi_2\Phi_2 & \cdot & Q\Phi_d\Phi_2 & \cdot & Q\Phi_n\Phi_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ Q\Phi_1\Phi_d & Q\Phi_2\Phi_d & \cdot & Q\Phi_d\Phi_d & \cdot & Q\Phi_n\Phi_d \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ Q\Phi_1\Phi_n & Q\Phi_2\Phi_n & \cdot & Q\Phi_d\Phi_n & \cdot & Q\Phi_n\Phi_n \end{bmatrix}_k$$

In the above, the subscript $d$ refers to the driving point of the test. That is, the main or reference exciter in the modal survey test is aligned parallel to the physical coordinate corresponding to the d'th location in the column matrix, q. Because the structure is directly excited at the driving location and both input force and response are directly measured, the data associated with this location would have good signal to noise ratio.

In Eq. (4-5), it is seen that the modal column (mode shape) of the k'th mode is proportional to any column of the residue matrix of k'th mode. Thus given the numerical estimate of one column of the residue, one knows the mode shape to within an arbitrary scale factor.

To normalize such that the scalar element of the mode shape at the driving point is unity (Method (a) of Sec. 4.2), first choose the column of $R^k$ that contains the driving point. Then

$$Q_k \Phi_{dk} \begin{bmatrix} \Phi_{1k} \\ \Phi_{2k} \\ \cdot \\ \Phi_{dk} \\ \cdot \\ \Phi_{nk} \end{bmatrix} = \begin{bmatrix} \hat{R}_{d1} \\ \hat{R}_{d2} \\ \cdot \\ \hat{R}_{dd} \\ \cdot \\ \hat{R}_{dn} \end{bmatrix}^k . \tag{4-6}$$

Assign $\Phi_{dk} = 1 + i0$. Then the estimate for $Q_k$ is

$$\hat{Q}_k = \hat{R}^k_{dd}. \tag{4-7a}$$

The properly scaled estimate for the mode shape is then

$$\hat{\Phi}_k = \frac{1}{\hat{R}^k_{dd}} \begin{bmatrix} \hat{R}_{d1} \\ \hat{R}_{d2} \\ \cdot \\ \hat{R}_{dd} \\ \cdot \\ \hat{R}_{dn} \end{bmatrix}^k . \tag{4-7b}$$

To normalize such that $Q_k = 1/2\lambda_k$ (Method (b) of Section 4.1), assign $Q_k = 1/2\hat{\lambda}_k$ in Eq. (4-6). Then

$$\hat{\Phi}_{dk} = (2\hat{\lambda}_k \hat{R}^k_{dd})^{\frac{1}{2}} \tag{4-8a}$$

and the properly scaled mode shape is:

$$\hat{\Phi}_k = \left(\frac{2\hat{\lambda}_k}{\hat{R}_{dd}}\right)^{\frac{1}{2}} \begin{bmatrix} \hat{R}_{d1} \\ \hat{R}_{d2} \\ \cdot \\ \hat{R}_{dd} \\ \cdot \\ \hat{R}_{dn} \end{bmatrix}^k . \tag{4-8b}$$

## 5.0 MODAL IDENTITIES

### 5.1 Bessel's and Parseval's Equations

Bessel's and Parseval's equations offer a convenient means of establishing certain modal identities that are useful for validating completeness of mode sets and modal truncation. The form of the equations that appear in standard references is not general enough for the situation at hand. The desired forms are derived from first principles, in this section.

Consider the real-valued arbitrary vector $\mathbf{G}$ of the inner product space as defined in Chapter 3.3. It may be represented in the form given in Eq. (3-26) (repeated below for convenience):

$$\mathbf{G} = \sum_{k=1}^{n} (\alpha_k \Upsilon_k + \alpha_k^* \Upsilon_k^*). \tag{5-1}$$

The Fourier coefficients, $\alpha_k$, are obtained by multiplying Eq. (5-1) by $\Gamma_r^T A$ and use of the orthogonality relationships, Eqs. (3-15). Likewise an expression for $\alpha_k^*$ is obtained by multiplication by $\Gamma_r^{*T} A$. The following are obtained.

$$\alpha_k = \frac{\Gamma_k^T A \mathbf{G}}{\Gamma_k^T A \Upsilon_k}; \qquad \alpha_k^* = \frac{\Gamma_k^{*T} A \mathbf{G}}{\Gamma_k^{*T} A \Upsilon_k^*}. \tag{5-2}$$

Similarly, $\mathbf{G}$ may be represented in terms of $\Gamma_k$ as

$$\mathbf{G}^T = \sum_{k=1}^{n} (\beta_k \Gamma_k^T + \beta_k^* \Gamma_k^{*T}). \tag{5-3}$$

The corresponding Fourier coefficients, derived by post-multiplication by $A\Upsilon_k$ and $A\Upsilon_k^*$ as above, are

$$\beta_k = \frac{\mathbf{G}^T A \Upsilon_k}{\Gamma_k^T A \Upsilon_k}; \qquad \beta_k^* = \frac{\mathbf{G}^T A \Upsilon_k^*}{\Gamma_k^{*T} A \Upsilon_k^*} \tag{5-4}$$

Interrelationships between the $\alpha_k$'s and $\beta_k$'s may be derived, but are not needed herein.

Consider the inner product,

$$\{\mathbf{G}^T - \sum_{k=1}^{n} (\beta_k \Gamma_k^T + \beta_k^* \Gamma_k^{*T})\} A \{\mathbf{G} - \sum_{k=1}^{n} (\alpha_k \Upsilon_k + \alpha_k^* \Upsilon_k^*)\} \geq 0. \tag{5-5}$$

The above inner product is real, and greater than or equal to zero because $A$ is positive - definite and the right and left multiplying vectors are equal. The expression equals zero provided the basis and dual, $\Upsilon_k$ and $\Gamma_k$, are complete. If they are incomplete, due to modal truncation for example, then the inner product is greater than zero. (If truncation is done, the dependence between the bases $\Upsilon_k$ and $\Gamma_k$ must be taken account of.)

The following result is achieved by multiplying Eq. (5-5) out, and simplifying with the orthogonality relations and the relationship $\Gamma_k^T A \Upsilon_k = \Upsilon_k^T A \Gamma_k$:

$$\mathbf{G}^T A \mathbf{G} \geq \sum_{k=1}^{n} \{(\Gamma_k^T A \Upsilon_k) \alpha_k \beta_k + (\Gamma_k^{*T} A \Upsilon_k^*) \alpha_k^* \beta_k^*\} \tag{5-6}$$

The above is the equivalent of Bessel's inequality. If the bases are complete, the equality holds and the relation is referred to as Parseval's Equation. The relations can be established to be valid for $n$ equal to infinity[14,15].

Consider now a second different arbitrary function $\bar{\mathbf{G}}$, contained in $E^{2n}$:

$$\bar{\mathbf{G}} = \sum_{k=1}^{n} (\bar{\alpha}_k \Upsilon_k + \bar{\alpha}_k^* \Upsilon_k), \tag{5-7}$$

where $\bar{\alpha}_k$ and $\bar{\alpha}_k^*$ are as per Eq. (5-2) with $\mathbf{G}$ replaced by $\bar{\mathbf{G}}$. Also

$$\bar{\mathbf{G}}^T = \sum_{k=1}^{n} (\bar{\beta}_k \Gamma_k^T + \bar{\beta}_k^* \Gamma_k^T), \tag{5-8}$$

where $\bar{\beta}_k$ and $\bar{\beta}_k^*$ are as per Eq. (5-4) with $\mathbf{G}$ replaced by $\bar{\mathbf{G}}$. As per Eq. (5-6),

$$\bar{\mathbf{G}}^T A \bar{\mathbf{G}} \geq \sum_{k=1}^{n} \{(\mathbf{\Gamma}_k^T A \mathbf{\Upsilon})\bar{\alpha}_k \bar{\beta}_k + (\mathbf{\Gamma}_k^{*T} A \mathbf{\Upsilon}_k^*)\bar{\alpha}_k^* \bar{\beta}_k^*\}. \tag{5-9}$$

The vector $(\mathbf{G} + \bar{\mathbf{G}})$ may also be represented in terms of $\mathbf{\Upsilon}_k$ and $\mathbf{\Gamma}_k$ basis vectors. The corresponding coefficients are $(\alpha_k + \bar{\alpha}_k)$ and $(\beta_k + \bar{\beta}_k)$. Bessels Inequality for $(\mathbf{G} + \bar{\mathbf{G}})$ is

$$(\mathbf{G} + \bar{\mathbf{G}})^T A (\mathbf{G} + \bar{\mathbf{G}}) \geq \sum_{k=1}^{n} \{(\mathbf{\Gamma}_k^T A \mathbf{\Upsilon}_k)(\alpha_k + \bar{\alpha}_k)(\beta_k + \bar{\beta}_k) + \mathbf{\Gamma}_k^{*T} A \mathbf{\Upsilon}_k^* (\alpha_k^* + \bar{\alpha}_k^*)(\beta_k^* + \bar{\beta}_k^*)\}. \tag{5-10}$$

Multiplying Eq. (5-10) out and reduction using Eqs. (5-6) and (5-9) leads to

$$\bar{\mathbf{G}}^T A \mathbf{G} \geq \frac{1}{2} \sum_{k=1}^{n} \{\mathbf{\Gamma}_k^T A \mathbf{\Upsilon}_k (\bar{\alpha} \beta_k + \alpha_k \bar{\beta}_k) + \mathbf{\Gamma}_k^{*T} A \mathbf{\Upsilon}_k^* (\bar{\alpha}_k^* \beta_k^* + \alpha_k^* \bar{\beta}_k^*)\}. \tag{5-11}$$

Eq. (5-11) is the equivalent of the general form of Bessel's Equation.

### 5.2 Identities Involving Modal Linear Momentum

Consider $\mathbf{\Phi}_k$ in terms of components:

$$\mathbf{\Phi}_k^T = [\theta_k^T, \phi_k^T, \psi_k^T] \tag{5-12}$$

where $\theta_k, \phi_k$, and $\psi_k$ are each $N \times 1$ column matrices and correspond to the $U, V$, and $W$ coordinates of the displacement. Then

$$\mathbf{\Gamma}_k^T = [-\lambda_k \theta_k^T, -\lambda_k \phi_k^T, -\lambda_k \psi_k^T, \theta_k^T, \phi_k^T, \psi_k^T] \tag{5-13a}$$

$$\mathbf{\Upsilon}_k^T = [\lambda_k \theta_k^T, \lambda_k \phi_k^T, \lambda_k \psi_k^T, \theta_k^T, \phi_k^T, \psi_k^T] \tag{5-13b}$$

$$\mathbf{\Gamma}_k^T A \mathbf{\Upsilon}_k = -\lambda_k / Q_k. \tag{5-13c}$$

Let the arbitrary $\mathbf{G}$ of Eq. (5-1) assume the value, $\mathbf{J}_x$, where

$$\mathbf{J}_x^T = [\Sigma^T, 0^T, 0^T, 0^T, 0^T, 0^T], \tag{5-14}$$

each $0$ is understood to be an $N \times 1$ column matrix, and $\Sigma$ is an $N \times 1$ column matrix,

$$\Sigma^T = [1, 1, 1, ...., 1]. \tag{5-15}$$

The Fourier coefficients for the above $\mathbf{J}_x$, obtained from Eqs. (5-2), (5-4) and (5-13), are

$$\alpha_k = Q_k \Sigma^T M^N \theta_k; \qquad \alpha_k^* = Q_k^* \Sigma^T M^N \theta_k^*$$

$$\beta_k = -Q_k \Sigma^T M^N \theta_k; \qquad \beta_k^* = -Q_k^* \Sigma^T M^N \theta_k^*. \tag{5-16}$$

Likewise

$$\mathbf{J}_x^T A \mathbf{J}_x = \Sigma^T M^N \Sigma = \sum_{i=1}^{N} m^i. \tag{5-17}$$

Thus $\mathbf{J}_x^T A \mathbf{J}_x$ equals the total mass, $m$, of the structure. Substitution of Eqs. (5-16) and (5-17) into Bessel's Inequality, Eq. (5-6), results in a desired identity,

$$m \geq \sum_{k=1}^{n} \{\lambda_k Q_k (\Sigma^T M^N \theta_k)^2 + \lambda_k^* Q_k^* (\Sigma^T M^N \theta_k^*)^2\}. \qquad (5-18)$$

Define the quantities, $P_{xk}, P_{yk}$, and $P_{zk}$ by

$$P_{xk} = \sum_{i=1}^{N} m^i \theta_k^i = \Sigma^T M^N \theta_k \qquad (5-19a)$$

$$P_{yk} = \sum_{i=1}^{N} m^i \phi_k^i = \Sigma^T M^N \phi_k \qquad (5-19b)$$

$$P_{zk} = \sum_{i=1}^{N} m^i \psi_k^i = \Sigma^T M^N \psi_k. \qquad (5-19c)$$

$P_{xk}$, $P_{yk}$, and $P_{zk}$ are the $0x, 0y$ and $0z$ components (Fig. 1) of the amplitude of the total linear momentum of the $k^{th}$ vibrational mode of the structure. Then Eq. (5-18) may be written

$$\sum_{k=1}^{n} \{\lambda_k Q_k P_{xk}^2 + \lambda_k^* Q_k^* P_{xk}^{*2}\} \leq m. \qquad (5-20a)$$

That is, the sum of the modal linear momentum components in the $0x$ direction (multiplied by scale factors $\lambda_k Q_k$) is bounded, and is less than the total mass of the structure.

Similarily, defining $\mathbf{J}_y$ and $\mathbf{J}_z$ as

$$\mathbf{J}_y^T = [0^T, \Sigma^T, 0^T, 0^T, 0^T, 0^T] \qquad (5-15b)$$

$$\mathbf{J}_z^T = [0^T, 0^T, \Sigma^T, 0^T, 0^T, 0^T] \qquad (5-15c)$$

and, with these as the arbitrary $\mathbf{G}$'s in Eq. (5-6), one obtains

$$\sum_{k=1}^{n} \{\lambda_k Q_k P_{yk}^2 + \lambda_k^* Q_k^* P_{yk}^{*2}\} \leq m \qquad (5-20b)$$

$$\sum_{k=1}^{n} \{\lambda_k Q_k P_{zk}^2 + \lambda_k^* Q_k^* P_{zk}^{*2}\} \leq m. \qquad (5-20c)$$

A similar procedure with the general form of Bessel's inequality (Eq. (5-11)) and $\mathbf{G} = \mathbf{J}_x, \mathbf{G} = \mathbf{J}_y$, leads to

$$\sum_{k=1}^{n} \{\lambda_k Q_k P_{xk} P_{yk} + \lambda_k^* Q_k^* P_{xk}^* P_{yk}^*\} \leq 0. \qquad (5-20d)$$

With the other combinations of $J_x$, $J_y$, and $J_z$ for $G$ and $G$ in Eq. (5-11), there results

$$\sum_{k=1}^{n} \{\lambda_k Q_k P_{xk} P_{zk} + \lambda_k^* Q_k^* P_{xk}^* P_{zk}^*\} \leq 0, \qquad (5-20e)$$

$$\sum_{k=1}^{n}\{\lambda_k Q_k P_{yk} P_{zk} + \lambda_k^* Q_k^* P_{yk}^* P_{zk}^*\} \le 0. \tag{5-20f}$$

The above six relations can be expressed in a single matrix relation

$$\sum_{k=1}^{n}\{\lambda_k Q_k \mathbf{P}_k \mathbf{P}_k^T + \lambda_k^* Q_k^* \mathbf{P}_k^* \mathbf{P}_k^{*T}\} \le \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}, \tag{5-21}$$

where $\mathbf{P}_k^T = \{P_{xk}, P_{yk}, P_{zk}\}$.

### 5.3 Identities Involving Modal Angular Momentum

Define the $6N \times 1$ column matrices,

$$\mathbf{L}_x^T = [0^T, -Z^T, Y^T, 0^T, 0^T, 0^T] \tag{5-22a}$$

$$\mathbf{L}_y^T = [Z^T, 0^T, -X^T, 0^T, 0^T, 0^T] \tag{5-22b}$$

$$\mathbf{L}_z^T = [-Y^T, X^T, 0^T, 0^T, 0^T, 0^T] \tag{5-22c}$$

where $X, Y$, and $Z$ position matrices of the particles of the structure as defined in Eq. (2-2).

Let the arbitrary vector, $\mathbf{G}$ of Eq. (5-1) assume the value $\mathbf{L}_x$. $\mathbf{L}_x^T A \mathbf{L}_x$ equals $Z^T M^N Z + Y^T M^N Y$, which equals $\sum_{i=1}^{N} m^i (y^{i2} + z^{i2})$, the moment of inertia, $I_{xx}$. The corresponding Fourier coefficients work out to

$$\alpha_k = Q_k (Y^T M^N \psi_k - Z^T M^N \phi_k).$$

The quantity in parentheses further reduces to

$$\sum_{i=1}^{N} m^i (y^i \psi_k^i - z^i \phi_k^i)$$

which is the amplitude of the total angular momentum of the $k^{th}$ vibrational mode about the 01 axis, $H_{xk}$. With this recognition,

$$\alpha_k = Q_k H_{xk} \qquad \alpha_k^* = Q_k^* H_{xk}^*$$

$$\beta_k = -Q_k H_{xk} \qquad \beta_k^* = -Q_k^* H_{xk}^*. \tag{5-23}$$

The quantity, $\mathbf{L}_x^T A \mathbf{L}_x$, reduces to $Z^T M^N Z + Y^T M^N Y$, which further reduces to

$$\sum_{i=1}^{N} m^i (y^{i2} + z^{i2})$$

which is equal to $I_{xx}$, the mass moment of inertia of the total structure about the 01 axis. Substitution of the above results into Eq. (5-6) yields the identity

$$\sum_{k=1}^{n}\{\lambda_k Q_k H_{xk}^2 + \lambda_k^* Q_k^* H_{xk}^{*2}\} \le I_{xx}. \tag{5-24a}$$

Similar procedures with the $\mathbf{G}$ and $\bar{\mathbf{G}}$ chosen to be various combinations of $\mathbf{L}_x$, $\mathbf{L}_y$, and $\mathbf{L}_z$ lead to

$$\sum_{k=1}^{n}\{\lambda_k Q_k H_{yk}^2 + \lambda_k^* Q_k^* H_{yk}^{*2}\} \leq I_{yy} \qquad (5-24b)$$

$$\sum_{k=1}^{n}\{\lambda_k Q_k H_{zk}^2 + \lambda_k^* Q_k^* H_{zk}^{*2}\} \leq I_{zz} \qquad (5-24c)$$

$$\sum_{k=1}^{n}\{\lambda_k Q_k H_{xk} H_{yk} + \lambda_k^* Q_k^* H_{xk}^* H_{yk}^*\} \leq I_{xy} \qquad (5-24d)$$

$$\sum_{k=1}^{n}\{\lambda_k Q_k H_{xk} H_{zk} + \lambda_k^* Q_k^* H_{xk}^* H_{zk}^*\} \leq I_{xz} \qquad (5-24e)$$

$$\sum_{k=1}^{n}\{\lambda_k Q_k H_{yk} H_{zk} + \lambda_k^* Q_k^* H_{yk}^* H_{zk}^*\} \leq I_{yz}. \qquad (5-24f)$$

In the above,

$$H_{xk} = Y^T M^N \psi_k - Z^T M^N \phi_k \qquad (5-25a)$$

$$H_{yk} = Z^T M^N \theta_k - X^T M^N \psi_k \qquad (5-25b)$$

$$H_{zk} = X^T M^N \phi_k - Y^T M^N \theta_k. \qquad (5-25c)$$

The above six relations of Eq. (5-24) can be expressed in a vector manner similar to that of Eq. (5-21).

### 5.4 Identities Involving Modal Linear and Angular Momentum

With $\mathbf{G}$ and $\bar{\mathbf{G}}$ chosen as $\mathbf{J}_x, \mathbf{J}_y$, or $\mathbf{J}_z$ and paired with the appropriate $\mathbf{L}_x, \mathbf{L}_y$ or $\mathbf{L}_z$, the following identity can be obtained:

$$\sum_{k=1}^{n}\{\lambda_k Q_k \mathbf{H}_k \mathbf{P}_k^T + \lambda_k^* Q_k^* \mathbf{H}_k^{*T} \mathbf{P}_k^{*T}\} \leq m \begin{bmatrix} 0 & c_z & -c_x \\ -c_z & 0 & c_y \\ c_x & -c_y & 0 \end{bmatrix}. \qquad (5-27)$$

In the above,

$$c_x = \sum_{i=1}^{N} m^i x^i / m; \qquad c_y = \sum_{i=1}^{N} m^i y^i / m; \qquad c_z = \sum_{i=1}^{N} m^i z^i / m \qquad (5-28)$$

where $c_x, c_y$, and $c_z$ are the coordinates of the center of mass of the structure.

## 6.0 DISCUSSION AND CONCLUSIONS

The foregoing has developed the natural modes and modal model for an elastic structure with linear viscous damping, via a formulation which gives a level of visibility into the system that is comparable to that of the classical normal modes formulation of the undamped case. The transfer functions and normalizations of use for experimental modal parameter estimation are given special attention. The counterparts of mass-properties-related modal identities recently obtained for undamped modes are obtained also.

Complex numbers and variables are used herein in order to be compatible with earlier-obtained well-known forms of transfer functions and other expressions. However, it should be noted that the complete formulation could be done in terms of real-valued modal vectors and real-valued modal variables (or stated another way, the appropriate vector space for this case is a real-valued inner product space). The use of complex quantities is a matter of convenience and not necessity. The term 'complex modes' often used in this case, is somewhat of a misnomer because the modes are definable and physically interpretable solely with real numbers. 'Damped natural modes' would appear to be a more suitable terminology.

The arrangement of the physical-variable equations in state vector form employed herein is that introduced in Refs. 9 and 10, rather than the usual arrangement of Ref. 3. The arrangement offers two advantages: it leads naturally to a relatively uncomplicated derivation of mass-properties-related modal identities; and, it can be easily generalized further to include gyroscopic forces if desired.

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## 9.0 NOTATION

| | |
|---|---|
| $A$ | Parameter matrix $(2n \times 2n)$, Eq. (2-10) |
| $a_k$ | Real number (scalar) |
| $B$ | Parameter matrix $(2n \times 2n)$, Eq. (2-10) |
| $b_k$ | Real number (scalar) |
| $C$ | Damping matrix $(n \times n)$ |
| $C_{xx}, C_{xy}$,etc. | Damping matrices $(N \times N)$ |
| $c_k$ | modal damping |
| $d_k$ | Modal scale factor (complex scalar) |
| $\mathbf{F}$ | Force matrix $(2n \times 1)$ |
| $\mathbf{f}$ | Force vector $(n \times 1)$ |
| $f_x, f_y, f_z$ | Force components $(N \times 1)$ |
| $\mathbf{G}, \bar{\mathbf{G}}$ | Arbitrary (real) vectors in $E^{2n}$ $(2n \times 1)$ |
| $\mathbf{H}_k$ | Modal angular momentum vector |
| $H(t), \bar{H}(s)$ | Transfer function matrices $(n \times n)$ |
| $H_{xk}, H_{yk}, H_{zk}$ | Components of $\mathbf{H}_k$ (complex scalars) |
| $I; I_{xx}$,etc. | Moment of inertia matrix $(3 \times 3)$ and components |
| $i$ | $\sqrt{-1}$ |
| $\mathbf{J}_x, \mathbf{J}_y, \mathbf{J}_z$ | Summation matrix $(2n \times 1)$, Eq. (5-14) |
| $K$ | Stiffness matrix $(n \times n)$ |
| $K_{xx}, K_{xy}$, etc. | Stiffness matrices $(N \times N)$ |
| $k_k$ | modal stiffness |
| $\mathbf{L}_x, \mathbf{L}_y, \mathbf{L}_z$ | Defined in Eq. (5-14) $(2n \times 1)$ |
| $M$ | Mass matrix $(n \times n)$, Eq. (2-6) |
| $M^N$ | Mass matrix $(N \times N)$, Eq. (2-4) |
| $m$ | Total mass of structure |
| $m^i$ | Mass of i'th point of structure |
| $m_k$ | modal mass |
| $N$ | Number of mass particles |
| $n$ | $n = 3N$.Order of Eq. (2-6) |
| $0x, 0y, 0z$ | Coordinate axes, Fig. 1 |
| $\mathbf{P}_k$ | Linear modal momentum vector (complex, $(3N \times 1)$) |
| $P_{xk}, P_{yk}, P_{zk}$ | Components of $\mathbf{P}_k$ (complex scalars) |
| $p_k$ | Modal coordinate, Eq. (3-44) |

| | |
|---|---|
| $\mathbf{Q}$ | State variable $[\dot{\mathbf{q}}^T, \mathbf{q}^T](2n \times 1)$ |
| $Q_k$ | Modal normalization constant (complex scalar) |
| $\mathbf{q}, \mathbf{q}_0, \dot{\mathbf{q}}_0$ | Deformation vector variable and initial conditions $(n \times 1)$ |
| $R^k$ | Residue matrix (complex $(n \times n)$) |
| $s$ | Laplace variable |
| $\mathbf{S}_k$ | Real part of $Q_k \mathbf{\Phi}_k$ |
| $\mathbf{T}_k$ | Imaginary part of $Q_k \mathbf{\Phi}_k$ |
| $t$ | time |
| $U, V, W$ | Deformation matrices $(N \times 1)$ |
| $u^i, v^i, w^i$ | Deformaion components of *ith* mass point |
| $X, Y, Z$ | Position matrices collection of mass points $(N \times 1)$ |

| | |
|---|---|
| $\alpha_k, \beta_k$ | Complex Fourier coefficients (scalars) |
| $\mathbf{\Gamma}_k, \mathbf{\Upsilon}_k$ | Left and right eigenvectors (complex, $(2n \times 1)$) |
| $\varsigma_k$ | Modal damping factor |
| $\theta_k, \phi_k, \psi_k$ | Eigenvector components corresponding to 0x, 0y, 0z coordinates (complex, $(N \times 1)$) |
| $\lambda_k$ | Complex eigenvalue (scalar) |
| $\xi_k(t), \eta_k(t)$ | Real modal variables (scalars) |
| $\sigma_k, \nu_k$ | Real and imaginary parts of $\lambda_k$ |
| $\rho_k(t)$ | Complex modal variable (scalar), $\xi_k + i\nu_k$ |
| $\mathbf{\Phi}_k$ | Mode shape (complex, $(n \times 1)$) |
| $\Sigma$ | Summation matrix $(N \times 1)$ (Eq. (5-15)) |
| $\omega_k$ | Modal frequency (scalar) |
| $E^{2n}$ | Euclidean vector space |

**Subscripts and Superscripts**

| | |
|---|---|
| k | denotes kth mode |
| N | denotes number of mass particles |
| n | 3N |
| x,y,z | denotes components in (0x,0y,0z) coordinate system, Fig. 1 |
| s,i | denotes the s'th or i'th particle |
| * | denotes complex conjugate |
| u,l | upper and lower |

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