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# ANALYSIS OF INTERFERENCE EFFECTS IN SATELLITE ON-BOARD REGENERATIVE REPEATERS ${ }^{1}$ 

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#### Abstract

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Abstract This report provides an introduction to the problem of analyzing the performance of satellite on-board regenerative repeaters. Two specific group demodulator architectures are discussed: one based on a polyphase network followed by an FFT processor, and the other based on a chirp Fourier transform implemented using surface acoustic wave (SAW) devices. Due to imperfect filtering and channel separation in the group demodulator, intersymbol and interchannel interference are introduced into the demodulated output. To show how it is possible to derive error performance curves in the presence of intersymbol and interchannel interference, two different analysis methods are presented.


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## Chapter 1

## Introduction

The "switchboard in the sky" is a concept that has been developing in satellite communications over the last 20 years. By having a satellite that is a regenerative repeater with multi-beam antennas and baseband switching, the need for centralized control is eliminated and it becomes possible to have single-hop transmissions between small, portable earth terminals. Regeneration on-board the satellite also makes it possible to use different multiple access schemes on the up-link and the down-link. By decoupling the up-link and the downlink, both can be optimized for greater power efficiency and bandwidth utilization.

There have been numerous designs proposed for the on-board regenerative repeater, both in the digital and analog domains. When the up-link access method is FDMA and the signals are of equal bandwidth and uniformly spaced in frequency, the process of regenerating the carriers is very similar to a spectral analysis in which a wideband signal is filtered, sampled and then divided into uniform subbands. At the centre of each of these subbands will be one of the desired signals. In the context of on-board processing, this "spectral analyzer" is most often referred to as a group demodulator or a multi-carrier demodulator (MCD). The demultiplexing part of the process (separating out the signals) can be performed by a number of difference structures, including a polyphase filter network followed by an FFT processor [1-3]; a multi-stage filter bank in which half-band filters are arranged in a binary tree [4, 5]; an FFT processor followed by frequency domain filtering and an inverse FFT $[6,7]$; or possibly a chirp Fourier transform implemented using surface acoustic wave (SAW) devices [8-12].

In most of the work presented to date on multi-carrier demodulators, simulation has been the tool used to analyze the system's performance. This is primarily due to the fact that imperfect filtering and channel separation cause intersymbol and interchannel interference to appear in the demodulated output signal, making it difficult to derive an expression for the probability of error. The purpose of this report is to look at methods by which the performance of a group demodulator
can be evaluated. Two MCD architectures in particular will be analyzed; the digital polyphase filter bank/FFT processor combination, and the SAW-based chirp Fourier transform.

The subject of performance evaluation in the presence of interference is an old one. ${ }^{1}$ A number of the techniques that have been proposed involve computing the moments of the interference and then the moments are used in a Gram-Charlier series [14,15] for the probability of error or as part of a Gauss quadrature rule [16-18]. Another approach that shows fairly rapid convergence is a Fourier series method due to Beaulieú [19]. These latter two methods will be investigated here for the SAW-based group demodulator and the polyphase network.

In the next two sections of this report, we give a description of the two group demodulator architectures and derive a general expression for the probability of bit error that includes the effects of intersymbol and interchannel interference. In both cases, this expression involves integrating over the probability density function of the interference. This probability density function is unknown and virtually impossible to find. The analysis methods presented in section 4 get around this problem by using other characteristics of the interference that can be easily computed. In particular, the moments and the characteristic function of the interference can be computed without knowing the probability density function of the interference. We adapt the use of these two interference analysis methods to the two group demodulator architectures under discussion and show the sources of error associated with these computation methods. These methods are then used to compute a set of numerical results that are presented in section 5. Various parameters are adjusted to see what the effects are on the interference and the group demodulator's performance. Part of the purpose of some of these computations is to get a general feel for how well the computation methods work, how fast the computations carf be performed, and what are the relative accuracies of the methods for a given number of terms in the computation. Finally, conclusions and a discussion of future work are presented in section 6.

[^1]
## Chapter 2

## Analysis of the SAW Group Demodulator

The basic receiver structure of a SAW-based group demodulator is as depicted in Figure 2.1. The SAW chirp Fourier transform is an IF processor; consequently, it is followed by a down-conversion system and a pair of quantizers for the in-phase and quadrature components of the baseband signal. The baseband signal is then processed by a phase decoder subsystem that tracks and corrects carrier phase offsets and recovers the data. As we shall see, the QPSK/FDMA signals appear at the output of the chirp Fourier transform as a set of sequential pulses. Each channel is sampled in turn and the phase decoder subsystem is time shared between the channels.

As indicated in Figure 2.2, there are two ways to implement the chirp Fourier transform using SAW devices. The first configuration uses two multipliers to which down-chirps are applied and a SAW convolver which is an up-chirp. For obvious reasons, this is called the multiply-convolvemultiply (MCM) configuration. The second configuration uses two SAW convolvers and a single multiplier and is called the convolve-multiply-convolve (CMC) configuration. Strictly speaking, the


Figure 2.1: Block diagram a SAW-based group demodulator structure.
mathematical expressions for the outputs of the configurations show that they are not equivalent [9]. The MCM configuration provides a true short time Fourier transform of the input signal while the CMC configuration provides only an approximation. If we compare the two outputs quantitatively, however, they are virtually indistinguishable. The CMC configuration has several implementation advantages over the MCM configuration, most notably the fact that the SAW convolvers are of half the duration of that required by the MCM configuration. This means that devices of half the physical length are required for the CMC configuration. In addition, it is possible to incorporate any anti-aliasing pre-filtering into the frequency response of the first convolver, and if windowing is desired, it can be built into the frequency response of the second convolver. For the MCM configuration, separate circuitry would be required to perform these two functions. Thus, it can be said that from an analysis stand point that the MCM configuration is simpler to work with, but from the point of view of implementation, the CMC configuration is more desirable.

In the analysis that follows, the mathematical expression for the MCM configuration will be used. In terms of the actual performance, the CMC configuration will be equivalent.

### 2.1 The FDMA Signal

The input $x(t)$ to the CFT consists of $N$ QPSK/FDMA signals plus noise,

$$
\begin{equation*}
x(t)=\sum_{k=-N / 2}^{N / 2-1} \tilde{s}_{k}(t) e^{j \omega_{c} t}+z(t) \tag{2.1}
\end{equation*}
$$

assuming an even number of channels $N$. The complex envelope of the $k$-th QPSK signal can be described by the expression

$$
\begin{equation*}
\tilde{s}_{k}(t)=\sum_{n=-\infty}^{\infty} A_{k} h_{s}\left(t-n T-\gamma_{k}\right) e^{j\left[\omega_{k} t+\alpha_{k, n}+\phi_{k}\right]} \tag{2.2}
\end{equation*}
$$

where $A_{k}$ is the signal amplitude, $\omega_{k}=2 \pi\left(k+\frac{1}{2}\right) \Delta f$ is the angular frequency offset with respect to the center frequency $\omega_{c}$ of the group, and $\Delta f$ is the frequency spacing between carriers. The parameters $\gamma_{k}$ and $\phi_{k}$ represent random timing and phase offsets, respectively, while $\alpha_{k, n}$ is the modulating phase in the $n$-th symbol interval of the $k$-th channel. The function $h_{s}(t)$ is a pulse shaping function that is assumed here to be a square pulse,

$$
h_{s}(t)=\left\{\begin{array}{cc}
1 & -\frac{T}{2} \leq t<\frac{T}{2}  \tag{2.3}\\
0 & \text { otherwise }
\end{array}\right.
$$

where $T$ is the symbol duration.

### 2.2 The Chirp Fourier Transform

The output of the CFT is a time wayeform $y(\tau)$ in which the time variable $\tau$ plays the role of frequency. It is well known that the Fourier transform of the square pulse shape $h_{s}(t)$ is a sinc function that has very large sidelobes. Traditionally, windowing has been employed to reduce the large sidelobes in the output of a short-time Fourier transform. However, there are two costs associated with windowing: the signal strength is reduced and the width of the mainlobe is increased causing an increase in the noise variance. The Kaiser-Bessel window function,

$$
\begin{equation*}
w(t)=\frac{I_{0}\left(\pi \delta \sqrt{1-\frac{4 t^{2}}{T^{2}}}\right)}{I_{0}(\pi \delta)} \tag{2.4}
\end{equation*}
$$

provides one of the best trade-offs between sidelobe suppression and main lobe expansion. If KaiserBessel windowing is included in the CFT, the complex envelope of the output in the $m$-th interval $m T-\frac{T}{2} \leq t<m T+\frac{T}{2}$ is given by [9]

$$
\begin{equation*}
\tilde{y}(\tau)=\int_{m T-\frac{T}{2}}^{m T+\frac{T}{2}} \tilde{x}(t) e^{-j \mu(\tau-m T) t} w(t-m T) d t \tag{2.5}
\end{equation*}
$$

where $\mu$ is the chirp slope.
The contribution to the output $\tilde{y}$ made by the $k$-th input signal, $\tilde{s}_{k}(t)$, is given by

$$
\begin{equation*}
\tilde{S}_{k}(\tau)=\sum_{n=-\infty}^{\infty} A_{k} e^{j \phi_{k}} \int_{m T-\frac{T}{2}}^{m T+\frac{T}{2}} h_{s}\left(t-n T-\gamma_{k}\right) e^{j\left[\omega_{k} t+\alpha_{k, n}\right]} e^{-j \mu(\tau-m T) t} w(t-m T) d t \tag{2.6}
\end{equation*}
$$

With a square shaping pulse, and for $\gamma_{k}>0$, this reduces to

$$
\begin{align*}
& \tilde{S}_{k}(\tau)=A_{k} e^{j \phi_{k}}\left\{\int_{m T-\frac{T}{2}}^{m T-\frac{T}{2}+\gamma_{k}} e^{j\left[\omega_{k} t+\alpha_{k, m-1}\right]} e^{-j \mu(\tau-m T) t} w(t-m T) d t\right. \\
&\left.\quad+\int_{m T-\frac{T}{2}+\gamma_{k}}^{m T+\frac{T}{2}} e^{j\left[\omega_{k} t+\alpha_{k, m}\right]} e^{-j \mu(\tau-m T) t} w(t-m T) d t\right\} \tag{2.7}
\end{align*}
$$

while for $\gamma_{k}<0$,

$$
\begin{align*}
& \tilde{S}_{k}(\tau)=A_{k} e^{j \phi_{k}}\left\{\int_{m T-\frac{T}{2}}^{m T+\frac{T}{2}+\gamma_{k}} e^{j\left[\omega_{k} t+\alpha_{k, m}\right]} e^{-j \mu(\tau-m T) t} w(t-m T) d t\right. \\
&\left.\quad+\int_{m T+\frac{T}{2}+\gamma_{k}}^{m T+\frac{T}{2}} e^{j\left[\omega_{k} t+\alpha_{k, m+1}\right]} e^{-j \mu(\tau-m T) t} w(t-m T) d t\right\} \tag{2.8}
\end{align*}
$$

A closed form solution for the CFT output is not possible and we must resort instead to numerical integration to evaluate (2.7) and (2.8).

Figure 2.3 shows the output for a single channel $\tilde{S}_{k}(\tau)$ with a timing offset of $\gamma_{k}=0.3 T$. The system parameters are that of [10]: the time-bandwidth product of the Kaiser-Bessel window is
$\delta=1.85$, the signals are 64 kbps QPSK/FPMA signals such that $T=31.25 \mu \mathrm{~s}$, and the channel spacing is $\Delta f=96 \mathrm{kHz}$. For the chirp Fourier transform, the chirp slope is $\mu=0.091 \mathrm{MHz} / \mu \mathrm{s}$. The $\tau$-axis has been shifted by $\tau_{k}=\frac{\omega_{k}}{\mu}+m T$ so that the pulse is centered about zero. The figure shows that the output is dependent on the difference between successive data symbols ( $\alpha_{k, m}$ and $\alpha_{k, m-1}$ ). Depending on this difference, there can be substantial attenuation on the peak resulting in a degradation in bit-error-rate performance. The vertical lines at $\tau-\tau_{k}= \pm 1.055 \mu \mathrm{~s}$ and $\pm 2.11 \mu \mathrm{~s}$ indicate the centre positions for the adjacent channels $k \pm 1$ and $k \pm 2$, respectively. If we look at the sidelobe levels at these time instances, they are quite high and indicate that the signal on channel $k$ will interfere significantly with its neighbours. Of course, the converse is also true, timing offsets on the adjacent channels will contribute significant interference into channel $k$, and it is this effect that we wish to analyze.

To recover the data phase $\alpha_{k, m}$ for the $k$-th channel, the output of the chirp Fourier transform must be sampled at the instants $\tau_{k}=\frac{\omega_{k}}{\mu}+m T$. The sampled output of the chirp Fourier transform is then

$$
\begin{equation*}
\tilde{y}\left(\tau_{k}\right)=\tilde{S}_{k}\left(\tau_{k}\right)+\sum_{\substack{l=-N / 2 \\ l \neq k}}^{N / 2-1} \tilde{S}_{l}\left(\tau_{k}\right)+Z\left(\tau_{k}\right) \tag{2.9}
\end{equation*}
$$

where $\tilde{S}_{k}\left(\tau_{k}\right)$ represents the desired signal, $\sum_{l \neq k} \tilde{S}_{l}\left(\tau_{k}\right)$ is the interchannel interference and $Z\left(\tau_{k}\right)$ is a sample from the noise process

$$
\begin{equation*}
Z(\tau)=\int_{m T-T / 2}^{m T+T / 2} z(t) w(t-m T) e^{-j \mu(\tau-m T) t} d t \tag{2.10}
\end{equation*}
$$

If the input noise process $z(t)$ is zero mean, additive white Gaussian noise of variance $\sigma_{i}^{2}$, then $Z(\tau)$ will also be a zero mean, additive white Gaussian noise process but with variance

$$
\begin{equation*}
\sigma_{o}^{2}=\sigma_{i}^{2} \int_{-\frac{T}{2}}^{\frac{T}{2}} w^{2}(t) d t \tag{2.11}
\end{equation*}
$$

Let $W(f)$ denote the Fourier transform of the window function $w(t)$,

$$
\begin{equation*}
W(f)=\int_{-\frac{T}{2}}^{\frac{T}{2}} w(t) e^{-j 2 \pi f t} d t \tag{2.12}
\end{equation*}
$$

and let.

$$
\begin{equation*}
\Gamma_{k}(f)=\frac{1}{W(f)} \int_{-\frac{T}{2}}^{-\frac{T}{2}+\left|\gamma_{k}\right|} w(t) e^{-j 2 \pi f t} d t \tag{2.13}
\end{equation*}
$$

At the sampling instants $\tau_{k}=\frac{\omega_{k}}{\mu}+m T$, the output of the CFT for the desired channel $\tilde{S}_{k}(\tau)$ can then be shown to equal

$$
\begin{equation*}
\tilde{S}_{k}\left(\tau_{k}\right)=A_{k} e^{j \phi_{k}}\left\{\left(1-\Gamma_{k}(0)\right) e^{j \alpha_{k, m}}+\Gamma_{k}(0) e^{j \alpha_{k, m \pm 1}}\right\} W(0) \tag{2.14}
\end{equation*}
$$

while the interference terms take on the form

$$
\begin{equation*}
\tilde{S}_{l}\left(\tau_{k}\right)=A_{l} e^{j \phi_{l}}\left\{\left(1-\Gamma_{l}[(k-l) \Delta f]\right) e^{j \alpha_{l, m}}+\Gamma_{l}[(k-l) \Delta f] e^{j \alpha_{l, m \pm 1}}\right\} W[(k-l) \Delta f] . \tag{2.15}
\end{equation*}
$$

The factor $W(0)$ on the far right in (2.14) represents the loss in signal strength as the area under the curve of the window function is less than that of a rectangular window. This is a fixed loss that is independent of any error in the timing of the signal. The effect of the timing error is embodied in the term $\Gamma_{k}(0)$. The symbol $\alpha_{k, m \pm 1}$ is used to indicate that the previous symbol $\alpha_{k, m-1}$ influences the present output when $\gamma_{k}>0$ and, similarly, the next symbol $\alpha_{k, m+1}$ will influence the present output when $\gamma_{k}<0$. Since the data symbols are assumed to be independent, identically distributed (i.i.d.) random variables, we are more interested in the magnitude of the timing offset, $\left|\gamma_{k}\right|$, than its sign and $\alpha_{k, m \pm 1}$ can be thought of as simply a QPSK data symbol.

### 2.3 The Probability of Bit Error

To compute the probability of bit error for the $k$-th channel, we can split the sampled output signal $\tilde{y}\left(\tau_{k}\right)$ into in-phase and quadrature components. The problem of computing the probability of bit error is easily shown to be symmetric between the in-phase and quadrature components and it is necessary to consider only one of the two. We consider then only the in-phase component $\tilde{y}_{k}^{I}=\Re\left\{\tilde{y}\left(\tau_{k}\right)\right\}$ of the sampled output, which is given by

$$
\begin{equation*}
\tilde{y}_{k}^{I}=\tilde{S}_{k}^{I}\left(\tau_{k}\right)+\sum_{\substack{l=-N / 2 \\ l \neq k}}^{N / 2-1} \tilde{S}_{l}^{I}\left(\tau_{k}\right)+Z_{k}^{I} \tag{2.16}
\end{equation*}
$$

where $\tilde{S}_{k}^{I}\left(\tau_{k}\right), \tilde{S}_{l}^{I}\left(\tau_{k}\right)$, and $N_{k}^{I}=\Re\left\{Z\left(\tau_{k}\right)\right\}$ are the in-phase components of the desired signal, the interference signal on channel $l$, and the noise, respectively. Assuming perfect phase synchronization, ( $\phi_{k}=0$ ), the in-phase component of the desired signal is given by

$$
\begin{align*}
\tilde{S}_{k}^{I}\left(\tau_{k}\right) & =A_{k}\left\{\left[1-\Gamma_{k}(0)\right] \cos \left(\alpha_{k, m}\right)+\Gamma_{k}(0) \cos \left(\alpha_{k, m \pm 1}\right)\right\} W(0) \\
& =\frac{A_{k} W(0)}{\sqrt{2}}\left\{\left[1-\Gamma_{k}(0)\right] a_{k, m}+\Gamma_{k}(0) a_{k, m \pm 1}\right\} \tag{2.17}
\end{align*}
$$

where we have used $e^{j \alpha_{k, m}}=\frac{1}{\sqrt{2}}\left(a_{k, m}+j b_{k, m}\right)$ and $a_{k, m}, b_{k, m}$ represent independent binary data symbols taking on the values of $\{-1,+1\}$ with equal probability. In (2.17), only the in-phase data symbols $a_{k, m}$ and $a_{k, m \pm 1}$ are present and they are assumed to be independent random variables. Note also that $\Gamma_{k}(0)$ and $W(0)$ are always real quantities, as is $W[(k-l) \Delta f]$.

For the interference, $\tilde{S}_{l}\left(\tau_{k}\right)$, the in-phase component is given by

$$
\begin{align*}
& \tilde{S}_{l}^{I}\left(\tau_{k}\right)=\frac{A_{l} W[(k-l) \Delta f]}{\sqrt{2}}\left\{a_{l, m}\left[\left(1-\Gamma_{l}^{I}\right) \cos \phi_{l}-\Gamma_{l}^{Q} \sin \phi_{l}\right]-b_{l, m}\left[\Gamma_{l}^{Q} \cos \phi_{l}+\left(1-\Gamma_{l}^{I}\right) \sin \phi_{l}\right]\right. \\
&\left.+a_{l, m \pm 1}\left[\Gamma_{l}^{I} \cos \phi_{l}-\Gamma_{l}^{Q} \sin \phi_{l}\right]-b_{l, m \pm 1}\left[\Gamma_{l}^{Q} \cos \phi_{l}+\Gamma_{l}^{I} \sin \phi_{l}\right]\right\} \tag{2.18}
\end{align*}
$$

where

$$
\begin{equation*}
\Gamma_{l}^{I}=\Re\left\{\Gamma_{l}[(k-l) \Delta f]\right\}=\frac{1}{W[(k-l) \Delta f]} \int_{-\frac{T}{2}}^{-\frac{T}{2}+\left|\gamma_{l}\right|} w(t) \cos [2 \pi(k-l) \Delta f t] d t \tag{2.19}
\end{equation*}
$$

and

$$
\begin{equation*}
\Gamma_{l}^{Q}=\Im\left\{\Gamma_{l}[(k-l) \Delta f]\right\}=\frac{1}{W[(k-l) \Delta f]} \int_{-\frac{T}{2}}^{-\frac{T}{2}+\left|\gamma_{l}\right|} w(t) \sin [2 \pi(k-l) \Delta f t] d t \tag{2.20}
\end{equation*}
$$

Each interference term $\tilde{S}_{l}\left(\tau_{k}\right)$ is influenced by four binary data symbols, $a_{l, m}, b_{l, m}, a_{l, m \pm 1}$, and $b_{l, m \pm 1}$. All are assumed to be i.i.d. random variables taking on values of $\{-1,+1\}$ with equal probability and independent of one another.

Let $\eta$ denote the sum of the interference terms

$$
\begin{equation*}
\eta=\sum_{\substack{l=-N / 2 \\ l \neq k}}^{N / 2-1} \tilde{S}_{l}^{I}\left(\tau_{k}\right) \tag{2.21}
\end{equation*}
$$

The decision variable $\tilde{y}_{k}^{I}$ can then be expressed as

$$
\begin{equation*}
\tilde{y}_{k}^{I}=\frac{A_{k} W(0)}{\sqrt{2}}\left\{\left[1-\Gamma_{k}(0)\right] a_{k, m}+\Gamma_{k}(0) a_{k, m \pm 1}\right\}+\eta+Z_{k}^{I} \tag{2.22}
\end{equation*}
$$

If we assume that $a_{k, m}=-1$ and average over the two possible values for $a_{k, m \pm 1}$, then the conditional probability of bit error, assuming a fixed timing offset on channel $k$, can be shown to equal

$$
\begin{align*}
P_{b \mid \eta}\left(\gamma_{k}\right) & =\frac{1}{2} \operatorname{Pr}\left(-\frac{A_{k} W(0)}{\sqrt{2}}+\eta+Z_{k}^{I}>0\right)+\frac{1}{2} \operatorname{Pr}\left(-\frac{A_{k} W(0)}{\sqrt{2}}\left[1-2 \Gamma_{k}(0)\right]+\eta+Z_{k}^{I}>0\right) \\
& =\frac{1}{2} \operatorname{Pr}\left(Z_{k}^{I}>\frac{A_{k} W(0)}{\sqrt{2}}-\eta\right)+\frac{1}{2} \operatorname{Pr}\left(Z_{k}^{I}>\frac{A_{k} W(0)}{\sqrt{2}}\left[1-2 \Gamma_{k}(0)\right]-\eta\right) \tag{2.23}
\end{align*}
$$

If $Q(x)$ is the complementary distribution function for Gaussian noise [20, p. 49],

$$
\begin{equation*}
Q(x)=\frac{1}{\sqrt{2 \pi}} \int_{x}^{\infty} e^{-t^{2} / 2} d t \tag{2.24}
\end{equation*}
$$

then the conditional probability of error can be expressed as

$$
\begin{equation*}
P_{b \mid \eta}\left(\gamma_{k}\right)=\frac{1}{2} Q\left(\frac{A_{k} W(0)}{\sqrt{2} \sigma_{o}}-\frac{\eta}{\sigma_{o}}\right)+\frac{1}{2} Q\left(\frac{A_{k} W(0)}{\sqrt{2} \sigma_{o}}\left[1-2 \Gamma_{k}(0)\right]-\frac{\eta}{\sigma_{o}}\right) . \tag{2.25}
\end{equation*}
$$

The probability of bit error $P_{b}$ is finally obtained by averaging $P_{b \mid \eta}$ over the probability density functions $f_{\eta}(\eta)$ and $f_{\gamma_{k}}\left(\gamma_{k}\right)$ of the interference $\eta$ and the timing offset $\gamma_{k}$, respectively

$$
\begin{equation*}
P_{b}=\int_{-\frac{T}{2}}^{\frac{T}{2}}\left[\int_{-\infty}^{\infty} P_{b \mid \eta}\left(\gamma_{k}\right) f_{\eta}(\eta) d \eta\right] f_{\gamma_{k}}\left(\gamma_{k}\right) d \gamma_{k} \tag{2.26}
\end{equation*}
$$

For reasons of simplicity, most of the numerical examples to be presented later in the report will be computed assuming a fixed timing offset. Consequently, we will only want to evaluate the inner integral and will skip the integration over the probability density function $f_{\gamma_{k}}\left(\gamma_{k}\right)$.


Figure 2.2: Two possible methods of implementing a chirp Fourier transform with SAW devices.


Figure 2.3: Example of the output for a single channel with a timing offset of $\gamma_{k}=0.3 T$.

## Chapter 3

## Analysis of the Polyphase Structure

The motivation behind any group demodulator for satellite on-board processing is to arrive at a single unified structure that can demultiplex $N$ FDMA channels as a group with far smaller complexity than $N$ individual circuits for each channel. This can be done by exploiting the fact that the channels are uniformly spaced and have the same filtering requirements. In this section, we will look at the polyphase filter bank/FFT approach to demultiplexing the FDMA channels.

### 3.1 The FDMA Signals

The input to the multi-carrier demodulator will be a baseband signal consisting of $N$ FDMA signals and noise,

$$
\begin{equation*}
y(t)=\sum_{k=0}^{N-1} s_{k}(t)+z(t) . \tag{3.1}
\end{equation*}
$$

Assuming that the modulation is QPSK, the $k$-th FDMA signal takes the form

$$
\begin{equation*}
s_{k}(t)=\sum_{i=-\infty}^{\infty} A_{k} e^{j\left[2 \pi f_{k} t+\alpha_{k, i}+\phi_{k}\right]} h_{s}\left(t-i T_{b}-\gamma_{k}\right) \tag{3.2}
\end{equation*}
$$

where $A_{k}$ is the carrier amplitude, $f_{k}$ is the frequency offset associated with the channel, $\alpha_{k, i}$ is the data phase in the $i$-th symbol interval, and $h_{s}(t)$ is the transmitted pulse shape. The parameters $\gamma_{k}$ and $\phi_{k}$ represent random timing and phase offsets, respectively. In Fig. 3.1, we have a model for the baseband signal $y(t)$ when $N=8$. The 8 channels are arranged such that $f_{k}=\left(k+\frac{1}{2}\right) f_{c}$ where $f_{c}$ is the frequency spacing between channels. Normally, one would represent the baseband signals as being symmetrically distributed about the zero frequency mark, but the representation of Fig. 3.1 proves convenient as the total bandwidth occupied by the signals is $f_{s}=N f_{c}$, which is also the sampling rate.

It is assumed that all the channels use the same transmit filter $h_{s}(t)$ which, in general, will have a root-Nyquist characteristic to split the filtering requirements between transmitter and receiver.


Figure 3.1: Baseband model for $N=8$ channels as input to the demultiplexer.


Figure 3.2: Model for the demultiplexing of a single channel.

If the transmit filter takes on a root-raised cosine characteristic with a rolloff factor of $\beta$ then the signal will occupy a bandwidth of $(1+\beta) f_{b}$, where $f_{b}=1 / T_{b}$ is the symbol rate. The channel separation $f_{c}$ must then be greater than or equal to $(1+\beta) f_{b}$.

### 3.2 The Polyphase Network

For the derivation of the multi-carrier demultiplexer structure based on the polyphase network, we begin with the model of Fig. 3.2 for the demultiplexing of a single channel.

The process of isolating the signal in the $k$-th channel involves shifting the signal to baseband and filtering with a low-pass filter. Since the original signal was sampled at $N$ times the rate of a single channel, we can also decimate the output of the low-pass filter by a factor of $N$. The sampled impulse response of the low-pass filter is denoted by $h\left(n T_{s}\right)$, and it is assumed that the same filter characteristic is used in the demultiplexing of all of the channels, $0 \leq k \leq N-1$. We will also assume that $h\left(n T_{s}\right)$ is a finite impulse response filter with $N L$ taps. The equation for the
combined frequency shifting and filtering operations is then...

$$
\begin{align*}
x_{k}\left(m T_{c}\right) & =\sum_{n=1}^{N L} h_{s}\left(n T_{s}\right) y\left[(m N-n) T_{s}\right] e^{-j 2 \pi\left(k+\frac{1}{2}\right) f_{c}(m N-n) T_{s}} \\
& =e^{-j \pi m} \sum_{n=1}^{N L} h\left(n T_{s}\right) y\left[(m N-n) T_{s}\right] e^{j 2 \pi\left(k+\frac{1}{2}\right) n / N} \tag{3.3}
\end{align*}
$$

where we have made use of the fact that $N T_{s}=1 / f_{c}$ and $e^{-j 2 \pi k m}=1$ when $k$ and $m$ are integers. If we now express $n$ as

$$
n=\nu N-\rho
$$

and sum over both $\nu$ and $\rho$, we can rewrite (3.3) as

$$
\begin{equation*}
x_{k}\left(m T_{c}\right)=(-1)^{m} \sum_{\rho=0}^{N-1} \sum_{\nu=1}^{L} h\left[(\nu N-\rho) T_{s}\right] y\left[(m-\nu) N T_{s}+\rho T_{s}\right] e^{j 2 \pi\left(k+\frac{1}{2}\right)(\nu N-\rho) / N} \tag{3.4}
\end{equation*}
$$

For each value of $\rho$, and running across all values of $\nu$, each set of filter coefficients $h\left[(\nu N-\rho) T_{s}\right]$ can be associated with a separate filter

$$
\begin{equation*}
\bar{p}_{\rho}\left(\nu T_{c}\right)=\bar{p}_{\rho}\left(\nu N T_{s}\right)=h\left[(\nu N-\rho) T_{s}\right] e^{j \pi \nu}=(-1)^{\nu} h\left[(\nu N-\rho) T_{s}\right] \tag{3.5}
\end{equation*}
$$

In a similar fashion, the input signal can be written as $N$ separate, decimated signals

$$
\begin{equation*}
\bar{y}_{\rho}\left[(m-\nu) T_{c}\right]=\bar{y}_{\rho}\left[(m-\nu) N T_{s}\right]=y\left[(m-\nu) N T_{s}+\rho T_{s}\right] . \tag{3.6}
\end{equation*}
$$

If we combine (3.5) and (3.6), and let

$$
\begin{equation*}
W_{N}=e^{j \frac{\pi}{N}} \tag{3.7}
\end{equation*}
$$

then we obtain the final equation for the output

$$
\begin{align*}
x_{k}\left(m T_{c}\right) & =(-1)^{m} \sum_{\rho=0}^{N-1} W_{N}^{-2 k \rho} W_{N}^{-\rho} \sum_{\nu=1}^{L} \bar{p}_{\rho}\left(\nu T_{c}\right) \bar{y}_{\rho}\left[(m-\nu) T_{c}\right] \\
& =(-1)^{m} \sum_{\rho=0}^{N-1} W_{N}^{-2 k \rho} W_{N}^{-\rho}\left[\bar{p}_{\rho}\left(m T_{c}\right) \star \bar{y}_{\rho}\left(m T_{c}\right)\right] \tag{3.8}
\end{align*}
$$

where $\star$ denotes digital convolution. Working from right to left then, the set of operations defined by (3.8) are: a digital filtering operation $\left[\bar{p}_{\rho}\left(m T_{c}\right) \star \bar{y}_{\rho}\left(m T_{c}\right)\right]$, multiplication by a set of phase offsets $\left[W_{N}^{-\rho}\right]$, followed by a discrete Fourier transform $\left[\sum_{\rho=0}^{N-1} W_{N}^{-2 k \rho}\right]$, and finally a set of alternating sign changes on the output $\left[(-1)^{m}\right]$. Note that the filtering operation is actually performed at the lower output sampling rate $f_{c}=1 / T_{c}$ rather than the higher input sampling rate $f_{s}=N f_{c}$. Fig. 3.3 gives a block diagram of the overall structure defined by (3.8).


Figure 3.3: Polyphase network/FFT processor used to demultiplex $N$ channels.

The filters $\bar{p}_{\rho}\left(\nu T_{c}\right)$ are called polyphase filters, as the decimation process of (3.5) gives a set of filters with the same amplitude response but different phase responses [21]. Assuming that the original filter $h\left(n T_{c}\right)$ has a linear phase response, the filter $\bar{p}_{\rho}\left(\nu T_{c}\right)$ will have a linear phase response whose slope is a constant times $\rho / N$. The name polyphase network thus comes from the fact that different paths in the network have different phase responses but the same amplitude response.

The fact that the polyphase filters have finite impulse response means that there cannot be perfect separation between the channels and some interchannel interference or crosstalk will occur, as illustrated in Fig. 3.4. The design problem for the multi-carrier demultiplexer is thus a tradeoff between minimizing the number of taps $L$ in the filters and minimizing the degradation caused by the crosstalk.

### 3.3 Interpolation in the Channel Processor

Following the multi-carrier demultiplexer is the channel processor where symbol timing and carrier phase synchronization are performed. From the point of view of the analysis here, the most important part of the channel processor is the rate conversion stage where the input samples at rate $f_{c}=1 / T_{c}$ are converted to a set of samples at the symbol rate $f_{b}=1 / T_{b}$. This rate conversion can


Figure 3.4: Illustration of the crosstalk between channels that occurs due to the finite impulse response of the polyphase filters.


Figure 3.5: Timing relationships between input and output samples in the rate conversion filter.
be written as a simple digital filtering operation

$$
\begin{equation*}
r_{k}\left(l T_{b}\right)=\sum_{m} x_{k}\left(m T_{c}\right) g\left(l T_{b}-m T_{c}\right) \tag{3.9}
\end{equation*}
$$

where $g(t)$ is the impulse response of the interpolating filter. Very good descriptions of the rate conversion process can be found in [1], [21, Ch. 2], and [22]. The discussion given here is taken from [22].

The relationship between the timing of the input samples $x_{k}\left(m T_{c}\right)$ and the output samples $r_{k}\left(l T_{b}\right)$ is illustrated on the time line of Fig. 3.5. Here it is seen that there are several input samples for every output sample, but the timing of the two sets does not coincide. Several new indices need
to be defined. The first involves converting the signal index $m$ to a filter index [22]

$$
\begin{equation*}
i=\operatorname{int}\left[\frac{l T_{b}}{T_{c}}\right]-m \tag{3.10}
\end{equation*}
$$

where int $[z]$ is the largest integer not exceeding $z$. Next, since the objective is to interpolate between the input samples to produce an output sample at time $l T_{b}$, define the basepoint index $\beta_{l}$ as the index of the input sample just preceding the $l$-th output sample,

$$
\begin{equation*}
\beta_{l}=\operatorname{int}\left[\frac{l T_{b}}{T_{c}}\right] \tag{3.11}
\end{equation*}
$$

The normalized time difference between the input sample at time $\beta_{l} T_{c}$ and the output sample at $l T_{b}$ is referred to as the fractional interval

$$
\begin{equation*}
\mu_{l}=\frac{l T_{b}}{T_{c}}-\beta_{l} \tag{3.12}
\end{equation*}
$$

Then using the indices of (3.10), (3.11) and (3.12), the rate conversion filtering can be rewritten as

$$
\begin{equation*}
r_{k}\left(l T_{b}\right)=r_{k}\left[\left(\beta_{l}+\mu_{l}\right) T_{c}\right]=\sum_{\xi=I_{1}}^{I_{2}} x_{k}\left[\left(\beta_{l}-\xi\right) T_{c}\right] g\left[\left(\xi+\mu_{l}\right) T_{c}\right] . \tag{3.13}
\end{equation*}
$$

If $g(t)$ is a finite impulse response filter, $I_{1}$ and $I_{2}$ are fixed integers and $I=I_{2}-I_{1}+1$ is the number of filter taps.

Essentially what the filtering equation of (3.13) tells us is that we must count the number of input samples that arrive between output samples to obtain $\beta_{l}$. The value of $\mu_{l}$ must also be known to obtain the proper filter coefficients $g\left[\left(\xi+\mu_{l}\right) T_{b}\right]$. These two parameters will in fact be estimated by the timing recovery algorithm. In general, a finite number of quantization steps will be used for $\mu_{l}$ so that the filter coefficients can be stored in memory. In the initial stages of analysis, it is usually assumed that $\mu_{l}$ and $\beta_{l}$ are precisely known and then we attempt to determine the number of taps, $L$ for the polyphase filters and $I$ for the rate conversion filters, to minimize the probability of error. This is also a reasonable assumption given that symbol synchronous transmission is being considered for many on-board processing systems [23]. If symbol synchronous transmission is used, then there is no real timing recovery performed and the rate conversion filter works its way through a set pattern of filter coefficients. This would also imply that the timing offset $\gamma_{k}$ in the transmitted signal response for channel $k$ is either zero (perfect symbol synchronization) or a random quantity of small magnitude.

In the rate conversion process, we have the choice of performing some of the receiver filtering, or it can be used simply as a delay to interpolate between the input samples $x_{k}\left(m T_{c}\right)$. In the latter case, the ideal choice for the impulse response $g(t)$ is a sinc-function. However, there is no reason why the receiver filter cannot be split between the polyphase filters $\bar{p}_{\rho}\left(r T_{c}\right)$ and the rate conversion
filter. If the overall receiver filter characteristic is to be a root Nyquist response, then the polyphase filters and the rate conversion filter will each approximate a fourth-root Nyquist characteristic as was suggested in [1].

### 3.4 The Final Output

In order to analyze the system performance, we have to arrive at an expression for the output of the multi-carrier demultiplexer/demodulator that clearly shows for the $k$-th channel the desired symbol in the $i$-th symbol interval plus any unwanted interference terms. To arrive at such an expression, we must combine the FDMA input of (3.1) and (3.2) with the demultiplexing equation of (3.3) or (3.8) and the rate conversion process of (3.13). In terms of the analysis, it is easier to work with equation (3.3). To begin, we write the sampled input signal as
$y\left[(m N-n) T_{s}\right]=e^{j \pi m} \sum_{q=0}^{N-1} A_{q} e^{j \phi_{q}} \sum_{i=-\infty}^{\infty} e^{j \alpha_{q, i}} h_{s}\left[(m N-n) T_{s}-i T_{b}-\gamma_{q}\right] e^{-j 2 \pi\left(q+\frac{1}{2}\right) \frac{n}{N}}+z\left[(m N-n) T_{s}\right]$.
Substituting (3.14) into (3.3), we arrive at the demultiplexer output for the $k$-th channel,

$$
\begin{align*}
x_{k}\left(m T_{c}\right)= & A_{k} e^{j \phi_{k}} \sum_{i=-\infty}^{\infty} e^{j \alpha_{k, i}} \sum_{n=1}^{N L} h\left(n T_{s}\right) h_{s}\left[(m N-n) T_{s}-i T_{b}-\gamma_{k}\right] \\
& +\sum_{\substack{q=0 \\
q \neq k}}^{N-1} A_{q} e^{j \phi_{q}} \sum_{i=-\infty}^{\infty} e^{j \alpha_{q, i}} \sum_{n=1}^{N L} h\left(n T_{s}\right) h_{s}\left[(m N-n) T_{s}-i T_{b}-\gamma_{q}\right] e^{j 2 \pi(k-q) \frac{n}{N}} \\
& +\sum_{n=1}^{N L} z\left[(m N-n) T_{s}\right] h\left(n T_{s}\right) . \tag{3.15}
\end{align*}
$$

Next, proceeding through the rate conversion stage, the equation for the final output can be given by
$r_{k}\left(l T_{b}\right)=A_{k} e^{j \phi_{k}}\left[e^{j \alpha_{k, l}} u(0)+\sum_{\substack{i=-\infty \\ i \neq l}}^{\infty} e^{j \alpha_{k, i}} u\left(l T_{b}-i T_{b}\right)\right]+\sum_{\substack{q=0 \\ q \neq k}}^{N-1} A_{q} e^{j \phi_{q}} \sum_{i=-\infty}^{\infty} e^{j \alpha_{q, i}} v\left(l T_{b}-i T_{b}\right)+z_{k}\left(l T_{b}\right)$
where

$$
\begin{gather*}
u\left(l T_{b}-i T_{b}\right)=\sum_{\xi=I_{1}}^{I_{2}} \sum_{n=1}^{N L} h\left(n T_{s}\right) h_{s}\left[\left(\beta_{l}-\xi\right) T_{c}-n T_{s}-i T_{b}-\gamma_{k}\right] g\left[\left(\xi+\mu_{l}\right) T_{c}\right]  \tag{3.17}\\
v\left(l T_{b}-i T_{b}\right)=\sum_{\xi=I_{1}}^{I_{2}} \sum_{n=1}^{N L} h\left(n T_{s}\right) h_{s}\left[\left(\beta_{l}-\xi\right) T_{c}-n T_{s}-i T_{b}-\gamma_{q}\right] g\left[\left(\xi+\mu_{l}\right) T_{c}\right] e^{j 2 \pi(k-q) \frac{n}{N}} \tag{3.18}
\end{gather*}
$$

and

$$
\begin{equation*}
z_{k}\left(l T_{b}\right)=\sum_{\xi=I_{1}}^{I_{2}} \sum_{n=1}^{N L} z\left[\left(\beta_{l}-\xi\right) T_{c}-n T_{s}\right] h\left(n T_{s}\right) g\left[\left(\xi+\mu_{l}\right) T_{c}\right] e^{-j \pi\left(\beta_{l}-\xi\right)} e^{-j 2 \pi\left(k+\frac{1}{2}\right) n / N} \tag{3.19}
\end{equation*}
$$

If $z(t)$ is an additive white Gaussian noise process of zero mean and variance $\sigma_{i}^{2}$, then the output noise samples $z_{k}\left(l T_{b}\right)$ will be from a zero-mean Gaussian noise process but with variance

$$
\begin{equation*}
\sigma_{o}^{2}=\sigma_{i}^{2} \sum_{\xi=I_{1}}^{T_{2}}\left|g\left[\left(\xi+\mu_{l}\right) T_{b}\right]\right|^{2} \sum_{n=1}^{N L}\left|h\left(n T_{s}\right)\right|^{2} \tag{3.20}
\end{equation*}
$$

The first term inside the square brackets on the right of (3.16), $e^{j \alpha_{k, l}} u(0)$, is the desired symbol for the $k$-th channel and the $l$-th symbol interval. The second term in the brackets represents intersymbol interference, while the final summation is the interchannel interference or crosstalk from the $N-1$ other channels.

To evaluate the probability of bit error we need only consider the in-phase component, as the QPSK signal can be thought of as two binary signals in quadrature, and the probability of bit error is symmetric between the in-phase and quadrature components. As with the SAW group demodulator, let $e^{j \alpha_{k, l}}=\frac{1}{\sqrt{2}}\left(a_{k, l}+j b_{k, l}\right)$, where $a_{k, l}$ and $b_{k, l}$ are independent, identically distributed binary data symbols taking on values of $\{-1,+1\}$ with equal probability. If we also assume perfect phase synchronization, ( $\phi_{k}=0$ ) the in-phase component of the output can then be expressed as

$$
\begin{align*}
r_{k}^{I}\left(l T_{b}\right)= & A_{k}\left[\begin{array}{c}
a_{k, l} u(0)+\sum_{\substack{i=-\infty \\
i \neq l}}^{\infty} a_{k, i} u\left(l T_{b}-i T_{b}\right) \\
\bullet \\
\end{array}\right. \\
& +\sum_{\substack{q=0 \\
q \neq k}}^{N-1} A_{q} \sum_{i=-\infty}^{\infty}\left\{a_{q, i}\left[v^{I}\left(l T_{b}-i T_{b}\right) \cos \phi_{q}-v^{Q}\left(l T_{b}-i T_{b}\right) \sin \phi_{q}\right]\right. \\
& +z_{k}^{I}\left(l T_{b}\right)
\end{align*}
$$

where $v^{I}\left(l T_{b}-i T_{b}\right)$ and $v^{Q}\left(l T_{b}-i T_{b}\right)$ are the in-phase and quadrature components, respectively, of the filter response $v\left(l T_{b}-i T_{b}\right)$ in (3.18), and $z_{k}^{I}\left(l T_{b}\right)$ represents the in-phase component of the noise process in (3.19).

Let the combined intersymbol and interchannel interference be denoted by $\eta$,

$$
\begin{align*}
\eta= & \sum_{\substack{i=-\infty \\
i \neq l}}^{\infty} A_{k} a_{k, i} u\left(l T_{b}-i T_{b}\right) \\
+ & \sum_{\substack{q=0 \\
q \neq k}}^{N-1} A_{q} \sum_{i=-\infty}^{\infty}\left\{\begin{array}{l}
i q, i \\
i
\end{array}\right. \\
& \left.\quad \dot{b}_{q, i}\left[v^{I}\left(l T_{b}-i T_{b}\right) \cos \phi_{q}-v^{Q}\left(l T_{b}-i T_{b}\right) \sin \phi_{q}+v^{Q}\left(l T_{b}-i T_{b}\right) \cos \phi_{q}\right]\right\} \tag{3.22}
\end{align*}
$$

The decision variable for the in-phase component is then described compactly by

$$
\begin{equation*}
r_{k}^{I}\left(l T_{b}\right)=\dot{A_{k}} a_{k, l} u(0)+\eta+z_{k}^{I}\left(l T_{b}\right) . \tag{3.23}
\end{equation*}
$$

The probability of bit error is easily shown to be symmetric with respect to the data symbols $a_{k, l}$; that is, $P_{b \mid a_{k, l}=-1}=P_{b \mid a_{k, l}=+1}$. Therefore, if we assume that $a_{k, l}=-1$, the probability of bit error conditioned on the interference variable $\eta$, is given by

$$
\begin{equation*}
P_{b \mid \eta}=\int_{-\frac{T}{2}}^{\frac{T}{2}} \operatorname{Pr}\left[-A_{k} u(0)+\eta+z_{k}^{I}\left(l T_{b}\right)>0\right] f\left(\gamma_{k}\right) d \gamma_{k}=\int_{-\frac{T}{2}}^{\frac{T}{2}} Q\left(\frac{A_{k} u(0)-\eta}{\sigma_{o}}\right) f\left(\gamma_{k}\right) d \gamma_{k} \tag{3.24}
\end{equation*}
$$

where $f\left(\gamma_{k}\right)$ is the probability density function of the timing offset on channel $k$. Finally, to obtain the unconditional probability of bit error, we must average over the probability density function of the interference variable $\eta$. The next section discusses how this is done for the polyphase network and for the SAW-based group demodulator using two different analysis methods.

## Chapter 4

## Methods of Computing the BER in the Presence of Interference

Generally, for any symbol interval we can express the output for the $k$-th channel of the group demodulator as the sum of three terms

$$
\begin{equation*}
r_{k}=a_{k} h_{k}+\eta+z \tag{4.1}
\end{equation*}
$$

where $a_{k}$ represents a binary data symbol, $h_{k}$ is the impulse response amplitude at the sampling instant, $\eta$ is the interference term, and $z$ is additive white Gaussian noise. The interference $\eta$ is a random variable and will itself be a sum of many terms of intersymbol and/or interchannel interference. In evaluating the probability of bit error, we generally assume that $a_{k}=-1$ and try to solve an integral of the form

$$
\begin{equation*}
\text { - } \quad P_{b}=\int_{a}^{b} Q\left(\frac{h_{k}-\eta}{\sigma_{z}}\right) f_{\eta}(\eta) d \eta \tag{4.2}
\end{equation*}
$$

where $Q(x)$ represents the complementary distribution function of the noise, $f_{\eta}(\eta)$ is the probability density function of the interference and is nonzero in some interval $[a, b]$, and $\sigma_{z}^{2}$ is the variance of the noise. The central problem with solving (4.2) is that the probability density function of the interference is unknown and virtually impossible to find. Finding a method of approximating the integral for $P_{b}$ in the presence of intersymbol and cochannel interference has been the subject of a considerable amount of research. Two methods of approximating the integral in (4.2) will be discussed: one based on Gauss quadrature rules [16-18] and the second based on a Fourier series due to Beaulieu [19].

### 4.1 The Gauss Quadrature Rule Method

One approach to approximating the integral (4.2) would be to use a linear combination of the values of the function $Q(x)$ at specified points,

$$
\begin{equation*}
\int_{a}^{b} Q\left(\frac{h_{k}-\eta}{\sigma_{z}}\right) f_{\eta}(\eta) d \eta \approx \sum_{i=1}^{M} w_{i} Q\left(\frac{h_{k}-x_{i}}{\sigma_{z}}\right) . \tag{4.3}
\end{equation*}
$$

The $x_{i}$ are called the abscissas of the formula and the $w_{i}$ are called the weights. Together, the set $\left\{w_{i}, x_{i}\right\}_{i=1}^{M}$ is called a Gauss quadrature rule corresponding to the weight function $f_{\eta}(\eta)$, as such formulas were first studied by Gauss. The weight function $f_{\eta}(\eta)$ has to satisfy several conditions for the approximation of (4.3) to be valid [16]: the function $f_{\eta}(\eta)$ must be nonnegative and integrable over $[a, b]$ with

$$
\begin{equation*}
\int_{a}^{b} f_{\eta}(\eta) d \eta>0 \tag{4.4}
\end{equation*}
$$

and the integrals

$$
\begin{equation*}
\int_{a}^{b}|\eta|^{k} f_{\eta}(\eta) d \eta \tag{4.5}
\end{equation*}
$$

must be definite and finite. If this is true, then $M$ abscissas and weights can be found such that (4.3) is exact for $Q(x)$ a polynomial of degree $n \leq 2 M-1$. Of course, the $Q$-function is not a polynomial but the error in the approximation can be made arbitrarily small by choosing $M$ sufficiently large.

The central problem is, of course, finding the weights and abscissas of the quadrature rule. Benedetto et al. [16-18] have proposed using an algorithm due to Golub and Welsch [24] that computes the set $\left\{w_{i}, x_{i}\right\}_{i=1}^{M}$ using the first $2 M+1$ moments of the interference. The algorithm proceeds as follows:

The unknown weight function $f_{\eta}(\eta)$ can be approximated by a set of polynomials $p_{0}(\eta), p_{1}(\eta), \ldots$, that are orthonormal with respect to $f_{\eta}(\eta)$. These polynomials satisfy a three-term recurrence relation [24]

$$
\begin{equation*}
\eta p_{j-1}(\eta)=\beta_{j-1} p_{j-2}(\eta)+\alpha_{j} p_{j-1}(\eta)+\beta_{j} p_{j}(\eta) \tag{4.6}
\end{equation*}
$$

It is possible to find the coefficients $\alpha_{j}$ and $\beta_{j}$ for the recurrence relation by knowing only the first $2 M+1$ moments of the interference. First, form an $(M+1) \times(M+1)$ Hankel matrix $A$ of the moments,

$$
\begin{equation*}
A=\left\{a_{i j}\right\}_{i, j=1}^{M+1}, \quad a_{i j}=\Omega_{i+j-2} \tag{4.7}
\end{equation*}
$$

where

$$
\begin{equation*}
\therefore \Omega_{k}=\int_{a}^{b} \eta^{k} f_{\eta}(\eta) d x \tag{4.8}
\end{equation*}
$$

is the $k$-th moment of the interference, $0 \leq k \leq 2 M$. Next, we perform the Cholesky decomposition $A=B^{T} B$ of the Hankel matrix to obtain an upper triangular matrix $B=\left\{b_{i j}\right\}_{i, j=1}^{M+1}$,

$$
\begin{align*}
& b_{i i}=\sqrt{a_{i i}-\sum_{k=1}^{i-1} b_{k i}^{2}} \\
& b_{i j}=\frac{a_{i j}-\sum_{k=1}^{i-1} b_{k i} b_{k j}}{b_{i i}} i<j . \tag{4.9}
\end{align*}
$$

Once the upper triangular matrix $B$ is found, the coefficients $\alpha_{j}$ and $\beta_{j}$ of the three term recurrence relation are given by

$$
\begin{array}{ll}
\alpha_{j}=\frac{b_{j, j+1}}{b_{j, j}}-\frac{b_{j-1, j}}{b_{j-1, j-1}} & j=1,2, \cdots, M  \tag{4.10}\\
\beta_{j}=\frac{b_{j+1, j+1}}{b_{j, j}} & j=1,2, \cdots, M-1
\end{array}
$$

with $b_{0,0}=1$ and $b_{0,1}=0$. The weights and abscissa of the quadrature rule are then found from the eigenvalues and eigenvectors of the symmetric tridiagonal matrix

$$
J=\left[\begin{array}{ccccc}
\alpha_{1} & \beta_{1} & & &  \tag{4.11}\\
\beta_{1} & \alpha_{2} & \beta_{2} & & \\
& \ddots & \ddots & \ddots & \\
& & \beta_{M-2} & \alpha_{M-1} & \beta_{M-1} \\
& & & \beta_{M-1} & \alpha_{M}
\end{array}\right]
$$

Let $q_{i, 1}$ denote the first component of the $i$-th orthonormalized eigenvector of $J$ and let $\lambda_{i}$ denote the $i$-th eigenvalue. Then

$$
\begin{gather*}
w_{i}=q_{i, 1}^{2} \Omega_{0}  \tag{4.12}\\
x_{i}=\lambda_{i}
\end{gather*}
$$

where $\Omega_{0}$ is the mean or zeroth moment of the interference.
Although the algorithm may seem slightly complicated, the programming of (4.7)-(4.11) is straight forward. As for finding the eigenvalues and orthonormalized eigenvectors of the tridiagonal matrix $J$, this is a common procedure in matrix algebra, and most math function libraries such as IMSL or LAPACK contain predefined routines to perform this task. As for the moments of
the interference, they can be computed in a recursive manner using a method due to Prabhu [25]. Consider that the interference variable $\eta$ is the sum of $K$ separate interference terms,

$$
\begin{equation*}
\eta=\sum_{i=1}^{K} \eta_{i} \tag{4.13}
\end{equation*}
$$

and let $\zeta_{n}$ denote the partial sum of the first $n$ terms,

$$
\begin{equation*}
\zeta_{n}=\sum_{i=1}^{n} \eta_{i} . \tag{4.14}
\end{equation*}
$$

Then it is possible to show that the following recursion holds [25]

$$
\begin{equation*}
E\left[\zeta_{n}^{k}\right]=\sum_{i=0}^{k}\binom{k}{i} E\left[\zeta_{n-1}^{i}\right] E\left[\eta_{n}^{k-i}\right] \tag{4.15}
\end{equation*}
$$

where $E[\cdot]$ denotes expectation. The $k$-th moment is then given by

$$
\begin{equation*}
\Omega_{k}=E\left[\zeta_{K}^{k}\right]=\sum_{i=0}^{k}\binom{k}{i} E\left[\zeta_{K-1}^{i}\right] E\left[\eta_{K}^{k-i}\right] . \tag{4.16}
\end{equation*}
$$

The following subsections briefly discuss computing the moments of the interference for the two group demodulators. There are several sources of error in computing the probability of bit error with the Gauss quadrature rule. Bounds on these errors will also be discussed.

### 4.1.1 Evaluating the Moments of the Interference in the SAW Group Demodulator

To compute the moments of the interference in the SAW-based group demodulator, we must find the expected value of $\left\{\tilde{S}_{l}^{I}\left(\tau_{k}\right)\right\}^{\nu}$ for $l=k \pm 1, k \pm 2, \cdots, k \pm K / 2$ and $\nu=0, \cdots, 2 M$. Essentially, for a fixed $l$ and $\nu$, we must compute $\left\{\tilde{S}_{l}^{T}\left(\tau_{k}\right)\right\}^{\nu}$ for the 16 possible combinations of $a_{l, m}, b_{l, m}, a_{l, m \pm 1}$ and $b_{l, m \pm 1}$ using (2.18), and average each combination over the random phase $\phi_{l} \in[0,2 \pi)$ and the random timing offset $\gamma_{l}$,

$$
\begin{align*}
\left\{\tilde{S}_{l}^{I}\left(\tau_{k}\right)\right\}^{\nu}=\frac{1}{2 \pi}\left[\frac{A_{l} W[(k-l) \Delta f]}{\sqrt{2}}\right]^{\nu} & \int_{0}^{2 \pi} \int_{-\frac{\pi}{2}}^{\frac{T}{2}}\left\{a_{l, m}\left[\left(1-\Gamma_{l}^{I}\right) \cos \phi_{l}-\Gamma_{l}^{Q} \sin \phi_{l}\right]\right. \\
& -b_{l, m}\left[\Gamma_{l}^{Q} \cos \phi_{l}+\left(1-\Gamma_{l}^{I}\right) \sin \phi_{l}\right] \\
& +a_{l, m \pm 1}\left[\Gamma_{l}^{I} \cos \phi_{l}-\Gamma_{l}^{Q} \sin \phi_{l}\right] \\
& \left.-b_{l, m \pm 1}\left[\Gamma_{l}^{Q} \cos \phi_{l}+\Gamma_{l}^{I} \sin \phi_{l}\right]\right\}^{\nu} f_{\gamma_{l}}\left(\gamma_{l}\right) d \gamma_{l} d \phi_{l} \tag{4.17}
\end{align*}
$$

This need only be done once and can be accomplished numerically. Some savings in computation can be gained by noting that only the even order moments $E\left[\left\{\tilde{S}_{l}^{I}\left(\tau_{k}\right)\right\}^{\nu}\right]$ are nonzero. This fact can be found by noting that for each combination $a_{l, m}, b_{l, m}, a_{l, m \pm 1}$, and $b_{l, m \pm 1}$, the complementary
combination $-a_{l, m},-b_{l, m},-a_{l, m \pm 1}$, and $-b_{l, m \pm 1}$ yields a value of $\left\{\tilde{S}_{l}^{I}\left(\tau_{k}\right)\right\}^{\nu}$ that is of equal magnitude to the former but multiplied by the scalar $(-1)^{\nu}$. When $\nu$ is odd, the two terms cancel each other out and we are left with a moment of zero. The above observation also means that we need only compute 8 of the 16 possible combinations representing a further savings in computation.

Once the moments $E\left[\left\{\tilde{S}_{l}^{I}\left(\tau_{k}\right)\right\}^{\nu}\right]$ of the individual interference components have been computed, the moments $\Omega_{\nu}$ of the total interference can be easily computed using the recursions of (4.15) and (4.16). From there, the probability of bit error is obtained by using the above mentioned algorithm to compute the weights and abscissa of the quadrature rule.

There are two sources of error in this approach to computing the bit error rate. The first is due to using only $K$ of the $N-1$ possible interference samples. This can be made arbitrarily small by choosing $K$ sufficiently large, or if $N$ is small, computing over the entire set of interference samples. The second type of error is due to the truncation of the quadrature rule to $M$ samples by using only the first $2 M+1$ moments of the interference. Let $R_{M}$ denote this truncation error. If we assume a fixed timing offset of $\gamma_{k}$ on the desired channel, then from [16, Thm. 1], the magnitude of the truncation error can be shown to equal

$$
\begin{equation*}
\left|R_{M}\right|=\frac{\prod_{i=1}^{M} \beta_{i}^{2}}{2(2 M)!}\left[Q^{(2 M)}\left(\frac{A_{k} W(0)}{\sqrt{2} \sigma_{o}}-\frac{\xi}{\sigma_{o}}\right)+Q^{(2 M)}\left(\frac{A_{k} W(0)}{\sqrt{2} \sigma_{o}}\left[1-2 \Gamma_{k}(0)\right]-\frac{\xi}{\sigma_{o}}\right)\right] \tag{4.18}
\end{equation*}
$$

where $Q^{(2 M)}(\cdot)$ denotes the $2 M$-th derivative of the complementary distribution function, the $\beta_{i}$ are coefficients in the three term recurrence relation of (4.6), and $\xi$ is a quantity that lies within the range of the interference $\eta$. Let $Z(x)$ denote the normal distribution of unit variance,

$$
\begin{equation*}
Z(x)=\frac{1}{\sqrt{2 \pi}} e^{-x^{2} / 2} \tag{4.19}
\end{equation*}
$$

and rewrite (2.24) as

$$
\begin{equation*}
Q(x)=\int_{x}^{\infty} Z(t) d t . \tag{4.20}
\end{equation*}
$$

From [26, p. 934], the $n$-th derivative of the complementary distribution function $Q(x)$ is then given by

$$
\begin{equation*}
Q^{(n)}(x)=-Z^{(n-1)}(x)=(-1)^{n} Z(x) H e_{n-1}(x) \tag{4.21}
\end{equation*}
$$

where $H e_{n}(x)$ is a Hermite polynomial satisfying the recurrence relation [26, p. 782]

$$
\begin{equation*}
H e_{n+1}(x)=x H e_{n}(x)+n H e_{n-1}(x) \tag{4.22}
\end{equation*}
$$

The magnitude of the truncation error is then written as ...

$$
\begin{align*}
\left|R_{M}\right|= & \frac{\prod_{i=1}^{M} \beta_{i}^{2}}{2 \sqrt{2 \pi}(2 M)!\sigma_{o}^{2 M}}\left[\exp \left(-\frac{\left(A_{k} W(0)-\sqrt{2} \xi\right)^{2}}{4 \sigma_{o}^{2}}\right)\left|H e_{2 M-1}\left(\frac{A_{k} W(0)-\sqrt{2} \xi}{\sqrt{2} \sigma_{o}}\right)\right|\right. \\
& \left.+\exp \left(-\frac{\left(A_{k} W(0)\left[1-2 \Gamma_{k}(0)\right]-\sqrt{2} \xi\right)^{2}}{4 \sigma_{o}^{2}}\right)\left|H e_{2 M-1}\left(\frac{A_{k} W(0)\left[1-2 \Gamma_{k}(0)\right]-\sqrt{2} \xi}{\sqrt{2} \sigma_{o}}\right)\right|\right] . \tag{4.23}
\end{align*}
$$

It can be shown [26, p. 787] that

$$
\begin{equation*}
\left|H e_{n}(x)\right|<B e^{x^{2} / 4} \sqrt{n!} \tag{4.24}
\end{equation*}
$$

where $B \approx 1.086435$. Using (4.24), the truncation error can be bounded by

$$
\begin{equation*}
\left|R_{M}\right|<C\left[\exp \left(-\frac{\left(A_{k} W(0)-\sqrt{2} \xi\right)^{2}}{8 \sigma_{o}^{2}}\right)+\exp \left(-\frac{\left(A_{k} W(0)\left[1-2 \Gamma_{k}(0)\right]-\sqrt{2} \xi\right)^{2}}{8 \sigma_{o}^{2}}\right)\right] \tag{4.25}
\end{equation*}
$$

where

$$
\begin{equation*}
C=\frac{B \sqrt{(2 M-1)!} \prod_{i=1}^{M} \beta_{i}^{2}}{2 \sqrt{2 \pi}(2 M)!\sigma_{o}^{2 M}} . \tag{4.26}
\end{equation*}
$$

Again, this assumes a fixed timing offset $\gamma_{k}$ for the desired channel. If the timing offset $\gamma_{k}$ is random, then we must average the right hand side of (4.25) over the probability density function of $\gamma_{k}$, or more specifically, we must average the second exponential over the pdf of $\gamma_{k}$.

To maximize the bound in (4.25), it is easy to see that we must maximize $\xi$, or in other words,

$$
\begin{equation*}
\text { - } \quad \xi_{\max }=\sum_{\substack{l=k-K / 2 \\ l \neq k}}^{k+K / 2} \max _{l}, \tilde{S}_{l}^{I} I\left(\tau_{k}\right) \mid . \tag{4.27}
\end{equation*}
$$

With repeated use of the triangle inequality, we arrive at the expression

$$
\begin{gather*}
\left|\tilde{S}_{l}^{I}\left(\tau_{k}\right)\right| \leq \frac{A_{l} W[(k-l) \Delta f]}{\sqrt{2}}\left\{\left|\left(1-\Gamma_{l}^{I}\right) \cos \phi_{l}\right|+\left|\left(1-\Gamma_{l}^{I}\right) \sin \phi_{l}\right|+\left|\Gamma_{l}^{I} \cos \phi_{l}\right|+\left|\Gamma_{l}^{I} \sin \phi_{l}\right|\right. \\
\left.+2\left|\Gamma_{l}^{Q} \sin \phi_{l}\right|+2\left|\Gamma_{l}^{Q} \cos \phi_{l}\right|\right\} \tag{4.28}
\end{gather*}
$$

For a fixed timing offset $\gamma_{l}$, this expression reaches a maximum at $\phi_{l}=\frac{\pi}{4}$, yielding

$$
\begin{equation*}
\max _{\gamma_{l}, \phi_{l}}\left|\tilde{S}_{l}^{I}\left(\tau_{k}\right)\right|<\max _{\gamma_{l}} A_{l} W[(k-l) \Delta f]\left\{1+2\left|\Gamma_{l}^{I}\right|+2\left|\Gamma_{l}^{Q}\right|\right\} . \tag{4.29}
\end{equation*}
$$

For fixed timing offsets $\gamma_{l}$, we simply take the term on the far right of (4.29). However, in the general case where $\gamma_{l}$ is a random quantity, we must. evaluate $\left|\Gamma_{l}^{I}\right|+\left|\Gamma_{l}^{Q}\right|$ and find the value of $\gamma_{l}$ for which this sum reaches a maximum.

### 4.1.2 Evaluating the Moments of,the Interference in the Polyphase Structure

For the polyphase network, there are two sets of terms involved in the computation of the moments; one set for the intersymbol interference and one for the interchannel interference. Both the intersymbol and the interchannel interference involve infinite sums. These will have to be truncated in order to arrive at something that is computable. Let $R$ denote the number of terms of intersymbol interference that contribute significantly to the probability of bit error. We assume that of these $R$ symbols, $m_{L}$ occur before the $l$-th symbol interval and $m_{U}$ occur after the $l$-th symbol interval so that $R=m_{u}+m_{L}$. Similarly, for the $q$-th interfering channel, let $R_{q}=m_{q L}+m_{q U}+1$ denote the number of samples that significantly affect the probability of bit error. The interference variable $\eta$ of (3.22) is then approximated by

$$
\begin{align*}
& \eta^{*}=\sum_{\substack{i=l-m_{L} \\
i \neq l}}^{l+m_{U}} A_{k} a_{k, i} u\left(l T_{b}-i T_{b}\right) \\
& +\sum_{\substack{q=0 \\
q \neq k}}^{N-1} A_{q} \sum_{i=l-m_{q L}}^{l+m_{q} U}\left\{a_{q, i}\left[v^{I}\left(l T_{b}-i T_{b}\right) \cos \phi_{q}-v^{Q}\left(l T_{b}-i T_{b}\right) \sin \phi_{q}\right]\right. \\
& \left.\quad-b_{q, i}\left[v^{I}\left(l T_{b}-i T_{b}\right) \sin \phi_{q}+v^{Q}\left(l T_{b}-i T_{b}\right) \cos \phi_{q}\right]\right\} . \tag{4.30}
\end{align*}
$$

The total number of interference samples is then

$$
\begin{equation*}
K=R+\sum_{\substack{q=0 \\ q \neq k}}^{N-1} R_{q} . \tag{4.31}
\end{equation*}
$$

For each of the intersymbol interference terms, we want to evaluate the moment

$$
\begin{equation*}
E\left[\eta_{i}^{\nu}\right]=\frac{1}{2} A_{k}^{\nu} \sum_{a_{k, i}} \int_{-\frac{T}{2}}^{\frac{T}{2}}\left[a_{k, i} u\left(l T_{b}-i T_{b}\right)\right]^{\nu} f_{\gamma_{k}}\left(\gamma_{k}\right) d \gamma_{k} \tag{4.32}
\end{equation*}
$$

Since $a_{k, i}$ is a binary data symbol taking on the values $\{-1,+1\}$, it is not difficult to see that the odd order moments will be zero. Similarly, each of the $K-R$ interchannel interference terms will have a moment of the form

$$
\begin{align*}
& E\left[\eta_{i}^{\nu}\right]=\frac{1}{8 \pi} A_{q}^{\nu} \sum_{a_{q, i}, b_{q, i}} \int_{0}^{2 \pi} \int_{-\frac{T}{2}}^{\frac{T}{2}}\left(a_{q, i}\left[v^{I}\left(l T_{b}-i T_{b}\right) \cos \phi_{q}-v^{Q}\left(l T_{b}-i T_{b}\right) \sin \phi_{q}\right]\right. \\
&\left.-b_{q, i}\left[v^{I}\left(l T_{b}-i T_{b}\right) \sin \phi_{q}+v^{Q}\left(l T_{b}-i T_{b}\right) \cos \phi_{q}\right]\right)^{\nu} f_{\gamma_{q}}\left(\gamma_{q}\right) d \gamma_{q} d \phi_{q} \tag{4.33}
\end{align*}
$$

of which only the even order moments will be nonzero.

As with the SAW group demodulator, there are two types of truncation error that arise in using this method to evaluate the probability of bit error for the polyphase structure: one is due to using only the first $2 M+1$ moments and the second is due to truncation of the two infinite series for the intersymbol and interchannel interference.

The truncation error due to using only $2 M+1$ moments was derived in the previous section for the SAW-based group demodulator. For the polyphase structure, it is easily shown that the magnitude of this truncation error is equal to [16]

$$
\begin{equation*}
\left|R_{M}\right|=\frac{\prod_{i=1}^{M} \beta_{i}^{2}}{2(2 M)!} \int_{-\frac{T}{2}}^{\frac{T}{2}} Q^{(2 M)}\left(\frac{A_{k} u(0)-\xi}{\sigma_{o}}\right) f_{\gamma_{k}}\left(\gamma_{k}\right) d \gamma_{k} \tag{4.34}
\end{equation*}
$$

where $\xi$ is a quantity in the range of the interference variable $\eta$, and $\sigma_{o}^{2}$ is the output noise variance given by (3.20). Following the steps used in the previous section, the truncation error can be bounded by

$$
\begin{equation*}
\left|R_{M}\right|<C \int_{-\frac{T}{2}}^{\frac{T}{2}} \exp \left(-\frac{\left(A_{k} u(0)-\xi\right)^{2}}{4 \sigma_{o}^{2}}\right) f_{\gamma_{k}}\left(\gamma_{k}\right) d \gamma_{k} \tag{4.35}
\end{equation*}
$$

where $C$ is the constant

$$
\begin{equation*}
C=\frac{B \sqrt{(2 M-1)!} \prod_{i=1}^{M} \beta_{i}^{2}}{2 \sqrt{2 \pi}(2 M)!\sigma_{o}^{2 M}} \tag{4.36}
\end{equation*}
$$

and $B \approx 1.086435$. To maximize the bound of (4.35), we must maximize $\xi$, and this can be shown to occur when

$$
\begin{equation*}
\xi_{\max }=\max _{\gamma_{k}} \sum_{\substack{i=l-m_{L} \\ i \neq 0}}^{l+m_{U}}\left|A_{k} u\left(l T_{b}^{\bullet}-i T_{b}\right)\right|+\sqrt{2} \sum_{\substack{q=0 \\ q \neq k}}^{N-1} A_{q} \max _{\gamma_{q}} \sum_{i=l=m_{q L}}^{l+m_{q} U}\left|v^{I}\left(l T_{b}-i T_{b}\right)\right|+\left|v^{Q}\left(l T_{b}-i T_{b}\right)\right| \tag{4.37}
\end{equation*}
$$

The second type of truncation error, due to using a finite rather than an infinite series of interference terms, is more difficult to classify. However, in [17] it is shown how the effect of finite interference on the probability of bit error can be bounded within a given range. If $\eta$ represents the true interference variable and $\eta^{*}$ is the finite approximation, then it is possible to define a third variable

$$
\begin{align*}
\eta^{* *}= & \sum_{\substack{i<l-m_{L} \\
i>l+m_{U}}} A_{k} a_{k, i} u\left(l T_{b}-i T_{b}\right)  \tag{4.38}\\
& +\sum_{\substack{q=0 \\
q \neq k}}^{N-1} A_{q} \sum_{\substack{i<l-m_{L} \\
i>l+m_{U}}}\left\{a_{q, i}\left[v^{I}\left(l T_{b}-i T_{b}\right) \cos \phi_{q}-v^{Q}\left(l T_{b}-i T_{b}\right) \sin \phi_{q}\right]\right. \\
& \left.\quad-b_{q, i}\left[v^{I}\left(l T_{b}-i T_{b}\right) \sin \phi_{q}+v^{Q}\left(l T_{b}-i T_{b}\right) \cos \phi_{q}\right]\right\} \tag{4.39}
\end{align*}
$$

such that $\eta=\eta^{*}+\eta^{* *}$. It can then be shown:that, for a given timing offset $\gamma_{k}$, the true probability of bit error lies within the interval [17, Eq. 49]

$$
\begin{align*}
& \int_{a}^{b} Q\left(\frac{A_{k} u(0)-\eta^{*}}{\sigma_{o}}\right) f_{\eta^{*}}\left(\eta^{*}\right) d \eta^{*} \\
\leq & \int_{a}^{b} Q\left(\frac{A_{k} u(0)-\eta}{\sigma_{o}}\right) f_{\eta}(\eta) d \eta \\
\leq & {\left[1-\left(\sigma_{r} / \sigma_{o}\right)^{2}\right]^{-\frac{1}{2}} \int_{a}^{b} Q\left(\frac{A_{k} u(0)-\eta^{*}}{\sigma_{o}\left[1-\left(\sigma_{r} / \sigma_{o}\right)^{2}\right]^{-\frac{1}{2}}}\right) f_{\eta^{*}}\left(\eta^{*}\right) d \eta^{*} } \tag{4.40}
\end{align*}
$$

where $\sigma_{r}^{2}$ is a quantity such that

$$
\begin{equation*}
E\left\{\exp \left(A \eta^{* *}\right)\right\} \leq \exp \left(\frac{A^{2}}{2} \sigma_{r}^{2}\right) \tag{4.41}
\end{equation*}
$$

for all $A$. Following the derivation in [17], the value of $\sigma_{r}^{2}$ should be chosen such that

$$
\begin{align*}
& \sigma_{r}^{2}= \max _{\gamma_{k}} \sum_{\substack{i<l-m_{L} \\
i>l+m_{U}}} \mid A_{k} a_{k, i}\left[\left.u\left(l T_{b}-i T_{b}\right)\right|^{2}\right. \\
&+\max _{\phi_{q}, \gamma_{q}} \sum_{\substack{q=0 \\
q \neq k}}^{N-1} A_{q}^{2} \sum_{\substack{i<l-m_{L} \\
i>l+m_{U}}} \mid a_{q, i}\left[v^{I}\left(l T_{b}-i T_{b}\right) \cos \phi_{q}-v^{Q}\left(l T_{b}-i T_{b}\right) \sin \phi_{q}\right] \\
& \quad-\left.b_{q, i}\left[v^{I}\left(l T_{b}-i T_{b}\right) \sin \phi_{q}+v^{Q}\left(l T_{b}-i T_{b}\right) \cos \phi_{q}\right]\right|^{2} \\
& \leq \max _{\gamma_{k}} \sum_{\substack{i<l-m_{L} \\
i>l+m_{U}}} A_{k}^{2}\left[u\left(l T_{b}-i T_{b}\right)\right]^{2}+\sum_{\substack{q=0 \\
q \neq k}}^{N-1} A_{q}^{2} \max _{\gamma_{q}} \sum_{\substack{i<l-m_{L} \\
i>l+m_{U}}}\left[v^{I}\left(l T_{b}-i T_{b}\right)\right]^{2}+\left[v^{Q}\left(l T_{b}-i T_{b}\right)\right]^{2} . \tag{4.42}
\end{align*}
$$

The inequality of (4.40) is valid as long as the eye pattern of the modulation remains open (i.e., the interference is not too large). If that is true, then by choosing $R$ and $R_{i}$ sufficiently large so that $\sigma_{r}$ remains small, the upper and lower bounds of (4.40) can be made to come together. It is stated in [17] that if $\left(\sigma_{r} / \sigma_{o}\right)^{2} \approx 10^{-2}$, then the separation between the upper and lower bound will be of the order of 0.05 dB .

### 4.2 Beaulieu's Fourier Series Method

In Beaulieu's approach to computing the bit error rate in the presence of interference, the complementary distribution function $Q(x)$ is approximated by a Fourier series,

$$
\begin{equation*}
Q(x)=\sum_{m=-\infty}^{\infty} c_{m} e^{j m \omega x}+\varepsilon(x) \tag{4.43}
\end{equation*}
$$

where $\varepsilon(x)$ represents an error term. If this. Fourier series is substituted into (4.2) with the assumption that the noise is of unit variance (i.e., $\sigma_{z}^{2}=1$ ), the expression for the probability of bit error can be rewritten as

$$
\begin{equation*}
P_{b}=\int_{a}^{b} f_{\eta}(\eta)\left[\sum_{m=-\infty}^{\infty} c_{m} e^{j m \omega\left(h_{k}-\eta\right)}+\varepsilon\left(h_{k}-\eta\right)\right] d \eta \tag{4.44}
\end{equation*}
$$

The characteristic function $\Phi_{\eta}(\omega)$ of the interference variable $\eta$ is given by

$$
\begin{equation*}
\Phi_{\eta}(\omega)=E\left[e^{j \omega \eta}\right]=\int_{-\infty}^{\infty} f_{\eta}(\eta) e^{j \omega \eta} d \eta \tag{4.45}
\end{equation*}
$$

Consequently, if we rearrange the order of summation and integration and substitute the characteristic function $\Phi_{\eta}(\omega)$, the expression for the probability of bit error can be shown to simplify to

$$
\begin{equation*}
P_{b}=\sum_{m=-\infty}^{\infty} c_{m} e^{j m \omega h_{k}} \Phi_{\eta}(-m \omega)+\beta, \tag{4.46}
\end{equation*}
$$

where

$$
\begin{equation*}
\beta=\int_{a}^{b} f_{\eta}(\eta) \varepsilon\left(h_{k}-\eta\right) d \eta \tag{4.47}
\end{equation*}
$$

represents the error in using the Fourier series to approximate the probability of bit error.
The benefit of this approach to computing the probability of error is that, while the probability density function of the interference may be unknown, the characteristic function of the interference can be easily found. The key to the method is finding a suitable series for (4.43) with coefficients $c_{m}$ that are easily computed. Beaulieu [19] has done this by combining the usual Fourier series with a Chernoff bound that involves gating the unit variance normal distribution with a square wave of period $\mathcal{T}$. The approximate Fourier series for $Q(x)$ then takes the form [19]

$$
\begin{equation*}
Q(x)=\frac{1}{2}-\frac{2}{\pi} \sum_{\substack{m=1 \\ m \text { odd }}}^{\infty} \frac{e^{-m^{2} \omega^{2} / 2}}{m} \sin (m \omega x)+\varepsilon(x) \tag{4.48}
\end{equation*}
$$

where

$$
\begin{equation*}
|\varepsilon(x)|<Q\left(\frac{\mathcal{T}}{2}-|x|\right) \tag{4.49}
\end{equation*}
$$

and the angular frequency is $\omega=2 \pi / \mathcal{T}$. Using the series of (4.48), suitably truncated to $M$ terms, provides an efficient means by which the probability of bit error can be computed. As we shall see, the errors associated with this method can be quite tightly bounded.

### 4.2.1 Beaulieu's Method Applied;to the SAW Group Demodulator

Substituting the Fourier series of (4.48) in the conditional probability of bit error expression (2.25) for the SAW group demodulator, and using the characteristic function (4.45) for the interference term $\eta$, we arrive at an expression for the probability of bit error

$$
\begin{align*}
P_{b}=\frac{1}{2}-\frac{1}{\pi} \sum_{\substack{m=1 \\
m o d d}}^{M} & \frac{e^{-m^{2} \omega^{2} / 2}}{m}\left\{\sin \left[m \omega \frac{A_{k} W(0)}{\sqrt{2}}\right]\right. \\
& \left.+\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin \left[m \omega \frac{A_{k} W(0)}{\sqrt{2}}\left[1-2 \Gamma_{k}(0)\right]\right] f_{\gamma_{k}}\left(\gamma_{k}\right) d \gamma_{k}\right\} \Phi_{\eta}(-m \omega)+R_{M}+\beta \tag{4.50}
\end{align*}
$$

where $R_{M}$ is the error due to truncation of the series, and $\beta$ is the error associated with using the Fourier series approximation. The expression for the truncation error is straightforward to define; for $m>M$

$$
\begin{align*}
& R_{M}=-\frac{1}{\pi} \sum_{\substack{m=M+1 \\
m \text { odd }}}^{\infty} \frac{e^{-m^{2} \omega^{2} / 2}}{m}\left\{\sin \left[m \omega \frac{A_{k} W(0)}{\sqrt{2}}\right]\right. \\
&\left.+\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin \left[m \omega \frac{A_{k} W(0)}{\sqrt{2}}\left[1-2 \Gamma_{k}(0)\right]\right] f_{\gamma_{k}}\left(\gamma_{k}\right) d \gamma_{k}\right\} \Phi_{\eta}(-m \omega) \tag{4.51}
\end{align*}
$$

and it is not difficult to show that the magnitude of the truncation error can be bounded by [19]

$$
\begin{equation*}
\left|R_{M}\right| \leq \frac{2}{\pi} \sum_{\substack{m=M+1 \\ m \text { odd }}}^{\infty} \frac{e^{-m^{2} \omega^{2} / 2}}{m}<\frac{\sqrt{2 \pi} \mathcal{T}}{\pi^{2} M} Q\left(\frac{2 \pi M}{\mathcal{T}}\right) \tag{4.52}
\end{equation*}
$$

The error term $\beta$ due to using a Fourier series approximation can be found from (4.49),

$$
\begin{align*}
|\beta|< & \frac{1}{2} Q\left(\frac{\mathcal{T}}{2}-\frac{A_{k} W(0)}{\sqrt{2}}-\sum_{\substack{l=k-K / 2 \\
l \neq k}}^{k+K / 2} \max \left|\tilde{S}_{l}^{I}\left(\tau_{k}\right)\right|\right) \\
& +\frac{1}{2} \int_{-\frac{T}{2}}^{\frac{T}{2}} Q\left(\frac{\mathcal{T}}{2}-\frac{A_{k} W(0)}{\sqrt{2}}\left[1-2 \Gamma_{k}(0)\right]-\sum_{\substack{l=k-K / 2 \\
l \neq k}}^{k+K / 2} \max \left|\tilde{S}_{l}^{I}\left(\tau_{k}\right)\right|\right) f_{\gamma_{k}}\left(\gamma_{k}\right) d \gamma_{k} \tag{4.53}
\end{align*}
$$

The maximum value of the interference $\tilde{S}_{l}^{I}\left(\tau_{k}\right)$ was given in the previous section in (4.29). Direct substitution into (4.53) yields the final expression for the error term $\beta$.

It should be remembered that the derivation of the probability of bit error is based on the assumption that the noise is of unit variance. Thus, the signal and interference terms in the above expression for $\beta$ in effect represent a signal plus interference-to-noise ratio. This means that as
the signal-to-noise ratio increases, we want, to increase the value of $\mathcal{T}$ in order for the error term $\beta$ to remain small. On the other hand, equation (4.52) indicates that an increase in $\mathcal{T}$ requires a corresponding increase in the number of terms $M$ in the series for the truncation error $R_{M}$ to remain small. There is a tradeoff then between these two parameters, and,for a specified level of accuracy in the computations, there will be an optimum choice for $M$ and $\mathcal{T}$. Experience has shown, however, that if $\mathcal{T}$ is chosen too small and the error term $\beta$ is too close to the true probability of bit error, the series may not converge. Thus, it is generally safer to set $\mathcal{T}$ to a large value and use more terms in the series than to have chosen $\mathcal{T}$ too small to begin with and have to repeat the calculation because the series will not converge.

To complete the derivation, we must find an expression for $\Phi_{\eta}(-m \omega)$. Let $\eta_{l}=\tilde{S}_{l}^{I}\left(\tau_{k}\right)$. Then the characteristic function of the interference $\eta$ can be expressed as the product

$$
\begin{equation*}
\Phi_{\eta}(-m \omega)=\prod_{\substack{l=k-K / 2 \\ l \neq k}}^{k+K / 2} \Phi_{\eta_{l}}(-m \omega) \tag{4.54}
\end{equation*}
$$

and each term $\Phi_{\eta_{l}}(-m \omega)$ can be shown to take the form

$$
\begin{align*}
\Phi_{\eta_{l}}(-m \omega)=\frac{1}{2 \pi} \int_{0}^{2 \pi} & \int_{-\frac{T}{2}}^{\frac{T}{2}}\left\{\cos \left(\frac{m \omega A_{l} W[(k-l) \Delta f]}{\sqrt{2}}\left[\left(1-\Gamma_{l}^{I}\right) \cos \phi_{l}-\Gamma_{l}^{Q} \sin \phi_{l}\right]\right)\right. \\
& \cdot \cos \left(\frac{m \omega A_{l} W[(k-l) \Delta f]}{\sqrt{2}}\left[\Gamma_{l}^{Q} \cos \phi_{l}+\left(1-\Gamma_{l}^{I}\right) \sin \phi_{l}\right]\right) \\
& \cdot \cos \left(\frac{m \omega A_{l} W[(k-l) \Delta f]}{\sqrt{2}}\left[\Gamma_{l}^{I} \cos \phi_{l}-\Gamma_{l}^{Q} \sin \phi_{l}\right]\right) \\
& \left.\cdot \cos \left(\frac{m \omega A_{l} W[(k-l) \Delta f]}{\sqrt{2}}\left[\Gamma_{l}^{Q} \cos \phi_{l}+\Gamma_{l}^{I} \sin \phi_{l}\right]\right)\right\} f_{\gamma_{l}}\left(\gamma_{l}\right) d \gamma_{l} d \phi \tag{4.55}
\end{align*}
$$

A closed form solution for $\Phi_{\eta_{l}}(-m \omega)$ does not seem possible but it can be evaluated quite easily by numerical integration.

In summary, the probability of bit error can be well approximated and easily computed by

$$
\begin{align*}
P_{b} \approx \frac{1}{2}-\frac{1}{\pi} \sum_{\substack{m=1 \\
m o d d}}^{M} & \frac{e^{-m^{2} \omega^{2} / 2}}{m}\left\{\sin \left[m \omega \frac{A_{k} W(0)}{\sqrt{2}}\right]\right. \\
& \left.+\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin \left[m \omega \frac{A_{k} W(0)}{\sqrt{2}}\left[1-2 \Gamma_{k}(0)\right]\right] f_{\gamma_{k}}\left(\gamma_{k}\right) d \gamma_{k}\right\} \Phi_{\eta}(-m \omega) \tag{4.56}
\end{align*}
$$

with (4.54) and (4.55) used to compute $\Phi_{\eta}(-m \omega)$. The magnitude of the error in the approximation is bounded by

$$
\begin{equation*}
\mid \text { error }\left|<\left|R_{M}\right|+|\beta| .\right. \tag{4.57}
\end{equation*}
$$

### 4.2.2 Beaulieu's Method Applied, to the Polyphase Structure

For the polyphase structure, application of Beaulieu's Fourier series method yields a bit-error-rate expression of the form

$$
\begin{equation*}
P_{b}=\frac{1}{2}-\frac{2}{\pi} \sum_{\substack{m=1 \\ m \text { odd }}}^{M} \frac{e^{-m^{2} \omega^{2} / 2}}{m}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin \left[m \omega A_{k} u(0)\right] f_{\gamma_{k}}\left(\gamma_{k}\right) d \gamma_{k}\right] \Phi_{\eta}(-m \omega)+R_{M}+\beta \tag{4.58}
\end{equation*}
$$

where the truncation error $R_{M}$ is again bounded by (4.52). The error term $\beta$ can be bounded by

$$
\begin{equation*}
|\beta|<\int_{-\frac{T}{2}}^{\frac{T}{2}} Q\left(\frac{\mathcal{T}}{2}-A_{k} u(0)-\xi_{\max }\right) f_{\gamma_{k}}\left(\gamma_{k}\right) d \gamma_{k} \tag{4.59}
\end{equation*}
$$

where $\xi_{\max }$ is given by (4.37).
The characteristic function of the interference is the product of an infinite number of terms of intersymbol and interchannel interference. If we again truncate these to $R=m_{L}+m_{U}$ terms for the intersymbol interference and $R_{q}=m_{q L}+m_{q U}+1$ terms for each interfering channel, then the characteristic function can be approximated by

$$
\begin{equation*}
\Phi_{\eta}^{*}(-m \omega)=\prod_{\substack{i=l-m_{L} \\ i \neq l}}^{l+m_{U}} \Phi_{\eta_{i}}(-m \omega) \prod_{\substack{q=0 \\ q \neq k}}^{N-1} \prod_{j=l-m_{q} L}^{l+m_{q} U} \Phi_{\eta_{q, j}}(-m \omega) . \tag{4.60}
\end{equation*}
$$

The characteristic function of each intersymbol interference component is given by the simple expression

$$
\begin{equation*}
\Phi_{\eta_{i}}(-m \omega)=\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos \left[m \omega A_{k} u\left(l T_{b}-i T_{b}\right)\right] f_{\gamma_{k}}\left(\gamma_{k}\right) d \gamma_{k} \tag{4.61}
\end{equation*}
$$

while the characteristic function of the interchannel interference terms is given by

$$
\begin{align*}
\Phi_{\eta_{q}, j}(-m \omega)=\frac{1}{2 \pi} \int_{0}^{2 \pi} & \int_{-\frac{T}{2}}^{\frac{T^{*}}{2}} \cos \left(m \omega\left[v^{I}\left(l T_{b}-j T_{b}\right) \cos \phi_{q}-v^{Q}\left(l T_{b}-j T_{b}\right) \sin \phi_{q}\right]\right) \\
& \cdot \cos \left(m \omega\left[v^{I}\left(l T_{b}-j T_{b}\right) \sin \phi_{q}+v^{Q}\left(l T_{b}-j T_{b}\right) \cos \phi_{q}\right]\right) f_{\gamma_{q}}\left(\gamma_{q}\right) d \gamma_{q} d \phi_{q} \tag{4.62}
\end{align*}
$$

The use of only $K=R+\sum_{q \neq k} R_{q}$ interference terms introduces a third error term in the probability of bit error expression. The true characteristic function of the interference $\Phi_{\eta}(-m \omega)$ can be shown to be the product of two terms,

$$
\begin{equation*}
\Phi_{\eta}(-m \omega)=\Phi_{\eta}^{*}(-m \omega) \Phi_{\eta}^{* *}(-m \omega), \tag{4.63}
\end{equation*}
$$

where $\Phi_{\eta}^{*}(-m \omega)$ is the approximation of (4.60) and $\Phi_{\eta}^{* *}(-m \omega)$ is simply all the remaining terms in the infinite products,

$$
\begin{equation*}
\Phi_{\eta}^{* *}(-m \omega)=\prod_{\substack{i<l-m_{L} \\ i>l+m_{U}}} \Phi_{\eta_{i}}(-m \omega) \prod_{\substack{q=0 \\ q \neq k}}^{N-1} \prod_{\substack{j<l-m_{q} L \\ j>l+m_{q} U}} \Phi_{\eta_{q, j}(-m \omega)} \tag{4.64}
\end{equation*}
$$

The error in using only $K$ interference terms is easily shown to equal

$$
\begin{equation*}
\Delta=-\frac{2}{\pi} \sum_{\substack{m=1 \\ m \text { odd }}}^{M} \frac{e^{-m^{2} \omega^{2} / 2}}{m}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin \left[m \omega A_{k} u(0)\right] f_{\gamma_{k}}\left(\gamma_{k}\right) d \gamma_{k}\right] \Phi_{\eta}^{*}(-m \omega)\left[\Phi_{\eta}^{* *}(-m \omega)-1\right] \tag{4.65}
\end{equation*}
$$

There are ways to further simplify the term $\Phi_{\eta}^{* *}(-m \omega)$ (see [19]) but we will leave those for the moment. It is fairly clear that we want to choose $R$ and the $R_{q}$ such that $\Phi_{\eta}^{* *}(-m \omega)$ approaches 1. For this to happen, the filter coefficient $u\left(l T_{b}-i T_{b}\right), v^{I}\left(l T_{b}-i T_{b}\right)$, and $v^{Q}\left(l T_{b}-i T_{b}\right)$ must obviously approach zero.

## Chapter 5

## Numerical Results

### 5.1 SAW Group Demodulator

The computations that we must perform involve a number of numerical integrations, beginning with computing the quantities $W(0), \Gamma_{k}(0), W[(k-1) \Delta f], \Gamma_{l}^{I}$, and $\Gamma_{l}^{Q}$. For all the numerical integrations, an extended Simpson's rule [27, Eq. 4.1.14] of 20000 points has been used. Since the error in this integration rule is $O\left(\frac{1}{N^{4}}\right)$ for an $N$-point integration, 20000 points should be sufficient to minimize the error due to numerical integration. Indeed, a number of the computations were repeated with a 40000 point integration rule and there was no change in the computed bit-errorrates. For the Kaiser-Bessel window function, the zeroth order modified Bessel function $I_{0}(x)$ is computed using the polynomial approximations found in Abramowitz and Stegun [26, Eqs. 9.8.1-2]. The magnitude of the error in these approximations is on the order of $10^{-7}$. Both the approximation of the Bessel functions and the numerical integration will contribute errors to the estimate of the bit-error-rate, but we have attempted to minimize their effect. Double precision arithmetic was used throughout to also minimize problems of round-off and truncation.

For the first part of the analysis, it was assumed that only two interferers were present on channels $k-1$ and $k+1$, where channel $k$ is the desired channel. Although using only two interference signals with fixed timing offset is a somewhat idealized case, it will prove useful in that the computations can be compared against bench measurements with existing hardware [10]. As well, we can expect that most of the contribution due to interference will come from the adjacent channels.

The system parameters used in the computations are those of [10]: the time-bandwidth product of the Kaiser-Bessel window is $\delta=1.85$, the QPSK/FDMA signals are 64 kbps such that $T=$ $31.25 \mu \mathrm{~s}$., and the channel spacing is $\Delta f=96 \mathrm{kHz}$. For the chirp Fourier transform, the chirp slope is $\mu=0.091 \mathrm{MHz} / \mu \mathrm{s}$.

The first set of results are for zero timing offset on the desired channel, and equal, fixed timing
offsets on the two interfering channels. This case gives an indication of how the growth in the sidelobes caused by timing offsets can affect another signal that is itself well behaved. Tables 5.1 and 5.2 give the numerical results for: $\gamma_{k-1}=\gamma_{k+1}=0.1 T$ and $\gamma_{k-1}=\gamma_{k+1}=0.4 T$, respectively. In the tables, rather than quote $M$, we have quoted the number of non-zero terms in the series, $N_{M}=(M+1) / 2$.

For Beaulieu's method, the value of $\mathcal{T}$ was chosen such that

$$
\begin{equation*}
\frac{\mathcal{T}}{2}-\frac{A_{k} W(0)}{\sqrt{2}}-\max \xi<8 \sqrt{\lambda \frac{E_{b}}{N_{0}}} \tag{5.1}
\end{equation*}
$$

where

$$
\begin{equation*}
\lambda=\frac{[W(0)]^{2}}{\int_{-\frac{T}{2}}^{\frac{T}{2}} w^{2}(t) d t} \tag{5.2}
\end{equation*}
$$

is the inverse of the equivalent noise bandwidth of the window and represents the loss in $E_{b} / N_{0}$ associated with windowing. For the particular window used here, $\lambda=-1.16 \mathrm{~dB}$. The value of $M$ was then chosen such that

$$
\begin{equation*}
\frac{2 \pi M}{\mathcal{T}}<4 \sqrt{\lambda \frac{E_{b}}{N_{0}}} \tag{5.3}
\end{equation*}
$$

With only a single channel and perfect timing, the probability of bit error is given by $Q\left(\sqrt{2 \lambda E_{b} / N_{0}}\right)$. Since the probability of error with interchannel interference and non-ideal timing is always going to be greater than this value, using (5.1) and (5.3) ensures that the error terms $R_{M}$ and $\beta$ become vanishingly small.

For the Gauss quadrature rule method, a number of trials were run beginning with $M=3$ until no further change could be observed in the results. Eight significant figures are reported for the probability of bit error, and six significant figures are given for the truncation error, $\left|R_{M}\right|$. Again, it should be pointed out that the probability of bit error is not guaranteed to be accurate to eight significant figures due to approximation of the modified Bessel function and the various numerical integrations involved in computing the probability of bit error. However, if both methods use the same integration rules, and are therefore subject to the same types of errors, it is feasible to compare the two methods and see how closely they agree. Looking through the tables, we see that there is generally agreement between the two methods to at least four significant figures, and in many cases the two numbers are identical.

If the number of nonzero terms, $N_{M}$, of Beaulieu's Fourier series method is compared with the number of terms $M$ used in the Gauss quadrature rule, it is seen that far fewer terms are required by the latter to achieve similar accuracy. Indeed, in almost all instances where 7 or more terms are

Table 5.1: Numerical results for $K=2$ interfering channels, zero timing offset on the desired channel $\left(\gamma_{k}=0\right)$ and timing offsets of $\gamma_{k-1}=\gamma_{k+1}=0.1 T$.

|  | $E_{b} / N_{0}$ |  |  |  | $N_{M}$ | $\mathcal{T}$ | $P_{b}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\left\|R_{M}\right\|$ | Gauss Quadrature Rule Method |  |  |  |  |  |  |
| 0 | 6 | 16.0 | $1.19961238 \mathrm{e}-01$ | $2.88605 \mathrm{e}-06$ | 3 | $1.19961267 \mathrm{e}-01$ | $\left\|R_{M}\right\|$ |
| 1 | 6 | 18.0 | $9.36790404 \mathrm{e}-02$ | $2.5594942 \mathrm{e}-05$ | 3 | $9.36794914 \mathrm{e}-02$ | $6.59344 \mathrm{e}-08$ |
| 2 | 8 | 20.0 | $6.95561473 \mathrm{e}-02$ | $4.14536 \mathrm{e}-07$ | 3 | $6.95561235 \mathrm{e}-02$ | $1.25601 \mathrm{e}-07$ |
| 3 | 10 | 23.0 | $4.85276438 \mathrm{e}-02$ | $3.22474 \mathrm{e}-08$ | 3 | $4.85276441 \mathrm{e}-02$ | $2.36409 \mathrm{e}-07$ |
| 4 | 12 | 26.0 | $3.13462251 \mathrm{e}-02$ | $3.91273 \mathrm{e}-09$ | 3 | $3.13462254 \mathrm{e}-02$ | $4.38305 \mathrm{e}-07$ |
| 5 | 15 | 29.0 | $1.84023392 \mathrm{e}-02$ | $4.21233 \mathrm{e}-11$ | 3 | $1.84023392 \mathrm{e}-02$ | $7.97320 \mathrm{e}-07$ |
| 6 | 18 | 32.0 | $9.59306238 \mathrm{e}-03$ | $7.33817 \mathrm{e}-13$ | 4 | $9.59306238 \mathrm{e}-03$ | $1.02848 \mathrm{e}-09$ |
| 7 | 22 | 36.0 | $4.31303630 \mathrm{e}-03$ | $6.53498 \mathrm{e}-15$ | 4 | $4.31303630 \mathrm{e}-03$ | $2.23138 \mathrm{e}-09$ |
| 8 | 28 | 40.0 | $1.61250458 \mathrm{e}-03$ | $5.21978 \mathrm{e}-19$ | 4 | $1.61250459 \mathrm{e}-03$ | $4.66097 \mathrm{e}-09$ |
| 9 | 35 | 45.0 | $4.78946580 \mathrm{e}-04$ | $4.74770 \mathrm{e}-23$ | 4 | $4.78946580 \mathrm{e}-04$ | $9.28196 \mathrm{e}-09$ |
| 10 | 44 | 51.0 | $1.06756221 \mathrm{e}-04$ | $6.21259 \mathrm{e}-28$ | 4 | $1.06756221 \mathrm{e}-04$ | $1.74058 \mathrm{e}-08$ |
| 11 | 55 | 57.0 | $1.66339276 \mathrm{e}-05$ | $1.96299 \mathrm{e}-34$ | 5 | $1.66339276 \mathrm{e}-05$ | $5.73309 \mathrm{e}-11$ |
| 12 | 68 | 64.0 | $1.65883643 \mathrm{e}-06$ | $2.58927 \mathrm{e}-41$ | 5 | $1.65883643 \mathrm{e}-06$ | $1.14076 \mathrm{e}-10$ |
| 13 | 85 | 71.0 | $9.49292001 \mathrm{e}-08$ | $7.62159 \mathrm{e}-52$ | 6 | $9.49292072 \mathrm{e}-08$ | $4.78912 \mathrm{e}-13$ |

quoted for the Gauss quadrature, we could get away with using only 4 or 5 terms and still obtain accuracy to three or four decimal places. The Gauss quadrature rule method is also significantly faster computationally. For a given value of $m$ and a given timing offset, it is possible to compute the BER curve in between 1 and 2 minutes on a Sparc 10/30. For Beaulieu's method computing the same probability of error curve takes from about 30 minutes, for timing offsets less than $0.2 T$, to more than 1 hour when the timing offset approaches $0.5 T$. The reason for the large difference in computation time is that in the Gauss quadrature rule all the numerical integrations are performed up front in the computation of the moments. Once the moments have been obtained, the weights and abscissas can be found and serve to generate an entire BER curve. For Beaulieu's method, however, a numerical integration must be performed to determine the characteristic function of the interference for each term in the series and at each value of $E_{b} / N_{0}$. Thus the total number of integrations needed to obtain the BER curve is the sum of the $N_{M}$ values shown in the tables. In all fairness, however, we have not sought the optimum value for $\mathcal{T}$ and minimized the number of terms in the series. The values for the truncation error $\left|R_{M}\right|$ reported in the tables would seem to indicate that the number of terms could be reduced fairly significantly without sacrificing accuracy.

While on this subject, it should be stated that choosing the value of $\mathcal{T}$ is one of the most critical parts of computing with Beaulieu's method. In early attempts with the method, $\mathcal{T}$ was chosen rather small, and as the value of $E_{b} / N_{0}$ increased, the value of the argument in the error term $\beta$ began to approached zero. The result was that the series tended to diverge and $P_{b}$ would approach 1. After successive attempts at trying to keep $\mathcal{T}$ small and still have the series converge,

Table 5.2: Numerical results for $K=2$ interfering channels, zero timing offset on the desired channel $\left(\gamma_{k}=0\right)$ and timing offsets of $\gamma_{k-1}=\gamma_{k+1}=0.4 T$.

|  | Beaulieu's Fourier Series Method |  |  |  |  | Gauss Quadrature Rule Method |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $E_{b} / N_{0}$ | $N_{M}$ | $\mathcal{T}$ | $P_{b}$ | $\left\|R_{M}\right\|$ | $m$ | $P_{b}$ | $\left\|R_{M}\right\|$ |  |
| 0 | 6 | 18.0 | $1.26281963 \mathrm{e}-01$ | $2.55949 \mathrm{e}-05$ | 4 | $1.26282706 \mathrm{e}-01$ | $2.15779 \mathrm{e}-06$ |  |
| 1 | 7 | 20.0 | $1.01147412 \mathrm{e}-01$ | $8.64561 \mathrm{e}-06$ | 5 | $1.01147425 \mathrm{e}-01$ | $6.94119 \mathrm{e}-08$ |  |
| 2 | 8 | 22.0 | $7.80050408 \mathrm{e}-02$ | $3.41898 \mathrm{e}-06$ | 6 | $7.80049644 \mathrm{e}-02$ | $2.90355 \mathrm{e}-09$ |  |
| 3 | 10 | 25.0 | $5.75944807 \mathrm{e}-02$ | $2.99941 \mathrm{e}-07$ | 6 | $5.75944778 \mathrm{e}-02$ | $1.14692 \mathrm{e}-08$ |  |
| 4 | 13 | 28.0 | $4.04732255 \mathrm{e}-02$ | $2.87782 \mathrm{e}-09$ | 6 | $4.04732255 \mathrm{e}-02$ | $4.52122 \mathrm{e}-08$ |  |
| 5 | 16 | 31.0 | $2.69114036 \mathrm{e}-02$ | $4.21223 \mathrm{e}-11$ | 7 | $2.69114036 \mathrm{e}-02$ | $4.28846 \mathrm{e}-09$ |  |
| 6 | 19 | 35.0 | $1.68376720 \mathrm{e}-02$ | $3.71161 \mathrm{e}-12$ | 8 | $1.68376720 \mathrm{e}-02$ | $5.56986 \mathrm{e}-10$ |  |
| 7 | 24 | 39.0 | $9.86376756 \mathrm{e}-03$ | $3.87136 \mathrm{e}-15$ | 8 | $9.86376756 \mathrm{e}-03$ | $3.44597 \mathrm{e}-09$ |  |
| 8 | 30 | 44.0 | $5.38658785 \mathrm{e}-03$ | $3.41113 \mathrm{e}-18$ | 9 | $5.38658785 \mathrm{e}-03$ | $7.90798 \mathrm{e}-10$ |  |
| 9 | 38 | 49.0 | $2.73092380 \mathrm{e}-03$ | $5.61672 \mathrm{e}-23$ | 9 | $2.73092379 \mathrm{e}-03$ | $6.08894 \mathrm{e}-09$ |  |
| 10 | 47 | 55.0 | $1.27942882 \mathrm{e}-03$ | $1.72536 \mathrm{e}-27$ | 9 | $1.27942881 \mathrm{e}-03$ | $4.65067 \mathrm{e}-08$ |  |
| 11 | 59 | 62.0 | $5.50289451 \mathrm{e}-04$ | $1.33233 \mathrm{e}-33$ | 10 | $5.50289450 \mathrm{e}-04$ | $2.38484 \mathrm{e}-08$ |  |
| 12 | 74 | 69.0 | $2.15034956 \mathrm{e}-04$ | $4.35663 \mathrm{e}-42$ | 11 | $2.15034956 \mathrm{e}-04$ | $1.73003 \mathrm{e}-08$ |  |
| 13 | 92 | 77.0 | $7.50789961 \mathrm{e}-05$ | $1.07785 \mathrm{e}-51$ | 12 | $7.50789961 \mathrm{e}-05$ | $1.73204 \mathrm{e}-08$ |  |
| 14 | 116 | 87.0 | $2.28366454 \mathrm{e}-05$ | $8.34181 \mathrm{e}-64$ | 12 | $2.28366453 \mathrm{e}-05$ | $2.48765 \mathrm{e}-07$ |  |
| 15 | 145 | 97.0 | $5.84050468 \mathrm{e}-06$ | $1.44964 \mathrm{e}-79$ | 13 | $5.84050468 \mathrm{e}-06$ | $4.30447 \mathrm{e}-07$ |  |
| 16 | 183 | 109.0 | $1.19908625 \mathrm{e}-06$ | $1.07029 \mathrm{e}-99$ | 14 | $1.19908626 \mathrm{e}-06$ | $1.08521 \mathrm{e}-06$ |  |
| 17 | 230 | 122.0 | $1.86472756 \mathrm{e}-07$ | $5.14558 \mathrm{e}-125$ | 15 | $1.86472757 \mathrm{e}-07$ | $3.94697 \mathrm{e}-06$ |  |
| 18 | 289 | 137.0 | $2.04401871 \mathrm{e}-08$ | $7.84595 \mathrm{e}-156$ | 16 | $2.04401893 \mathrm{e}-08$ | $1.99024 \mathrm{e}-05$ |  |

the rather conservative approach of (5.1) was adopted. The general feeling obtained with experience is that it is better to choose $\mathcal{T}$ too large and use more terms in the series, than to set it too low and have to repeat the computation several times because the series didn't converge.

The values for the truncation error $\left|R_{M}\right|$ reported in the tables for Beaulieu's method were computed using the Maple Symbolic Computation system. The series approximation of the complementary error function in [26] breaks down for large arguments, and consequently, other means must be found to compute the truncation error. The Maple system claims to evaluate functions to any desired degree of accuracy, and in this case seems to function quite well. For large values of $E_{b} / N_{0}$, the truncation error is so many orders of magnitude smaller than the probability of bit error that it is inconsequential. Beaulieu has reported [19] that for the intersymbol and co-channel interference problems he has analyzed, the error bounds áre quite tight. We have no reason to believe that this is not the case here. For the Gauss quadrature rule method, the truncation bound tends to be somewhat loose for moderate to high values of $E_{b} / N_{0}$ and as the amount of interference becomes large. If we compare against the values obtained by Beaulieu's method, we see that in many cases the probability of bit error has converged to the same value, but the bound on truncation error of the Gauss quadrature rule is only two or three orders of magnitude smaller than the


Figure 5.1: Probability of bit error for $K=2$ interfering channels with zero timing offset on the desired channel ( $\gamma_{k}=0$ ).
probability of bit error. Increasing the number of terms $M$ in the quadrature rule will lower the truncation error, but the probability of bit error will not improve beyond what is given. Essentially, this indicates that by itself, the truncation error bound is not a very useful measure for determining what is a sufficient number of terms to achieve a desired accuracy. Instead, it is more useful to look directly at the probability of bit error and increase $M$ until the probability of bit error converges to the desired number of significant digits. Given the speed of the computation method, there is nothing particularly wrong with this approach. It takes about 20 minutes to compute the BER curves for all values of $M$ in the range from 3 to 20 ; hardly an onerous amount of computation time.

These results of Tables 5.1 and 5.2 are plotted in Fig. 5.1 along with those for other timing offsets. It is fairly clear from this figure that adjacent channel interference is not a problem so long as the timing offsets are kept to within about $10 \%$ of the symbol interval. Beyond that, the
adjacent channel interference causes significant degradation in the desired channel. The effect is even more severe when we consider the next case where the desired channel and the two interfering channels have equal timing offsets. Tables 5.3 and 5.4 show the results when all three channels, $k$, $k-1$ and $k+1$ have timing offsets of $0.1 T$ and $0.4 T$, respectively. If we compare Tables 5.1 and 5.3, we see that there is relatively little change in the probability of bit error and in the number of terms required by each of the computation methods. However, a comparison of Tables 5.2 and 5.4 tells quite a different story. In the latter table, we see that not only does the probability of bit error increase dramatically, but there is also more than a four-fold increase in the number of terms required by Beaulieu's method as the value of $E_{b} / N_{0}$ becomes large, although again we have not attempted to optimize the procedure. By comparison, the Gauss quadrature rule method still requires relatively few terms ( $M<20$ ), but we can see that as the value of $E_{b} / N_{0}$ becomes large, the method seems to break down a little as the truncation error bound becomes greater than one. This is a problem that we will elaborate more on shortly.

Table 5.3: Numerical results for $K=2$ interfering channels and timing offsets of $\gamma_{k}=\gamma_{k-1}=$ $\gamma_{k+1}=0.1 T$.

| $E_{b} / N_{0}$ | Beaulieu's Fourier Series Method |  |  |  |  | Gauss Quadrature Rule Method |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $N_{M}$ | $\mathcal{T}$ | $P_{b}$ | $\left\|R_{M}\right\|$ | $m$ | $P_{b}$ | $\left\|R_{M}\right\|$ |  |
| 0 | 6 | 16.0 | $1.22983590 \mathrm{e}-01$ | $2.88605 \mathrm{e}-06$ | 3 | $1.22983627 \mathrm{e}-01$ | $3.44186 \mathrm{e}-08$ |  |
| 1 | 6 | 18.0 | $9.65278928 \mathrm{e}-02$ | $2.55949 \mathrm{e}-05$ | 3 | $9.65284624 \mathrm{e}-02$ | $6.62602 \mathrm{e}-08$ |  |
| 2 | 8 | 20.0 | $7.21235285 \mathrm{e}-02$ | $4.14536 \mathrm{e}-07$ | 3 | $7.21235047 \mathrm{e}-02$ | $1.26383 \mathrm{e}-07$ |  |
| 3 | 10 | 23.0 | $5.07140125 \mathrm{e}-02$ | $3.22474 \mathrm{e}-08$ | 3 | $5.07140125 \mathrm{e}-02$ | $2.38266 \mathrm{e}-07$ |  |
| 4 | 12 | 26.0 | $3.30803159 \mathrm{e}-02$ | $3.91273 \mathrm{e}-09$ | 3 | $3.30803162 \mathrm{e}-02$ | $4.42648 \mathrm{e}-07$ |  |
| 5 | 15 | 29.0 | $1.96601419 \mathrm{e}-02$ | $4.21223 \mathrm{e}-11$ | 4 | $1.96601419 \mathrm{e}-02$ | $4.65727 \mathrm{e}-10$ |  |
| 6 | 18 | 32.0 | $1.04085423 \mathrm{e}-02$ | $7.33817 \mathrm{e}-13$ | 4 | $1.04085423 \mathrm{e}-02$ | $1.04472 \mathrm{e}-09$ |  |
| 7 | 22 | 36.0 | $4.77227353 \mathrm{e}-03$ | $6.53498 \mathrm{e}-15$ | 4 | $4.77227353 \mathrm{e}-03$ | $2.27592 \mathrm{e}-09$ |  |
| 8 | 28 | 40.0 | $1.82924120 \mathrm{e}-03$ | $5.21978 \mathrm{e}-19$ | 4 | $1.82924120 \mathrm{e}-03$ | $4.77870 \mathrm{e}-09$ |  |
| 9 | 35 | 45.0 | $5.60922239 \mathrm{e}-04$ | $4.74770 \mathrm{e}-23$ | 5 | $5.60922239 \mathrm{e}-04$ | $1.14507 \mathrm{e}-11$ |  |
| 10 | 44 | 51.0 | $1.30258508 \mathrm{e}-04$ | $6.21259 \mathrm{e}-28$ | 5 | $1.30258508 \mathrm{e}-04$ | $2.72582 \mathrm{e}-11$ |  |
| 11 | 55 | 57.0 | $2.13999362 \mathrm{e}-05$ | $1.96299 \mathrm{e}-34$ | 5 | $2.13999362 \mathrm{e}-05$ | $6.02931 \mathrm{e}-11$ |  |
| 12 | 68 | 64.0 | $2.28629112 \mathrm{e}-06$ | $2.58927 \mathrm{e}-41$ | 5 | $2.28629111 \mathrm{e}-06$ | $1.21593 \mathrm{e}-10$ |  |
| 13 | 85 | 71.0 | $1.43168783 \mathrm{e}-07$ | $7.62159 \mathrm{e}-52$ | 6 | $1.43168790 \mathrm{e}-07$ | $5.19298 \mathrm{e}-13$ |  |
| 14 | 107 | 80.0 | $4.62516982 \mathrm{e}-09$ | $3.84447 \mathrm{e}-64$ | 7 | $4.62516139 \mathrm{e}-09$ | $2.50663 \mathrm{e}-15$ |  |

To better appreciate the effects of the interference, in ${ }^{\circ}$ Fig. 5.2 we have plotted the probability of bit error for channel $k$ with different timing offsets but no interference. This is followed by Fig. 5.3, where we now have two interfering channels with timing offsets equal to that on channel $k$. The most notable effect of the interference is that error floors begin to appear when the timing offset is large on the desired channels and on the interfering channels. With only two interfering channels, the error floors do not begin to appear until the timing offsets approach $0.4 T$. However,

Table 5.4: Numerical results for $K=2$ interfering channels and timing offsets of $\gamma_{k}=\gamma_{k-1}=$ $\gamma_{k+1}=0.4 T$.

|  | Beaulieu's Fourier Series Method |  |  |  |  | Gauss Quadrature Rule Method |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $E_{b} / N_{0}$ | $N_{M}$ | $\mathcal{T}$ | $P_{b}$ | $\left\|R_{M}\right\|$ | $m$ | $P_{b}$ | $\left\|R_{M}\right\|$ |  |
| 0 | 6 | 18.0 | $2.28979092 \mathrm{e}-01$ | $2.55949 \mathrm{e}-05$ | 5 | $2.28979047 \mathrm{e}-01$ | $2.21115 \mathrm{e}-08$ |  |
| 1 | 7 | 20.0 | $2.07480374 \mathrm{e}-01$ | $8.64561 \mathrm{e}-06$ | 5 | $2.07480280 \mathrm{e}-01$ | $6.96219 \mathrm{e}-08$ |  |
| 2 | 8 | 22.0 | $1.86362813 \mathrm{e}-01$ | $3.41898 \mathrm{e}-06$ | 6 | $1.86362757 \mathrm{e}-01$ | $2.91462 \mathrm{e}-09$ |  |
| 3 | 10 | 25.0 | $1.66065811 \mathrm{e}-01$ | $2.99941 \mathrm{e}-07$ | 6 | $1.66065810 \mathrm{e}-01$ | $1.15243 \mathrm{e}-08$ |  |
| 4 | 13 | 28.0 | $1.46975974 \mathrm{e}-01$ | $2.87782 \mathrm{e}-09$ | 6 | $1.46975974 \mathrm{e}-01$ | $4.54860 \mathrm{e}-08$ |  |
| 5 | 16 | 31.0 | $1.29376769 \mathrm{e}-01$ | $4.21223 \mathrm{e}-11$ | 6 | $1.29376769 \mathrm{e}-01$ | $1.79133 \mathrm{e}-07$ |  |
| 6 | 19 | 35.0 | $1.13419027 \mathrm{e}-01$ | $3.71161 \mathrm{e}-12$ | 7 | $1.1341 .19027 \mathrm{e}-01$ | $2.13642 \mathrm{e}-08$ |  |
| 7 | 24 | 39.0 | $9.91242860 \mathrm{e}-02$ | $3.87136 \mathrm{e}-15$ | 8 | $9.91242860 \mathrm{e}-02$ | $3.48785 \mathrm{e}-09$ |  |
| 8 | 30 | 44.0 | $8.64199180 \mathrm{e}-02$ | $3.41113 \mathrm{e}-18$ | 8 | $8.64199180 \mathrm{e}-02$ | $2.15370 \mathrm{e}-08$ |  |
| 9 | 38 | 49.0 | $7.51891170 \mathrm{e}-02$ | $5.61672 \mathrm{e}-23$ | 9 | $7.51891170 \mathrm{e}-02$ | $6.20707 \mathrm{e}-09$ |  |
| 10 | 47 | 55.0 | $6.53128719 \mathrm{e}-02$ | $1.72536 \mathrm{e}-27$ | 10 | $6.53128719 \mathrm{e}-02$ | $2.56699 \mathrm{e}-09$ |  |
| 11 | 59 | 62.0 | $5.66894692 \mathrm{e}-02$ | $1.33233 \mathrm{e}-33$ | 10 | $5.66894692 \mathrm{e}-02$ | $2.45900 \mathrm{e}-08$ |  |
| 12 | 74 | 69.0 | $4.92325012 \mathrm{e}-02$ | $4.35663 \mathrm{e}-42$ | 11 | $4.92325012 \mathrm{e}-02$ | $1.79829 \mathrm{e}-08$ |  |
| 13 | 92 | 77.0 | $4.28588266 \mathrm{e}-02$ | $1.07785 \mathrm{e}-51$ | 11 | $4.28588266 \mathrm{e}-02$ | $2.11516 \mathrm{e}-07$ |  |
| 14 | 116 | 87.0 | $3.74781108 \mathrm{e}-02$ | $8.34181 \mathrm{e}-64$ | 12 | $3.74781108 \mathrm{e}-02$ | $2.64677 \mathrm{e}-07$ |  |
| 15 | 145 | 97.0 | $3.29894203 \mathrm{e}-02$ | $1.44964 \mathrm{e}-79$ | 14 | $3.29894203 \mathrm{e}-02$ | $5.46321 \mathrm{e}-08$ |  |
| 16 | 183 | 109.0 | $2.92843478 \mathrm{e}-02$ | $1.07029 \mathrm{e}-99$ | 14 | $2.92843478 \mathrm{e}-02$ | $1.19926 \mathrm{e}-06$ |  |
| 17 | 230 | 122.0 | $2.62534461 \mathrm{e}-02$ | $5.14558 \mathrm{e}-125$ | 16 | $2.62534461 \mathrm{e}-02$ | $7.27129 \mathrm{e}-07$ |  |
| 18 | 289 | 137.0 | $2.37927647 \mathrm{e}-02$ | $7.84595 \mathrm{e}-156$ | 16 | $2.37927646 \mathrm{e}-02$ | $2.34159 \mathrm{e}-05$ |  |
| 19 | 364 | 154.0 | $2.18084033 \mathrm{e}-02$ | $6.47841 \mathrm{e}-195$ | 18 | $2.18084033 \mathrm{e}-02$ | $4.33853 \mathrm{e}-05$ |  |
| 20 | 459 | 173.0 | $2.02183892 \mathrm{e}-02$ | $7.95780 \mathrm{e}-245$ | 18 | $2.02183898 \mathrm{e}-02$ | $1.96344 \mathrm{e}-03$ |  |
| 21 | 577 | 194.0 | $1.89524360 \mathrm{e}-02$ | $7.07711 \mathrm{e}-307$ | 18 | $1.89524362 \mathrm{e}-02$ | $8.17971 \mathrm{e}-02$ |  |
| 22 | 724 | 217.0 | $1.79507396 \mathrm{e}-02$ | $2.38729 \mathrm{e}-385$ | 19 | $1.79507380 \mathrm{e}-02$ | $9.50713 \mathrm{e}-01$ |  |
| 23 | 913 | 244.0 | $1.71627580 \mathrm{e}-02$ | $7.59627 \mathrm{e}-484$ | 19 | $1.71627016 \mathrm{e}-02$ | $3.96556 \mathrm{e}+01$ |  |
| 24 | 1146 | 273.0 | $1.65462980 \mathrm{e}-02$ | $4.31470 \mathrm{e}-608$ | 19 | $1.65460490 \mathrm{e}-02$ | $1.41539 \mathrm{e}+03$ |  |
| 25 | 1446 | 307.0 | $1.60667076 \mathrm{e}-02$ | $1.12135 \mathrm{e}-764$ | 19 | $1.60665917 \mathrm{e}-02$ | $4.16993 \mathrm{e}+04$ |  |

if the number of interfering channels were to increase, we might expect to see error floors at smaller timing offsets. This is the basis for the next set of computations presented.

Figs. 5.4 and 5.5 show the results when the desired channel has zero timing offset and the interfering channels all have the same timing offset of between $\gamma_{l}=0.2 T$ and $\gamma_{l}=0.5 T$. The cases of $\gamma_{l}=0$ and $\gamma_{l}=0.1 T$ have not been included, as Fig. 5.1 indicates that the adjacent channel interference has little or no effect on the probability of, bit error, and increasing the number of channels is not expected to change this. For $\gamma_{k}=\gamma_{l}=0.2 T$, Fig. 5.4 shows that increasing the number of channels does not substantially alter the performance. At a BER of $10^{-8}$, there seems to be no more than about 0.5 dB degradation when the number of interfering channels increases from $K=2$ to $K=10$, and there is almost no further degradation when $K$ increases beyond 10 . When the timing offset increases to $\gamma_{k}=\gamma_{l}=0.3 T$, however, the number of interfering channels


Figure 5.2: Bit-error-rate curves for a single channel with various fixed timing offsets.
does significantly affect the probability of bit error. For values of BER around $10^{-7}$, the error performance degrades by about $2-2.5 \mathrm{~dB}$ when the number of interfering channels increases to $K=10$, and there is a further 0.5 dB degradation when $K=20$. Clearly, when timing offsets are large, a significant number of channels can affect the performance of a given channel. When the timing offsets on the interfering channels increase to $\gamma_{l}=0.4 T$ and $\gamma_{l}=0.5 T$, the performance of the desired channel is limited by the interference as error floors appear. Similar results are found when the desired channel also has a timing offset as shown in Fig. 5.6. For $\gamma_{k}=\gamma_{l}=0.2 T$, the degradation is again only about 0.5 dB at $P_{b}=10^{-8}$ when we go from $K=2$ to $K=10$. However, for $\gamma_{k}=\gamma_{l}=0.3 T$, an asymptotic error floor begins to appear when $K=10$, and by the time we reach $K=20$, the asymptotic value of this error floor seems to be of the order of $5 \times 10^{-6}$. Previous results in Fig. 5.3 showed that the adjacent channel interference was sufficient to introduce a fairly large error floor when $\gamma_{k}=\gamma_{l}=0.4 T$, and so we have not bothered to compute the error probability for $K>2$ as the only effect would be to raise this error floor.


Figure 5.3: Probability of bit error for $K=2$ interfering channels and equal timing offset $\gamma_{k}=\gamma_{l}$, $l=k \pm 1$.

For $K=10$, the results shown in Figs. 5.4 and 5.6 were computed using both Beaulieu's method and the Gauss quadrature rule method. The remaining results for $K=20$ in Figs. 5.4 and 5.6, and the results shown in Fig. 5.6, were computed using the Gauss quadrature rule method alone. As the number of channels increases, the amount of computation time required by Beaulieu's method increases exponentially. Again, this is due to the large number of numerical integrations that must be performed. When $K=10$, the computation time increases to about 30 hours on a Sparc 10/30 for each case. It is for this reason that we did not pursue the case of $K=20$ interfering channels and larger timing offsets with this method. For the Gauss quadrature rule method, however, the computation time with $K=10$ and $m=25$ is of the order of 7 minutes. When the number of interfering channels increases to $K=20$ and the number of moments increases to $m=30$, the computation time is approximately 15-20 minutes. For a large number of interfering channels, the problem with the Gauss quadrature rule method is not one of computation time, but rather that


Figure 5.4: Probability of bit error for various numbers of interfering channels with zero timing offset on the desired channel and equal timing offsets of $\gamma_{l}=0.2 T$ and $\gamma_{l}=0.3 T$ on the interfering channels.
the method begins to break down as the amount of interference increases and $E_{b} / N_{0}$ becomes large. For timing offsets of $0.2 T$ and $0.3 T$, the method still works reasonabll well, but when the timing offset increases to $0.4 T$ or $0.5 T$, the method is slow to converge and the truncation error bound becomes positive and large. An increasing number of terms $M$ are required to accurately compute $P_{b}$, but there is a limit to this number as the highest order moments of the interference become too small and begin to underflow the double precision arithmetic. It should be stated, however, that these cases are interesting from a computational point of view but are not reasonable conditions for an operational system. Indeed, for the cases that we would be interested in for an operational system (i.e., small timing offsets), the Gauss quadrature rule method performs adequately and is quite fast.


Figure 5.5: Probability of bit error for various numbers of interfering channels with zero timing offset on the desired channel and equal timing offsets of $\gamma_{l}=0.4 T$ and $\gamma_{l}=0.5 T$ on the interfering channels.

### 5.2 Polyphase Network

For the digital group demodulator based on the polyphase network/FFT, several channel spacing and filtering scenarios were examined. In the first of these scenarios, it will be assumed that the spacing between carriers is 1.5 times the symbol rate of the signals ( $f_{c}=1.5 f_{b}$ ) and that the group demodulator has $N=8$ channels. The transmit filter will have a root raised cosine spectral characteristic, while the polyphase network and the rate conversion filters will each have a fourth root raised cosine characteristic. The rolloff factor of the raised cosine response was set to 0.4. Under ideal conditions, the cascade of these three filters would produce a full raised cosine spectral response.

To design the filters, the root raised cosine or fourth root raised cosine spectral characteristics are specified in the frequency domain, and then an inverse FFT is used to obtain the time domain


Figure 5.6: Probability of bit error for various numbers of interfering channels with equal timing offsets of $\gamma_{k}=\gamma_{l}=0.2 T$ and $\gamma_{k}=\gamma_{l}=0.3 T$ on the desired and interfering channels.
response. A large FFT of 8192 points was used in order to better approximate the true time function. This is then truncated to the desired number of filter taps or transmit filter coefficients as the case may be. For the polyphase filter and the rate conversion filter, the filter coefficients are normalized to have unit energy, so that for convenience, $\sigma_{o}^{2}=\sigma_{i}^{2}$ in (3.20).

To simplify the initial computations and estimate the number of taps required for the filters, the signals are all assumed to be time synchronized with the group demodulator. With perfect timing and a channel spacing of $f_{c}=1.5 f_{b}$, there are only two possibilities for the timing parameters $\beta_{l}$ and $\mu_{l}$ in (3.13): either $\beta_{l} T_{c}=l T_{b}$ and $\mu_{l} T_{c}=0$ or $\beta_{l} T_{c}=l T_{b}-T_{b} / 3$ and $\mu_{l} T_{c}=T_{b} / 3$. The rate conversion filter will alternate between these two sets of values from one symbol interval to the next, and with the assumption of perfect timing, will know exactly which state to be in during any symbol interval. In the analysis, the probability of bit error will be computed for each state, and the average of the two values taken. When $\beta_{l} T_{c}=l T_{b}$ and $\mu_{l} T_{c}=0$, the filtering equations can be
rewritten as

$$
\begin{equation*}
u\left(l T_{b}-i T_{b}\right)=\sum_{\xi=I_{1}}^{I_{2}} \sum_{n=1}^{N L} h\left(n T_{s}\right) g\left(\xi T_{c}\right) h_{s}\left[l T_{b}-i T_{b}-(n-N L / 2) T_{s}-\xi T_{c}\right] \tag{5.4}
\end{equation*}
$$

and

$$
\begin{equation*}
v\left(l T_{b}-i T_{b}\right)=\sum_{\xi=0}^{I} \sum_{n=1}^{N L} h\left(n T_{s}\right) g\left(\xi T_{c}\right) h_{s}\left[l T_{b}-i T_{b}-(n-N L / 2) T_{s}-\xi T_{c}\right] e^{j 2 \pi(k-q) \frac{n}{N}} \tag{5.5}
\end{equation*}
$$

Similar equations follow for $\beta_{l} T_{c}=l T_{b}-T_{b} / 3$ and $\mu_{l} T_{c}=T_{b} / 3$.
The taps of the prototype filter $h\left(n T_{s}\right)$ for the polyphase network are chosen such that $N L / 2$ is always the center tap. This accounts for the offset of $N L / 2$ in the transmit response $h_{s}(t)$ in (5.4) and (5.5). For the rate conversion filter, the taps are chosen such that $I_{1}=-I / 2$ and $I_{2}=I / 2-1$ when the number of filter taps $I$ is even, or $I_{1}=-(I-1) / 2$ and $I_{2}=(I-1) / 2$ when $I$ is odd. When $\beta_{l} T_{c}=l T_{b}$ and $\mu_{l} T_{c}=0$, the filter tap at $\xi=0$ will be the peak of the rate conversion filter response $g\left(\xi T_{c}\right)$, while for the second case, an offset of $\mu_{l} T_{c}=T_{b} / 3$ means that the peak of the root raised cosine response occurs at a point half way between the taps at $\xi=0$ and $\xi=1$.

Fig. 5.7 shows the first set of results where all the carriers are assumed to have the same power. The desired signal, for which the BER was computed, is assumed to be on channel $k=3$ in the middle of the group of signals. The number of taps $L$ in the polyphase filters and $I$ in the rate conversion filter were chosen to be equal to integer multiples of the channel spacing so that the impulse response would span an integer number of symbol intervals. The figure shows that with as few as 6 taps per filter, the error performance closely approximates that of an ideal QPSK signal. Unfortunately, the assumption of equal power on all the carriers is a somewhat idealistic one. A scenario that we would more commonly want to design for is where the desired channel is faded with respect to the other carriers. In such a case, the interchannel interference could have a much more significant impact on the performance of the desired signal, and we want to choose the number of filter taps to be large enough to provide sufficient isolation from the adjacent channels. Fig. 5.8 gives one example of this where the desired signal is experiencing a fairly severe flat fade of 9 dB with respect to the remaining channels. By choosing the number of taps to be $L=I=9$, the performance will be degraded by no more than 1 dB at an error rate of $10^{-8}$.

Generally, windowing is used to reduce the sidelobes of a finite impulse response filtering. For the present case, this would imply that windowing would lower the amount of interchannel interference. To see what improvements in error performance are possible, Hamming windowing was applied to the polyphase and rate conversion filters, and this last computation was repeated. The results shown in Fig. 5.9 indicate that significant improvements can be obtained in the performance with


Figure 5.7: Probability of bit error for an 8 channel digital group demodulator with a channel spacing of 1.5 times the baud rate. $L$ is the number of taps per polyphase filter and $I$ is the number of taps in the rate conversion filter. All the carriers are assumed to be of equal power.
$L=I=6$ taps by using windowing, and for $L=I=9$ taps, we can come to within about 0.5 dB of the theoretical performance at an error rate of $10^{-8}$.

If we increase the channel spacing, we would expect that for a given error performance, the number of required taps would be reduced. To this end, we increased the channel spacing to 1.75 times the symbol rate and set the rolloff factor of the filters to 0.5 . Hamming windowing was also used for the polyphase and rate conversion filters. With a 9 dB fade on the desired channel, Fig. 5.10 shows that we can get away with $L=I=8$ taps and still get the same performance as with a channel spacing of $f_{c}=1.5 f_{b}$ and $L=I=9$ taps. If we are willing to accept a slightly larger degradation, $L=I=7$ taps can be used with a channel spacing of 1.75.

To obtain these results, both the Gauss quadrature rule and the Fourier series method were used. For the numerical integrations over the phase in (4.33) and (4.62), a 2000 point integral was


Figure 5.8: Probability of bit error for an 8 channel digital group demodulator with a channel spacing of 1.5 times the baud rate. $L$ is the number of taps per polyphase filter and $I$ is the number of taps in the rate conversion filter. The desired carrier is assumed to be experiencing a fade of 9 dB relative to the remaining channels.
used with the alternative extended Simpson's rule [27, Eq. 4.1.14]. For the number of contributing intersymbol and interchannel interference samples, $m L$ and $m U$ were chosen to equal 20. Although the error associated with the truncated series has not been computed, it is expected to be very small. Similar results and timing were experienced as with the SAW-based group demodulator. For the Gauss quadrature rule, rather than search for the smallest number of moments, we simply set $M=15$ for all the cases. The resultant truncation was in almost all instances vanishingly small compared to the probability of bit error. Execution time was of the order of $7-8$ minutes on a Sparc 10/30 for each case. For the Fourier series method, a comparable set of inequalities to those of (5.1) and (5.3) were used to choose $M$ and $T$ (i.e., the same upper limits were used). The average computation time was more than 6 hours for each set of values for $L$ and $I$. Eight significant figures were recorded in each of the computations, and there was agreement to at least


Figure 5.9: Probability of bit error for an 8 channel digital group demodulator with a channel spacing of 1.5 times the baud rate. The desired carrier is assumed to be experiencing a fade of 9 dB relative to the remaining channels. Hamming windowing has been applied to the polyphase and rate conversion filters to reduce the sidelobes.
four significant figures between the two methods.


Figure 5.10: Probability of bit exror for an 8 channel digital group demodulator with a channel spacing of 1.75 times the baud rate. The desired carrier is assumed to be experiencing a fade of 9 dB relative to the remaining channels. The filters were designed with a rolloff factor of 0.5 and Hamming windowing was applied to the polyphase and rate conversion filters.

## Chapter 6

## Conclusions

Two methods of analysis in interference environments have been presented and applied to two different group demodulator structures for satellite on-board processing. In each of these structures, the demodulated signal contains some amount of intersymbol and/or interchannel interference. In general, the probability density function of this interference is difficult to find but certain other measures, such as the moments of the interference or its characteristic function, can be readily computed. Using these measures, the Gauss quadrature rule method and Beaulieu's Fourier series method allow for the efficient computation of the probability of bit error. When we compare the two methods in terms of computational speed, the Gauss quadrature rule method proved to be significantly faster. This is offset somewhat by the fact that the Gauss quadrature rule method tends to break down when the interference and the signal-to-noise ratio become large. Given sufficient computing time, however, virtually any desired level of accuracy can be achieved with Beaulieu's Fourier series method. In general, it can be said that both methods work very well for small to moderate amounts of interference. Given that these are the conditions under which any system is likely to be designed and operated, either method could be used to predict system performance with a high degree of accuracy. In such circumstances, the one thing that favours the Gauss quadrature rule method is its speed.

The numerical results that have been presented have concentrated on fairly idealized conditions mostly for purposes of demonstration. The general goal of this report has been to develop the analysis and prove the efficacy of the two computation methods. The next stage of this work will be to broaden the analysis to include such issues as quantization and the use of error-control coding. To analyze quantization effects, some simplifying assumptions will have to be made to make the problem manageable, particularly since the group demodulator includes many additions and multiplications where re-quantization would take place. Some changes will also be required to analyze a system with error-control coding. In particular, the analysis done thus far has assumed
that the symbols are uncorrelated, but cqding will introduce some correlation from symbol to symbol. As a first approximation, we might ignore this correlation and simply modify the error expressions derived here to include coaing. In such a situation, we would expect that whatever coding gain is possible on a Gaussian channel would also be available here. Further work needs to be done to look into this in more detail.

## Bibliography

[1] F. Takahata et al., "A PSK group modem for satellite communications", IEEE J. Selected Areas Commun., vol. SAC-5, pp. 648-661, May 1987.
[2] W. F. Yim, C. C. D. Kwan, F. P. Coakly, and B. G. Evans, "Multi-carrier demodulators for on-board processing satellites", Int. J. Sat. Commun., vol. 6, pp. 243-251, 1988.
[3] C. Loo and M. Umehira, "Performance estimation and design of group demodulator for satellite FDMA/TDM transmission", in IEEE Conf. Rec. Globecom'89, November 1989, pp. 1110-1114.
[4] H. Göckler, "A modular multistage approach to digital FDM demultiplexing for mobile SCPC satellite communications", Int. J. Sat. Commun., vol. 6, pp. 283-288, 1988.
[5] H. Göckler and H. Eyssele, "Study of on-board digital FDM-demultiplexing for mobile SCPC satellite communications - Parts I and IT", European Trans. Telecomm. Systems, vol. 3, pp. 7-30, 1992.
[6] J. M. Kappes and S. I. Sayegh, "Programmable demultiplexer/demodulator processor", in IEEE Conf. Rec. MILCOM'90, 1990, pp. 7.3.1-5.
[7] S. I. Sayegh, J. M. Kappes, and S. J. Campanella, "On-board multi-carrier demultiplexer/demodulator", in Proc. Int. Conf. Dig. Sat. Commun., 1992, pp. 433-438.
[8] T. Kohri, M. Morikura, and S. Kato, "A 400ch SCPC signal demodulator using chirp transform and correlation detection scheme", in IEEE Conf. Rec. GLOBECOM'8', 1987, pp. 8.4.1-6.
[9] C. Loo, "Performance analysis of a SAW-based group demultiplexer for on-board processing communications satellites", IEEE Trans. Commun., vol. COM-41, pp. 1112-1116, July 1993.
[10] C. Loo and M. D. Shaw, "Performance analysis and measurement of a SAW-based group demodulator for on-board processing in communications satellites", Int. J. Sat. Commun., vol. 11, pp. 243-252, Sept.-Oct. 1993.
[11] K. Kobayashi, T. Kumagai, and S. Kato, "A group demodulator employing multi-symbol chirp Fourier transform", in Proc. Of the 4th Int. Symp. On Personal, Indoor and Mobile Radio Commun. (PIMRC'93), Yokohama, Japan, Sept. 1993, pp. B2.4.1-5.
[12] K. Kobayashi, T. Kumagai, and S. Kato, "New group demodulator for bandlimited and bit asynchronous FDMA signals", Electronic Letters, vol. 30, no. 10, pp. 751-752, May 121994.
[13] C. W. Helstrom, "Calculating error probabilies for intersymbol and cochannel interference", IEEE Trans. Commun., vol. COM-34, pp. 430-435, May 1986.
[14] E. Y. Ho and Y. S. Yeh, "A new approạch for evaluating the error probability in the presence of intersymbol and additive Gaussian noise", Bell Sys. Tech. J., vol. 49, pp. 2249-2265, November 1970.
[15] O. Shimbo and M.I. Celebiler, "The probability of error due to intersymbol interference and Gaussian noise in digital communication systems", IEEE Trans. Commun., vol. COM-19, pp. 113-119, April 1971.
[16] S. Benedetto, G. De Vincentiis, and A. Luvison, "Error probability in the presence of intersymbol interference and additive noise for multilevel digital signals", IEEE Trans. Commun. Tech., vol. COM-21, pp. 181-190, 1973.
[17] S. Benedetto, E. Biglieri, and V. Castellani, "Combined effects of intersymbol, interchannel, and co-channel interference in M-ary CPSK systems", IEEE Trans. Commun. Tech., vol. COM-21, pp. 997-1008, 1973.
[18] S. Benedetto, E. Biglieri, A. Luvison, and V. Zingarelli, "Moment-based performance evaluation of digital transmission systems", IEE Procedings-Part $I$, vol. 139, no. 3, pp. 258-266, June 1992.
[19] N. C. Beaulieu, "The evaluation of error probabilities for intersymbol and cochannel interference", IEEE Trans. Commun., vol. COM-39, pp. 1740-1749, December 1991.
[20] J. M. Wozencraft and I. M. Jacobs, Principles of Communication Engineering, John Wiley and Sons, New York, NY, 1965.
[21] R. E. Crochiere and L. R. Rabiner, Multirate Digital Signal Processing, Prentice-Hall, Englewood Cliffs, NJ, 1983.
[22] F. M. Gardner, "Interpolation in digital modems-Part I: Fundamentals", IEEE Trans. Commun., vol. COM-41, pp. 501-507, March 1993.
[23] F. Ananasso and E. Del Re, "Clock and carrier synchronization in FDMA/TDM user-oriented satellite systems", in Proc. IEEE Int. Conf. Commun., Seattle, WA, June 1987, pp. 41.7.1-5.
[24] G. H. Golub and J. H. Welsch, "Calculation of Gauss quadrature rules", Math. Comput., vol. 23, pp. 221-230, 1969.
[25] V. K. Prabhu, "Some considerations of error bounds in digital systems", Bell Sys. Tech. J., vol. 50, pp. 3127-3151, December 1971.
[26] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover Publications, New York, 1972.
[27] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, Numerical Recipes in C, Cambridge University Press, New York, 1988.



[^0]:    ${ }^{1}$ This work was carried out under a Memorandum of Agreement with the Canadian Institute for Telecommunications Research (CITR).

[^1]:    ${ }^{1}$ See the paper of Helstrom [13] for an extensive bibliography.

