Communications Research Centre

MOTION EQUATIONS FOR DUAL SPIN SATELLITES

by

F.R. Vigneron, T.W. Garrett and L.R. Eisenhauer

DEPARTMENT OF COMMUNICATIONS
MINISTÈRE DES COMMUNICATIONS

CRC TECHNICAL NOTE NO. 630

CANADA

OTTAWA, DECEMBER 1970

COMMUNICATIONS RESEARCH CENTRE

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70-443

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ABSTRACT

Motion equations are developed for two configurations of dual spin satellites. Each configuration consists of an asymmetric platform, a motor driven symmetric rotor, a platform damper, and a rotor damper. The differences between the configurations lie in the asymmetries of the platform and the degree of complexity of the dampers. The equations describe the satellite behaviour in the 'earth-seeking' and 'earth-pointing' modes of operation. Computer programs are listed for

a) assessing the stability of the solution for the earth seeking mode.

b) solving the motion equations for the 'free spin' mode, and

c) solving the motion equations when pulsed, attitude correction torques are applied during the 'earth-pointing' mode.

1. INTRODUCTION

Presented herein is a development of the motion equations for dual-spin satellites of interest in the Canadian Telesat program. The equations describe satellite behaviour in the 'earth-pointing' mode and the 'earth-seeking' mode, in which the platform (i.e., the antenna) swings through large seeking angles. A computer method for assessing stability of the 'earth-pointing' mode of operation is described.

2. CONFIGURATIONS OF INTEREST

The mathematical models of two configurations, a simple one and a more complex one, will be given for circular orbit flight. The first model, shown in Figure 1 and designated as M1, consists of an asymmetric platform, a motor driven symmetric rotor, a platform damper, and a very simple spherical rotor-damper. The second model, shown in Figure 2 and designated as M2, consists of an asymmetric platform, a motor driven symmetric rotor, a platform damper, which is modelled after one to be used in practice, and a rotor damper to simulate the first mode of fuel sloshing and structural damping.

Model M2 has the generality required for preliminary model designs of interest in the Telesat program. Model M1 is particularly useful in analytical work, since the equations of motion for it turn out to be relatively uncomplicated.

3. MOTION EQUATIONS

In section 3.1, general motion equations applicable to flexible satellites are developed from variational principles. They are then applied to derive linearized equations for the M1 and M2 configurations in sections 3.2 and 3.3.

3.1 GENERAL MOTION EQUATIONS

Consider a deformable body near the earth, as is shown in Figure 3. Moving axes (O'x'y'z') are affixed to some fibers of the body and remain so during deformation. Axes (Oxyz) are assigned to the body so that they always remain parallel to (O'x'y'z') and so that the point O always coincides with the instantaneous mass center of the body, i.e., for the vector $\underline{r}$ from O to any point within the body,

$$\int_M \underline{r}\, dm = 0 \,, \qquad \text{.....(3-1)}$$

where M is the total mass of the body, and dm is an increment of mass at $\underline{r}$. The vector $\underline{r}'$ has components in (O'x'y'z')

$$\underline{r}' = \underline{\rho} + \underline{r} \,, \qquad \text{.....(3-2)}$$

where $\underline{\rho}$ is the distance to the mass center. Combining equations (3-2) with (3-1) leads to

$$\underline{\rho} = \frac{1}{M} \int_M r' \, dm \qquad \text{.....(3-3)}$$

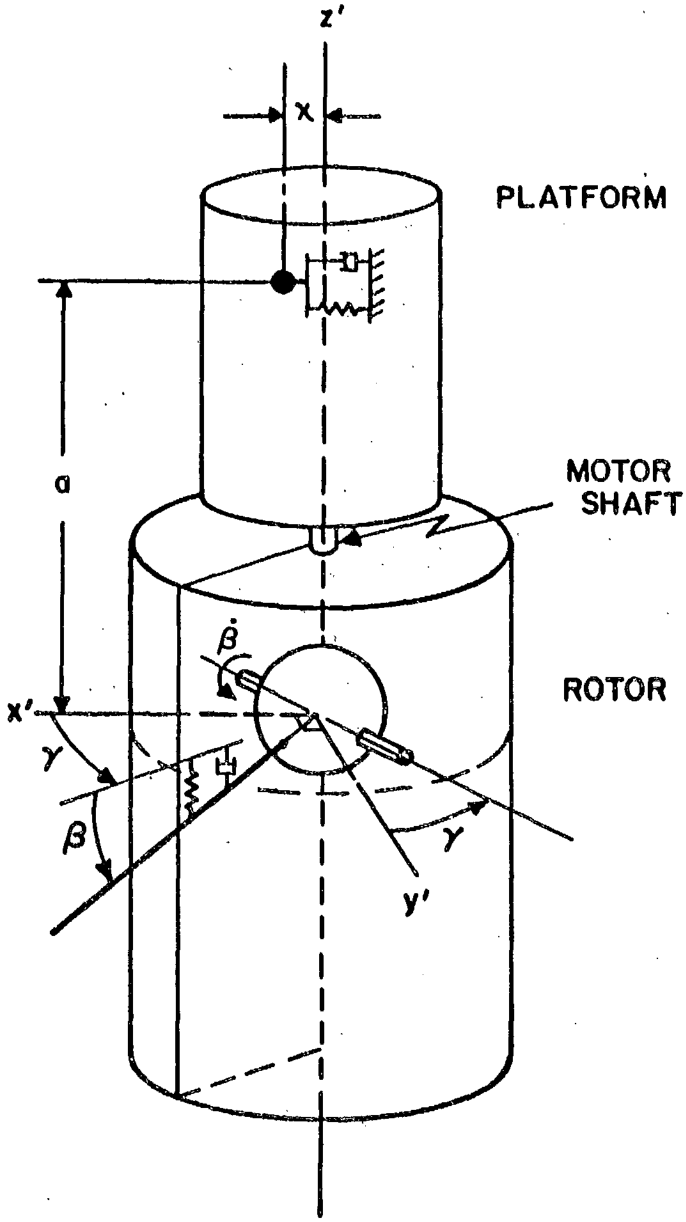

Figure 1.

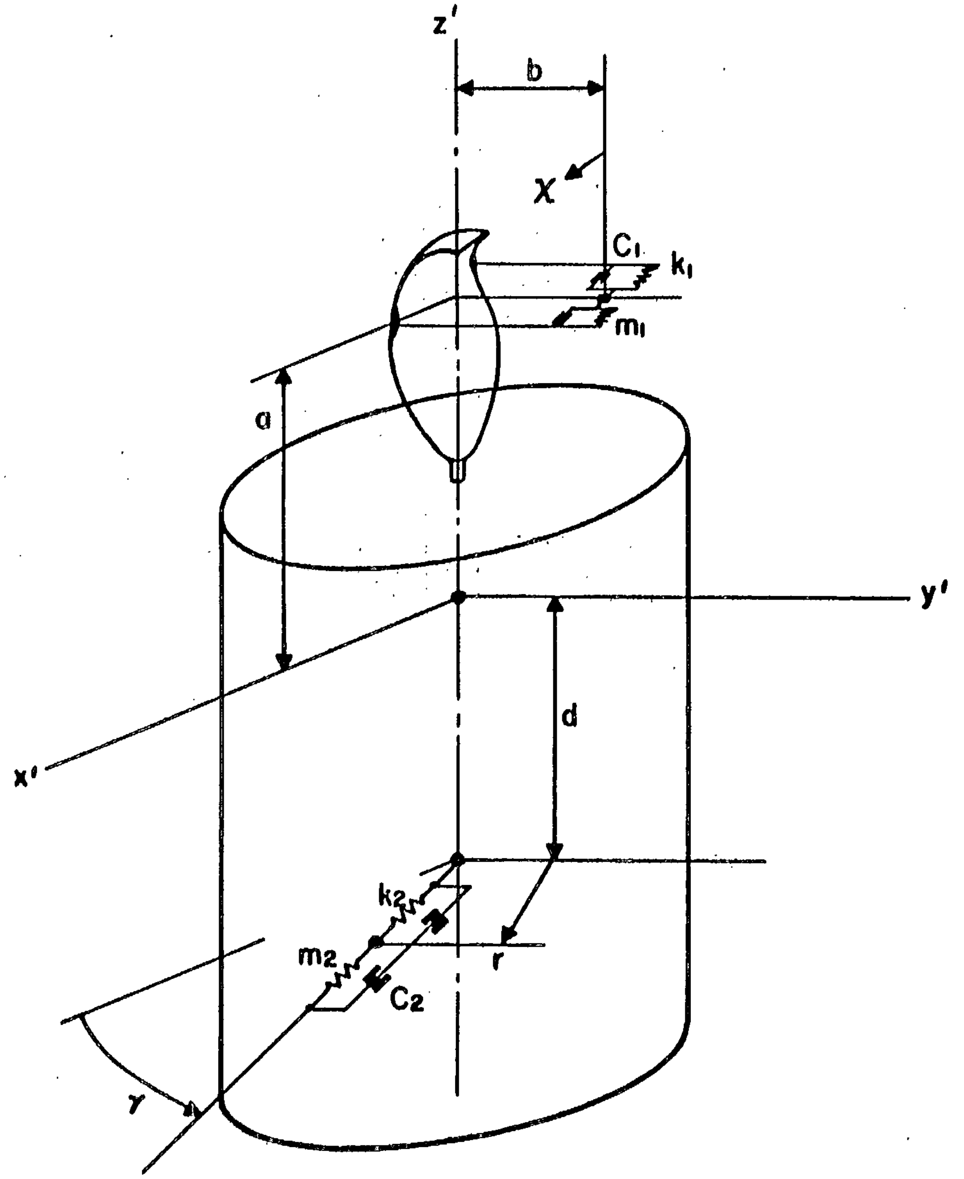

Figure 2.

The motion of the mass center, 0, will be assumed to follow a Keplerian circular orbit, as is the usual practice in satellite mechanics. Hence, the vector $\underline{R}_0$ from the center of the earth, E, to 0 has constant magnitude. The orbital angular rate is given by

$$\Omega^2 = GM_E/R_0^3,$$

where GM_E is the earth's gravitational constant.

The angular position of the axes (Oxyz) relative to the earth based inertial system (EXYZ) will be specified by the Euler angles (ψ, θ, ϕ,) and Ω as shown in Figure 4. The orbit will be assumed to lie in the XY plane (without loss of generality) and the Euler angles will be defined relative to orbital axes ($0\ \xi_1\ \xi_2\ \xi_3$) by the right hand rotation scheme (i) ψ about $0\xi_1$, (ii) θ about $\underline{B}_2$ (iii) ϕ about $\underline{C}_3$, where $\underline{B}_2$ and $\underline{C}_3$ are shown in Figure 4.

The kinematical relations between the velocities ($\dot{\psi}$, $\dot{\theta}$, $\dot{\phi}$) and the velocities (ω_x, ω_y, ω_z) resolved in the axes (Oxyz) are

$$\omega_x = \dot{\psi}\cos\phi\ \cos\theta + \dot{\theta}\sin\phi + \Omega(-\cos\phi\ \sin\theta\ \cos\psi + \sin\phi\ \sin\psi) \quad .....(3\text{-}4a)$$

$$\omega_y = \dot{\psi}\sin\phi\ \cos\theta + \dot{\theta}\cos\phi + \Omega(\ \sin\phi\ \sin\theta\ \cos\psi + \cos\phi\ \sin\psi) \quad .....(3\text{-}4b)$$

$$\omega_z = \dot{\psi}\sin\theta + \Omega\ \cos\theta\ \cos\psi + \dot{\phi}. \quad .....(3\text{-}4c)$$

The kinetic energy of the body is

$$T = \frac{1}{2}\int_M (\dot{\underline{R}}_0 + \dot{\underline{r}}) \cdot (\dot{\underline{R}}_0 + \dot{\underline{r}})dm,$$

where the superscript dot denotes vector differentiation with respect to the inertial (EXYZ) frame. The term $(\dot{\underline{R}}_0 + \dot{\underline{r}})$ may be expanded in the form,

$$\dot{\underline{R}}_0 + \mathring{\underline{r}} + \underline{\omega} \times \mathring{\underline{r}}\ ,$$

where $\underline{\omega}$ denotes (ω_x, ω_y, ω_z) resolved in (Oxyz), and the superscript $(\mathring{\ })$ denotes differentiation with respect to the (Oxyz) frame. Combining the above two expressions yields, after some manipulation,

$$T = \frac{1}{2} M\ \dot{\underline{R}}_0 \cdot \dot{\underline{R}}_0 + \frac{1}{2}\ \underline{\omega} \cdot \underset{\sim}{I}\underline{\omega} + \underline{\omega} \cdot \underline{\Gamma} + \int_M \mathring{\underline{r}} \cdot \mathring{\underline{r}}\ dm\ , \quad .....(3\text{-}5a)$$

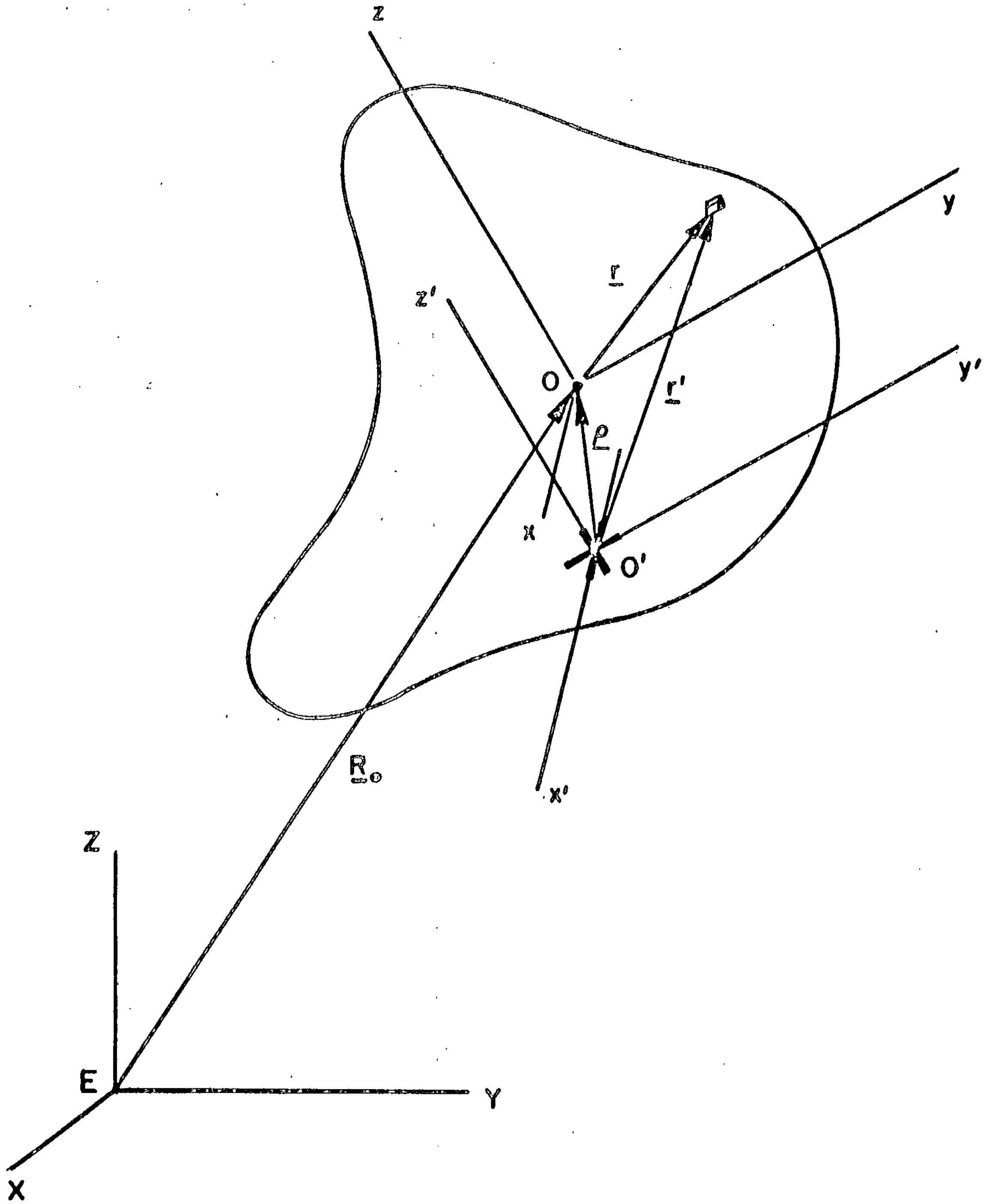

Figure 3.

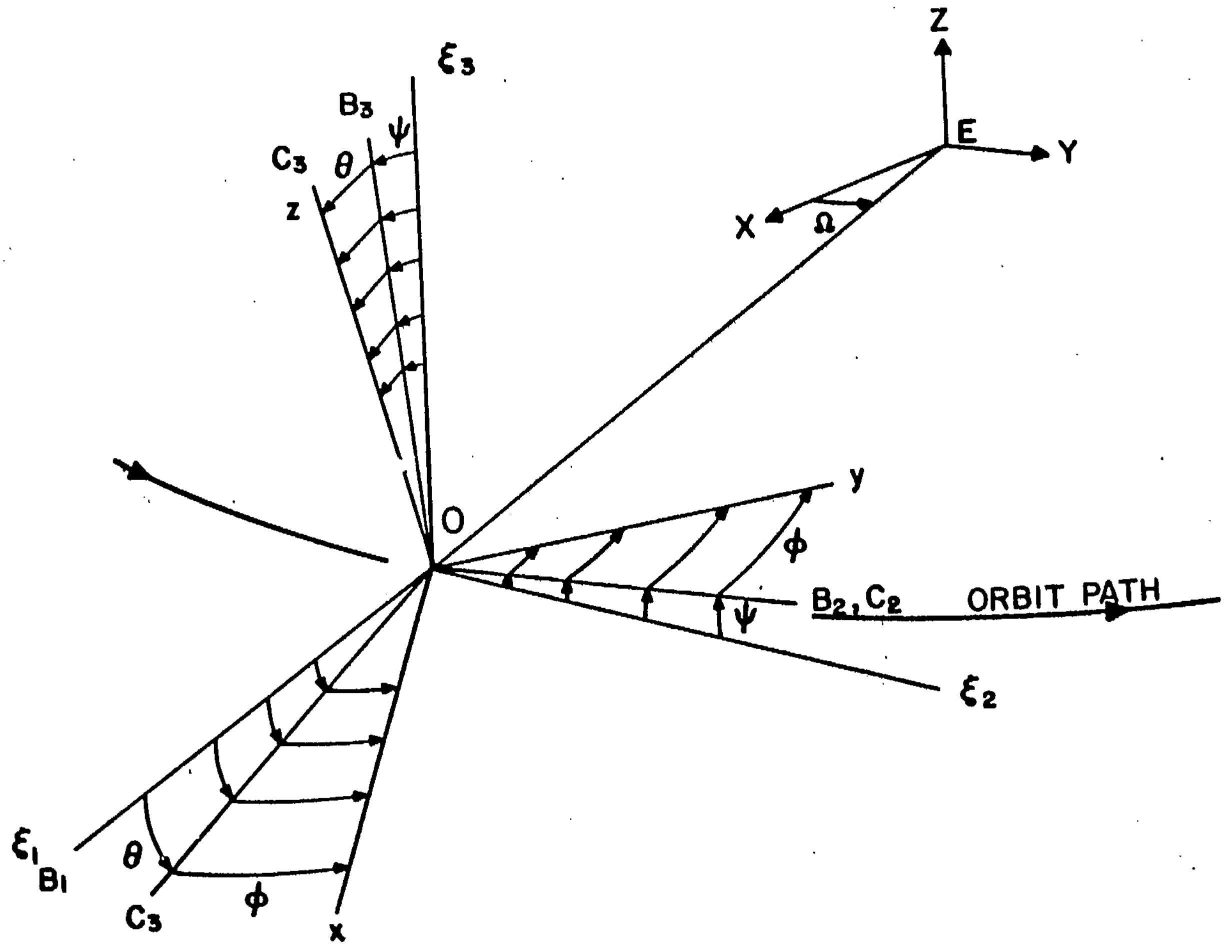

Figure 4.

where

$$\underset{\sim}{I} = \int_M (r^2 \underset{\sim}{1} - \underline{r} \otimes \underline{r})\, dm , \qquad \text{.....(3-5b)}$$

$$\underline{\Gamma} = \int_M (\underline{r} \times \overset{\circ}{\underline{r}})\, dm . \qquad \text{.....(3-5c)}$$

The symbol (~) denotes a second order tensor, and $\otimes$ is defined by

$$(\underline{u} \otimes \underline{v})\underline{w} = \underline{u}(\underline{v} \cdot \underline{w}) .$$

$\underset{\sim}{I}$ is called the inertia tensor, and $\underline{\Gamma}$ the angular momentum of deformation.

The gravitational potential and torque are given by

$$V_G = - \frac{GM_E}{2R_0^3} [\mathrm{tr}\ \underset{\sim}{I} - 3\ \underline{e}_1 \cdot (\underset{\sim}{I}\underline{e}_1)] \qquad \text{.....(3-6)}$$

$$\underline{L}_G = 3 \frac{GM_E}{R_0^3} (\underline{e}_1 \times \underset{\sim}{I}\ \underline{e}_1) , \qquad \text{.....(3-7)}$$

where $\underline{e}_1$ is a unit vector along EO directed from the center of the earth. In terms of (ψ, θ, ϕ), the vector $\underline{e}_1$ has components

$$\ell_1 = \cos\phi\ \cos\theta \qquad \text{.....(3-8a)}$$

$$\ell_2 = -\sin\phi\ \cos\theta \qquad \text{.....(3-8b)}$$

$$\ell_3 = \sin\theta \qquad \text{.....(3-8c)}$$

resolved in (Oxyz).

To calculate the components of $\underset{\sim}{I}$ and $\underline{\Gamma}$ (which are taken about point O, the moving mass center), it is often convenient to work from a frame (O'x'y'z'), where O' is fixed on some part of the body, but does not remain fixed on the mass center. For this purpose, one may derive the following parallel axes tranformations:

$$\underset{\sim}{I} = \underset{\sim}{I}' - M (\rho \cdot \rho \underset{\sim}{1} - \underline{\rho} \otimes \underline{\rho}) \qquad \text{.....(3-9a)}$$

$$\underline{\Gamma} = \underline{\Gamma}' - M (\underline{\rho} \times \overset{\circ}{\underline{\rho}}) \qquad \text{.....(3-9b)}$$

$$\int_M \overset{\circ}{\underline{r}} \cdot \overset{\circ}{\underline{r}}\, dm = \int_M (\overset{\circ}{r}{}' \cdot \overset{\circ}{r}{}')\, dm - M (\overset{\circ}{\rho} \cdot \overset{\circ}{\rho}) , \qquad \text{.....(3-9c)}$$

where the prime superscript denotes components taken about O'.

Motion equations applicable to all classes of flexible satellites may be deduced from the above T and appropriate V (of which V_G embodies only those forces due to gravity) by using Hamilton's Principle[1]. When the representation of T and V is expressed in terms of discrete coordinates, Hamilton's Principle reduces to Lagrange's equations.

It is convenient in rotational problems to retain the Euler equations (in a generalized form) in the variables (ω_x, ω_y, ω_z), rather than working directly in terms of coordinates (ψ, θ, ϕ), as is usual in the variational approach (i.e., it is usual to substitute equations (3-4) into (3-5) to eliminate $\underline{\omega}$). The desired equations, which are the Lagrange equations corresponding to the variable (ω_x, ω_y, ω_z) may be shown to be [2,3]

$$\frac{d}{dt}\,(\underset{\sim}{\underline{I}}\underline{\omega} + \underline{\Gamma}) + \underline{\omega}\mathrm{x}(\underset{\sim}{\underline{I}}\underline{\omega} + \underline{\Gamma}) = \underline{L} \qquad \ldots\ldots (3\text{-}10)$$

where $\underline{L}$ is the external torque.

Equations (3-4) may be inverted to obtain,

$$\dot{\psi} = \omega_x \cos\phi \sec\theta - \omega_y \sin\phi \sec\theta + \Omega\tan\theta \cos\psi \qquad \ldots\ldots(3\text{-}11a)$$

$$\dot{\theta} = \omega_x \sin\phi + \omega_y \cos\phi - \Omega\sin\psi \qquad \ldots\ldots(3\text{-}11b)$$

$$\dot{\phi} = \omega_z - \omega_x \cos\phi \tan\theta + \omega_y \sin\phi \tan\theta - \Omega\sec\theta \cos\psi. \qquad \ldots\ldots(3\text{-}11c)$$

When the deformations of the body are represented by continuous coordinates, Hamilton's Principle will provide partial differential equations and appropriate boundary conditions to define the motion of the continuum. When the parts of the body may be described by discrete generalized coordinates, Lagrange's equations will provide the balance of the equations required to supplement (3-10) and (3-11).

3.2 MOTION EQUATIONS FOR MODEL M1

Consider model M1, depicted in figure 1, composed of a platform, a rotor, and two dampers. Axes (0 x' y' z') are assigned to the body so that when the damper springs are in their undeformed state, the axis 0'z' is a common rotation axes of the two bodies and is also a principal axis. The point 0' coincides with the mass center of the composite body, and the axes 0'x' and 0'y' are fixed in the platform. The rotor rotates with respect to the platform about the 0'z' axis by an angle γ. The rotation rate $\dot{\gamma}$ is supplied by an internal torquing motor. The platform damper is in static equilibrum with its mass in the position (0, 0, a) in (0'x'y'z'), and is constrained to oscillate in the 0'x' direction with a displacement χ. The rotor damper is fixed in the rotor, but able to rotate about a transverse axis of the rotor through an angle β. An additional set of axes (0xyz) is assigned to be parallel to the (0'x'y'z') body fixed axes, so that 0 coincides with the instantaneous mass center of the configuration as the dampers oscillate. In the following development, it will be assumed that the satellite axis Oz remains oriented approximately normal to the orbit plane, and that the dampers execute small motions. Accordingly the equations of motion will be linearized in the variables ψ, θ, χ and β. As a consequence, ω_x and ω_y may also be regarded as being small, as seen by inspection of the linearized forms of equations (3-4a) and (3-4b). The resulting equations will then provide an approximate behaviour of the dual spin satellite in its 'earth-seeking' mode, in which ϕ may be large, and also in its 'earth-pointing' mode, in which ϕ is small.

The platform damper of mass m has position vector

$$\underline{r}' = \chi\underline{e}_1' + a\,\underline{e}_3' ,$$

where the primes shall henceforth represent quantities resolved in the (O'x'y'z') frame. Using the above relation in equation (3-3), and the fact that O' coincides with the mass center when χ is zero leads to

$$\underline{\rho} = \frac{m}{M}\chi\underline{e}_1'$$

The components of $\underset{\sim}{I}$ shall be denoted by

$$\underset{\sim}{I} = \begin{pmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{pmatrix}$$

From expressions (3-5b) and (3-9a), it is found that,

$$A = A' - M(\rho_2^2 + \rho_3^2)$$

$$F = F' - M\rho_1\,\rho_2$$

and so forth, where (ρ_1, ρ_2, ρ_3) are components of $\underline{\rho}$. Utilizing the above expressions gives

$$A = A_0 ,$$

where A_0 is the moment of inertia, calculated in the (O'x'y'z') frame, of the platform, rotor, and damper in its undeformed state. It will be assumed that the rotor is symmetric, so that A_0 will always be constant as the rotor spins. Similarly,

$$B = B_0 + m(1-\mu)\chi^2$$

$$C = C_0 + m(1-\mu)\chi^2$$

$$D = 0$$

$$E = ma\chi$$

$$F = 0,$$

where $\mu = m/M$, and the axes $(0'x'y'z')$ are principal axes of the platform and rotor combination when $\chi = 0$. Likewise, the components of $\underline{\Gamma}$ are found to be,

$$\Gamma_1 = -I_s\dot{\beta}\sin\gamma$$

$$\Gamma_2 = I_s\dot{\beta}\cos\gamma + ma\dot{\chi}$$

$$\Gamma_3 = C_R\dot{\gamma} ,$$

where I_s is the inertia of the spherical damper, and C_R is the moment of inertia of the rotor (which includes I_s). The last term of expression (3-5a) takes the form, after using equation (3-9c),

$$\int_m \underline{\dot{r}} \cdot \underline{\dot{r}}\, dm = \frac{1}{2} m \dot{\chi}^2 (1 - \mu) + \frac{1}{2} C_R\dot{\gamma}^2 + \frac{1}{2} I_s\dot{\beta}^2 .$$

The kinetic energy then becomes,

$$T = \frac{1}{2} \{C_0 + m(1-\mu)\chi^2\}\, \omega_z^2 - ma\chi\omega_x\omega_z - I_s\omega_x\dot{\beta}\sin\gamma$$
$$+ I_s\omega_y\dot{\beta}\cos\gamma + ma\omega_y\dot{\chi} + \omega_z C_R\dot{\gamma}$$
$$+ \frac{1}{2} m(1-\mu)\dot{\chi}^2 + \frac{1}{2}C_R\dot{\gamma}^2 + \frac{1}{2}I_s\dot{\beta}^2$$

The above expression is linearized in Θ and ψ, which also implies linearization in ω_x and ω_y (see equation (3-4)).

The gravitational torque given by equation (3-7) (together with (3-6)) has the component form

$$L_{1G} = 3\Omega^2[(C - B)\ell_2\ell_3 + D(\ell_3^2 - \ell_2^2) - E\ell_1\ell_2 + F\ell_1\ell_3]$$

together with two additional expressions obtained by permuting symbols. These expressions become, after linearization in Θ, ψ, χ_s and β,

$$L_{1G} = 3\Omega^2[-(C_0 - B_0)\theta\sin\phi + ma\chi\cos\phi\sin\phi]$$

$$L_{2G} = 3\Omega^2[(A_0 - C_0)\theta \cos\phi + ma\chi \cos^2\phi]$$

$$L_{3G} = -3\Omega^2(B_0 - A_0) \sin\phi \cos\phi$$

The gravitational potential, V_G, becomes, after linearization in (θ, ψ),

$$V = -\frac{\Omega^2}{2} [A_0(1 - 3 \cos^2\phi) + (B_0 + m(1 - \mu)\chi^2) (1 - 3 \sin^2\phi)$$
$$+ C_0 + m(1 - \mu)\chi^2 + 6 ma\chi \cos\phi]$$

The potential for the damper springs is

$$V_S = \frac{1}{2} k_1\chi^2 + \frac{1}{2} k_2\beta^2$$

Employing the above relationships in equations (3-10), and linearizing in θ, ψ and χ yields,

$$A_0\dot{\omega}_x - ma\dot{\chi}\omega_z - ma\chi\dot{\omega}_z - I_s\ddot{\beta} \sin\gamma$$
$$- I_s\dot{\beta}\dot{\gamma} \cos\gamma + (C_0 - B_0)\omega_y\omega_z + C_R\dot{\gamma}\omega_y$$
$$- (I_s\dot{\beta} \cos\gamma + ma\dot{\chi})\ \omega_z = 3\Omega^2[-(C_0 - B_0)\theta \sin\phi$$
$$+ ma\chi \cos\phi \sin\phi] + L_1 \tag{3-12a}$$

$$B_0\dot{\omega}_y + I_s\ddot{\beta} \cos\gamma - I_s\dot{\beta}\dot{\gamma} \sin\gamma + ma\ddot{\chi} + (A_0 - C_0)\ \omega_x\omega_z$$
$$- ma\omega_z^2\chi - I_s\dot{\beta}\omega_z \sin\gamma - C_R\dot{\gamma}\omega_x$$
$$= 3\Omega^2[(A_0 - C_0)\theta \cos\phi + ma\chi \cos^2\phi] + L_2 \tag{3-12b}$$

$$\frac{d}{dt} [C_0\omega_z + C_R\dot{\gamma}] = -3\Omega^2(B_0 - A_0) \sin\phi \cos\phi + L_3\ , \tag{3-12c}$$

where L_1, L_2, and L_3 are external torque components about axes (Oxyz) excluding gravity torque, (e.g. gas jet torques).

The following equation for the rotor may be obtained by using the above T and V in Lagrange's equations with γ as a generalized coordinate.

$$\frac{d}{dt}\left[C_R\ (\omega_z + \dot{\gamma})\right] = -c\dot{\gamma} + T_0 + T_1\ \sin\phi + T_2\dot{\phi} + L_{3_R}, \quad \ldots\ldots(3\text{-}13)$$

where L_{3_R} is the external torque applied to the rotor, the generalized force (in Lagrange's equation) is taken to be $-c\dot{\gamma}$, i.e., a bearing friction torque, plus, $(T_0 + T_1\sin\phi + T_2\phi)$ a control system torque which is applied by the internal motor and which may be designed to make the platform seek the earth pointing direction.

Equations of motion of the dampers are obtained by using χ and β as generalized coordinates with the preceeding T,V, and V_s in Lagrange's equations:

$$m(1-\mu)\ddot{\chi} + ma\dot{\omega}_y + ma\omega_z\omega_x + c_1\dot{\chi} + [k_1 - m\omega_z^2(1-\mu) - \Omega^2(1-\mu)(2-3\sin^2\phi)]\chi$$
$$- 3\Omega^2\ ma\theta\ \cos\phi = 0 \quad \ldots\ldots(3\text{-}14)$$

$$I_s\ddot{\beta} - I_s\dot{\gamma}\ \cos\gamma\ \omega_x - I_s\ \sin\gamma\ \dot{\omega}_x - I_s\dot{\gamma}\ \sin\gamma\ \omega_y + I_s\ \cos\gamma\ \dot{\omega}_y + c_2\dot{\beta} + k_2\beta = 0. \quad \ldots\ldots(3\text{-}15)$$

In the above equations, generalized forces due to damping in the rotors are included (i.e. $-c_1\dot{\chi}$ and $-c_2\beta$).

The kinematical relations (3-11) become,

$$\dot{\theta} = \omega_x\ \sin\phi + \omega_y\ \cos\phi - \Omega\psi \quad \ldots\ldots(3\text{-}16a)$$

$$\dot{\psi} = \omega_x\ \cos\phi - \omega_y\ \sin\phi + \Omega\theta \quad \ldots\ldots(3\text{-}16b)$$

$$\dot{\phi} = \omega_x - \Omega\ , \quad \ldots\ldots(3\text{-}16c)$$

Equations (3-12), (3-13), (3-14), (3-15) and (3-16) form a set of equations which may be solved for the set of variables $\omega_x\ \omega_y\ \omega_z$, θ, ϕ, ψ, γ, χ and β. Since these linear equations have coefficients which are periodic in both γ and ϕ, they are solvable by analytical means. These equations are also general enough to enable the study of the 'earth-seeking' performance where ϕ is not necessarily small.

3.3 MOTION EQUATIONS FOR MODEL M2

The motion equations for the second more complex model are derived in exactly the same manner as that outlined in Section 3.2, subject to the following limitations:

(a) linearization in ψ, θ, χ and r (the rotor damper radial deformation)

(b) the rotor is assumed to be symmetric when the damper is in its undeformed state (i.e. $A_R = B_R$, $D_R = E_R = F_R = 0$, where the subscript "R" denotes the rotor)

(c) The mass center of the undeformed satellite is assumed to lie on the principal O'z' axis of the rotor.

The equations will be quoted directly without further development.

The Euler equations become

$$\frac{d}{dt}\Big[A_0\omega_x - F_p\omega_y - (E_p + m_1 a\chi - m_2 dr\cos\gamma)\,\omega_z$$

$$+ m_2 d\,(\dot{r}\sin\gamma + r\dot{\gamma}\cos\gamma)\Big] + (C_0 - B_0)\omega_y\omega_z$$

$$+ (D_p - m_2 dr\sin\gamma)\,\omega_z^2 + F_p\,\omega_z\omega_x + C_R\dot{\gamma}\omega_y$$

$$- \omega_z\,\{m_1 a\dot{\chi} - m_2 d(\dot{r}\cos\gamma - r\dot{\gamma}\sin\gamma)\}$$

$$= 3\Omega^2\,\big[-(C_0 - B_0)\theta\sin\phi - (D_p - m_2 dr\sin\gamma)\sin^2\phi$$

$$+ (E_p + ma\chi - m_2 dr\cos\gamma)\sin\phi\cos\phi$$

$$+ F_p\theta\cos\phi\big] + L_1 \qquad \text{..... (3-17a)}$$

$$\frac{d}{dt}\Big[B_0\omega_y - D_p\omega_z + m_2 dr\omega_z\sin\gamma - F_p\omega_x$$

$$+ m_1 a\dot{\chi} - m_2 d\,(\dot{r}\cos\gamma - r\dot{\gamma}\sin\gamma)\Big]$$

$$+ (A_0 - C_0)\omega_z\omega_x - (E_p + m_1 a\chi - m_2 dr\cos\gamma)\omega_z^2$$

$$- F_p\omega_z\omega_y + m_2 d\omega_z\ (\dot{r}\ \sin\gamma + r\dot{\gamma}\ \cos\gamma)$$

$$- C_R\dot{\gamma}\omega_x = 3\Omega^2\ [(a_0 - C_0)\theta\ \cos\phi + (E_p + m_1 a\chi$$

$$- m_2 dr\ \cos\gamma)\ \cos^2\phi + F_p\theta\ \sin\phi$$

$$- (d_p - m_2 dr\ \sin\gamma)\ \sin\phi\ \cos\phi] + L_2 \qquad .....(3\text{-}17b)$$

$$\frac{d}{dt}\ [C_0\omega_z + 2m_2 e\omega_z r - E_p\omega_x - D_p\omega_y$$

$$+ C_R\dot{\gamma} - m_1 b\dot{\chi} + 2m_2 e\dot{\gamma}r] + E_p\omega_z\omega_y$$

$$- D_p\omega_z\omega_x = 3\Omega^2\ [-(B_0 - A_0)\ \sin\phi\ \cos\phi$$

$$- (F_0 + m_1 b\chi + m_2 er\ \sin 2\gamma)\ \cos 2\phi$$

$$- D_0\theta\ \cos\phi - E_0\theta\ \sin\phi\ ] + L_3\ . \qquad .....(3\text{-}17c)$$

These equations are sufficiently general to account for unbalance of the platform (i.e. $D_p = E_p = F_p$ are non-zero), a feature which may be a characteristic of the Telesat design.

The rotor equation becomes

$$\frac{d}{dt}\ [C_R(\omega_z + \dot{\gamma}) + 2m_2 e\omega_z r + 2m_2 er\dot{\gamma}]$$

$$= -3\Omega^2 m_2 er\ \sin 2\gamma - c\dot{\gamma} + T_0 + T_1\ \sin\phi + T_2\dot{\phi} + L_{3R}. \qquad .....(3\text{-}18)$$

For the platform damper,

$$m_1\ \frac{d}{dt}\ [\dot{\chi}\ (1 - \mu_1) + a\omega_y = b\omega_z + \mu_2\Omega\ \sin\gamma r - \mu_2\ (\dot{r}\ \cos\gamma$$

$$- r\dot{\gamma}\sin\gamma)]\ - m_1\omega_z^2\{(1 - \mu_1)\chi - \mu_2 r\ \cos\gamma\} + m_1 a\omega_z\omega_x$$

$$+ m_1\omega_z\ \mu_2\ (\dot{r}\ \sin\gamma + r\dot{\gamma}\ \cos\gamma) + c_1\dot{\chi} + k_1\chi$$

$$+ \Omega^2 m_1 [\{ (1 - \mu_1)\chi - \mu_2 r \cos\gamma \} (2 - 3 \sin^2\phi)$$

$$+ 3\mu\theta \cos\phi - 3(b - \mu_2 r \sin\gamma) \cos\phi \sin\phi] = 0 . \qquad ..... (3\text{-}19)$$

For the rotor damper,

$$m_2 \frac{d}{dt} [\mathring{r} (1 - \mu_2) - \mu_1 \mathring{\chi} \cos\gamma + \omega_x d \sin\gamma - \omega_y d \cos\gamma - \omega_z \mu_1 \chi \sin\gamma]$$

$$- m_2 \omega_x d (\omega_z + \dot{\gamma}) \cos\gamma - m_2 \omega_y d (\omega_z + \dot{\gamma}) \sin\gamma$$

$$+ m_2 \chi \mu_1 \cos\gamma (\omega_z^2 + \omega_z \dot{\gamma}) - m_2 \mathring{\chi} (\omega_z + \dot{\gamma}) \mu_1 \sin\gamma$$

$$- m_2 \{e + r(1 - \mu_2)\} (\omega_z + \dot{\gamma})^2 + c_2 \dot{r} + k_2 r$$

$$+ \Omega^2 [\{(e + r(1 - \mu_2)\} \{2 - 3\sin^2(\phi + \gamma)\}$$

$$+ 3\mu_1 \chi \cos\gamma \sin^2\phi - 3 d\theta \cos (\phi - \gamma)] = 0 . \qquad ..... (3\text{-}20)$$

Equation (3-17) to (3-20) are supplemented by the kinematical relations, (3-16).

3.4 SOLUTION OF EQUATIONS

The equations of sections 3.2 and 3.3 are solvable by Runge-Kutta or a related numerical procedure. To do this efficiently, the equations are expressed in matrix form. One defines a column matrix variable *,

$$\underline{x} = (\omega_x, \omega_y, \omega_z, \psi, \theta, \phi, \chi, \mathring{\chi}, \beta, \mathring{\beta}, \gamma, \dot{\gamma})$$

and, with elementary operations, casts the problem into the form

$$\underset{\sim}{A}\mathring{\underline{x}} = \underset{\sim}{B}\underline{x} + \underline{L}$$

* $\underline{x}$ as defined here applies to equations of section 3.2. In section 3.3, one must replace "β" with "r".

where $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are (12x12) matrices with periodic coefficients. The equivalent form,

$$\dot{\underline{x}} = \underset{\sim}{A}^{-1}\underset{\sim}{B}\underline{x} + \underset{\sim}{A}^{-1}\underline{L}$$

may be readily handled by matrix and integration subroutines.

The procedure outlined above will be adopted in later sections.

4. SPECIAL FORMS OF EQUATIONS

4.1 MODEL M1 IN FREE-SPIN MODE

In the recent literature much attention has been given to determining stability criteria for dual-spin satellites which are in 'free-spin', i.e., cases in which V_G and $\underline{L}_G$ are neglected. For model M1, the equations defining this situation are obtained by setting Ω equal to zero. An analytical treatment of the resulting equations by the Method of Averaging is presented in reference 4.

4.2 EQUATIONS FOR MODEL M1 IN THE 'EARTH POINTING' MODE

In this section the equations of section 3.2 will be specialized to describe satellite performance during the 'earth-pointing' mode of operation, which is hopefully the normal mode of operation.

Consider equations (3-12c), (3-13) and (3-16c), involving only the variables ω_z, $\dot{\gamma}$, and ϕ. An 'earth-pointing' solution to these is

$$\phi = 0,\ \omega_z = \Omega,\ \dot{\gamma} = \dot{\gamma}_0,\ L_3 = L_{3_R} = 0 \qquad (4\text{-}1)$$

which may be achieved when the control torque is selected so that $T_0 = c\dot{\gamma}_0$. In practice, the torque components T_1 and T_2 are selected so as to make the above solution asymptotically stable. Assume that the perturbed solution about the stable one, (4-1), may be represented by

$$\phi = \phi_1$$

$$\omega_z = r + r_1$$

$$\dot{\gamma} = \dot{\gamma}_0 + \dot{\gamma}_1 ,$$

where ϕ_1, r_1, and $\mathring{\gamma}_1$ are small. Substitution of these into equations (3-12a), (3-12b), (3-14), and (3-15), and linearization in ω_x, ω_y, Θ, χ, ϕ_1, r_1, $\mathring{\gamma}_1$ χ, and β yields,

$$A_0\mathring{\omega}_x - 2ma\mathring{\chi}\Omega - I_s\ddot{\beta}\sin\gamma_0 - I_s\mathring{\beta}(\mathring{\gamma}_0 + \Omega)\cos\gamma_0$$

$$+ \{(C_0 - B_0)\Omega + C_R\mathring{\gamma}_0\}\omega_y = L_1 \qquad \text{.....(4-2a)}$$

$$B_0\mathring{\omega}_y + I_s\ddot{\beta}\cos\gamma_0 - I_s\mathring{\beta}(\mathring{\gamma}_0 + r)\sin\gamma_0 + ma\ddot{\chi} - 4mar^2\chi$$

$$- \{(C - A_0)r + C_R\mathring{\gamma}_0\}\omega_x + 3r^2(C_0 - A_0)\Theta = L_2 \qquad \text{.....(4-2b)}$$

$$m(1 - \mu)\ddot{\chi} + c_1\mathring{\chi} + [k_1 - 3mr^2(1 - \mu)]\chi - 3ma\Omega^2\Theta$$

$$+ ma\mathring{\omega}_y + ma\Omega\omega_x = 0 \qquad \text{.....(4-3)}$$

$$I_s\ddot{\beta} - I_s\mathring{\gamma}_0\cos\gamma_0\omega_x - I_s\sin\gamma_0\mathring{\omega}_x - I_s\mathring{\gamma}_0\sin\gamma_0\omega_y$$

$$+ I_s\cos\gamma_0\mathring{\omega}_y + c_2\mathring{\beta} + k_2\beta = 0 \qquad \text{.....(4-4)}$$

where $\gamma_0 = \mathring{\gamma}_0 t$.

Also, equations (3-16a) and (3-16b) become,

$$\mathring{\Theta} = \omega_y - \Omega\psi \qquad \text{.....(4-5a)}$$

$$\mathring{\psi} = \omega_x + \Omega\theta \qquad \text{.....(4-5b)}$$

Introduce the dimensionless variables

$$\tau = \Omega t$$

$$W_x = \omega_x/\Omega$$

$$W_y = \omega_y/\Omega$$

$$\xi = \chi/a$$

$$\alpha = \dot{\gamma}_0/\Omega \ , \gamma \ = \alpha\tau$$

$$\eta = \xi' = \dot{\chi}/a\Omega$$

$$\nu = \beta' = \dot{\beta}/\Omega,$$

where the primes denote differentiation with respect to τ. Also, define the dimensionless parameters,

$$\Delta_{CA} = (C_0 - A_0)/B_0, \qquad \Delta_{CB} = (C_0 - B_0)/A_0,$$

$$R_A = ma^2/A_0, \qquad R_B = ma^2/B_0$$

$$I_A = I_S/A_0, \qquad I_B = I_s/B_0$$

$$J_A = C_R/A_0, \qquad J_B = C_R/B_0$$

$$\overline{c}_1 = c_1/m_1\Omega, \qquad \overline{c}_2 = c_2/I_s\Omega$$

$$\overline{k}_1 = k_1/m\Omega^2 \qquad \overline{k}_2 = k_2/I\Omega^2$$

$$\mathcal{L}_1 = L_1/A_0\Omega^2, \qquad \mathcal{L}_2 = L_2/B_0\Omega^2$$

Then the preceding equations take the form

$$\xi' = \eta$$

$$\beta' = \nu$$

$$W_x' - 2R_A\eta - I_A\sin\gamma\ \nu' + I_A(\alpha + 1)\ \cos\gamma\ \nu + (\Delta_{CB} + J_A\alpha)\ W_y = \mathcal{L}_1$$

$$W_y' + I_B\cos\gamma\ \nu' - I_B(\alpha + 1)\sin\gamma\ \nu + R_B\eta' - (\Delta_{CA} + J_B\alpha)\ W_x - 4R_B\xi + 3\Delta_{CA}\Theta = \mathcal{L}_2$$

$$(1 - \mu)\eta' + W_y' + \overline{c}_1\eta + \{\overline{k} - 3(1 - \mu)\}\xi - 3\Theta + W_x = 0$$

$$\nu' - q\cos\gamma\ W_x - \sin\gamma\ W_x' - \alpha\sin\gamma\ W_y + \cos\gamma\ W_y' + \overline{c}_2\nu + \overline{k}_2\beta = 0$$

$$\Theta' = W_y - \psi$$

$$\psi' = W_x + \Theta$$

Define a column matrix by

$$\underset{\sim}{x} = [\Theta,\ \psi,\ W_x,\ W_y,\ \xi,\ \beta,\ \eta,\ \nu]^T,$$

Then the above set of equations may be cast in the form

$$\underset{\sim}{A}\underset{\sim}{x}' = \underset{\sim}{B}\underset{\sim}{x} + \mathcal{L}, \qquad \text{......(4-6)}$$

where $\mathcal{L}$, $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are,

$$\mathcal{L} = [\mathcal{L}_1,\ \mathcal{L}_2,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\ ]^T$$

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & -I_A\sin\gamma \\ 0 & 0 & 0 & 1 & 0 & 0 & R_B & I_B\cos\gamma \\ 0 & 0 & 0 & 1 & 0 & 0 & 1-\mu & 0 \\ 0 & 0 & -\sin\gamma & \cos\gamma & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{B} = \begin{bmatrix} 0 & 0 & 0 & -\Delta_{CB}-J_A\alpha & 0 & 0 & 2R_A & I_A(\alpha+1)\cos\gamma \\ -3\Delta_{CA} & 0 & \Delta_{CA}+J_B\alpha & 0 & 4R_B & 0 & 0 & I_B(\alpha+1)\sin\gamma \\ 3 & 0 & -1 & 0 & -\overline{k}_1+3(1-\mu) & 0 & -\overline{c}_1 & 0 \\ 0 & 0 & \alpha\cos\gamma & \alpha\cos\gamma & 0 & -\overline{k}_2 & 0 & -\overline{c}_2 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

5. STABILITY ANALYSIS OF EQUATION 4-6

The coefficients of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ of equation (4-6) are periodic with period $(2\pi/\alpha)$. The stability of the solutions may be assessed by using Floquet Theory, when L_1 and L_2 the externally applied control torques are zero. As outlined in reference 5, the boundedness or otherwise of the solution depends on the change over one period of the (8x8) matrix $\underset{\sim}{W}$ which is generated from the differential equation.

$$\underset{\sim}{W}' = \underset{\sim}{A}^{-1} \underset{\sim}{B} \underset{\sim}{W} \qquad \text{.....(5-1)}$$

with initial condition $\underset{\sim}{W}(\underset{\sim}{0}) = \underset{\sim}{1}$, the unit matrix. The solutions are bounded as $\tau \to \infty$ if and only if the modulus of each of the eigenvalues of $\underset{\sim}{W}(2\pi/\alpha)$ is less than or equal to unity, and if, for any eigenvalue λ_k for which $|\lambda_k| = 1$, the multiplicity of λ_k is equal to the nullity of the matrix $\{\underset{\sim}{W}(2\pi/\alpha) - \underset{\sim}{W}(0)\}$.

The stability of the solution of equations (4-6) may be assessed by digital computer by solving $\underset{\sim}{W}$ numerically from the sixty-four first order equations embodied in (5-1), and then testing the eigenvalues of $\underset{\sim}{W}(2\pi/\alpha)$ in accordance with the above stated criteria.

6. CONCLUDING REMARKS

The preceding work documents the equations of motion for dual spin satellites of interest in the Telesat program. The equations are of sufficient generality to efficiently provide preliminary engineering design information on dynamic performance.

At the date of writing (Oct./69) computer programs have been written for various phases of the study based on Model M1. All may be readily converted to Model M2 as required. Presently available are:

a) TELESAT 1 - a program which assesses stability according to the procedure outlined in section 5,

b) TELESAT 2 - a program which solves the motion equations for the 'free spin' mode, outlined in section 4.1,

c) TELESAT 3 - a program which solves motion equations when pulsed, attitude correction torques are applied during the 'earth-pointing' mode of operation.

Computer listings, descriptions, and card decks for the above programs are on file. These may be made available by authors on request.

7. REFERENCES

1. Langhaar, H.L. *Energy Methods in Applied Mechanics*. John Wiley and Sons, New York, 1962. Chapter 7.

2. Whittaker, E.T. *Analytical Dynamics*. Dover Publications Incorporated, New York, 1944. pp. 41-44.

3. Vigneron, F.R. *Stability of a Freely Spinning Flexible Satellite of Crossed-Dipole Configuration*. CASI Transactions, Vol. 3, No. 1, 1970 pp. 8-19.

4. Vigneron, F.R. *Stability of a Dual Spin Satellite with Two Dampers*. Proceedings of the AIAA 3rd Communications Satellite Systems Conference, AIAA Paper No. 70-431, Los Angeles, April 6-8, 1970.

5. Cesari, L. *Asymptotic Behaviour and Stability Problems in Ordinary Differential Equations*. Academic Press Incorporated, New York, 1963.

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