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THE ZOOM TRANSFORM

by

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ABSTRACT

The zoom transform is the discrete Fourier transform of the discrete Fourier transforms of successive sections of a sampled signal. An analysis of the zoom transformation is presented, which shows that this process is approximately equivalent to a single Fourier transformation of all the signal samples. The predictions of the analysis have been verified by computer tests, which are described. Correction factors are derived for the phase and amplitude of the zoom transform. In comparison with the large single transformation, the zoom technique offers important savings in storage capacity requirements, and, in applications where high-resolution analysis of only selected regions of the spectrum is required, the amount of computation needed may be greatly reduced.

1. INTRODUCTION

A technique has recently been proposed by Dr. E. Shaw[1] for increasing the resolution of the Fast Fourier Transform (FFT) beyond that determined by the block size of the basic analyser. In principle, the resolution that can be obtained is limited only by the stationarity and duration of the signal. Other methods for achieving this result are known; for example, Gentleman and Sande[2] have described a technique for obtaining the Mm-point transform by performing a certain sequence of M-point and m-point transformations. However, in the latter method the computer must store all Mm signal samples simultaneously, and when the product Mm is large, the storage capacity needed may be prohibitive, especially if the analysis must be conducted in real time without loss of data, so that double-buffering is called for. In the present method only M signal samples need be stored at one time, where M is the block size of the basic FFT analyser. The method described by Gentleman and Sande also requires the complete Mm-point transform to be computed, even if only one or a few small regions of the spectrum are of interest. A novel feature of the technique proposed by Shaw is that the high resolution analysis can be performed on any arbitrarily selected set of frequency cells. This feature appears to be of particular value in cases where the basic analyser is not sufficiently fast to perform the full Mm-point transform in real time, provided that the cells of interest are known, or can be determined by inspecting the results of the M-point analysis. The ability of the proposed technique to reveal the detail of a limited region (of the spectrum) led Shaw to name his technique the 'zoom transform', by analogy with the somewhat similar characteristics of the zoom camera.

The zoom procedure is to compute the M-point transforms of m successive blocks of M signal samples, and then to carry out the m-point transformation of the elements of these transforms corresponding to the frequency cell or cells of interest. The result is claimed to be the Mm-point transform of the signal for the frequency interval covered by the selected cells.

There are established methods for performing the high-resolution analysis of a selected region of the spectrum, as in the zoom transformation. In one, the selected region, of bandwidth W, is isolated by bandpass filtering of the input signal; the output of the filter is then heterodyned to baseband, sampled at the rate 2W, and transformed. The need for heterodyning can be avoided: after bandpass filtering, the signal is quadrature-sampled and transformed. Quadrature sampling may be carried out either directly on the bandpass signal[3], or by sampling both the bandpass signal and its Hilbert transform[4]; in either case the minimum sampling rate is W, half that of the first method, but of course each quadrature sample consists of a pair of values.

The Fourier transform of the Hilbert transform has a very simple relation to the Fourier transform of the signal; it is the same but for a phase shift of $\pi/2$,[5]. This observation shows the connection between the zoom transformation and the second method discussed above. The successive outputs in a given cell of the M-point transform may be regarded as the quadrature samples of a bandpass signal. The basic FFT analyser serves as the bandpass filter, or rather as a comb of such filters, since each frequency cell represents one filter. While the definition of the zoom transform as the transform of a sequence of transforms is scarcely perspicuous, this analogy with a known technique makes the validity of the method almost obvious. The present investigation was undertaken to supply an analytical basis for the zoom transformation, and to test it by means of computer experiments.

2. DESCRIPTION OF THE ZOOM TECHNIQUE

The N-point discrete Fourier transform of a signal $x_0(t)$ sampled at the rate of S samples/sec is defined by:

$$X(\sigma) = \sum_{\tau=0}^{N-1} x(\tau) \exp(-2\pi i\tau\sigma/N) \qquad .....(1)$$

where τ and σ are time and frequency indices, respectively, which are related to the time t and frequency f by the relations:

$$\tau = St \quad , \quad \sigma = Nf/S, \qquad .....(2)$$

and $x(\tau) = x_0(\tau/S)$. The time interval covered by the section of signal analysed (the block time) is $t_B = N/S$ secs, and the corresponding frequency resolution of the transform is $\Delta f = S/N$ Hz. For a real signal, the frequency range covered by the transform is S/2 Hz.

If N = Mm, the N-point transform given in equation (1) can be written in the form:

$$X(\sigma) = \sum_{r=0}^{m-1} \sum_{s=0}^{M-1} x(rM + s) \exp(-2\pi i s\sigma/Mm) \exp(-2\pi i r\sigma/m) \quad .....(3)$$

$$= \sum_{r=0}^{m-1} y(r;\sigma) \exp(-2\pi i r\sigma/m) \quad .....(4)$$

where $$y(r;\sigma) = \sum_{s=0}^{M-1} x(rM + s) \exp(-2\pi i s\sigma/Mm). \quad .....(5)$$

If we set $y(r;\sigma) = Y_r(\rho), \rho = \sigma/m$, we observe that, for integral values of ρ, equation (5) has the form of an M-point Fourier transform: $Y_r(\rho)$ is the transform of the $(r + 1)^{th}$ M-point subsection of the original N-point section of the signal. Similarly, equation (4) is the m-point transform of m successive M-point transforms. It follows that if the signal is a sinusoid of frequency $f_0 = S\sigma_0/N$, where σ_0/m, and therefore σ_0, are integers, the exact value of the Mm-point transform $X(\sigma)$ can be obtained by computing m successive smaller (M-point) transforms, and then computing the single m-point transform of the m outputs of the cell of frequency index σ_0/m. This is the basic concept of the zoom FFT.

The exact equivalence just demonstrated has been established only under extremely restrictive conditions. In the first place, the signal frequency must be equal to the centre frequency of one of the cells of the N-point transform, and secondly, the signal frequency must be equal to the centre frequency of one of the cells of the M-point transform; i.e., σ_0/m must be an integer. Evidently the second condition implies the first, but not vice versa. Does the technique remain valid if these conditions are not met? Suppose that the first condition is satisfied but not the second. It is obvious without analysis that in this case the technique is not valid, in the sense that the result of the zoom procedure cannot be exactly equal to the result of performing a single N-point transformation. In the latter operation all but two of the Fourier exponentials are orthogonal to the signal, and the output in the corresponding frequency cells is precisely zero. In the M-point transformation, on the other hand, none of the exponentials is orthogonal to the signal, so that the signal energy is distributed over all M cells according to the shape of the spectral window. The subsequent m-point transformation applied to the output of a single cell clearly cannot recover the energy that is not in that cell.

The conclusion we have just reached is that the basic M-point FFT analyser, regarded as a bandpass filter, distorts the signal. But so does any other form of bandpass filter. As will shortly be seen, the errors introduced by the FFT filter can be approximately corrected. The quality of the approximation is examined in the following sections.

3. ANALYSIS

The M-point transform of a signal $x(t)$ is defined as in equation (1), with the time and frequency indices τ,σ given in terms of the sampling rate and the block size as in equation (2). The complex exponential signal $x_0(t) = \exp(2\pi i f_0 t)$ becomes $x(\tau) = \exp(2\pi i\tau\sigma_0/M)$ when expressed in terms of the indices; the index σ_0 is not restricted to integral values. The transform of this signal is:

$$X(\sigma) = \sum_{\tau=0}^{M-1} \exp(2\pi i\tau\sigma_0/M)\exp(-2\pi i\tau\sigma/M) \qquad .....(6)$$

$$= \sum_{\tau=0}^{M-1} \exp\{2\pi i\tau(\sigma_0 - \sigma)/M\}. \qquad .....(7)$$

The result of performing the finite summation may be expressed in the following form:

$$X(\sigma) = \begin{cases} \exp\{i\pi(\sigma_0 - \sigma)(M-1)/M\}\left[\sin\{\pi(\sigma_0 - \sigma)\}/\sin\{\pi(\sigma_0 - \sigma)/M\}\right], & \sigma \neq \sigma_0 \\ M, & \sigma = \sigma_0. \end{cases} \qquad .....(8)$$

The right-hand side of equation (8) can be simplified; the upper expression includes the lower if its value at $\sigma = \sigma_0$ is understood to be the limit to which it tends as σ approaches σ_0.

Now consider the real signal $x(t) = A\cos(2\pi f_0 t + \alpha)$, which can be written as the sum of two exponential signals:

$$x(t) = \frac{1}{2}A\exp(i\alpha)\exp(2\pi i f_0 t) + \frac{1}{2}A\exp(-i\alpha)\exp(-2\pi i f_0 t).$$

Making use of the result given in equation (8), and recalling that the principle of superposition applies, we can express the M-point transform of $x(t)$ in terms of the frequency index σ, as follows:

$$X(\sigma) = \frac{1}{2}A\exp(i\alpha)\exp\{i\pi(\sigma_0 - \sigma)(M-1)/M\}\left[\sin\{\pi(\sigma_0 - \sigma)\}/\sin\{\pi(\sigma_0 - \sigma)/M\}\right]$$

$$+ \frac{1}{2}A\exp(-i\alpha)\exp\{-i\pi(\sigma_0 + \sigma)(M-1)/M\}\left[\sin\{\pi(\sigma_0 + \sigma)\}/\sin\{\pi(\sigma_0 + \sigma)/M\}\right]. \qquad .....(9)$$

This result can be applied to investigate the zoom transformation in the cases where one (or both) of the conditions given in Section 2 is violated. With respect to the M-point transformation, the signal $A\cos(2\pi f_0 t + \alpha)$, sampled at S samples/sec, has a frequency index $\sigma_0 = Mf_0/S$. Let the integer closest to σ_0 be σ_1, so that σ_0 may be expressed as $\sigma_1 + \Delta$, where $|\Delta| \leq 1/2$. The maximum amplitude of the M-point transform of this signal will then occur in cell σ_1, and it is therefore on the contents of this cell that the subsequent zoom operations must be performed. The component of frequency index σ_1 of the $(r + 1)^{th}$ M-point transform, calculated from equation (9), is:

$$y(r;\sigma_1) = \frac{1}{2} A \exp (i\alpha) \exp \{ i\pi\Delta(M - 1)/M \} [\sin (\pi\Delta)/\sin (\pi\Delta/M)] \exp (2\pi i r\delta_0/m)$$

$$+ \frac{1}{2} A \exp (-i\alpha) \exp \{ -i\pi(2\sigma_1 + \Delta)(M - 1)/M \} [\sin (\pi\Delta)/\sin \{ \pi(2\sigma_1 + \Delta)/M \}]$$

$$\exp (-2\pi i r\delta_0/m). \qquad .....(10)$$

$y(r;\sigma_1)$ is therefore a time-varying signal consisting of the sum of two complex exponentials with different complex amplitudes. The quantity δ_0 (= $m\Delta$) is defined and used in the exponentials in order to express them in the standard index notation. The time index r and the frequency index δ are related to the true time and the incremental frequency f' by the equations: $r = Rt$, $\delta = mf'/R$, where R is the sampling rate for the input to the m-point transformation, i.e., the reciprocal of the block time for the M-point transformation.

The m-point transform of $y(r;\sigma_1)$ is the zoom transform of the signal:

$$Z(\delta) = \frac{1}{2} A \exp (i\alpha) \exp \{ i\pi\delta(M - 1)/Mm \} \exp \{ i\pi(\delta_0 - \delta)(Mm - 1)/Mm \}$$

$$[\sin (\pi\delta_0/m)/\sin (\pi\delta_0/Mm)]\cdot$$

$$[\sin \{ \pi(\delta_0 - \delta)\} /\sin \{ \pi(\delta_0 - \delta)/m \}] + \frac{1}{2} A \exp (-i\alpha) \exp \{ i\pi\delta(M - 1)/Mm \}\cdot$$

$$\exp \{ -i\pi(2m\sigma_1 + \delta_0 + \delta)(Mm - 1)/Mm \} [\sin (\pi\delta_0/m)/\sin \{ \pi(2m\sigma_1 + \delta_0)/Mm \}]\cdot$$

$$[\sin \{ \pi(\delta_0 + \delta)\} /\sin \{ \pi(\delta_0 + \delta)/m \}]. \qquad .....(11)$$

Consider, for comparison, the true value of the discrete transform, which is obtained by means of a single Mm-point transformation. This is given by an equation similar to (9), written in terms of the appropriate block size and frequency index. The frequency index of the signal in this case is $\gamma_0 = Mmf_0/S$.

The true transform of the signal is:

$$T(\gamma) = \frac{1}{2} A \exp (i\alpha) \exp \{ i\pi(\gamma_0 - \gamma) (Mm-1)/Mm \} [\sin \{ \pi(\gamma_0 - \gamma)\} /\sin \{ \pi(\gamma_0 - \gamma)/Mm \}]$$

$$+ \frac{1}{2} A \exp (-i\alpha) \exp \{ -i\pi(\gamma_0 + \gamma)(Mm-1)/Mm \} [\sin \{ \pi(\gamma_0 + \gamma)\} /\sin \{ \pi(\gamma_0 + \gamma)/Mm \}].$$

$$.....(12)$$

The comparison of equations (11) and (12) will be facilitated if both are expressed in terms of the same frequency indices. The cell sizes in the zoom and the Mm-point transforms are the same--i.e., unit increments in γ and δ represent the same frequency increment Δf(=S/Mm)--while the cell size in the M-point transform is m times as great. The centre frequency of the zoom cell $\delta = 0$ is $S\sigma_1/M$. The cell of index δ in (11) therefore corresponds to the cell of index γ in (12) when δ and γ satisfy the conditions $\gamma = m\sigma_1 + \delta$, and $|\delta| \leq m/2$. In terms of the index δ, equation (12) takes the form:

$$T(\delta) = \frac{1}{2} A \exp(i\alpha) \exp\{i\pi(\delta_0 - \delta)(Mm - 1)/Mm\}\left[\sin\{\pi(\delta_0 - \delta)\}/\sin\{\pi(\delta_0 - \delta)/Mm\}\right]$$

$$+ \frac{1}{2} A \exp(-i\alpha) \exp\{-i\pi(2m\sigma_1 + \delta_0 + \delta)(Mm - 1)/Mm\}\Big[\sin \pi(\delta_0 + \delta)/$$

$$\sin\{\pi(2m\sigma_1 + \delta_0 + \delta)/Mm\}\Big]. \quad .....(12a)$$

4. PROPERTIES OF THE ZOOM TRANSFORM

Each of the transforms Z and T is the sum of two terms, and in each the first term is the direct contribution of the positive signal frequency; the second term represents the influence of the negative frequency component. For all but the lowest frequencies (i.e., unless σ_1 is very small) the magnitude of the second term would therefore be expected to be much less than that of the first, and the equations confirm this expectation. If we ignore, for the present, the case of signals of very low frequency we may therefore confine our attention to the first term in equations (11) and (12a).

Comparing the two equations, we observe that the zoom procedure introduces a phase error, represented by the factor $\exp\{i\pi\delta(M - 1)/Mm\}$. This is of no consequence in applications where the purpose of the analysis is to find the power spectrum, or where only the relative phase of two signals of the same frequency is to be determined. In other cases this error can be eliminated completely by an obvious procedure.

The zoom procedure also leads to amplitude errors. The equations agree only when $\delta_0 = 0$; that is, when the signal frequency coincides with the centre frequency of a cell of the M-point transform (which implies that it also coincides with the centre frequency of a cell of the zoom and Mm-point transforms). This result was predicted in Section 2. When $\delta_0 \neq 0$, the zoom method gives amplitudes that are low, by an amount that increases with $|\delta_0|$. The reduction is described by the first square-bracketted factor in equation (11), which shows that, in the worst case, the amplitude is reduced to about two-thirds of the correct value, and the magnitude of the power spectrum to about 40%. It is possible to cancel this error, at least approximately. The indicated procedure is to multiply the contents of the zoom cells by the factors $M\,[\sin(\pi\delta/Mm)/\sin(\pi\delta/m)]$ for non-zero values of δ. These correction factors could be precomputed and stored.

When δ_0 is not an integer, the second square-bracketted factor in equation (11) also leads to errors in amplitude, and these errors cannot be cancelled. There is a similar factor in equation (12a), but the two factors are not identical. The effect is to spread the spectrum: a single sine wave contributes to the output not only in the cell appropriate to the frequency of the wave, but in every other cell as well. This spreading is particularly undesirable in the zoom transform, for two reasons. First, the amplitude in a given cell diminishes with the number of cells between that cell and the cell in which the signal frequency lies, and since the total number of zoom cells is likely, in practice, to be rather small, the amplitudes in all the zoom cells may be appreciable. In the second place, the discrete transform is cyclic, as is apparent from its definition, and therefore a signal whose frequency falls close to either boundary of a major cell (i.e., a cell of the M-point transformation) may indicate significant amplitudes at frequencies near that of the other boundary. For most applications, however, exact agreement with the equivalent single transform is not required, and these difficulties can be largely overcome by avoiding very small values of m, and using a tapered data window in the second (zoom) transformation to reduce the spreading of the spectrum. It should be remembered that tapering introduces severe phase distortion, so that its use is not appropriate for absolute measurements of phase.

A further property of the zoom transform can be deduced from equation (10): aliassing exists between certain frequencies. Suppose $\delta_0 = m/2$; the phase of the signal $\exp(2\pi i r \delta_0/m)$ then advances by π between adjacent M-point blocks, a condition indistinguishable from the case $\delta_0 = -m/2$, when the phase is retarded by π between blocks. The impossibility of distinguishing these frequencies must be reflected in the transform, and so it is: the cells $m/2$ and $-m/2$ are, in fact, the same cell (see Fig. 1, and recall the cyclic nature of the discrete transform). Thus signals of frequencies corresponding to $\delta_0 = \pm m/2$ both fall in the same cell of the zoom transform. Aliassing is not confined to signals of these exact frequencies--any value of δ_0 in the range $m/2 - 1/2 < |\delta_0| \leq m/2$ leads to a transform having its maximum in cell $m/2$, and so causes ambiguity. However, this ambiguity is easily resolved. If δ_0 is close to $+ m/2$ the amplitude of the M-point transform will be large in cells σ_1 and $\sigma_1 + 1$, while if δ_0 is in the neighbourhood of $- m/2$ the large values will occur in cells $\sigma_1 - 1$ and σ_1. The only case likely to be troublesome is when signals of both frequencies are present. If this situation is suspected the best procedure is to increase the value of m to obtain still finer resolution.

We now return to consider the second term in equations (11) and (12a), which we have so far ignored. We observe that while the second term of (12a) is zero for every integral value of δ_0, the corresponding term of (11) is zero only if $\delta_0 = 0$. The two expressions, therefore, are not equivalent, as is otherwise obvious. To simplify the discussion we define three ranges of frequency.

First, let us consider the middle and upper frequencies (the imprecision of this definition will be removed shortly). In equation (11), the ratio of the first square-bracketted factor of the first term to the corresponding factor of the second term is of the order of M, for mid-range values of σ_1. If M is 1000, say, the component of the power spectrum arising from the first term is about six orders of magnitude greater than that from the second. This, of course, was the justification for neglecting the second term in the preceding discussion.

Second, consider the range of frequencies covered by the 'zero' frequency cell $\sigma_1 = 0$. If the zoom transform for this cell is modified by the phase and amplitude correction factors discussed above (the factors are the same for both terms) the result is a good approximation to the true transform: the phase error is nil and the amplitude error is small. The second term is as important as the first, but it is not an error term in this case.

Finally, consider the low frequencies, where σ_1 is small but not zero. For frequencies in this range, the magnitude of the second term in (11) may be of the same order as that of the first term. Moreover, the energy represented by the second term is largely concentrated in a single cell, $\delta = -\delta_1$ (where δ_1 is the integer closest to δ_0), while the first term has its maximum in the cell δ_1, as it should. This means that the zoom spectrum indicates false signal components at the low frequency end. However, this conclusion is not quite so disastrous as it may appear. The reason becomes clear when we re-examine the amplitudes of the two terms. The worst case is when σ_1 is small and δ_1 large and negative. When $\sigma_1 = 1$, i.e., when we are examining the first cell of the M-point transform, the amplitude of the false component cannot exceed 1/3 of the amplitude of the true component. When $\sigma_1 = 3$, the maximum value of the ratio is 1/11. The squares of these ratios apply to the power spectrum, so that at frequencies as low as those of the third cell of the M-point transform, the energy associated with the false component is less than 1 per cent of the energy that appears at the correct frequency. At higher frequencies the energy ratio decreases as $1/f^2$. This consideration enables us to define the low-frequency and high-frequency ranges more precisely: the former is the range in which the second term of equation (11) cannot be ignored, and the latter is the remainder of the range covered by the transform. What can or cannot be ignored is, of course, decided by the accuracy that is required (and significant) in the particular application.

Since the false components are contributed by the second term in equation (11), they have their origin in the negative-frequency part of the spectrum of the bandpass signal. The basic (M-point) FFT analyser represents a comb of band-pass filters with rather poor roll-off characteristics, and if the roll-off could be improved the mutual influence of each half of the spectrum on the other would be reduced. This is easily done: all that is necessary is to taper the data window of the M-point transformation. The effect of tapering on the phase should be remembered. Of course, if the low-frequency range is of primary concern, a preferable procedure is to decimate the data (with appropriate low-pass filtering) and perform the M-point transformation on the resulting sequence of samples, without a subsequent zoom transformation.

It is worth pointing out that the zoom technique requires the m blocks of M signal samples to be contiguous in time: no samples may be dropped between blocks. The effect of dropping samples is to change the apparent value of δ_0. If n samples are lost between successive blocks the apparent value of δ_0 becomes $\left\{ \delta_0 + n\ (m\delta_0/M) \right\}$. The error in δ_0 represents an error in the incremental frequency, and this, of course, defeats the purpose of the zoom procedure.

5. COMPUTER TESTS OF THE ZOOM TECHNIQUE

A series of computer experiments was carried out to test the predictions of the analysis of the previous sections. The computer used was the XDS Sigma-7. The FFT routine FASTFORM was used to obtain the transforms, and the computation was carried out in double-precision floating-point arithmetic. The experiments are described below.

5.1 FALSE SPECTRAL LINES

The theoretical prediction that the zoom transform indicates false signal components had not been anticipated. A computer test of this prediction therefore provides a useful check on the validity of the analysis. The block sizes chosen were M = 256, m = 16. If the sampling rate S is taken to be 256 samples/sec, the resolution of the M-point transform is 1 Hz, and that of the m-point transform is 1/16 Hz. The signal frequency selected was 9/16 Hz. The choice of a non-integral value ensures that the false component is not zero; the value chosen is the centre frequency of one of the cells of the 4096-point transformation to which the zoom analysis is approximately equivalent. The 4096-point transform may be computed from equation (12a), and the square modulus of the result gives the power spectrum. Alternatively, the power spectrum may be found by use of the discrete analog of Parseval's Theorem, without the use of equation (12a). The theoretically correct transform gives the following power spectrum (to three significant figures):

$$|X(\sigma)|^2 = \begin{cases} 4.19 \times 10^6 , & \sigma = 9 \quad (f = 9/16 \text{ Hz}) \\ = 0 , & \text{otherwise} \end{cases} \qquad \text{.....(13)}$$

Since the FFT routine FASTFORM has been thoroughly tested, and found to give results in accordance with the theory, a computer verification of equation (13) was not carried out. The computer values for $\sigma \neq 9$ would not be exactly zero, of course, because of round-off error, but they should be many orders of magnitude smaller than the main peak.

The theoretical value of the power spectrum as given by the zoom method applied to the 1 Hz cell of the 256-point transform was computed from equation (11). The parameter values are $A = 1$, $\alpha = 0$, $\sigma_1 = 1$, $\delta_0 = -7$. The true signal component should therefore appear in the cell $\delta = -7$, corresponding to the frequency 1-7/16 Hz (see Fig. 1). The output in all other cells should be zero. The power spectrum, calculated to slide-rule accuracy from the theoretical zoom transform, is:

$$|X(\delta)|^2 = \begin{cases} 2.13 \times 10^6 , & \delta = -7 \quad (f = \frac{9}{16} \text{ Hz}) \\ 1.67 \times 10^5 , & \delta = \ \ 7 \quad (f = 1\frac{7}{16} \text{ Hz}) \\ 0 , & \text{otherwise} \end{cases} \qquad \text{.....(14)}$$

Equations (13) and (14) illustrate the amplitude error of the zoom transform as well as the presence of the false component.

The results of the zoom transformation given by the Sigma-7 computer are:

$$|X(\delta)|^2 = \begin{cases} 2.14 \times 10^6, & \delta = -7 \quad (f = \frac{9}{16}\text{ Hz}) \\ 1.67 \times 10^5, & \quad\; = 7 \quad (f = 1\frac{7}{16}\text{ Hz}) \\ \sim 10^{-16}, & \text{otherwise} \end{cases} \qquad \ldots\ldots(15)$$

The computer therefore yields values in exact agreement with the predictions of equation (11). This quantitative agreement confirms the validity of the analysis of Section 3, and demonstrates the existence of the false component. It should be noted that the signal frequency in this example was chosen to give a false component of almost the maximum possible magnitude.

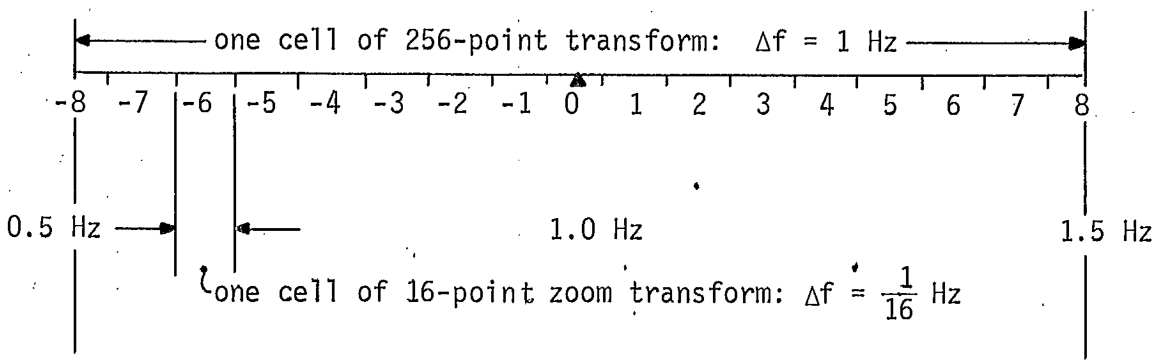

Fig. 1. Location of zoom cells within a major cell.

5.2 ALIASSING

It was noted in Section 4 that the two half-cells labelled -8 and 8 in Figure 1 actually appear as the same cell in the zoom transform, so that signals of frequencies falling in these half-cells cannot be distinguished. This was demonstrated by applying the zoom technique to the 64 Hz cell, with input signals of frequency 63.52 Hz and 64.48 Hz. Three tests were made: one with each signal separately, and one with both signals together. In all cases the maximum output appeared in cell 8, as expected.

5.3 AMPLITUDE AND PHASE CORRECTION

It has been observed that the zoom transform gives incorrect values of the amplitude and phase except in special cases. Section 4 indicated the desirability of using a tapered data window and of avoiding very small values of m (the number of points in the zoom transform), and factors were also introduced to cancel the phase error and part of the amplitude error. Computer experiments were carried out to demonstrate the effect of the correction factors and of tapering, using a small value of m.

In the first test a signal of frequency 60.375 Hz was used, with M = 256 and m = 8; the sampling rate, as before, was assumed to be 256 samples/second. This signal frequency is the centre frequency of one of the zoom cells, so that the second square-bracketted factor in equation (11) should contribute no error, and the amplitude correction factor, which in general is only approximate, should be exact. The transform was computed by the zoom method, with and without the use of the correction factors, and also by the exact 2048-point single-transform method. The results are presented in Table 1. As expected, the corrected zoom transform is exact, except for the presence of the small false component in cell -3.

For the second test the signal frequency was changed to 59.72 Hz, a value that is eccentrically located in its cell; the cell boundaries are 59.6875 and 59.8125 Hz. The small value of m and the absence of tapering aggravate the spectral spreading effects discussed in Section 4. The test was otherwise the same as the first one, and the results are given in Table 2. The corrected zoom transform gives the phase very accurately, but the amplitude is less satisfactory. The main peak occurs in the correct cell, with an amplitude only 2-3% low, but the spreading is worse than in the true transform. The reason, clearly, is that the signal frequency is less than two cell-widths removed from the 59.5 Hz lower boundary of the 60 Hz major cell, so that the spreading continues cyclically into the neighbourhood of the 60.5 Hz upper boundary. The spreading completely conceals the presence of the small false component, which should appear in cell 2. While the amplitudes given by the zoom transform are not in very close agreement with the exact values, the discrepancies should not be over-emphasized: in the power spectrum the amplitudes are squared, and the main peak in cell -2 dominates the spectrum, as it should.

TABLE 1 - Transform Values for Signal Frequency 60.375 Hz

Cell Centre Frequency (Hz)	Cell Number	Exact Method		Zoom Method			
				Uncorrected		Corrected	
		Amplitude	Phase	Amplitude	Phase	Amplitude	Phase
59.500	-4	10^{-6}	10^{-4}	10^{-6}	4.703	10^{-6}	10^{-2}
59.625	-3*	10^{-6}	10^{-4}	3.71	0.299	4.73	1.473
59.750	-2	10^{-6}	10^{-4}	10^{-6}	5.504	10^{-6}	10^{-3}
59.875	-1	10^{-6}	10^{-4}	10^{-6}	5.892	10^{-6}	10^{-4}
60.000	0	10^{-6}	10^{-5}	10^{-6}	10^{-5}	10^{-6}	10^{-5}
60.125	1	10^{-6}	10^{-5}	10^{-6}	0.391	10^{-6}	10^{-4}
60.250	2	10^{-6}	10^{-5}	10^{-6}	0.783	10^{-6}	10^{-4}
60.375	3	1024.0	10^{-8}	803.04	1.174	1024.0	10^{-8}
60.500	4	10^{-6}	3.142	10^{-6}	4.703	10^{-6}	3.139

* Cell occupied by false signal (zoom method only).

The second test points to the need for controlling the spreading of the spectrum. A tapered data window is indicated, and for the third test a triangular taper was applied, with other conditions the same as in the second test. The results are exhibited in Table 3. As regards the amplitude, the tapering is very effective; significant amplitudes occur only in the correct cell and its nearest neighbour on either side. The phase, however, is heavily distorted. Both the amplitude and phase effects of the triangular data window can be predicted from the corresponding spectral window, which is given in reference 6 for a data window centred on zero. (The lag and (power) spectral windows of the reference are identical to our data window and (complex) spectral window, apart from the translation of the data window, which introduces a phase modification of the spectral window, and must be taken into account.) The phase distortion due to the use of a tapered data window cannot be corrected unless the exact frequency of the signal is known.

TABLE 2 - Transform Values for Signal Frequency 59.72 Hz

Cell Centre Frequency (Hz)	Cell Number	Exact Method		Zoom Method			
				Uncorrected		Corrected	
		Amplitude	Phase	Amplitude	Phase	Amplitude	Phase
59.500	-4	126.44	2.385	120.00	0.821	188.49	2.385
59.625	-3	293.25	2.387	260.73	1.213	332.47	2.387
59.750	-2	930.04	5.530	815.83	4.747	906.15	5.530
59.875	-1	180.28	5.531	164.30	5.140	168.60	5.531
60.000	0	99.95	5.532	99.95	5.532	99.95	5.532
60.125	1	69.21	5.533	80.86	5.924	82.98	5.533
60.250	2*	52.96	5.535	79.87	0.029	88.72	5.530
60.375	3	42.92	5.536	85.96	0.429	109.61	5.539
60.500	4	36.10	5.537	120.00	0.821	188.49	5.539

* Cell occupied by false signal (zoom method only).

TABLE 3 - Transform Values for Signal Frequency 59.72 Hz

Cell Centre Frequency (Hz)	Cell Number	Exact Method		Zoom Method with Taper			
				Uncorrected		Corrected	
		Amplitude	Phase	Amplitude	Phase	Amplitude	Phase
59.500	-4	126.44	2.385	14.36	3.962	22.56	5.526
59.625	-3	293.25	2.387	535.64	1.213	683.04	2.387
59.750	-2	930.04	5.530	854.00	4.747	948.54	5.530
59.875	-1	180.28	5.531	195.66	1.998	200.78	2.390
60.000	0	99.95	5.532	8.18	5.532	8.18	5.532
60.125	1	69.21	5.533	16.26	2.777	16.70	2.386
60.250	2	52.96	5.535	2.24	6.096	2.48	5.314
60.375	3	42.92	5.536	27.22	0.439	34.70	5.549
60.500	4	36.10	5.537	14.36	3.962	22.56	2.398

5.4 RESOLUTION OF DOUBLETS

One of the principal uses anticipated for the zoom technique is to examine the fine structure of the power spectrum. A computer experiment was performed to demonstrate this use. A signal consisting of two sine waves was generated by the computer and its Fourier transform computed by the zoom method. The frequencies were chosen such that the waves should not be resolved by the first (256-point) transformation, but should be resolved by the zoom (16-point) transformation. The sampling rate was taken to be 256 samples/sec, as before, and on this basis the signal frequencies were 99.8 and 100.05 Hz. Both frequencies lie in the 100 Hz cell of the 256-point transform, the boundaries of which are 99.5 and 100.5 Hz. In the zoom transform, however, the two frequencies lie in different cells; the 99.8 Hz wave occupies cell number -3, which is bounded by the frequencies 99.78125 and 99.84375 Hz, and the 100.05 Hz wave occupies cell number 1, which covers the range 100.03125-100.09375 Hz. The predicted outcome of the experiment, therefore, was that the power spectrum computed from the first transform should exhibit a strong signal in only one cell, the 100 Hz cell, but the zoom spectrum should show that the signal actually consists of two components of estimated frequencies 99.8125 and 100.0625 Hz (the centre frequencies of cells -3 and 1). The results of the computer experiment confirmed the prediction exactly: the doublet was not resolved by the first transformation, but was correctly analysed by the zoom transformation.

6. SUMMARY

The results of the analysis and the computer tests presented in this report show that the zoom procedure gives an approximation to the true Fourier transform that is good enough to permit the use of the unmodified zoom transform in many applications. By the introduction of correction factors that are derived in this report, the phase error can be eliminated and the estimate of amplitude improved. If the phase is not of interest, or if only the relative phase of two signals of the same frequency is required, it is advisable to use a tapered data window, together with the amplitude correction factor.

The zoom transform indicates the presence of false signal components, which are large at very low frequencies, although of negligible magnitude throughout most of the frequency range of the transform. A method for reducing the magnitude of the false components has been indicated.

An inconvenience of the zoom technique is the aliassing of signal components whose frequencies lie close to either boundary of the cell selected for high-resolution analysis. The consequent ambiguity in the frequency can be removed, in many cases, by a very simple check, and in other cases by the use of higher resolution.

The outstanding merit of the zoom technique is that it provides very high resolution at a cost in data storage capacity that is only a fraction of the capacity required for the equivalent single transform. In applications where only a few cells of the initial transform contain information of interest, the zoom method also yields large savings in the amount of computation required.

7. REFERENCES

1. Shaw, E. *Oral presentation of zoom transform technique.* Computing Devices of Canada, Ltd., Ottawa, 9 January 1970.

2. Gentleman, W.M., and G. Sande. *Fast Fourier transforms--for fun and profit.* Proceedings, AFIPS Fall Joint Computer Conference, 1966.

3. Grace, O.D., and S.P. Pitt. *Quadrature sampling of high-frequency waveforms.* Journal of the Acoustical Society of America, Vol. 44, No. 5, 1454-5, 1968.

4. Woodward, P.M. *Probability and information theory with applications to radar.* Pergammon Press, London, 1953.

5. Goldman, S. *Information theory.* Prentice-Hall, New York, 1953.

6. Blackman, R.B., and J.W. Tukey. *The measurement of power spectra.* Dover, New York, 1958.