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ON AVERAGING IN BURG MEM SPECTRAL ANALYSIS OF SINUSOIDAL SIGNALS – THEORY

by

R.W. Herring

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ABSTRACT

When independent records of sampled data from a common source are available, one method for combining these data while using Burg MEM spectral analysis is, at each iteration of the Burg algorithm, to compute separately sets of numerator and denominator terms for the reflection coefficient, and then to compute the reflection coefficient as the quotient of the averaged numerator and denominator terms. The effect of this form of averaging for the case of multiple records comprising two noise-free complex sinusoids is investigated analytically. It is shown that when the relative phase between the sinusoidal components varies randomly from record to record, bias terms in the first and second reflection coefficients are inversely proportional to the number of records processed.

1. INTRODUCTION

A recent paper [1] has indicated that line-splitting in Burg maximum-entropy method (MEM) spectral estimation for the case of two complex sinusoids is caused by intrinsic biases in the estimated values of the first and second reflection coefficients. These biases were shown to be a function both of the relative phase between the complex sinusoids at the middle of the data record, and of the number of cycles of the difference frequency between the two sinusoids contained within the length of the data record. In [1] it was assumed that only one set of data was available and the phase difference between the complex sinusoids was fixed, but that any additive noise was stochastic. Therefore, the expected values of the Burg reflection coefficients were explicit functions of this phase difference.

In certain circumstances, independent records of sampled data derived from the same source and containing fixed-frequency sinusoidal components are available, and it is reasonable to assume that the phase differences between the sinusoidal components vary on a record-to-record basis. This topic has been considered in a theoretical and simulation study by Gabriel [2] for the case of sets of records of data from a sampled-aperture antenna array. In [2] the received signals were modelled as radar returns from two or more targets occurring at slightly different bearings and moving at different velocities, so that a variation in the relative phase of the signal components from record to record was induced by the relative motion of the targets, due to the doppler effect. It was shown that a form of averaging was effective in improving angular spectrum estimates based on direct evaluations of the aperture correlation matrix.

Here, the effects of a particular form of averaging on the biases described in [1] are shown analytically. Specifically, when K independent records, each comprising N samples of the sum of two complex sinusoids, are available, the reflection coefficient at each iteration of the Burg algorithm can be computed by averaging the K numerator and denominator terms computed from each independent set of data, and then computing the reflection coefficient as the quotient of these averages [3,4]. This averaging is shown to reduce the expected magnitudes of the biases which cause line splitting when the SNR is high, provided that the relative phase between the complex sinusoids varies from record to record in a uniformly distributed random manner (cf. [2]). However, the analysis also shows that subdividing K records of length N into 2K records of length N/2 is counter-productive, at least when noise is absent.

2. THEORY

Assume that K records of complex data, each record of length N, are available. Let $x_k(n)$, $n = 0,1,\ldots,N-1$ denote the kth record. Following the examples of Burg [3] and Moorcroft [4] and using a notation based on [1], the Burg algorithm can be written in a parallel lattice form:

$$f_{M,k}(n) = f_{M-1,k}(n) - \beta(M,M)\, b_{M-1,k}(n-1) \tag{1a}$$

$$b_{M,k}(n) = b_{M-1,k}(n-1) - \beta^*(M,M)\, f_{M-1,k}(n) \tag{1b}$$

$$n = M, M+1, \ldots, N-1$$
$$M = 1,2,\ldots,P \le N-1$$

$$f_{0,k}(n) = x_k(n) = b_{0,k}(n) \tag{1c}$$

$$n = 0,1,\ldots,N-1$$

where
$$\beta(M,M) = NUM(M,M)/DEN(M,M), \tag{2}$$

$$NUM(M,M) = \frac{1}{K(N-M)} \sum_{k=1}^{K} \sum_{n=M}^{N-1} b^{*}_{M-1,k}(n-1)\, f_{M-1,k}(n), \tag{3a}$$

$$DEN(M,M) = \frac{1}{2K(N-M)} \sum_{k=1}^{K} \sum_{n=M}^{N-1} \left\{ |b_{M-1,k}(n-1)|^2 + |f_{M-1,k}(n)|^2 \right\}, \tag{3b}$$

and P is the order of the lattices.

Equations (1a-c) indicate that a distinct lattice is required for each of the K data records, but, at each stage of each lattice, the same reflection coefficient $\beta(M,M)$ is used in all the lattices. The series $f_{M,k}(n)$ are respectively the forward and backward error signals in the kth lattice at the Mth stage.

Equations (2) and (3a-b) state explicitly how each of the reflection coefficients $\beta(M,M)$ is to be computed: by separately averaging the numerator terms and denominator terms which can be calculated from each lattice, to yield the values of NUM(M,M) and DEN(M,M) respectively. The quotient of these averages is then computed to produce a single estimate of the Mth reflection coefficient, according to (2). Burg [3] has proven that, in general, the reflection coefficients thus computed satisfy the relation

$$|\beta(M,M)|^2 \leq 1 \tag{4}$$

so that the corresponding covariance matrix is non-negative definite.

If there are K records of data, each comprising N samples of a signal consisting of two complex sinusoids with no additive noise (infinite SNR), then $x_k(n)$ can be written as [1]

$$x_k(n) = A_o \exp[j(n\omega_o + \phi_k)]$$
$$\bullet \{r \exp[j(n\Delta\omega + \Delta\phi_k)] + r^{-1} \exp[-j(n\Delta\omega + \Delta\phi_k)]\} \tag{5}$$

One of the complex sinusoids has amplitude $A_o r$, angular frequency $\omega_o + \Delta\omega$ and initial phase $\phi_k + \Delta\phi_k$; the other has amplitude $A_o r^{-1}$, angular frequency $\omega_o - \Delta\omega$ and initial phase $\phi_k - \Delta\phi_k$.

Substitution of (5) into (3a) and (3b) for M = 1 yields

$$\mathrm{NUM}(1,1) = A_o^2(r^2 + r^{-2})\ \exp(j\omega_o)$$

$$\bullet\left\{\cos\Delta\omega + j\ \rho(r)\ \sin\Delta\omega + \frac{2G(N-1,\Delta\omega)}{r^2 + r^{-2}}\ \mathrm{SUM}\ (\Delta\phi\mathrm{mid},K)\right\} \tag{6a}$$

and

$$\mathrm{DEN}(1,1) = A_o^2\ (r^2 + r^{-2})$$

$$\bullet\left\{1 + 2\ \frac{G(N-1,\Delta\omega)}{r^2 + r^{-2}}\ \cos\Delta\omega\mathrm{SUM}(\Delta\phi\mathrm{mid},K)\right\} \tag{6b}$$

where $\rho(r)$ is defined as

$$\rho(r) = (r^2-r^{-2})/(r^2+r^{-2}), \tag{7}$$

$$G(N,\Delta\omega) = \sin(N\Delta\omega)/[N\ \sin(\Delta\omega)] \tag{8}$$

is the common grating-function frequency response of a normalized uniformly weighted discrete Fourier transform of N data,

$$\Delta\phi\mathrm{mid},k = (N-1)\Delta\omega + 2\Delta\phi_k \tag{9}$$

is the phase difference between the two complex sinusoidal components, reckoned at the middle of the kth record, and

$$\mathrm{SUM}(\Delta\phi\mathrm{mid},K) = \frac{1}{K}\sum_{k-1}^{K} \cos(\Delta\phi\mathrm{mid},K) \tag{10}$$

is an estimate of the average value of the cosines of the phase differences. Examination of (10) shows that if $\Delta\phi$mid,k is uniformly and randomly distributed over the interval 0 to 2π, then the statistically expected value for SUM($\Delta\phi$mid,K) is zero, with variance 1/2K.

Applying (2) to (6a) and (6b) yields for the first reflection coefficient:

$$\beta(1,1) = \exp(j\omega_o)$$

$$\bullet\left\{\frac{\cos\Delta\omega + j\rho(r)\ \sin\Delta\omega + [2\ G(N-1,\Delta\omega)/(r^2 + r^{-1})]\ \mathrm{SUM}\ (\Delta\phi\mathrm{mid},K)}{1 + [2G(N-1,\Delta\omega)/(r^2 + r^{-2})]\ \cos\ \Delta\omega\ \mathrm{SUM}\ (\Delta\phi\mathrm{mid},K)}\right\} \tag{11}$$

Equation (11) should be compared with (43) of [1], which is the result for the single record case.

If SUM($\Delta\phi$mid,K) is replaced in (11) by its statistically expected value of 0, the result is a perfect estimate of the first reflection coefficient as given by the known-autocorrelation (KA) case for infinite SNR, $\alpha(1,1)$ (cf. (30) of [1]):

$$\alpha(1,1) = \exp(j\omega_o)\ [\cos\Delta\omega + j\rho(r)\sin\Delta\omega]. \tag{12}$$

It is more correct, however, to estimate the expectation of (11) after the quotient has been computed. When either of the conditions $\sqrt{2K} >> 1$ or $|G(N-1,\Delta\omega)| << 1$ is met, the approximation $1/(1+x) \approx (1-x+x^2)$ can be applied to (11) to get an expected value for $\beta(1,1)$ of

$$<\beta(1,1)> = \exp(j\omega_o)\Big\{[1 - B_1\ (N,\Delta\omega,r,K)\sin^2\Delta\omega]\ \cos\Delta\omega$$
$$+ j\rho(r)[1 + B_1\ (N,\Delta\omega,r,K)\cos^2\Delta\omega]\ \sin\Delta\omega\Big\} \tag{13}$$

where

$$B_1\ (N,\Delta\omega,r,K) = 2[G(N-1,\Delta\omega)/(r^2+r^{-2})]^2/K \tag{14}$$

is a small bias term which varies inversely with K. In (14), $[\mathrm{SUM}(\Delta\phi\mathrm{mid},K)]^2$ has been replaced by its expected value of 1/2K.

Further analysis shows that, to the same degree of approximation, the variance of $\beta(1,1)$, VAR[$\beta(1,1)$] is given by

$$\mathrm{VAR}[\beta(1,1)] = B_1\ (N,\Delta\omega,r,K)\ \sin^2\Delta\omega[\sin^2\Delta\omega + \rho^2(r)\cos^2\Delta\omega] \tag{15}$$

and the ratio of the squared magnitude of the difference $<\beta(1,1)>-\alpha(1,1)$ to the variance of $\beta(1,1)$ is given by

$$\frac{|<\beta(1,1)> - \alpha(1,1)|^2}{\mathrm{VAR}[\beta(1,1)]} = B_1\ (N,\Delta\omega,r,K)\ \cos^2\Delta\omega. \tag{16}$$

Proceeding now to the second reflection coefficient, the result (11) can be applied in (1a-b) and (3a-b) to derive the following results for NUM(2,2) and DEN(2,2):

$$\begin{aligned}
\mathrm{NUM}(2,2) = \Big\{ & -4A_o^2 \sin^2\Delta\omega \exp(j2\omega_o)/ \\
& [1 + 2\ \mathrm{SUM}(\Delta\phi\mathrm{mid},K) \cos\Delta\omega\ G(N-1,\Delta\omega)/(r^2 + r^{-2})]\Big\} \\
\bullet \Big\{ & 1 - 2\ \mathrm{SUM}(\Delta\phi\mathrm{mid},K)\ G(N-2,\Delta\omega)/(r^2 + r^{-2}) \\
& -2\ \mathrm{SUM}(\Delta\phi\mathrm{mid},K)\ G(N-1,\Delta\omega) \\
& \{\mathrm{SUM}(\Delta\phi\mathrm{mid},K)\ G(N-2,\Delta\omega)\ [\cos\Delta\omega + j\rho(r) \sin\Delta\omega] \\
& -2 \cos\Delta\omega/(r^2 + r^{-2})\} \\
& +\mathrm{SUM}^2(\Delta\phi\mathrm{mid},K)\ G^2(N-1,\Delta\omega)\ \{[\cos 2\Delta\omega + j\rho(r)\sin 2\Delta\omega] \\
& -2\ \mathrm{SUM}(\Delta\phi\mathrm{mid},K)\ G(N-2,\Delta\omega)/(r^2 + r^{-2})\}\Big\}
\end{aligned} \tag{17a}$$

and

$$\begin{aligned}
\mathrm{DEN}(2,2) = \Big\{ & 4A_o^2 \sin^2\Delta\omega/ \\
& [1 + 2\ \mathrm{SUM}(\Delta\phi\mathrm{mid},K) \cos\Delta\omega G(N-1,\Delta\omega)/(r^2 + r^{-2})]\Big\} \\
\bullet \Big\{ & 1 - 2\ \mathrm{SUM}(\Delta\phi\mathrm{mid},K)\ G(N-2,\Delta\omega)/(r^2 + r^{-2}) \\
& -2\ \mathrm{SUM}(\Delta\phi\mathrm{mid},K) \cos\Delta\omega\ G(N-1,\Delta\omega) \\
& [\mathrm{SUM}(\Delta\phi\mathrm{mid},K)\ G(N-2,\Delta\omega) - 2/(r^2 + r^{-2})] \\
& +\mathrm{SUM}^2(\Delta\phi\mathrm{mid},K)\ G^2(N-1,\Delta\omega) \\
& [1 - 2\ \mathrm{SUM}(\Delta\phi\mathrm{mid},K) \cos 2\Delta\omega\ G(N-2,\Delta\omega)/(r^2 + r^{-2})]\Big\}
\end{aligned} \tag{17b}$$

Equations (17a) and (17b) can be combined using (2) to derive a result for $\beta(2,2)$ which, for $K = 1$, is the same as (47) of [1].

If SUM($\Delta\phi$mid,K) is replaced in (17a) and (17b) by its expected value of 0 and the quotient is then computed, the result again is a perfect estimate of the KA reflection coefficient for infinite SNR, $\alpha(2,2)$ (cf. (34) of [11]:

$$\alpha(2,2) = -\exp(j2\omega_o). \tag{18}$$

By following the same approach as taken in deriving (13), the expected value for $\beta(2,2)$ when either $\sqrt{2K} \gg 1$ or both $|G(N-1,\Delta\omega)| \ll 1$ and $|G(N-2,\Delta\omega)| \ll 1$ are valid is found to be

$$<\beta(2,2)> = -\exp(j2\omega_o)\left\{1 - B_2(N,\Delta\omega,r,K)\right\} \tag{19}$$

where

$$B_2(N,\Delta\omega,r,K) = \left\{G^2(N-1,\Delta\omega)\ \sin\Delta\omega[\sin\Delta\omega - j\rho(r)\cos\Delta\omega] + j\rho(r)G(N-1,\Delta\omega)G(N-2,\Delta\omega)\sin\Delta\omega\right\}/K \tag{20}$$

is another small bias term.

Similar to the case of the first reflection coefficient, further analysis shows that $VAR[\beta(2,2)]$ is given by

$$VAR[\beta(2,2)] = 2G^2(N-1,\Delta\omega)\sin\Delta\omega[1+\rho^2(r)]/K \tag{21}$$

and the ratio of the squared magnitude of the difference $<\beta(2,2)>-\alpha(2,2)$ to the variance of $\beta(2,2)$ is given by

$$\frac{|B_2(N,\Delta\omega,r,K)|^2}{VAR[\beta(2,2)]} = 2G^2(N-1,\Delta\omega)\sin^2\Delta\omega[1 + \rho^2(r)]/K \tag{22}$$

3. DISCUSSION

Consideration of (14) and (20) shows that, since $|G(N-1,\Delta\omega)|$ is proportional to $1/(N-1)$, the bias terms are roughly proportional to $1/[(N-1)^2K]$. This result is important because, for example, it shows that doubling K at the expense of halving N, by processing K records of length N as 2K disjoint records of length N/2, nearly doubles the expected magnitudes of the bias terms.

Similarly, consideration of (15) and (21) shows that the variances of the first two reflection coefficients are also roughly proportional to $1/[(N-1)^2K]$. This means that not only the magnitudes of the biases, but also their variances increase when K is increased in a tradeoff with N.

Finally, (16) and (22) show that the ratios of the squared magnitudes of the biases to the variances of the Burg estimates of the reflection coefficients have the same $1/[(N-1)\ K]$ dependence. This indicates that, subject to the constraint that the signal to noise ratio be very high, whenever it is possible to arrange a tradeoff between N and K the choice should be to make the record length N as great as possible at the expense of K. Equations (6a-b) and (17a-b) are consistent with this assertion, since it can be seen that the maximum magnitudes of the terms which give rise to the biases are inversely proportional to N when $\Delta\omega$ is non-zero.

In general, it appears to be best to process data records which are as long as possible when the signal to noise ratio is high and the Burg MEM is used.

4. CONCLUSIONS

It has been shown that the suggested form of averaging for multiple sets of noise-free sampled data comprised of two complex sinusoids yields expected values for the first and second Burg reflection coefficients with rather simple form when K, the number of sets of data, is large. In this case, the expected values of the first and second reflection coefficients are given by (13) and (19) respectively. These expected values are biased from the result for the known-autocorrelation case by additive terms proportional to 1/K.

For the suggested averaging technique to be effective, it is essential that the relative phases of the sinusoidal components vary from record to record in a uniformly distributed random manner. Such can be the case when successive frames of data from a linear sampled-aperture array are available [2], or when systematic constraints make only non-contiguous time domain records available [4].

It has been shown that segmenting long records with high signal to noise ratio in order to increase the number of records available for averaging is theoretically counter-productive.

5. ACKNOWLEDGEMENT

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