by s.c.(Garg)

Report prepared for the Department of Communications, Govt. of Canada


This report was prepared for the Department of Communications and is the Final Report as required under the terms of DSS Contract Number 07SU.36001-8-4792. The encouragement and support of the scientific authority, Dr. Rolf Mamen of the Communications Research Centre, is gratefully acknowledged. The able assistance of Mr. Jean de Lafontaine in the collection and summarization of several references, and his help in preparing the final report are much appreciated.
Page
Symbols ..... -ii-
Introduction ..... 1.

1. Orbital Perturbations ..... 1
2. Perturbation Modelling ..... 5
a) The Geopotential ..... 5
b) Iuni-Solar Attractions ..... 7
c) Atmospheric Drag ..... 8
3. Analytical Approaches: A Survey ..... 27
a) Osculating Orbital Elements ..... 29
b) Spatial Curves ..... 38
c) Graphical Procedures ..... 43
4. Discussion ..... 45
a) The Three Analytical Approches ..... 45
b) Anelytic vs. Other Methods ..... 47
c) On the Use of the Various Methods ..... 48
Conclusion ..... 49

Unless otherwise stated, the following symbolism will be used throyrhont,

| $r$ | geocentric radius |
| :---: | :---: |
| h | altitude above the Earth's surface |
| $\theta$ | true anomaly |
| E | eccentric anomaly |
| a | semi-major axis of the orbit |
| e | eccentricity of the orbit |
| V | inertial velocity of the satellite |
| T | orbital period |
| N | number of revolution |
| $\mu$ | gravitational constant ( $=$ GM) |
| $\beta$ | reciprocal of scale-height |
| H. | scale-height |
| $\rho$ | atmospheric density |
| t | time |
| $C_{D}$ | drag coefficient |
| $\pm$ | Earth's flattening |
| Subscripts: |  |
| 0 | initial value |
| $p$ | perigee value (or PE) |
| I | end of life value |
| E | equatorial value (Earth) |
| A | atmospheric value |

In the past two decades, as the number of satellites in terrestrial orbit increased, scientistics were concerned with a new (sometimes dangerous) situation: satellite suffered orbit contraction and eventually, re-entered the atmosphere. Very often, complete desintegration occurs but there remains a small but non-negligible probability of aircraft hazard or ground impact ( $O A-2$ ). This report will gather the various means of predicting orbit decay that have been developed for . bounded, near-earth orbits (perigee altitude below 600 km ), assuming an uncontrolled satellite. A survey of the different mathematical modelling of orbit perturbations is also included.

1. ORBTTAL PERTURBATTONS

In the simplest possible situation, the Earth and the orbiting satellite are considered as being point mass interacting in vacuo without any other celestial body nearby. It is well known that the resulting orbit is a conic section, lying in a fixed plane with respect to the attracting body inertial frame. Since negative total energy is considered here, the orbit is an ellipse (or a circle, as a particular case of the ellipse. The unusual case of a rectilinear orbit is excluded.). Orbital elements ( $a, e, i, \omega, \Omega, \theta$ ) or any (rectangular, spherical, etc) coordinates completely describe the state of the satellite in its trajectory.

In our "real world", non-symmetric finite masses, other physical bodies (atmosphere, sun, moon, charged particles) and electromagnetic fields perturb the orbit which then differs from an ellipse.

These alterations to the nominal ellipse may be classified into two classes:

## a) Environmental Perturbations <br> b) Particular Perturbations

Environmental perturbations are those perturbations caused by the physical bodies exterior to, and surrounding the satellite. They are usually function of the spatial position. Table I gives the main environmental perturbations affecting earth satellite.

TABLE I: ENVIRONMENTAL PERTURBATIONS
a) Solar wind and radiation pressure
b) Earth's magnetic field
c) Magnetohydrodynamical drag (LP-7)
d) Meteoroids impacts
e) Relativistic effects
f) Luni-Solar attractions
g) Non-symmetric mass distribution of the Earth
h) Atmospheric drag.

Solar wind and radiation pressure (LP-38, LP-13) have their greatest importance on high altitude satellite with large area-to-mass ratio. Since we are dealing with near-earth satellite (and the majority of the satellites launched so far are "compact" in nature) these perturbations may be neglected compared to air drag. Similarly, the influence of earth's magnetic field and that of the ionospheric charged particles are neglected in nearly all the papers since the order of magnitude involved is very sma11. Of course, relativistic effects are excluded for the same reason. Unless a large meteoroid hits the satellite, ending its life on the spot, small meteoroids impacts are most of the time neglected. Furthermore, since we are considering perturbations that can lead to orbit decay, the above mentioned disturbances to satellite trajectory are clearly useless for our purposes and are thus discarded.

The most important factor that must be considered here is obviously the atmospheric drag. This perturbation leads to decreasing value in orbit eccentricity and semi-major axis. This may end the life of the satellite (LP-13). Atmospheric rotation causes the orbital inclination to increase and periodic oscillations in $\Omega$ while atmospheric
oblateness alters the argument of perigee (LP-13). It has been shown (LP-52) that the diurnal bulge may alter the lifetime of a satellite for $4 \%$ (circular, 200 km - high orbit) and up to $20 \%$ (at 500 km ). It has been suggested (LP-45) that unpredicted solar activity causes the upper atmospheric density to alter the orbit of Skylab in such a maner that it's re-entry into the atmosphere is now expected 3 years earlier than it was first predicted. In (LP-22), an analysis of in-track and out-of-plane dispersions caused by drag is given.

Even if luni-solar attractions and the non-spherical mass distribution alone may not lead to orbit decay and re-entry, their effect. combined to that of the air drag may be substantial. Coupling between Earth oblateness and drag is discussed in (LP-28). Luni-solar attraction causes periodic radial perturbations (LP-17) and, for orbits with $e<0.25$ (LP-13), it may induce perturbations in perigee height, where the effect of drag is the highest. All orbital elements, except a, are affected by luni-solar attractions and their perturbations are usually small and periodic. However, high-eccentricity orbits are subjected to large oscillations in eccentricity (LP-13).

Because of the importance of the position of the perigee (where drag is usually the highest), its precession induced by the moon and, to a lesser extent, by the sun represents the major factor to be considered in this third-body perturbation problem.

This precession (proportional to a,e, and cosi) (LP-47), may be up to $0.18^{\circ}$ per year (Vanguard I) and thus, may be important for a long-1ife satellite.

The non-symmetric mass distribution of the Earth brings large secular variations in $\Omega$ and $\omega$ and small periodic perturbations in $e, i$, $\Omega$ and $\omega$ (LP-13). The perigee height is also affected by $J_{3}$ harmonics.

Clearly, drag forces are likely to increase as the radial distance of the satellite decreases (some exceptions occur when the satellite velocity decreases faster than the increase in density, see Lp-13 and Breakwell, J.V. (1962). Astronaut dal Sci. Rev. 4,21) because density increases. Therefore perigee height becomes a very important parameter. Actually, the importance of luni-solar attractions and geopotential perturbations comes from the fact that, in some cases, they can displace the perigee in some critical regions where the density
is higher (e.g. diurnal bulge) and decay rates in orbital elements would then increase.

Particular perturbationsare those which depend on the particular spacecraft concerned. They alter the effects that the environmental perturbations may have on the satellite motion. They are related to the physical aspect and the properties of the given satellite. Table II gives the most important parameters causing these particular perturbations.

TABLE II: PARTICULAR PERTURBATIONS
a) Finite satellite dimensions and non-symmetrical mass distribution
b) Non-symmetrical shape of the satellite
c) Large area-to-mass ratio
d) Variable mass
e) Ablation and break-up of the satellite.

The finite dimensions of a satellite and its non-symmetrical mass distribution bring about the problem of the attitude determlnation of an uncontrolled object in a decaying orbit. In this case, variable cross-sectional area (SP-3, SP-6), variable drag coefficient (SP-4) and gravity-gradients may lead to large trajectory dispersions. Along the same line of ideas, non-symmetrical shape may generate lift and increase dispersion. A third instance, which may be thought of as a particular case of the first one, occurs for balloon-type satellites which are more strongly affected by solar radiation pressure (LP-17). In this case, (LP-53), radiation pressure may be of the order of drag deceleration.

Decay rates may be increased when mass is lost as time goes on, slnce the drag force has then a greater effect. Variable mass of satellites may be caused by boiling-off of propellant (L.p-52), rupture of appendage due to meteoroid impacts, etc. Finally, the severe temperature conditions experienced by a re-entering satellite may cause loss of mass by ablation and break up (OA-2) is always a probable outcome. The dynamics of the airflow around the satellite and the
chemical reactions involved need thus to be analyzed in some detail. since it is known that temperature differences and Mach numbers greatly affect $C_{D}(S P-1, S P-2, S P-5)$. Errors in prediction are directly proportional to uncettainty in $C_{D}$ : the evaluation of $C_{D}$ and $A$ is thus critical (LP~52).

Although some discussion on satellite properties variations are included in this survey, most of the papers in the literature consider the satellite as a point mass with constant physical parameters and being large enough to experience atmospheric drag. Therefore, this survey will-mainly deal with the three important environmental. perturbations: airdrag, non-spherical mass distrlbution of the Earth and luni-solar attraction. Obviously this choice is pertinent: environmental disturbances may be analyzed in with general expressions while particular perturbations are dependent on the given satellite.

The following analysis will have three parts. First, a survey of the different mathematical models expressing the above mentioned perturbations will be discussed along with their conformity with the "real world". Then, a survey of the various analytical expressions for predicting orbit decay and lifetime will be included. It is worth noting that "analytical expressions" here exclude other methods like numerical integration of a particular orbit trajectory and satellite tracking. Finally, a general assessment of the state-of-the--art will be given, along with the weakness and the usefulness of the theories so far developed.
2. PERTURBATION MODELLING

Proper consideration of perturbing forces implies adequate mathematical modelling of the actual phenomenon in question. This rather challenging task has been carried out to some degree of success and it is the purpose of this section to bring into light the contribution of a few theorists.
a) The Geopotential

It is known that, due to the non-symmetric distribution of the Earth's mass, its potential field is not strictly of a $1 / r-f o r m$. In general, the potential is written.in terms of Legendre polynomials $\left(\mathrm{P}_{\mathrm{nm}}\right)$ :

$$
\Phi(r)=\frac{G M}{r}\left[1+\sum_{n=2}^{\infty}\left(\frac{r_{E}}{r}\right)^{r} \sum_{m=0}^{n}\left(C_{n m} \cos m \lambda+S_{n m} \sin m \lambda\right) P_{n m}(\sin \phi)\right](2.1)
$$

where dependence on longitude ( $\lambda$ ) and latitude ( $\varnothing$ ) is shown. $C_{n m}$ and $S_{n m}$ are the cosine and sine gravitational coefficients. Other forms are sometimes used, for instance (LP-1.2):

$$
\begin{align*}
\Phi(r)= & \frac{G M}{r}\left[1-\sum_{n=2}^{\infty} J_{n}\left(\frac{r_{E}}{r}\right)^{n} P_{n}^{(0)}(\sin \phi)+\right. \\
& \left.\sum_{n=2}^{\infty} \sum_{m=1}^{n} J_{n}^{(m)}\left(\frac{r_{E}}{r}\right)^{n} P_{n}^{(m)}(\sin \phi) \cos m\left(\lambda-\lambda_{n}^{(m)}\right)\right] \tag{2.2}
\end{align*}
$$

where $J_{n}, J_{n}{ }^{(m)}$ and $\lambda_{n}{ }^{(m)}$ are constants, specific to the Earth's potential.

The evaluation of the spherical ( $\mathrm{J}_{\mathrm{n}}$ ) and sectorial harmonics ( $\mathrm{J}_{\mathrm{n}}(\mathrm{m})$ ) has been done by many (King-Hele, Cook and Rees, (1963), Geophys. J. 8, 119; Newton, Hopfield and K1ine, (1961), Nature, London 190, 617; Kosai, Smithsontion Tnst: Astrophys. Observatory, Spechal Rept. 72 (1961)).

The dependence of the earth potential on longitude is usually neglected since the order of magnitude of this influence is largely smaller than the influence of drag. Then, the geopotential takes the following form:

$$
\begin{equation*}
\Phi(r)=\frac{G M}{r}\left[1-\sum_{n=2}^{\infty} J_{n}\left(\frac{r_{E}}{r}\right)^{n} P_{n}(\sin \phi)\right] \tag{2,3}
\end{equation*}
$$

The spherical harmonics $J_{n}$ represent the slight variation of the Earth's mass from spherical symmetry. The first term in the infinite series $\left(J_{2}\right)$ is the one that takes into account the oblate mass distribution of the Earth; $J_{3}$ stands for its "pear-shape", etc (LP-13).

From the values obtained by King-Hele, Cook and Rees, it is clear that:

$$
\begin{equation*}
\left|\frac{J_{n}}{J_{2}}\right|<0.003 \text { for } n \geqslant 3 \tag{2,4}
\end{equation*}
$$

and consequently, spherical harmonics of order greater than 2 are usually neglected and the most common geopotential model encountered in this survey considers the oblateness of the Earth only:

$$
\begin{equation*}
\Phi(r)=\frac{G M}{r}\left[1+\frac{3}{2} J_{2} \frac{r_{E}^{2}}{r^{2}}\left(\frac{1}{3}-\sin ^{2} \phi\right)\right] \tag{2.5}
\end{equation*}
$$

The effect of Earth's oblateness on the motion of satellite is a very well understood problem (L P-4, LP -33, LP-6, LP-28, LP-35, LP-36). For a more detailed analysis of geopotential perturbations of orbit eccentricity, Kosai (ref. 11 and 12 in LP-4), Croopnick (LP-34) and Guttman (LP-12) present good analysis.
b) Luni-Solar Attraction

The tidal forces cause oscillations in some orbital elements as well as precession of the orbital plane. Obviously, the greater the distance of the trajectory from the earth centre, the greater these attractions. The resulting perturbations are significant only for eccentricity larger than approximately 0.5.

A simple approach to this problem considers the sun and the moon as rings of matter around the Earth (L P-47), having the same total mass. Then, average torques on the orbit may be calculated by the gyroscopic theory and precession rates are obtained. Here, the precession rate must be small compared to the rotation of the disturbing bodies around the Earth.

A more complicated, but more accurate technique consists in including into the equations of motion the various attracting forces exerted by the Sun and the Moon (LP-38, LP -37).

Very often, however, theories deal with low eccentricity orbits and consequently, lunf-solar perturbations are excluded.
c) Atmospheric Drag

The atmospheric drag represents the most difficult orbital perturbation to model because it depends on many parameters whose properties and variations are usually known to a very poor accuracy.

Various mathematical forms representing the air drag force exist (LP-18, LP -44) but the more widely used is the following:

$$
\begin{equation*}
F_{D}=\frac{-1}{2} C_{D} A p V_{A}^{2} \tag{2.6}
\end{equation*}
$$

$$
\begin{aligned}
\left(v_{\mathrm{A}}=\right. & \text { velocity of air } \\
& \text { relative to, the } \\
& \text { satellite })
\end{aligned}
$$

and it follows that the drag deceleration is:

$$
\begin{equation*}
a_{0}=-\frac{1}{2}\left(\frac{C_{0} A}{m}\right) \rho V_{A}^{2} \tag{2.7}
\end{equation*}
$$

where $A$ is the cross-section area of the satellite. A is often called the projected area or the effective area of the satellite and represents the equivalent area normal to the flow. For simplicity in the equation, A is usually assumed to be constant and equal to some reference area $(A=A r e f$.$) appropriate to the satellite size. Then, C_{D}$ takes into account the changes in effective area so that the following is verified.

$$
\begin{equation*}
C_{0}=\frac{-F_{0}}{\frac{1}{2} p v_{A}^{2} A_{\text {ref }}} \tag{2.8}
\end{equation*}
$$

It is not an easy task to determine $C_{D}$, especially in orbital conditions. The evaluation of the density $P$ is an even more complicated task. Fur thermore, since the Earth's atmosphere is not stationary, the proper value of $v_{A}$ in equation 2.7 is usually different from the satellite's velocity and discrepancies must be considered.

The following sections will describe various approaches to the evaluation of these three parameters; density ( $p$ ), relative velocity $\left(\mathrm{V}_{\mathrm{A}}\right)$ and drag coefficient $\left(\mathrm{C}_{\mathrm{D}}\right)$.
i) Density

In general, the density is dependent on the following entities:

$$
\begin{equation*}
p=p(h, E, S) \tag{2.9}
\end{equation*}
$$

The symbols used in 2.9 are described here:
$h=a l t i t u d e$ above Earth's surface
$E=E a r t h ' s$ figure and rotation
$S=$ Sun $^{\prime} \mathrm{s}$ radiations
These dependences will be evaluated separately.
Dependence on Altitude
Due to the attraction of the Earth, it is obvious that the density of the atmosphere will vary with altitude in a more or less predictable fashion. However, when changes in altitude are small for a particular analysis, the density may be considered constant (LP-22). In order to improve this approach, lst order approximation may be obtained by assuming the atmospheric density constant and then, using the principle of variation of parameters, $\rho$ may then be allowed to vary with altitude (LP-20).

A simple evaluation of the density as a function of altitude is detailed here.


Consider a vertical cylinder of fixed cross-section area $A$ and the atmosphere as a perfect gas and isothermal. Then:

$$
\begin{align*}
P V & =k m  \tag{2.10}\\
P & =k p \tag{2.11}
\end{align*}
$$

The difference in pressure between level (1) and level (2) is the pressure caused by the infinitesimal weight of the air inside the volume element Adh. Then,

$$
\begin{equation*}
d P=-\frac{\text { weight }}{\text { Area }}=-\frac{(p A d h) g}{A}=-\rho g d h \tag{2.12}
\end{equation*}
$$

where $g$ is assumed constant ( $9.8 \mathrm{~m} / \mathrm{sec}^{2}$ ) for the range of altitudes where atmospheric density is nonmegligible. Using 2.11, we have:

$$
\begin{equation*}
\frac{d p}{p}=-\frac{g}{k} d h \tag{2.13}
\end{equation*}
$$

which yields the exponential model of the variation of density with altitude:

$$
\begin{equation*}
p=p_{0} e^{-\beta\left(h-h_{0}\right)} \text { or } p=p_{0} e^{-\beta\left(r-r_{0}\right)} \tag{2.14}
\end{equation*}
$$

where $\beta, P_{o}$ and $r_{o}$ are matching coefficients, $\beta$ being the reciprocal of the "scale-height". This derivation is in its simplest form. However, more detailed derivations yield the same result (Eq. 2:14) but with a different value for $\beta$.

Using standard atmosphere models (like ARDC, CIRA, Jac chia) the constant $H, P_{o}$ and $r_{o}$ may be selected so that equation 2.14 fits the experimental curve. Occasionally, the atmosphere is broken up in say, $n$ layers. Within each of these layers, the matching coefficients are fided and the $i^{\text {th }}$ set of coefficients apply as long as the satellite is in this particular $i^{\text {th }}$ layer. The "layer technique" brings a greater conformity to the real atmosphere at the expense of numerical clumsiness. Therefore, the exponential model of density variation is largely used without this "layer technique" (LP-5, LP-6, LP-19, LP-43, LP-28, LP-31, LP-13, LP-14, LP-15, LP-16 and many others). King-Hele (LP-13) avoided this technique by considering variations in the coefficient $H$ with respect to altitude. In the COSPAR International Reference Atmosphere (ref: Kallmann-Bijl, et al., CIRA 1.961, Amsterdam: North-Holland) it is shown that the variation of: $H$ with altitude is approximately linear:
 $h=400 \mathrm{~km}(\mathrm{LP}-13)$.

However, varying scale-height complicates matters since $H$ appears in the exponential term. King-Hele avoided this complex situation by assuming the form:

$$
\begin{equation*}
p=\rho_{0}\left[1+b\left(r-r_{0}\right)^{2}\right] \exp \left[-\frac{\left(r-r_{0}\right)}{H_{0}}\right] \tag{2.15}
\end{equation*}
$$

where 'o' denotes reference (usually perigee) values and $b$ and $H_{o}$ are constant over one orbital revolution, but $H_{o}$ is of the form $H=H_{i n i t i a l}+0.1\left(r-r_{i n i t i a l}\right)$. This approach gives relatively good results for altitude variation of density in a spherical atmosphere. It must be noted that equations 2.14 and 2.15 are quite useful when differential equations including $\rho$ must be integrated: they lead to the Bessel functions of imaginary argument, as shown by King-Hele (LP-13).

Apart from the exponential model, there exists another law which is currently used in orbit analyses. It gives good fits of the real variation in air density with altitude and can al so be employed under a "layer technique" form. It is called the "power-law" model of the form:

$$
\begin{equation*}
p=\rho_{0}\left(\frac{r_{0}-r_{E}}{r-r_{E}}\right)^{\eta} \tag{2.16}
\end{equation*}
$$

$$
\eta=\text { constant to be determined. }
$$

Due to the assumptions made in deriving the exponential law (Michielson, ref. 2 in LP-18), equation 2.14 is strictly valid for low altitudes whereas the power-law model is accurate over a wider range of altitude. According to Billik (LP-18), comparisons between 2.14 and 2.16 favour this last model. A table of $\eta$ vs. altitude is given in (LP-18).

Introducing the altitude $h=r-r$ into 2.16 one gets:

$$
p=p_{0}\left(\frac{h_{0}}{h}\right)^{n}, \quad h_{0}=r_{0}-r_{E}
$$

or

$$
\begin{equation*}
p h^{\eta}=\rho_{0} h_{0}^{\eta}=\text { constant } \tag{2.17}
\end{equation*}
$$

Equation 2.17 is similar to the atmosphere model used by Egorov (L P-1); He used:

$$
\begin{equation*}
\rho r^{\alpha}=\rho_{0} r_{0}^{\alpha}=\text { constant } \tag{2.18}
\end{equation*}
$$

where

$$
\alpha=\frac{g r}{(d P / d \rho)}=\text { constant }
$$

Finally, as a particular case of the power-law model, the atmospheric density may take the form:

$$
\begin{equation*}
p=k(\lambda+\sigma / r) \tag{2.19}
\end{equation*}
$$

where $k=2 m / \Lambda C_{D}$ and $\lambda, \sigma$ are matching coefficients. Equation 2.19 has the advantage of being very simple and easy to handle in mathematical manipulations. However, its accuracy is acceptable in very small range so that the "layer-technique" has to be used for some realistic prediction.

As a last comment to this sub-section, it must be pointed out that any theory that gives lifetime as a function of an integral of the form:

$$
\begin{equation*}
\int_{r_{0}}^{r} \rho f(r) d r \cdot o r \int_{r_{0}}^{r} \frac{f(r)}{p} d r \tag{2.20}
\end{equation*}
$$

(where $f$ is any function of the radius $r$ ) is bound to be quite acceptable since integral 2.20 can be tabulated using experimental data and consequently, the modelling inaccuracies introduced by Equations 2.14, 2.16 or 2.19 are avoided.

Apart from.altering the value of $\mathrm{v}_{\mathrm{A}}$ in equation 2.7 (discussed in sub-section ii on velocity), the atmospheric rotation, induced by the Earth's rotation, has the effect of "displacing" local density profiles in a west-to-east direction. This rotation causes the "diurnal bulge" to lag some $30^{\circ}$ behind the sub-solar point (see Sun's radiations effects).

The geometrical shape of the Earth is not spherical but, at the first approximation, oblate with flattening $f$

$$
\begin{equation*}
f=\frac{r_{E}-r_{P}}{r_{E}} \simeq \frac{1}{298.25} \tag{2.21}
\end{equation*}
$$

where $p$ stands for polar value.
Since the air density is dependent on the altitude above the Earth's surface, according to the preceding section, it follows that on a sphere around the Earth, the density is greater at the equatorial plane than at the poles since altitudes would be larger at the latter region. Assuming the atmospheric flattening to be the same as that of the Earth, the radial distance $\sigma$ from the Earth's constant centre to the oblate spheroid of contact density can be written in the form (LP-13):

$$
\begin{equation*}
\sigma=\sigma_{E}\left[1+f \sin ^{2} \phi+o\left(f^{2}\right)\right] \tag{2.22}
\end{equation*}
$$

$$
\begin{aligned}
\sigma_{E} & =\text { equatorial radius of the oblate spheroid } \\
\emptyset & =\text { geocentric latitude }
\end{aligned}
$$

According to King-Hele, the $O\left(f^{2}\right)$ term contributes to less than 0.03 km , and can be neglected. Tncluding this radial distance in an atmospheric model would then take into account the oblate figure of the atmosphere.

Another means of getting around this problem is to introduce in the equations a correcting factor $F_{A O}$ for atmospheric oblateness defined as (L P-24):

$$
\begin{equation*}
F_{A O}=\frac{1}{2 \pi} \int_{0}^{2 \pi} p_{o} / p_{s} d u \tag{2.23}
\end{equation*}
$$

where

$$
\begin{aligned}
& P_{0}=\text { density model for oblate atm } \\
& P_{s}=\text { density model for spherical atm. }
\end{aligned}
$$

This factor is mainly useful for circular orbits in which case it takes the form:

$$
F_{A O}=1-K / 2+3 K^{2} / 16-15 K^{3} / 288+\ldots
$$

$$
\text { where } \quad K=\frac{r_{f} f_{5 n^{2} i}}{H}
$$

$$
\begin{aligned}
& H=\text { scale -height } \\
& i=\text { orbital inclination. }
\end{aligned}
$$

However, the constant density spheroid introduced by KingHell seems to be the most adequate procedure in dealing with the oblateness influence. It is consequently widely used (L P-3, LP -4, for instance).

Dependence on the Sun (Heating \& Radiation)

King-Hele (LP-13) gives a very complete outline of the main effects of the Sun's activity upon atmospheric density, which become non-negligible at height above 180 km . A brief summary of these influences is shown in Table III.

CAUSE

Diurnal heating
Axial rotation of the Sun

Elliptic trajectory of the Earth around the Sun and Inclination of the Earth's orbit to the plane of the Sun's equator

Sun's activity, number of sunspots

Flares, unknown (LP-13, LP-40) (LP-13, LP40)

EFFECT

Diurnal bulge, winds (AP-26)
Periodic (27 days) variations in $\rho$.

Semi-annual density variations (AP-10). (AP-10)

Large (10-11 year) variations in $\rho$. (LP-52)

Unpredicted variations in $P$.

Each of these effects will be roughly described now.

DIURNÁL BULGE (AP-6, AP-19, AP-26)

The constant heating of the atmosphere by the sun causes the former to expand in a big hump on the sunlit side of the Earth. This "diurnal bulge", as it is often referred to, may be as high as 100 km at $500 \mathrm{kam} . a l t i t u d e ~(L P-13)$ and causes the densities to be different as one leaves the dark side of the earth toward the sunlit side on a constant altitude trajectory. This effect becomes appreciable at heights above 250 km . For average solar activity, the ratio $f=P_{\text {day }} / P_{\text {night }}$ on the constant altitude surface may reach the factor of 8 (LP-13). Due to the rotation of the atmosphere the bulge is not situated directly below the sun's position but lags about 25 to $30^{\circ}$ (LP-52). Therefore, the maximum air density at a given altitude occurs around 14 h . local time. The declination of the bulge centre, however, is the same as that of the sun.

The diurnal bulge has small effects on high eccentricity orbits but it must be considered for all near-earth satellite with e<0.2.

Jacchia (in LP-52) gave an empirical equation expressing the variation of the atmospheric density due to the diurnal bulge:

$$
\begin{align*}
& P=P_{0}(z)\left[1+0.19\left(e^{0.0055 z}\right.\right.  \tag{2.25}\\
&\left.=1.9) \cos ^{6} \frac{\phi}{2}\right] \\
& P_{0}(z)=\text { altitude dependence of } p \\
& z=\text { altitude } \\
& \emptyset=\text { anginlar distance from the bulge. }
\end{align*}
$$

Equation (2.25) supposes symmetry about the Earth-bulge centreline. Comparison of 2.25 to the 1959 APDC atmospheric model (LP-52) shows that it is in relatively good agreement with actual densities for the range $1.17 \mathrm{~km}<z<700 \mathrm{kn}$. For $\mathrm{z} \doteq 117 \mathrm{~km}$, the factor ( $e^{0.0055 z}-1.9$ ) equal zero and below this value of $z$, the heating effect of the sun is distributed on both dark and sunlit sides of the Earth. The diurnal bulge effect is then absent.

A more theoretical approach was made by Cook and King-Hele (LP-16). They assumed a sinusoidal variation of the density with angular displacement from the bulge centre. Including the altitude variation, they proposed:

$$
\begin{equation*}
p=p_{0}(1+F \cos \phi) \exp \left[-\left(r-r_{0}\right) / H\right] \tag{2.26}
\end{equation*}
$$

where $F$ and $H$ are constant at the first approximation and $\emptyset$ demotes the angular distance from the bulge centre. $F$ was taken as:

$$
\begin{equation*}
F=\frac{f-1}{f+1} \quad \text { where } \quad f=\frac{p_{\max }}{\rho_{\min }}=\frac{p_{\text {OAF }}}{p_{\text {NIGHT }}} \tag{2.27}
\end{equation*}
$$

From 2.27, F may be thought of as the eccentricity of the ellipse obtained by the intersection of a plane (containing the Earth-bulge centreline) and the constant diurnal density surface. In fact, $F$ can be written:

$$
\begin{equation*}
F=\frac{P_{\max }-P_{\min }}{P_{\max }+P_{\min }} \tag{2.28}
\end{equation*}
$$

which is similar to the eccentricity of the orbit of a satellite where:

$$
\begin{equation*}
e=\frac{r_{A P}-r_{P E}}{r_{A P}+r_{P E}} \tag{2.29}
\end{equation*}
$$

It is quite reasonable, as King-Hele and Cook put it (LP-16), to conclude that the importance of the diurnal bulge effect grows with the ration $F / Z$ where $Z$ is directly proportional to the orbit eccentricity. Like Eq. $2.25,2.26$ assumes the horizontal cross-section of the bulge to be circular. It is know, however, that the diurnal bulge does not show such a symmetry and its variation with latitude has been shown to be larger than it was expected a few years ago (AP-28).

A second-order approximation (LP-16) could be as follows:

$$
\begin{equation*}
p \sim 1.8+4.8 \cos ^{4} \frac{\phi}{2} \tag{2.30}
\end{equation*}
$$

which gives a better fit than 2.26. However, mathematical simplicity is lost. Therefore, equation 2.26 gives an acceptable expression which takes consideration of the bulge while keeping the complexity of mathematical manipulations to a minimum. Thus; its use is widely spread (LP-10, LP-4, L.P-3). Daily variations in the solar radiatifons also induces variations in winds above 200 km ( $\mathrm{AP}-26$ ). These are usually neglected.

## ROTATION OF THE SUN

Experimental data on the orbital acceleration of artificial Earth satellites showed periodic oscillations in their motion. The period was roughly 27 days and it was then assumed that the cause was the Sun's rotation (LP-26). As explained in (LP-13), particular sources on the Sun $^{\text { }} s$ surface reappear every 27 days (the period of rotation of the Sun with respect to the Earth). These particular sources affect the atmospheric densities so that satellite orbits are perturbed.

This 27-day recurrence of solar activity is not exact and is fallible (LP-13) and thus no mathematical modelling of this phenomenon is discussed.

## SEMI-ANNUAL CHANGES

The semi-annual variations of the atmospheric density correspond to two superimposed effects (LP-13). First, the variable distance from the Earth to the Sun (maximum in July) causes low densities in early Jul.y (LP-13). Second, the inclination of the Earth's orbit ( $7^{\circ}$ ) to the plane of the Sun's equator induces increasing density every 6 months, when the Earth crosses this plane. For a more detailed analysis, the reader is referred to Cook (AP-10). This variation is usually ignored for short term predictions. For long term predictions, average value of the density over one year are used.

## SUNSPOTS NUMBER

Sunspots, which show periodic fluctuations in their number, are known to have a ruajor influence on the density of the atmosphere (AP-6, LP-52, LP-13). The ratio of densities during high solar activity to those of low activity can be quite high; up to 100 at 600 km altitude (LP-13). The variation in predicted lifetime may be affected by $-60 \%$ to $150 \%$ (LP-52) of the nominal value. However, the period of this activity is 10 to 11 years. Short term predictions are usually not affected.

The degree of the activity of the Sun and the number of sunspots are very difficult to predict and scientists must rely. on past cycles and extrapolate these data to get an acceptable prediction of solar activity. Even then, large inaccuracies are still present: and sometimes, they lead to drastic and unexpected oritit contraction (LP-45).

Finally for long-term predictions, graphical methods proved to be quite acceptable in dealing wisth the ll-year variations of density (LP-49, LP-52).

## SPORADIC VARIATIONS

Even if all the above parameters influencing density were taken into account, unpredicted density variations occur daily and locally around the Earth's sphere. These are mainly due to ephemeral solar disturbances ( $L P-\frac{13}{37}$ ) and solar flares (LP-40). These variations are usually smaller than $10 \%$ but may sometimes reach large values when large flares occur (LP-13). These spolfic variations are the most unpredictable fluctuations in air density and, at the same time, the ones that may lead to the largest trajectory dispersions. Obviously, a deterministic approach to this problem isivain, and one must then rely on a stochastic approach (LP-50).

As a concluding remark to this section on atmospheric models, one must be aware of the fact that, the best model to the atmosphere that can be obtained by combining such expressions as Eqs. 15; 16, 22 and 26 still contain non-negligible discrepancies but, what is more important, it becomes in some cases largely inaccurate when unpredictable random fluctuations of the density occur. It must then be kept in mind that any conclusion on the probable life duration of a satellite has inherent in it, a probable large degree of inaccuracy. While the orbit determination theory is very well developed few analysis including the random behaviour of density profiles can be obtained from the current literature.

The upper-atmosphere of the Earth is not stationary but rotates at about the same angular velocity as the Earth (AP-21-, $\mathrm{AP}-4, \mathrm{AP}-22, \mathrm{AP}-26)$. Let us define the ratio:

$$
\begin{aligned}
& \Lambda=\frac{W_{A}}{W E} \\
W_{A} & =\text { angular velocity of the atmosphere } \\
W_{E} & =\text { angular velocity of the Earth. }
\end{aligned}
$$

The numerical value of $\Lambda$ can be determined by orbit analysis, (see AP-4, AP-22, AP-26, AP-29, LP-10, LP-13; AP-21) since the rotational motion of the atmosphere causes the orbit inclination and the right ascension of the node to vary. It can be shown (LP-10, LP-13) that $\Delta i$ and $\Delta \Omega$ are directly proportional to $\Lambda$.

King-Hele and Scott, in (AP-22), obtained a mean value for :

$$
\begin{equation*}
\Lambda_{\text {mean }}=1.27 \pm 0.18 \tag{2.32}
\end{equation*}
$$

for height near 200 km to 300 km . This corresponds to an average west-to-east wind of about $100 \mathrm{~m} / \mathrm{sec}$ in mid-1atitudes. However, $\Lambda$ as a slight dependence on altitude, as shom by King-Hele, Scott and Walker (AP-4). Their results are in Table IV.

TABLE IV: VALUES OF $\Lambda$

| $\Lambda$ | Height (km) |
| :--- | :--- |
| 1.1 | 200 |
| 1.25 | 250 |
| 1.4 | 350 |

with standard deviation ranging between 0.05 ( 200 km ) and 0.1 ( 350 km ), (see the Table in AP-4).

Due to the importance of the effect of a rotational atmosphere $(\mathrm{AP}-21, \mathrm{AP}-29, \mathrm{LP}-10), \mathrm{IP}-13)$, it is then required to include the proper value of $v_{A}$ in Eq. 2.7. The most expeditive way of considering this problem has been worked out by king-Hele (LP-13). Assuming the path of the satellite to be horizontal (at perigee) when the air rotation has its influence, he showed that the inclusion of a factor $F$ in the drag equation 2.7 approximate the effect of rotation:

$$
\begin{equation*}
a_{D}=-\frac{1}{2} \rho \mathrm{v}^{2} F C_{D} A \tag{2.33}
\end{equation*}
$$

where $v$ is now the satellite velocity and $F$ is a constant

$$
\begin{equation*}
F=\left(1-\frac{r_{p O}}{V_{P O}} W_{A} \cos i_{0}\right)^{2} \tag{2.34}
\end{equation*}
$$

Fis usually in the interval $[0.9,1.1]$.

Obviously, a more general approach would include the actual velocity of the spacecraft relative to the ambient air in Eq. 2.7. In t!is case, the problem becomes greatly complicated since $v_{A}$ depends on the orientation of the velocity vector of the satellite with respect to the west-to-east direction of the atmospheric' winds.

Sporadic winds may also influence the trajectory dispersion in high altitudes. At low altitudes, the velocity of the satellite is large compared to these winds and may thus be neglected.

## iii) Coefficient of Drag CD-

A rigorous mathematical approach to the evaluation of the drag coefficient is probably the most complicated task involved in orbit determination theory. Probably due to the fact that $C_{D}$ depends on so many parameters, few theoretical determination of $C_{D}$ exists while a lot of experimental work has been done. Our poor knowledge of the actual surface interaction taking place is another
reason explaining this relative ignorance of analytical expressions for $C_{D}$.

Unfortunately, the evaluation of a proper drag coefficient can not be left aside. Flow conditions experienced by a decaying satellite vary greatly when it goes fron a sustained orbit to reentry, causing fluctuations in $C_{D}$. Between 120 km and 160 km , the mean free pathe of the air molecules may vary by a factor of 25 and, depending on the dimensions of the satellite and other parameters, the conditions may be that of a free-molecular flow or transition ("slip") flow (LP-13). $C_{D}$ is consequently affected. Variations in the drag coefficient lead to large dispersions in the trajectory of a satellite (LP-28, SP-4). Comparisons of solutions with variable and constant drag coefficients show (SP-4) this importance. For stabilized spacecraft $C_{D}$ may vary by a factor of 5 or 6 (LP-52) which is carried out in the orbital lifetime predictions.

In the general case, the coefficient of drag depends on the following:
-The properties of the surrounding medium (temperature, density, chemical properties and composition, velocity .of the particles).
-The nature of the interaction between the surface of the body and the medium.
-The chemical composition of the body, its surface lattice structure.
-The shape, the dimension and the attitude motion of the body.
-The velocity of the medium relative to the body.

The properties of the surrounding medium, espectially its flow conditions, greatly affects $C_{D}$ (SP-1, SP-2, SP-5). These properties have their importance when one studies the nature of the interaction between the medium and the body. The mechanism of this interaction is the grey area in the analysis. Two extremes

- Specular reflection or mirror-like reflection where only the normal linear momentum is altered and an impulsion normal to the surface of the body is transmitted.
- Diffuse reflection in which case the incident particle is absorbed by the surface and transfers all of its momentum to the body. After a period of time (usually unknown), the particle is re-emitted in a direction completely independent of the fncident direction with a velocity proportional to the temperature of the surface it is leaving. We then speak of the "accomodation" of the incident particle to the surface of the body.

Although the exact definition of the interaction is practically unknown, it is recognized that something like a mixture of both cases is occurring. In other words, we usually have partial accomodation. As stated above, the work involved in this area is matnly experimental and empirical. Three parameters are usually used in this empirical approach (LP-52). These are:

- Thermal or energy accomodation coefficient ( $\alpha$ )
- Normal ( $\sigma^{\prime}$ ) and tangential ( $\sigma$ ) momentum accomodation coefficients.

They are defined as:

$$
\begin{equation*}
\alpha=\frac{E_{i}-E_{r}}{E_{i}-E_{w}} \quad \sigma^{\prime}=\frac{P_{i}-P_{r}}{P_{i}-P_{w}} \quad \sigma=\frac{T_{i}-T_{r}}{T_{i}} \tag{2.35}
\end{equation*}
$$

where
$E=$ energy of the particles
$P=$ normal momentum of the particles relative to the surface
$J^{\prime}=$ tangential momentum of the particles rel. to the surface
$i=$ incident particles
$r$ " $=$ re-emitted particles (actual value)
$\mathrm{W}=$ re-emitted particles (value when equilibrium with the body surface is reached).

In air, experimental work (LP-52) shows that the accomodation coefficient is very close to unity but for other gas mixtures, it may be as low as 0.1. A large degree of uncertainty exists. Ladner and Ragsdale (LP-52) give graphs of the value of $C_{D}$ with the uncertainty range:

$$
\begin{aligned}
\sigma^{\prime} & \approx 1.0 \\
0.85<\sigma & <1.0 \\
0<\alpha & <1.0
\end{aligned}
$$

In the same graphs, the dependence of $C_{D}$ on altitude is also shown.

The importance of the chemical composition of the body and the nature of its lattice structure is also stressed in Ladner and Ragsdale's paper.

The dimensions and the shape of a satellite influences the drag coefficient as well. As an example (LP-13), a satellite lm. long at a given altitude in free molecular flow may find itself in transition flow conditions where its dimensions increased to 50 m . With constant reference area $A_{r e f}$, the attitude motion of the satellite will present different cross-sectional area to the incident flow and $C_{D}$ will vary accordingly ( $\mathrm{SP}-3, \mathrm{SP}-6$ ).

Finally, the velocity range greatly affects the drag coefficient as shown in (SP-1), (SP-2), (SP-5). A mathematical form used by Hunziker (SP-4) is the following:

$$
\begin{equation*}
C_{D}=C_{D_{0}}+k \frac{v_{S}}{V} \tag{2.38}
\end{equation*}
$$

$$
\begin{aligned}
& C_{D_{0}}=\text { constant } \\
& K=\text { parameter (matching coefficient) } \\
& \mathrm{V}_{\mathrm{S}}=\text { air thermal speed } \\
& \mathrm{V}=\text { satellite speed. }
\end{aligned}
$$

Apart from Ladner and Ragsdale, other analytical approaches to the determination of $C_{D}$ are given (Cook (LP-8), Nocilla (LP-42)). along with values of $C_{D}$ for different shapes. They all explain our poor knowledge of the actual surface interaction mechanism. Williams shows how to consider the attitude motion of a satellite in determining $C_{D}(S P-3, S P-6)$. The basic steps are:
a) Find (experimentally) the curve of $C_{D}$ vs. the angle of attach $\alpha$
b) Fit this curve with a series of the form:

$$
\begin{equation*}
C_{0}(\alpha)=\sum_{j=0}^{N} b_{j} \cos ^{j} \alpha \tag{2.39}
\end{equation*}
$$

c) Knowing the attitude history of the satellite, find $\alpha$ as a function of the orbital elements
d) Find $\left\langle C_{D}\right\rangle$, the averaged value of $C_{D}(\alpha)$ over one complete revolution of the satellite.

Although $\left\langle C_{D}\right\rangle$ represents a very acceptable value to be used in the equations, this method supposes some geometrical
symmetry and the knowledge of the attitude of the spacecraft as time goes on. The first assumption is very often verified while the second, in our case, is invalid: our starting assumption was the consideration of uncontrolled satellites. However, this may represent a simplification, rather than an unsolvable problem, since most of the uncontrolled satellites would tumble at a constant rate about the axis of largest moment of inertia. An average of the two extreme cases should then be acceptable. This averaging technique is valid for most of the satellites launched so far (see LP-13 for exception, like Tiros). For tumbling satellites, the reference area, $A_{r e f}$, is usually taken as the total area divided by 4 (LP-52).

The best method of evaluation of $C_{D}$ remains the experimental technqiues. In this case, different values of $C_{D}$ are obtained for different conditions. However, for shapes like cylinders, spheres. and cones (which roughly represent the possible shapes of most of the satellite), the normally accepted values gathered in the literature are quite the same:

$$
\begin{aligned}
& \text { Shamberg: } C_{D}=2.5 \pm 20 \% \quad \text { (Ref. } 12 \text { in LP-52) } \\
& \text { King-Hele: } \quad C_{D}=2.2 \pm 5 \% \quad \text { (LP-13) } \\
& \text { Cook: } \quad C_{D}=2.12 \text { (average, for Sputnik 3) (LP-8) }
\end{aligned}
$$

The mathematical modelling of the drag coefficient is still at an early stage of development and a lot of research has still to be done in this area. Fortunately, in our case, apart from some odd cases, the fluctuations in the averaged value of the coefficient of drag for a tumbling satellite are usually small for the conditions encountered in a decaying orbit. Variations in $\left\langle C_{D}\right\rangle$ need not be included in first order approximations. In any case, an extremely precise value of $C_{D}$ is useless when the density may vary by a factor of $10 \%$ in an unpredicted way.

We are now well prepared to consider the various ways of predicting the orbit contraction and lifetime of Earth satellites. However, a short discussion on the main "mathematical tools" utiilized by theorists must be first inserted here. A few words ant the various "analytical approaches" will also be included to present the outline of this section.

## MATHEMATICAL "TOOLS"

Obviously, the equations of motion for a satellite subject to air drag and gravitational perturbations have no closed-form exact solution. Only approximate expressions can be obtained at the expense of accuracy and/or range of application. This brings about the necessity of some mathematical simplifing manipulations in order to get an approximate closed-form expression of the solution. It is the purpose of this sub-section to present the most common tools encountered in this survey.

It is important, at this point, to repeat the fact that the discussion of tracking and numerical integration of a particular orbit are not included here. However, graphical methods of predicting Iffetime are part of this report, even if they require some complicated numerical analysis. The difference between graphical methods and numerical integration of a particular orbit lies in the fact that the former are applicable to any satellite (which make them similar to analytical closed-form expressions) whereas, the latter is used for a particular satellite under given conditions and must be repeated for any other orbit. Furthermore, orbit integration by numerical analysis represents a tedious, and costly way of predicting lifetimes.

The following represent the various analytical tools used by most of the theorists.
i) Neglect of small order terms
ii) Orbital averaging
iii) Asymptotic expansion in nonlinear mechanics
iv) Perturbation (variational) solutions
v) Variation of parameters.

The first of this short list is the most widely used. In fact, terms in $e^{n}, n \geq 2$ for small eccentricity orbits are usually neglected (LP-13); for near-circular orbits, $v^{2}$ is usually taken as $\mu / r(L P-46)$; quantities like $p, a, e, ~ e t c$. are often assumed constant over a small interval of time (usually the period T) (IP-13, LP-20, LP-1) and so ón. Orbital averaging techniques (MA-3, MA-6) give the mean motion of a satellite and, by their very nature, fail to give oscillations within orbital revolutions (LP-13, LP-3, LP-4). The averaging technqiues is also known as the $K \cdots B$ method (Kryloff and Bogoluiboff). The asymptotic method is an extension of the $K-B$ method by Bogluiboff and Mitropolsky. It is a powerful tool especially shaped for celestial mechanics problems (Ref. 9, of LP-31). Zee used it in many papers ( $\mathrm{L} P-33, \mathrm{LP}-5, \mathrm{LP}-6, \mathrm{LP}-37, \mathrm{LP}-38$ ) and Brofman used a modified version of it (LP-31). This method gives good lst order approximation expressions. Linearization about a reference solution can also give acceptable solutions for a small range of applicability ( $L P-20$ ), and so is the variation of parameters technqiue. An unusual technique, by Morduchen and Volpe (LP-27) could be called a "reversed process": instead of using an atmospheric model and solving the equations of motion in order to get the trajectory, they supposed a trajectory and, by mathematical manipulations, they derived an expression for the required atmospheric model that would result in their "trial trajectory". This technique has the advantage of giving exact solutions but the atmospheric models obtained may not fit the actual atmospheric conditions. Some trajectories gave relatively good models.

## ANALYTICAL APPROACHES

Theories developed in the literature survey tend to approach the oxinital decay prediction in three diflirent way, according to their "analytical approach"x to the problem. For clarity in the upconing discussion, the following three classes will be analyzed separately:

FFootnote bottom of Pg .29
a). Osculating Orbital Elements Approach
b) Spatial Curve Approach
c) Graphical Approach.
(a) Osculating Orbital Elements Approach

Thís method is based on the Lagrange Planetary equations which give the rate of variation in the orbital elements of an orbit due to the perturbing forces acting on the satellite. The minimur time interval considered is the orbital period $T$, within which, all fluctuations in the orbital elements are assumed to be very small. Ideally, changes of the orbital elements are obtained by fntegration of these equations, when the proper perturbation models has been inserted first. The lifetime (from now on denoted by $t_{L}$ ) is usually taken as any one of the following conditions:

> Condition (a) Time required for e to be zero.
> Condition(b) Time required for $r_{p}<r_{c}$ where $r_{c}$ is a critical perigee radius ( $120-150 \mathrm{~km}$ or $\mathrm{r}_{\mathrm{E}}$ )
> Condition (c) Time required for T to be $\sim 87 \mathrm{~min}$. (equivalent to $\left.a_{L} \sim 6510 \mathrm{~km},(L P-13)\right)$.

Condition (a) is mainly used for elliptic orbits and (c) for circular orbits. Condition (b) may be used in either case.

It can be seen from these conditions that $\Delta \mathrm{a}$ and $\Delta \mathrm{e}$ as function of time $t$ (or number of revolutions $N$ ) are important to be known. From $\Delta a$ and $\Delta e, \Delta r_{p}$ and $\Delta T$ can be obtained and conditions

* In the following sections of this report, "techniques" or "tools" will refer to any mathematical manipulatious used in simplifying the equations, (i) to (v) above. "Approach" will refer to any one of the three above mentioned classes ( $\langle, b$ and $c$ ) and "nethods" will refer to: tracking, numerical integration (of a particular orbit) and analytical methods. Clearly, the three classes of approach just mentioned constitute this last. method.
( $a, b$ or $c$ ) are applied for lifetime prediction. The orbital elements a and $e$ are the two variables that are directly affected by air drag: from this fact comes their importance. Other elements ( $\Omega, \omega$, i) vary at slower rates and are also affected by gravitational perturbations. In many theories, they are assumed as constant and only planar considerations of the orbit decay follows.

The various theories developed on the grounds of this analytical approach will now follow. The logical sequerice used goes from the simplest case to the more conform theory to actual conditions.

The simplest case in treating decaying orbits assumes a stationary, spherically symmetric atmosphere with exponential density distribution (equation 2.14) and a simple $1 / r$ potential field. Lunisolar perturbations are excluded.

For small eccentricities, Parkyn (LP-43) obtained the $\quad . \dot{c}$ average rates of change for $a$, $e$ and $r_{e}$ and showed that when $e \doteq 0$, 5 or 6 hours were left in the satellite lifetime. His approach gave an upper limit for the lifetine which may include up to $25 \%$ error.

For intermediate eccentricities (when $P$ (apogee) $\neq 0$ and $P$ (APOGEE) $\neq P$ (PERIGEE)), Newton (LP-23) used the simple atmospheric model given by Eq. 2.19. Through variation of parameters, he fitted an ellipse to the decaying orbit and derived the equations for $e$, $a$, and $\omega$ describing this ellipse which has the form:

$$
\begin{equation*}
\frac{1}{r}=k[1+a \cos \theta-b \sin \theta]=k[1+e \cos (\theta+\omega)] \tag{3.1}
\end{equation*}
$$

where $k$, $a$ and $b$ are functions of $\mu, \theta$, $h_{u}^{u}$ and $e_{u}$ where $h_{u}$ represents the angular momentum for the unperturbed ellipse and $e_{u}$ its eccentricity. The atmospheric matrking coefficients, $\sigma$ and $\lambda$ (Eq. 2.19), are also included in $a, b$ and $k$. Holding allı quantities constant except $\theta$, herived the rate of change of the eccentricity, the semimajor axis and the argument of perigee of the ellipse that best fits the decaying orbit:
$d a / d \theta=-2 \lambda a-\left[2 \sigma a\left(1+e^{2}\right) /\left(1-e^{2}\right)\right]$

$$
\begin{aligned}
& d e / d \theta=-e\left[\lambda a\left(1-e^{2}\right)+2 \sigma\right] \\
& d \omega / d \theta=-\left[(2+e)\left(\lambda h_{u}^{2} / \mu\right)+2 \sigma\right]\left[\lambda h_{u}^{2} / \mu+2 \sigma\right] / e
\end{aligned}
$$

This derivation is useful because of its simplicity. The decaying trajectory is well defined. However, the atmospheric model is too simple to be realistic for long-range predictions. It also shows that the decay in the semi-major axis is proportional to the density at the average altitude (a) rather than at perigee attitude. In fact, neglecting $e^{2}$ terms in the first of 3.2 yields:

$$
\begin{equation*}
d a / d \theta=-2 a^{2}(\lambda+\sigma / a)=-2 a^{2} \rho(a) \tag{3.3}
\end{equation*}
$$

Still in this simple situation where the atmospheric density is spherically symmetric and exponentially decreasing with a $1 / r$ - potential field, King-Hele did a tremendous work (L P-13) in his book: "Theory of Satellite Orbits in an Atmosphere". The basic steps in this book are:
i) Use equations forda/dt and de/dt (from Lagrange)
ii) Express them as $d a / d E$ and de/dE, functions of $a, e, E, p, \delta$.
iii) Define $x=$ ae and find $d x / d E$
iv) Incorporate the density model $\rho$
v) Integrate over one revolution and get $\Delta a, \Delta x$.
vi) Compute $r_{p}=r_{p}(e), T=T(e), e=e(t), t_{L}=t_{L}(e)$ and then $r_{p}=r_{p}(t), T=T(t), \dot{T}=\dot{T}(t), t_{L}=t_{L}(\dot{T})$

Here, $\quad \delta=\frac{C_{Q} A_{r e f} F}{m}$
where F is given by 2.34.

Different mathematical tools were used for different situations:
i) Normal e and $3<\beta \times<30$ (i.e., e $>0.02$ ): Phase 1
ii) Normal e and $0<\beta x \leq 3$ (i.e., e $<0.02$ ): Phase 2
iii) Circular orbits
iv) High eccentricity orbits: $0.2<\varepsilon<0.9 *$

Normal e refers to eccentricity smaller than 0.2 .

For normal e, terms in $e^{n}, n \geq 2$ or 3 are usually discarded. Phase 1 allows one to use the asymptotic expansion in the Bessel functions of imaginary argument but they must be kept unchanged in Phase 2. For high eccentricity orbit, a new variable is defined and other low order terms are neglected.

Initially, all lifetimes are deffued as the time Laken for e to drop to zero (condition (a) for lifetime definition) except for circular orbits where condition (c) is used. Then, corrections are brought. In Phase 2, condition (a) Implies an infinite drop in perigee height and consequently, condition (b) makes more sense, with $r_{e}=120 \mathrm{~km}-150 \mathrm{~km}$ or, equivalently, when $e=0.1 \mathrm{e}_{\mathrm{o}}$. In Phase l , $t$ is the time required for Phase 2 to be reached plus the lifetimes in Phase 2. For high-eccentricity orbits, $t_{L}$ is the time required for Phase 2 to be reached plus the lifetimes in Phase 2 and Phase 1.

Atmospheric oblateness is shown as an important perturbing factor for small eccentricity orbits in some particular orientations. High eccentricity orbits are virtually not affected.

Variable scale heights complicate the theory but some rules have been derived by King-Hele to simplify the expressions. The expected lifetime is very altered when e>0.02. Substraction of a constant coefficient can given an approximate lifetime in this case. When e<0.02, King-Hele derived the "constant-H" formula where the value of $H$ is evaluated at a height kH above or below perigee and this constant II value is substituted into the lifetime expressions

[^0]derived $\mathrm{E}_{\mathrm{m}}$ in the constant $-H$ theory. The parameter $K$ takes different value for different situations.

Finally, an analysis of the variation in the orbital elements $\Omega$, $i$ and $\omega$ due to air drag and atmospheric rotation is included.

Thé various results are included in Talbes IV, V and VI.
The nice advantage in King-Hele's work is the independence of his expressions upon day-to-day and yearly density variations. Indeed, since $t_{L}$ and $\dot{T}$ are both affected by the value $\delta \rho_{P O}=C_{D} A_{r e f} F \rho_{P O} / m$, expressing $t_{L}$ as function of $\dot{T}$ brings cancellation of this erroneous factor. The same thing occurs when $a, r_{p}$ and $T$ are expressed as functions of eccentricity. However, eccentricity vs. time is dependent on $\delta \rho_{p o}$. For long-lived satellites, the accuracy of this theory is not better than $10 \%$, as King-Hele himself claims it is. For short term prediction, these equations furnish a very good estimate of the probable lifetime. However, assuming a $1 / r$-potential field may occasionally lead to large discrepancies as pointed out by Santora (LP-4). Secular variations in $\Omega$ and $i$ and oscillations in $\omega$ induced by the geopotential have been known to have a great influence on the magnitude of air drag. Furthermore, for light balloon type satellites, King-Hele's theory needs refinement as suggested by Jupp (LP-7). As a last remark, King-Hele's theory is quite acceptable when lifetime predictions are concerned but actual trajectory description are not obtained with great accuracy, mainly due to: the fact that the Osculating Orbital Elements Approach neglect oscillations occurring within one orbital period.

Later Cook and King-Hele extended this theory to include the diurnal bulge (LP-16) according to Eq. 2.26. The degree of severity of the day-tomight effect is shown to be proportional to $F / Z$, where $F$ is given by 2.28 and $Z=a e / H$.

The changes in orbital inclination of a satellite due to a rotating atmosphere including day-to-night variations in air density has been analyzed by King-Hele and Walker (LP-10) where resonances. conditions are also treated. It is demonstrated that $\Delta_{i} / \Delta T$ is directly proportional to $\Lambda$ (Eq. 2.31 ) and (sin i) (LP-10).

Initial Eccentricity

## Total Lifetime

$$
0.02<e_{0}<0.2 \text { (Phase 1) } \quad t_{L}=-\frac{e_{0} T_{0}}{\dot{T}_{0}}\left[F\left(e_{0}\right)+O\left(H / 2 a_{0}\right)\right]
$$

$$
0<\beta a_{0} e_{0} \leq 3 \quad\left(\text { Phase 2) } \quad t_{L}=-\left(\frac{3 e_{0} T_{0}}{T_{0}}\right) \frac{I_{0}\left(z_{0}\right)}{I_{1}\left(z_{0}\right)}\left[1+2 e_{0} \frac{I_{1}\left(z_{0}\right)}{I_{0}\left(z_{0}\right)}-\frac{q e_{0} z_{0}}{40}+0(0.008)\right]\right.
$$

$$
e_{o}>0.2
$$

$$
t_{L}=-\frac{e_{0} T_{0} F^{\prime}\left(e_{0}\right)}{T_{0}}\left[1+0\left(\frac{4 H}{5 r_{p 0}}\right)\right]
$$

$$
e_{0}=0
$$

$$
t_{L}=\because \frac{m H T_{0}}{2 \pi \delta \rho_{0} a_{0}^{2}}
$$

where

$$
\begin{aligned}
& F\left(e_{0}\right)=\frac{3}{4}\left[1+\frac{7 e_{c}}{6}+\frac{5 \dot{-}_{0}^{2}}{16}+\frac{H}{2 a_{0} e_{0}}\left(1+\frac{11 e_{0}}{12}+\frac{3 H}{4 a_{0} e_{0}}+\frac{3 H^{2}}{4 a_{0}^{2} e_{0}^{2}}\right)+O\left(e_{0}^{3}, \frac{H^{4}}{2 a_{0}^{4} e_{0}^{4}}\right)\right] \\
& F^{\prime}\left(e_{0}\right)=\frac{3+e_{0}}{\left(1+e_{0}\right)\left(1-e_{0}\right)^{1 / 2}}-3-\frac{1}{\sqrt{2}} \ln \left(\frac{\sqrt{2}+\sqrt{\left(1-e_{0}\right)}}{(\sqrt{2}+1) \sqrt{1+e_{0}}}\right) \\
& m=1-\exp \left[-\beta\left(a_{0}-a_{L}\right)\right] \quad a_{L} \sim 6510 \mathrm{~km} . \\
& \text { In }=\text { Bessel function of imaginary argument } Z=a e / H \\
& \text { ( } \delta \text { is given in 3.4). }
\end{aligned}
$$

TABLE V: OBLATE ATMOSPHERE (King-Hele)

Phase $1 \quad t_{L} \simeq\left(t_{L}\right)_{\text {SPH.ATM. }}\left[\begin{array}{c}\left.1+2 f \sin ^{2} i \cos 2 \bar{u}\right] \exp \left[c \cos 2 \omega_{0}-\cos 2 \bar{u}\right]\end{array}\right]$
Phase $2 \quad$ "t $t_{L} \simeq\left(t_{L}\right)_{S P H, A T M}\left[1+c \cos 2 \bar{\omega} \frac{I_{2}\left(z_{0}\right)}{I_{0}\left(z_{0}\right)}+o\left(\frac{c^{2}}{2}\right)\right]$

$$
e=0 \quad t_{L}=\left(t_{L}\right)_{S P H \cdot A T M}\left(\frac{1}{1+c^{2} / 4}\right)
$$

$$
\text { large e } \quad t_{L} \simeq\left(t_{L}\right)_{\text {SM. ATM. }}
$$

where

$$
()_{\text {SSH }} \text { ATM. }=\text { spherical atmosphere value }
$$

W = average value of argument of perigee
$\mathrm{f} \quad=$ flattening (2.21)
c $\quad=1 / 2$ f $\quad r_{\text {po }} \sin ^{2} i$

Phase $1 \quad t_{i}=-\frac{3 e_{0} T_{0}}{T_{0}}\left[1+\frac{7 e_{0}}{6}+\frac{H_{p o}}{2 a_{0} e_{0}}-\gamma\left(\frac{7}{16}-\frac{3 H_{p o}}{4 e_{0} a_{0}}\right)+\dot{o}\left(\frac{5 e_{0}^{2}}{16}, \frac{3 H^{2}}{8 a_{0}^{2} e_{0}^{2}}, \gamma^{2}\right)\right]$

Phase $2 \quad t_{L}=-\left(\frac{3 \epsilon_{0} T_{0}}{4 \dot{T}_{0}}\right) \frac{I_{0}\left(z_{0}\right)}{I_{1}\left(z_{0}\right)}\left[1-\gamma\left\{2+z_{0}-z_{0}^{2}-\left(z_{0}^{2}+z_{0} / 2\right)\left(y_{0}-\frac{1}{y_{0}}\right)\right\}+0\left(\gamma_{1}^{2}, t\right)\right]$
$e=0 \quad t_{L}=\frac{H_{0}^{*} T_{0} m}{2 \pi \delta a_{e}^{2} P_{0}}\left[1+\frac{\gamma}{2}\left(\frac{a_{0}-a_{L}}{H_{0}^{*}}\right)^{2}\left(\frac{1}{m}-1\right)\right]$
where

$$
\begin{aligned}
& y_{0}= I_{o}\left(Z_{o}\right) / I_{1}\left(Z_{0}\right) \\
& X= d H / d r \approx 0.1 \\
& *= \text { denotes values evaluated at one scale-height } \\
& \text { below initial height ie., } a_{o}^{*}=a_{o}-H_{o}^{*} \\
& \text { and } H_{o}^{*}=H\left(r=a_{o}^{*}\right)
\end{aligned}
$$

So far, gravitational pertubations have not been included in the above theories. Santora (LF-3) attacked the problen. In his paper, he first found out the decay rates in $T$, $e$ and $\omega$ for an oblate, rotating atmosphere including the diurnal density effects (using Eqs. 2.14, 2.22, 2.26 and 2.34). Auxiliary equations are derived to take into ascount the gravitational perturbations in perigee height and perigee argument and also, for the proper selection of the scaleheight H when diurnal effects are considerable. A new scale-height equation, applicable to diurnal density, follows the same variation as the density and is due to Santora:

$$
H=H_{0}+S \cos \phi
$$

where

$$
\begin{aligned}
H_{o} & =\frac{H_{\max }+\mathrm{H}_{\min }}{2}=H_{\text {AVERAGE }} \\
\mathrm{S} & =\frac{\left(\mathrm{H}_{\max }-\mathrm{H}_{\min }\right)}{\mathrm{H}_{\max }+\mathrm{H}_{\min }}
\end{aligned}
$$

$\emptyset=$ angular displacement from the bulge centre.

A modified Jacchia 1967 atmosphere model is used and, like King-Hele's work, this theory is valid for e $<0.2$. The "kH" (or constant-H) theory due to King-Hele is shown to be inadequate when oblateness and bulge effects are combined.

Santora has thus succeeded in combining many perturbing forces together in a single mathematical development. A particular case to this unified theory (from Santora again) considers near circular orbits (LP -4 ). This special case is treated separately because of some particular effects (like the short term gravitational perturbations and the different density profile involved). An exponential, oblate and diumal atmosphere model and a $(7,0)$ gravity field are assumed.

Santora's work represents one of the most realistic approches to the determination of the actual trajectory of a decaying satellite in the Osculating Orbital Elements Approach. His equations can be directly used in orbital. lifetime predictions. Comparison with numerically integrated trajectory showed $2 \%$ discrepancy with his equations taking the $1 / 50^{\text {th }}$ of the computer time needed in numerical integration.

Zee (LP-6, LP-33) produced a "borderline" case in which he derived the equations of a spatial curve in spherical coordinates and then, "translated" his results in terms of orbital elements. Strictly speaking; the Osculating Orbital Elements Approach was not used and these papers will be discussed in the next section.

Finally, Lubowe (LP-28) contributed to the advancement of the theory by showing the importance of oblateness - drag coupling ( $J_{2}$ D coupling) in the orbit prediction of a satellite and thus, the influence of tesseral harmonics are shown to be important in any realistic approach to our problem.
(b) Spatial Curve Approach

Contrarily to the Osculating Orbital Elements Approach, the Spatial Curve Approach supposes approximations applied directly to the equations of motion, rather than to the Lagrange's equations. Therefore, spatial curves representing the trajectory of the satellite are obtained. They usually have the form:

$$
\begin{aligned}
& r=r(t) \\
& r=r(\theta), \quad \theta=\theta(t) \\
& r=r(e), \quad e=e(\epsilon)
\end{aligned}
$$

Some papers consider planar motion only and the complexity of the theory increases when 3 -dimensional motion is anlayzed.

Assuring a spherically symmetrical atmosphere model, where density varies according to Eq. 2.18 , Egorov derived the time
required for an orbit to decay from an initial radius, rod down to a radius r. The velocity of the satellite is assumed circular at all. times and the geopotential follows the simple $1 / r$ model. He obtained

$$
\begin{equation*}
\hat{t}=\frac{1-\hat{r} \alpha+\frac{1}{2}}{(\alpha+1 / 2) B} \tag{3.5}
\end{equation*}
$$

where

$$
\begin{aligned}
& t=\text { non-dimensional time }=\sqrt{\mu / r_{0}^{3}} t \\
& r=\text { non-dimensional radius }=r / r_{o} \\
& B=C_{D} A P_{o} r_{o} / m \\
& \alpha=\text { defined in Eq. } 2.18
\end{aligned}
$$

Equation 3.5 gives a reliable approximation if the orbit satisfies particular conditions like:

- trajectory always nearly circular,
- bulge effects negligible (bulge line perpendicular to the orbital plane),
- small altitude variations,
- short-term predictions.

The "layer-technique" can somewhat improve the reliability of Eq. 3.5. In any case, due to the great number of approximations, it is not very close to the real situation but its simplicity makes it a useful estimate when raw predictions are needed for particular orbits.

Back in a spherically symmetrical and exponentially varying model, Zee ( $\mathrm{LP}-5$ ), using the techniques of the variation of parameters, obtained the equation of an ellipse in which e and h (angular momentum) are functions of the true anomaly:

$$
\begin{equation*}
\frac{1}{r}=\frac{C}{h^{2}}+\frac{1}{k_{1} h} \cos \left(\theta+k_{2}\right) \tag{3.6}
\end{equation*}
$$

( $\mathrm{K}_{1}, \mathrm{~K}_{2}, \mathrm{c}$ are constants)
where $h$ decreases with time (or $\theta$ ). The first term represents the mean motion. The second tern represents the oscillation about the mean path with an amplifying coefficient, $1 / K_{1} h$. Equation 3.6 shows that even an initially circular orbit does not strictly remain circular and that oscillations about the mean path increase as time goes on. This theory is valid for low eccentricity orbits and asymptotic expansions approximations (Boguluiboff, Mitropolsky) are used. Extending this theory, Zee (LP-6) later considered the oblateness in the mass distribution of the Earth. Using similar techniques, a spatial curve is obtained for initially small eccentricity orbits. A similar theory is also given in (LP-33) in which case, low radial thrust is the pertur ing force.

A very detailed analysis of the decay of nearly circular orbits is due to Brofman (LP-31).

The perturbing forces are very simplfied: spherical Earth, spherical and stationary atmosphere, exponential density decay and planar motion only are considered. However, the mathematical tool used (modified Mitropolsky) deserve mentioning this work, the solution obtained is of the form

$$
\begin{equation*}
r=\sum_{n=0}^{N} k^{r_{1}} r_{n} \tag{3.7}
\end{equation*}
$$

where $r_{n}$ is the $n{ }^{\text {th }}$ order solution to the planar equations of mtoion. Both mean and oscillatory trajectories are shown in this solution.

Variational equations are used by Perkins (LP-20) to demonstrate that air drag makes the satellite spiralling with a sinusoidal modulation:

$$
\begin{equation*}
\Delta r=-K(I-\sin \bar{t}) \quad\left(e_{0}=0\right) \tag{3.8}
\end{equation*}
$$

where

$$
\begin{array}{ll}
\dot{\bar{t}} & =v_{0} t / r_{0} \\
K & =g_{E}\left(\frac{c_{0} A}{w}\right) p_{0} r_{0}^{2} \quad w=w e i g h t
\end{array}
$$

For initially, elliptic trajectory, he obtained:

$$
\left.\begin{array}{c}
\Delta r=A(1-\cos \bar{t})-K e^{-c}[a(\bar{t}-\sin \bar{t})+b / 2(\sin \bar{t}-\bar{t} \cos \bar{t})+ \\
d \\
2^{2}-1(\sin \bar{t}-\sin 2 \bar{t} \\
2
\end{array}\right)+\underset{3^{2}-1}{f}\binom{\sin \bar{t}-\sin 3 \bar{t})+\cdots}{3} .
$$

Here, the orbital lifetime of circular satellites is presented in the form

$$
\begin{equation*}
\bar{t}_{L}=\frac{p_{0}}{k} \int_{h_{i}}^{h_{1_{0}}} \frac{d h_{1}}{p} \tag{3.10}
\end{equation*}
$$

This expression is very useful since the integral can be obtained graphically, if no atmosphere model is judged pertinent. However, if exponential density is assumed, 3.10 reduces to:

$$
\begin{equation*}
\bar{t}_{L}=\frac{1}{\beta K}\left[1-e^{\beta\left(h_{L}-h_{0}\right)}\right] \tag{3.11}
\end{equation*}
$$

and for the power la wo model ( $\rho \sim h^{n}$ )

$$
\begin{equation*}
\bar{i}_{L}=\left(\frac{1}{1-\eta}\right) \frac{h_{0}}{K}\left[1-\left(\frac{h_{L}}{h_{0}}\right)^{1-\eta}\right] \tag{3.12}
\end{equation*}
$$

Because the exponential model is not valid over the whole atmospheric range (with $\beta=1 / H=$ constant), Perkins used 3.12 for an approximate lifetime expression $\left(h_{L}=0\right)$ :

$$
\begin{equation*}
\bar{t}_{L}=\left(\frac{1}{1-n}\right) \frac{h_{0}}{K} \tag{3.13}
\end{equation*}
$$

The periodic oscillations in the altitude of a satellite under drag conditions, as derived in the above theories, was also demonstrated by Karrenberg, et al:

$$
\begin{align*}
& \frac{r}{r_{0}} \simeq 1+2 \alpha(S-\sin S)  \tag{3.14}\\
& \frac{V}{V_{0}} \simeq 1-\alpha(S-\sin S) \tag{3.15}
\end{align*}
$$

where

$$
\begin{aligned}
\alpha & =\text { ratio of applied-to-gravitational acceleration } \\
r_{0} S & =\text { arc length traversed by the satellite. }
\end{aligned}
$$

From these two equations, in-track and out-of-plane dispersion expressions due to uncertainties in the mean atmospheric density are derived.

In-track angular displacement $\xi:$

$$
\begin{equation*}
\xi=\xi_{0}-\frac{3}{2} \alpha S^{2}+4 \alpha(1-\cos S) \tag{3.16}
\end{equation*}
$$

In-track displacement d:

$$
d=-\begin{gather*}
3  \tag{3.17}\\
2
\end{gather*} r_{0} \alpha\binom{v_{0} t}{r_{0}}^{2}+4 \alpha r_{0}\left[1-\cos \binom{v_{0} t}{r_{0}}\right]
$$

Cross-track displacement y:

$$
y \sim \psi \frac{d}{r_{0}}
$$

where is the angle between inertial and relative (to Earth) velocities of the satellite. In this treatment, spherical Earth, spherical and stationary atmosphere and $1 / r$ field are assumed. The density is considered constant in the range for which dispersions are measured. This constitutes its major weakness.

Finally, the reader is referred to (LP-46) for an interesting discussion of the so-called "satellite-paradox" where it is shown that the effect of drag on a satellite affects its motion as if the drag force were reversed and were accelerating the satellite.

## c) Graphical Approach

Graphical approaches usually provide a series of graphs obtained by numerical analysis. Each graph may contain many curves (one for each value of $\mathrm{C}_{\mathrm{D}} \mathrm{A} / \mathrm{m}$, for instance). Sometimes, tables are. added when correction factors are needed. A combination of graphs may also be given in which, an iterative process determines the predicted lifetime of a satellite.

Vlasov (LP-49) used such a combination of graphs: one for constant ${ }_{L}$ lines, one for constant eccentricity lines, onc for constant $\sigma=C_{D} A / m g$ lines and one that takes into account the yearly variations of the mean atmospheric density due to the sunspot cycle. Exponential atmosphere model is used and no gravitational perturbations are included.

A very extensive work has been done by Ladner and Ragsdale (LP-52). Their graphs were constructed through numerical integration (Runge--Kutta and Simpson's Rule) of the modified Sterne's equations (similar to Lagrange's but for perigee height only). The modifications comprise variable mass and variable drag coefficient due to attitude stabilization. Although the theory is mainly applicable to circular orbits only, the effect of various parameters ( $\rho, C_{D}$, change in mass, etc) are discussed in terms of their influence on the nominal lifetime (i.e., when all theses variations are neglected).

First the influence of the diurnal bulge on lifetime is
assessed. The bulge is modelled according to Jacchia's empirical formula (Eq. 2.25). It is shown that maximum lifetime occurs when the bulge line (from Earth's centre to bulge centre) is perpendicular to the orbital plane and minimum lifetime occurs when this line is in the orbital plane (maximum drag conditions). The maximum percent loss in lifetime due to diurnal effects ranges from $4 \%(200 \mathrm{~km})$ to $20 \%$ ( 500 km ) for a circular orbit. The influence on solar activity (10-11 year cycle) is also analyzed and uncertainty in the prediction of $P_{\text {mean }}$ may lead to $+150 \%$ and $-60 \%$ uncertainty in lifetime predictions. However, because the authors assumed a fixed geometry (fixed orbital plane and fixed Sun) the predictions obtained are good for short periods of time only (<1 week).

A detailed analysis of the drag coefficient is given and the effect of mass loss (due to boiloff of propellant) on lifetime is assessed. Uncontrolled and attitude-stablized spacecraft (both earth-fixed and space-fixed) are considered. The final result is given here:

$$
\begin{equation*}
\left.t_{L}=t_{L}^{*}[1+\Delta(11 \text { yr })-\Delta(\text { (BULGE })]+(\text { var. roc })\right] \pm U \tag{3.18}
\end{equation*}
$$

where

$$
\begin{aligned}
t_{\dot{L}}^{\dot{*}} & =\text { nominal lifetime (given in a graph) } \\
\Delta(11 \mathrm{yr}) & =\text { variations due to the solar cycle in density } \\
\Delta(\text { BULGE }) & =\text { variations due to the diurnal bulge } \\
\Delta(\text { var.mass }) & =\text { variations due to mass loss } \\
U & =\text { uncertainty due to } C_{D} \text { and solar flare. }
\end{aligned}
$$

All variations, $\Delta(11 \mathrm{yr}),. \Delta(B U L G E)$ and $\Delta$ (var .mass) are given in graphs as percent loss of lifetime. Therefore, reading 4 graphs and adding a few numbers give the estimated lifetime.

As a final note to this quick literature survey, it can be remakeed that the general behaviour of a satellite, subject to the three environmental perturbations defined earlier, has been very well analyzed. The complexity of the equations increases rapidly as the number of perturbing forces and their degree of sophistication in representing the "real world" increase. However, unpredictable variations in some parameters make any high degree of refinement somewhat useless.

Short-term predictions can be obtained with an acceptable degree of uncertainty. Unless scientists discover some deterministic equations expressing the variations in the Sun's activity and its influence on the atmospheric density, long-term prediction will always be given with a large degree of uncertainty.

Finally, the reader is referred to (LP-19) and (LP-20) for another appreciation of the available theories dealing with satellite lifetime.

## 4. DISCUSSION

This section will compare the three analytical approaches described in Section 3. Then, a comparison of the analytical method with the tracking and the numerical integration methods will be given. A possible scheme for actually predicting the time of re-entry of a satellite is also given.
(a) The Three Analytical Approaches

The Osculating Oruital Elements Approach is very well suited to lifetime predictions, The equations expressing the estimated lifetime are usually simple (King-Hele) and easy to evaluate. Their accuracy, given the sporadic behaviour of the density, is very acceptable. However, the equations are valid for sustained orbits only (where the orbital elements can still have some meaning) where the variations within each revolution must be small (according to the basic assumptions of the method). Consequently, upon re-entry,
this approach, technically, cease to be valid. But as long as lifetime prediction is concerned, this is not a major drawback: normal re-entry time is of the order of a few minutes (OA-2, OA-4) and thus, would not affect the estimated lifetime. However, the predicted trajectory would have no meaning during re-entry. Even during sustained orbits, the Orbital Elements Approach gives average value for every period and fails to give the oscillatory motion depicted in (LP-5), (LP-20), (LP-31) and others. Those oscillations imply different density profiles and the trajectory predicted by this approach may be quite different from the actual. This is why the theorists using this approach give at the most, the evolution of the perigee height but no expressions on the motion of the satellite on its orbit. Overall, because of its relative mathematical simplicity, this method remains very useful, as far as lifetime prediction goes.

The weakness of the Osculating Orbital Elements Approach is not present in the Spatial Curve method. The mean trajectory of the satellite, along with its oscillations about this mean path are given in three-dimensional coordinates (in the more general case) by this approach. Therefore, it is a powerful approach and lifetime prediction as well as position estimation can be made as accurate as the density of the atmosphere permits. It is also (technically) valid for both sustained orbit and re-entry trajectory. Since the actual motion of the satellite is described in this method, the density profiles encountered are more realistic than in the previous. approach. The main drawback to thịs method is its mathematical complexity and the awful task of getting the time as the independent variable. Generally, it is mot possible to obtain the position variables as an explicit function of the time. Parametric equations are usually given. Consequently, closed-form equations (like Tables IV, $V$ and $V I$ ) for the lifetime cannot be obtained.

Finally, graphical techniques give the easiest lifetime evaluations: it can be done without any computations. Obviously, the trade-in is the acouracy. Furthermore, the graph:i cimmot repreacmi all the possible cases and consequently, a satellite may have some parameter. such that its numerical value is between two constantlines on the graph. This leads to further errors in reading the graphs. Also, even if lifetime estimates are readily obtained, one must not forget that the graphs represent long and costly computer work.

As a concluding remark to this comparison, one can get an easy rough estimate of the lifetime through graphical techniques. If a better estimate is required (with a few percent error) the Opbitit Elements Approach is the answer. At the expense of mathematical simplicity, the estimates that are likely to be the best are those given by the Spatial Curve Approach though sone computer usage may be needed.
(b) Analytic vs. Other Methods

A brief comparison between analytical methods and other methods is now presented.
i) Analytical Methods

These methods, described in this report, were seen to give easy and relatively accurate estimates of the lifetime. However, long-term predictions tend to be meaningless. For the more sophisticated theories, computer assistance may be needed but there are lifetime equations that can be very easily evaluated. They are general by their very nature and can be applied to any satellite with great success.
ii) Tracking Methods

Tracking is a very porerful method in estimating the actual position of a satellite. The actual density profiles and the effective coefficient of drag can be inferred from observations. Short-term predictions can be obtained by computer analysis of the data obtained by this method. However, uncertainties in the equipment, in the location of the station and the restricted number of stations are the major problems associated with this method. Furthermore, since it is not possible to keep track of the trajectory of all the satellites (it would be too expensive and rather useless), the application of this method must be restricted to selected satellites only.

## iii) Numerical Integration of a Particular Orbit

Computer integration implies the substitution of the algebraic variables ( $C_{D}, A_{r e f}, P_{o}, r_{o}, m_{,} \Lambda$, etc) by their numerical value before any analysis can be done. Consequently, the results obtained are very specific and they only apply for the particular satellite chosen and the particular conditions existing. When lifetime estimates are required, the integrations of the trajectory over the entire life of the satellite involves long conputing time and this method becomes very expensive. Depending on the particular software, it may take up to 6 hours for a 1900 - revolution lifetime (Zee). However, the trajectory obtained is quite accurate, so is the lifetime. In one particular case (LP-2), the difference between computed and actual time of entry were so close that the error represented less than $1 \%$ of the remaining lifetime at the time of the estimation..
(c) On the Use of the Various Methods

From the above considerations, it makes sense to use these methods in the following sequence.

First the analytic method can be used to obtain a rough estimate of the lifetime. Since it would be too expensive and useless to integrate the trajectory of the satellite by computer means, the analytic approximation would tell the analyst when to start the numerical integration.

Near the end of the satellite life, corlputer integration may be used. Due to the proximity of the re-entry, more exact evaluations are needed. The computing time involved would be far less than the reentry duration simulated.

Finally, tracking could be used as a complement to estimate the actual density profiles and the effective drag coefficient involved. The state of the satellite would be used in both analytic and numerical methods (where initial conditions are needed). For
instance, in the early stage the tracking method can provide the value of $\dot{T}$ which, once substituted in King-Hele's equations yields the analytical estimation of the total lifetime. At the final stage, the effective values of $C_{D}$ and density could be inserted in the numerical analysis for a more accurate prediction.

## CONCLUSION

The various ways of modelling the impotant perturbing forces acting on a satellite under drag deceleration have been reviewed. Many satisfactory expressions were derived but many other perturbations remain unmodelled, mainly because of the unpredictable nature of the parameters on which they depend.

Some equations, expressing the lifetimes of a satellite, were taken from the existing literature and discussed. While short term predictions can be relatively accurate, long term estimations showed larged uncertainty because of the sporadic behaviour of the atmospheric density.

In conclusion, the best theory (including the more exact models in its equations) will always have inherent in its results a certain degree of uncertainty, by its deterministic nature. In consideration of the problems caused by density and drag coefficient estimation, the theorist must change his attitude. Instead of asking: what is the lifetime?, he must ask himself: how accurately can it be predicted?

# SATELLITE ORBIT DECAY <br> AND 

RE-ENTRY PREDICTION

LITERATURE SURVEY

COitTEITS
A. Legend
B. Lists of papers, reports and books of which a complete photocopy has been prepared.
C.

Lists of papers, reports and books of which only the first page(s) has been photocopied. (Since these papers either had a large number of pages or were not relatively important, a conplete photocnpy was not made).

## A - LEGEND

i) Classification of papers:

LP: Lifetine Prediction and Orbit Decay
Atmospheric Drag
Third Body and Other Perturbations
SP: $\quad$ Structural and Physical Properties of
Satellites (ablation, drag, coefficients, etc.)
AP: Atmospheric Properties.
(Density, rotation, scaleheight var, solar influence)

RH: Re-entry Mechanics
Ballistic Re-Entry (missiles)
Trajectory Dispersion
0A: Orbit Analysis and Determination
MA: Mathematical Considerations

- : Miscellaneous Papers
ii) Codes:
* : Refers to a book
+ : Refers to a report

B - PAPERS

## $L P$

1. "The simplest relation for an estimate of the lifetime of artificial planetary satellites in near-circular orbits".
D. Ya. Fgorov, Cosmic Research, May 71, 923-5
2. 

"Decay of a highly eccentric satellite".
G. Janin, E.A. Roth, Celestial Mechanics, 14 (1976) 141-9
3.
"Satellite Drag Perturbations in an Oblate Diurnal Atmosphere.
F.A. Santora, AIAA J. Vol.13, No.9 (1975) 1212-6
4.
"Drag Perturbations of Near-Circular Orbits in an Oblate Diurnal Atmosphere"
F.A. Santora, AIAA J. Vol.14, ilo.9 (1976) 1196-1200
5.
"Trajectories of satellites under the influence of Air Drag".

Chong-Hung Zee, AIAA Astrod.Conf. Aug 19-21, (1963), 1-5
"Trajectories of satellites under the combined influences of earth oblateness and air drag.".

Chong-Hung, Zee. Celestial Nech. 3(1971) 148~168
9.
"Some Investigations into the Atmospheric Drag Problem".
Alan H. Jupp. Celestial Mech. 14 (1976) 335-9
"The aerodynamic drag of near earth satellites".
G.E.Cook. C.P. \#523 (1960), Ministry ? Avacion, Aevencentical Res. C...ucil Current
"Changes in the Inclination of Satellite Orbits to the Equator"
R.H. Merson, D.G. King-Hele, R.H.A. Plimimer. Nature, Jan 24 (1959), 239-40
"The change in satellite orbital inclination caused by a rotating atmosphere with day-to-night density variation".
D.G. King-Hele, D.M.C. Walker. RAE Tech. Rep. 70208 (\$्रR 22999) Nov. (1970).
19.
"The magnetohydrodynamical drag on artificial satellite". E. Aerts. Ann. Geophys. t. 25, fasc.4, (1969), p.981-5
"Infiuence of Tesseral Harmonics on Nearly Circular Plar and Equatorial Orbits".
P.T. Guttman. AIAA J. Vol.3, No.2,Feb (1965) 330-7
"Theory of Satellite Orbits in an Atmosphere" (I, II).
D.G. King-Hele. Butterworths Publ. Col. London, 1964
"The contraction of satellite orbits under the influence of air drag

II - High-eccentricity orbits ( $0.2 \leq \mathrm{e}<1$ )".
D.G. King-Hele. Proc. Roy. Soc. 267A, p.541-557 (1962)
'IV - With scale height dependent on altitude".
D.C. King-Hele, G.E. Cook. Proc. Roy. Soc., 275A, p. 357 (1963)
" $V$ - With day-to-night variation in air density".
G.E. Cook, D. G. King-Hele. Phil. Trans. Roy. Soc. Vol. 259A, p. 33 (1965)
"VI - Near-circular orbits with day-to-night variation in air density".
G.E. Cook, D.G. King-Hele. Proc. Roy. Soc., 303n, p. 17-35 (1968)
"Survey of current literature on satellite lifetimes". B.H. Billik. ARS J., ilov, 1962, (32), p. 1641
"Satellite Lifetines".
B.H. Billik. ARS J., Dec 1962 (32), p. 1926
20.
"In Analytical Solution for Flight Time of Satellites in Eccentric and Circular Orbits".

F.M. Perkins. Astronaut. Acts, Vol. IV, Fasc. 2, p. 113 (1958)

"Órbital Perturbations and Stationkeeping of Communication Satellites".
S.K. Shrivastava. J. Spacecraft \& R., Vol. 15, No.2, (1978), p. 67
"Variation of Satellite Position with Uncertainties in the Mean Atmospheric Density".
H.K. Karrenberg, F. Levin, D.H. Lewis. ARS J. Vol. 32 Apr (1962), p. 576
"Motion of a Satellite in an Atmosphere of Low Gradient".
R.R. Newtoll. ARS J., Vot. 32, May (1962), p. 770
"Atmosphere-0blateness Correction-Factor for Circular Satellite Orbits".
V.A. Lee. ARS J., Vol. 32, Jan (1962), p. 102
"Effect of air drag on the orbit of the Russian earth satellite 1957B: Comparison of theory and observation".
D.G. King-Hele, D.C.M. Leslic. Nature, Vol. 181, June (1958), p. 1761
"Two atmospheric effects in the orbital acceleration of artificial satellites".
L.g. Jacchia. Nature, Vol. 183, Feb (1959), p. 526
"Exact Analytical Solutions for Orbits of Bodies with Atrospheric Drag".
M. Morduchow, G. Volpe. AIAA J., Vol. 11, Ho. 3, (1.973), p. 381
"Satellite Orbit Prediction in the Presence of Atmospheric Drag".
A.G. Lubowe. Astronautica Acts, Vol. 15, p.143-8 (1970)
"The Effect of a Meridional Wind on a Satellite Orbit".
D.f. King-Hele. Proc. Roy. Soc., Vot. 295A, (1966), p. 261
"The change in satellite orbital inclination caused by a rotating atmosp. with. day-to-night density variation".
D.G. King-Hele, D.li.C. Walker. Celestial Mechanics, 5, (1972), p. 41-54
"Approximate analytical solutions for satellite orbits subjected to small thrust or drag".
W. Brofman. AIAA J., Vo1. 5, Mo.6, (1967), p. 1121
"Motion of a satellite under the influence of a constant normal thrust".
H. Lass, C.B. Solloway. ARS J., Vol. 32, Jan (1962), p. 97
"Trajectories of satellites under the combined influences of low radial thrust and earth oblateness".

Chong-Hung, Zee. Astronautica Acts, Vol. 14, p. 289-98, (1969)
"Orbit Prediction in the Presence of Gravitational Anorial jes".
S.R. Croopnick. AIAA J., Vol. 10, 110.7, (1972), p. 867
"Effect of earth oblateness and equator ellipticity on a synchronous satellite".

Chong-Hung, Zee. Astronaut. Acts, Vol. 16, p. 143 (1971)
"Effect of earth's oblateness on the rotational motion of an artificial earth satellite".
Y.A. Sarychev. ARS J., (Supplement), Yol. 32, Hay (1962), p. 834
"Effects of the Sun and the Moon on a Mear-Equatorial Synchronous Satellite".

Chong-Hung, Zee. Astronaut. Acts, Vol. 17, p. 891 (1972)
"Effect of the Sun, the Moon and Solar Radiation Pressure on a llear-Equatorial Synchronous Satellite".

Chong-Hung, Zee. Astronaut. Acts, Vol. 18, p. 281, (1973).
"Corpuscular Radiation and the Acceleration of Artificial Satellites".
L.G. Jacchia. Nature, Vol. 183, June (1959), p. 1662
"Effect of Solar Flares on Earth Satellite 1957B".
T. Nonweiler. Nature, Vol. 182, Aug (1958), p. 468
"Earth Satellites and Related Orbit and Perturbation Theory". (Part 1 to 4 incl.)
S. Herrick. Space Technology, Chap. 5, (1959)
"Theoretical Datermination of the Aerodynamic Forces on Satellites".
S. Hocilla. Astronaut. Acts, Vol. 17, (1972), p. 245
"Elliptic Orbits in a Frictional Atmosphere".
D.G. Parkyn. An. J. Phys., (1958), p. 436
"Atmospheric Drag on the Satellite".
R. Jastrow, C.A. Pearse. J. Geophys. Res, Vol. 62, No. 3, (1952), p. 413
"The Skylab is Falling and Sunspots are Behind it All".
R.J. Smith. Science, Vol. 200, 7 Apr (1978), p. 28
"Satellite Paradox".
B.D. Mills, Jr., Arm. J. Phys., (1959), p. 115
"Lunar-Solar Perturbations of an Earth Satellite".
L. Blitzer. Am. J. Phys., (1959), p. 634
"Synchronous and Resonant Satellite Orbits Associated with Equatorial Ellipticity".
L. Blitzer. ARS J., (1962), p. 1016
49. "Nomogram for determining the lifetine of an artificial earth satellite?"
Yu. S. VLASOV

COSMIC RES., Vol. 5, No.2, (1967), p. 173
50. "Methods of estimating minimum lifetime of an artificial satellite down to a given height".
R.F. APPAZOV, B.D. BUILIN

COSMIC RES., Vol. 11, No. 6 (1973), p. 745
51. "Interaction of cosmic bodies with planetary atmospheres".
V.A. KHOKHRYAKOV

COSMIC RES., Vol. 15, No. 2 (1977), p. 169
52. "Earth Orbital Satellite Lifetime".
J.E. LADNER, G.C. RAGSDALE

NASA TN D-1995, January (1964).
$L P=61$
$\because$ Rasuidi Lubelite ffetion Extination airhw s. Fuchs (efoel)
 frad Jet.ch, brembett M4, 2071 , Cug 1975

 NASA - CR146801, An' 76.
"Drag coefficients of spheres in continuury and rarefied flows".
C.B. Henderson. AIAA J., Vol. 14, No. 6, (1976), p. 707-8
"Sphere drag at transonic speeds and high Reynolds nünbers".
A.B. Bailey, R.F. Starr. AIAA J., Vol. 14, No. 11, p. 1631 (1976)
"Drag coefficients for astronomical observatory satellites".
R.R. Williams. J. Spacecraft, Vo. 12, No. 2, p.74-8
"Effects of the variation of drag coefficients on the ephemeris of earth satellites".
R.R. Hunziker. Astronaut. Acta, Vol. 15, p. 161 (1970)
"Sphere Drag Coefficients for a Broad Range of Mach and: Reynolds Nurbers".
A.B. Bailey, J. Hiatt. AIAN J., Vol. 10, Ho. 11, (1972), p. 1436
"Atmospheric Drag Cross Section of Spacecraft Solar Arrays".
R.R. Milliams. J. Spacecraft, Vol. 9, Ho. 8, Aug (1972), p. 565
"The Thermal Acconodation Coefficient: A Critical Survey".
H.Y. Wachman. NRS J., Vol. 32, Jan (1962), p. 2
1.
2.
3.
4.
5.
7.
8.
"Temperature, density and composition in the disturbed thermosphere from Esro 4 Gas Analyzer measurements: A global model".
[.G. Jacchia, J.W. Slowey, V. von Zahn. J. of Geophys. Rech. Feb 1 (1977), p. 684-8
$\square$
B.A. Mirtov, A.G. Starkova

Cosmic Rech. 14(1), Jan-Feb (1976), p.43-6
"Seasonal Variations in the Latitudinal Structure of Atrospheric Disturbances".
G.W. Prolss, U. von Zahn
J. of Geophys. Res. Dec 1 (1977), Vol. 83, No. 35
"Diurnal Variations in the Thermosphere II, Temperature, Composition and Winds".
H.G. Mayr, I. Harris
J. of Geophys: Res. July (1977), Vol. 82, No. 19
9.
10.
11.
12.
13.
4.
5.
16.
.
"The large semi-annual variation in exospheric density:
A possible explanation".
G.E. Cook

Planet. Space Sci. Vol. 15, (1967), p. 627
"The variation of air density at a height of 220 km in the first half of 1966".
D.G. King-Hele, E. Quinn

Planet Space. Sci. Vol. 15, (1967), P. 1067
"Density of the atmosphere at heights between 200 km and $400 \mathrm{~km} .$, from analysis of artificial satellite orbits".
D.G. King-Hele Nature, Vol. 183, May (1959), p. 1224
"Variations in exospheric density during 1967-68, as revealed by Echo $2^{\prime \prime}$.
G.E. Cook

Planet. Space Sci. Vol. 18, p. 387 (1970)
"The semi-annual variation in the upper atmosphere during 1967 and 1968".
G.E. Cook

Planet. Space Sci. Vol. 18 (1970), P. 1573
"Measurement of solar and diurnal effects in the high atmosphere by artificial satellites".
H.A. Martin, H. Priester

Nature, Vol. 185, Feb (1960), p. 600
"Density of the upper atmosphere from analysis of satellite orbits: Further results".
D.G. King-Hele

Mature, Vol. 184, Oct (1959), p. 1267
17.
18.
24.
8.
"Upper atniosphere density measurements by the Sputnik III and San Marco satellites".

N. Bellomo<br>Astronaut. Acta, Vol. 18 (1974), p. 289

"Variation of upper-atmosphere density with latitude and season: Further evidence from satellite orbits".
D.G. King-Hele, D.M.C. Walker Nature, Vol. 185, March (1960), p. 727
"Irregularities of satellite drag and diurnal variations in density of the air".
G.V. Groves

Nature, Vol. 182, Nov (1958), p. 1533
"Upper-atrosphere rotational speed and its variation with height".
D.G. King-Hele, D.H. Scott, D.M.C. Walker Planet. Space Sci, Vol. 18, (1970), p. 1433
"Rotational speed of the upper atmosphere from the orbits of satellites 1966-51 A, B, and C".
D.G. King-Hele, D.W. Scott
Nature, $\quad(1967)$, p. 1110 (manis,170)
"A revaluation of the rotational speed of the upper atmosphere".
D.G. King-Hele, D.W. Scott Planet. Space Sci. Vol. 14, (1966), p. 1339
"Air density at heights near 180 km in 1968 and 1969, from the orbits of 1967-31A".
D.G. King-Hele, D.M.C. Walker

Planet. Space Sci, Vol. 19, (1971), p. 297
"Atriospheric density and rotation below 195 km from a high resolution drag analysis of the satellite OV1-15 (1968-059A)".
B.K. Ching
J. Geophys. Res. Vol. 76, Ho. 1 (1971), p. 197
25.
26.
27.
29.
30.
"Semi-annual variation in the heterosphere: A reappraisal".
L.G. Jacchia
J. Geophys. Res., Vol.76, Ho. 19 (1971), p. 4602
31.
"Seasonal variations of atmospheric-ionospheric disturbances".
G.W. Prolss
J. Geophys. Res. Vol. 82, No. 10 (1977), p. 1635
34. Ihe Jempertiure atove the I hermoreuse" LGfacchia, Space Rexearch $E$, N. Molkent Publece, Austidiom $1905^{\circ}$, $15^{1152}-1124$.

## RM

1. "Effect on roll rate of mass and aerodynamic asymmetric for ballistic re-entry bodies".
«L.S: Glover
J. Spacecraft, March-April (1965), p. 220-5
2. 

"Exact solution for dynamic oscillations of re-entry bodies".
E.V. Laitone, N.X. Vinh AIAA, Vol. 10, No. 7,July (1972), p. 925-7
3.
"Sorie exact analytical solutions of planetary entry".
W.H.T. Loh

AIAA J., April (1963), p. 836-42
4.
5.
6.
8.
9.
"A second order theory of entry mechanics into a planetary atmosphere".
H.H.T. Loh

National IAS-ARS Joint Meeting, June 13-16, 1961 Paper No. 61-116-1810
"Flight with life modulation inside a planetary atmosphere".
II.X. Vinh, N.A. Bletsos, A. Busemann, R.D. Culp AIAN J., Vol. 15, No. 11, Hov (1977), p. 1617-23
"Re-entry vehicle stagnation region feat-transfer in particle environment".
D.T. Hove, II.C.L. Shih

AIAA J., July (1977), p. 1002-1005
"On the dynamic response of missiles with varying roll rates".
B.Z.I1. Have!, R.M. Brach

AIAA J., Vol. 14, No. 1, Jan (1976), p. 9-10
"Response of a symmetric missile in a spin varying environment".
B.Z.M. Haveh AIAA J.,

May (1976), p. 689-90
10.
"Hissile stability using finite elements - an unconstrained variational approach".
J.J. Wu

AIAA J., March (1976), p. 313-19
"Approximate equations for irpact dispersion resulting from winds and deviations in density".
L.S. Glover
J. Spacecraft, Vo1. 9, No. 7, (1972), p. 483
"Tumbling bodies entering the atmosphere".
K.L. Remmler

ARS J., Vol. 32, Jan (1962), p. 92
"Terminal dynamics of atmospheric entry capsules".
P. Jaffe

AIAA J., Vol. 7, No. 6, (1969), p. 1157
14.
"A general theory on space and re-entry similar trajectories".
L. Broglio

AIAA J., Vol. 2, Ho. 10, (1964), p. 1774
"Analysis of re-entry vehicle behaviour during boundarylayer transition".
G.J. Chrusciel

AIAA J., Vol.13, 110.2 (1975), p. 154
16.
"Minimurl ballistic factor missile shapes".
S.C. Jain, V.B. Tawakley

Astronaut. Acta, Vol. 16, p. 277 (1971)
"Ballistic re-entry at small angles of inclination".
N.H.T. Loh

ARS J., Vol. 32, llay (1962), p. 718
"Angular motion of a re-entering symmetric missile".
C.H. Murphy

AIAN J., Vol. 3, Ho. 7 (1965), p. 1275
19.
20.
21.
22.
21.
.
"Inducation and wave drag on long cylindrical satellites".
N.S. Venkataraman, W.A. Gustafson

AIAA J., Vol.9, No. 5 (1971), p. 944

## .

"A higher order theory of ballistic entry". W.H.T. Loh, celicin cle athorautiacal Lience, Vol XI Weitein Periodicali Co:, 1963 , 1529-5\%4
"Similar solutions in re-entry lifting trajectories".
L. Broglio

Universita di Roma, TH. Ho.3, 1959
(AFOSR TN. 60-678)
"Ablation and breakup of large meteoroids during atmospheric entry".
B. Baldwin, Y. Sheaffer
J. Geophys. Res., Vol.76, No. 19 (1971), p. 4653
1.
2.
"The decay of Cosmos 253 rocket over England".
D.G. King-Hele, Halker, P.E.L. Meirinck Tech. Memo Space 119, Jan (1969), p. 7
3.
"Effects of atmospheric winds and aerodynamic life on the inclination of the orbit of the S3-1 satellite".
B.K. Ching, D.R. Hickman, J.M.Straus
J. of Phys. Rech. Vol.83, Ho.10, Apr (1977), p. 1474-80
4.
"The last minutes of satellite 1957B (Sputnik 2)".
D.G. King-Hele, D.M.C. Walker
llature, Vol. 182, Aug (1958), p. 426
5.
"Use of artificial satellites to explore the earth's gravitational field: Results from Sputnik 2 (19578)".
R.H. Merson, D.G. King-Hele
llature, Vol.132, Sept (1958), p. 640
"The orbit of 1968-59A, and its use in upper-atmosphere research".
D.G. King-Hele

Planet Space Sci. Voi. 18, (1970), p. 1585
7.
"The orbit of Cosmos 307 rocket and its use in atmospheric research".
H. Hiller

Planet Space Sci., Vol. 20, (1972), p. 891
"Approximate statistics of random vector magnitudes".
H. Hagar, Jr.
AIAA J., Vol. 14, Ho.5, p. 700-02
1.
2.
3.
4.
5.
6.
7.
.

$$
3 .
$$

"A, practical note on the use of Lambert's equation".
G.S. Gedeon

AIAA J., Jan (1965), Vol. 3(1), p. 149-50
"Averaging techniques and their application to orbit determination systems".
C.E. Velez, A.J. Fuchs

AIAA J., Vol.13, No.1, Jan (1975), p.12-16
C. Uphoff

AIAA J., Vol. 11, No. 11 (1973), p. 1512
"Celestial mechanics of artificial satellites".
T.E. Sterne

Sky \& Telescope, (1957), pp.66-68
"Ground control of drag sacellites utilizing on-board accelerometer data".
A.J. Fuchs, C.E. Velez
J. Spacecraft, Vol. 13(3), March (1976), p.188-192

## LP

I. "Longitudinal dynamics of a lifting vehicle in orbital flight".
B. Etkin
J. Aerosp. Sci., Vol.28, No.10, (1961), p. 779
II. "The effect of the earth's oblateness on the orbit of a near satellite".
D.G. King-Hele

Proc. Roy. Soc., (1958), p. 49
III. $\quad$ Sunlight pressure perturbations on satellite orbits".
I.I. Shapiro
IV. "Motion of an artificial satellite about is center of mass".
V.V. Beletskii

Mech. of Space Flight Series, NASA, U.S.^.
I. "Comparison of air densities obtained from orbital decay and instruments".
G.E. Cook

Trans: Phil. Soc. of London, 252A, (1967)
II.
III. "Variations in exospheric density at heights near 1100 km , derived from satellite orbits".
G.E. Cook

RAE TR-67180
IV.
"Variations in exospheric density during 1967-8 as revealed by Echo 2".
G.E. Cook

RAE TR-69127
V. "The seni-annual variation in the upper atmosphere during 1967 and 1968".
G.E. Cook

RAE TR-69254
"The semi-annual variation in air density at a height of 1100 km from 1964 to 1967".
G.E. Cook, D.W. Scott

RAE TR-68162
VII. "The serii-annual variation in the upper atmosphere: A review".
G.E. Cook

RAE TR-69074
VIII.
"Thermospheric observations combining chemical seeding and ground-based techniques. I. winds, turbulence, and the parameters of the neutral atmosphere".
K.H. Lloyd, C.H. Low, B.J. McAvaney, D. Rees, R.G. Roper Planet Space Sci., Vo.. 20 (1972), p. 761
IX. "Air density at heights near 150 km in 1970, from the orbit of Cosmos 316 (1969-108A)".
D.G. King-Hele, D.M.C. Walker

Planet Space Sci., Vol.19,(1971), p. 1637
X. $\quad$ Exospheric densities near solar minimum derived from the orbit of Echo 2".
G.E. Cook, D.W. Scott

Planet Space Sci., Vol.14, (1966), p. 1149
XI. $\quad$ Direct measurements of nightime therrospheric winds and temperatures 1. Seasonal variations during geomagnetic quiet periods".
G. Hernandez, R.G. Roble
J. Geophys. Res. Vo]. 8, No.13, (1976), p. 2065

## OA

I.
"Analysis of the orbit of Cosmos 316 (1969-108A)".
D.G. King-Hele

RAE TR-72062
1.
"Tables of $Z$ functions for atmosphere entry analyses".
D.R. Chapman, A.K. Kapphahn NASA TR-R106, (1961).

## IV.

"A solution for atmospheric entry trajectories deviating from equilibrium glide".
M. Hanin

UTIAS Report Ho. 111 (1966)
V.
"On a relatively cool transition from a satellite orbit to an equilibrium glide".
B. Etkin

UTIAS Report No. 75 (1961)
VI.
"Theoretical investigations of the dynamics of bodies entering the atmosphere".
B. Etkin

UTIAS Report No. 80 (1962)

> VII:
VIII.
"Dynamic stability of high-drag planetary entry vehicles at transonic speeds".
W. Marko

AIAA Paper No. 69-105
"The dynamic motion of a missile descending through the atmosphere".
H.R. Friedrich, F.J. Dore

$$
\text { S. Aeronambial Scenes, Sept } 195 ; \text {, } 188
$$

IX.
X. "Atmosphere entry".
A.J. Eggers, Jr. editor, IASA AIAA selected reprints series, Vol. 1.
XI. $\quad$ The effect of the subsolar atmospheric bulge on satellite re-entry latitudes".

?VII.
"A study of ballistic re-entry trajectories at a velocity of $50,000 \mathrm{ft} / \mathrm{sec}$. ( $15240 \mathrm{~m} / \mathrm{sec})^{\prime \prime}$.
J.T. Suttles, L.C. Coltrane HASA T.N. D-4011
XI.

"Effect of spin on the velocity of a re-entry body".
A.H. Nayfeh

AIAA J., Vot.8, No. 5 (1970), p. 978
"Nonlinear resonances in the motion of rolling re-entry bodies".
A.H. Nayfeh, W.S. Saric

AIAA Paper No. 71-47 (1971)
"Stability of flight paths of lifting vehicles during entry into planetary atmospheres".
J.H. Fine

UTIAS Tech. Note No. 48 (1961)
"Dynamic stability of vehicles traversing ascending or descending paths through the atmosphere".
M. Tobak, H.J. Allen

MACA T.N. 4275, (1958).
"Oscillatory motion about centre of gravity for a spacecraft in planetary atmosphere".
W.H.T. Loh

IAF Paper 1lo. AD4
"Gas dynamics of the flight and explosion of meteorites".
V.P. Korobeinikov, P.I. Chushkin, L.V. Shurshalov Astronautica Acts, Vol.17, (1972), p. 339
R.H. Gooding RAE TR-69104
I.
"Dynamic stability of space vehicles Vol. XI - entry disturbance".
F.D. Steketee NASA CR-945 (1967)


[^0]:    * Here; $e<0.9$ allors the analyst to neglect luni-solar perturbations.

