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ORBIT DETERMINATION AND PREDICTION MATHEMATICS

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1.0 INTRODUCTION

The mathematics in this document are used in several computer programs written by SED. The most recent programs are:

- ORBIT PREDICTION SOFTWARE (OPS)
  which is an interactive program on a SIGMA 9 computer
- ORBIT DETERMINATION, PREDICTION, AND CORRECTION PROGRAM (ODAP, ODAI, ODPAC)
  which is an interactive program on a Sigma 9 computer; the latest version is ODPAC
- ODAP for SARSAT
  which is a file-driven program on an HP 1000 computer

1.1 Orbit Prediction

The orbit of a satellite is defined by a set of orbital elements at a specified time. These are two basic sets of orbital elements used herein; spacecraft position (r_x, r_y, r_z) and velocity (v_x, v_y, v_z) and a set of velocity space elements (C_{g1}, C_{g2}, C_{g3}, R_{f1}, R_{f2}, λ). In addition, other sets of orbital elements may be used for input or output.

For orbit prediction, we start with a set of orbital elements at a specified time and predict the orbital elements at another time. We may then use the orbital elements to predict other things such as tracking station look angles.

There are many forces which affect the motion of a satellite. The ones considered herein are those which are most significant for an earth orbiting satellite; that is the central body attraction of the earth, the nonsphericity of the earth, the solar and lunar gravity, solar radiation pressure and atmospheric drag.

1.2 Orbit Determination

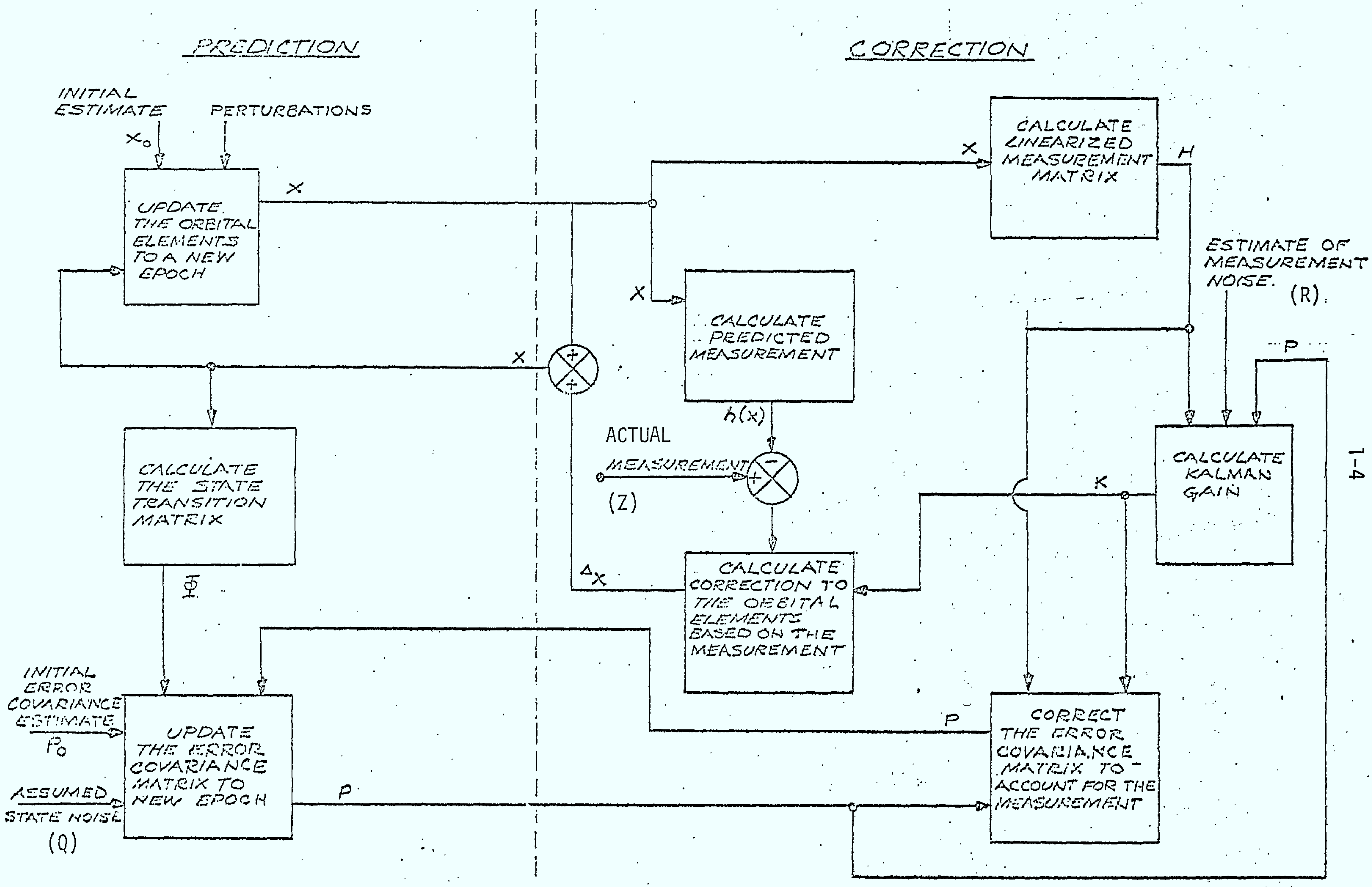
The objective of orbit determination is to find accurate values for a set of orbital elements which describe the orbit of the satellite, using observations of the satellite. Many types of observations are possible depending on the stations which are tracking the satellite. The analytics are developed for azimuth, elevation, range and range-rate.

There are two basic approaches to orbit determination, real-time and batch. In real-time orbit determination, a new estimate of orbital elements may be obtained each time a measurement is taken. In batch orbit determination, many measurements are obtained over a time span, and a new estimate of orbital elements is not produced until all measurements are available. The orbit estimator described herein uses real-time orbit determination.

The orbit estimator uses a process of prediction and correction. It is started with an initial estimate of spacecraft orbital elements, X_o. This estimate is updated to the time of a measurement and a predicted measurement h (X) is generated. The difference between the actual measurement and the predicted measurement is calculated, and the difference is used to improve estimates of the orbital elements. The new estimate of orbital elements is then updated to the time of the next measurement and the process is repeated.

There are several factors which determine how the orbital elements are corrected. An estimate of the measurement noise, the measurement noise covariance matrix, Q, is a stored constant. An estimate of the accuracy of the orbital elements, the error covariance matrix, R, is stored and updated when necessary. A measurement matrix, H, which maps orbital elements onto corresponding measurements is calculated each time a measurement is taken. These factors are all combined to form a Kalman gain matrix, K, which is multiplied by the difference between predicted and actual measurement to give the correction that is applied to the orbital elements.

Figure 1.2/1 shows the data flow within the orbit estimator.



BLOCK DIAGRAM
OF THE ORBIT ESTIMATOR
FIGURE 1.2/1

2.0 DEFINITIONS

2.1 Fundamental Reference Frames

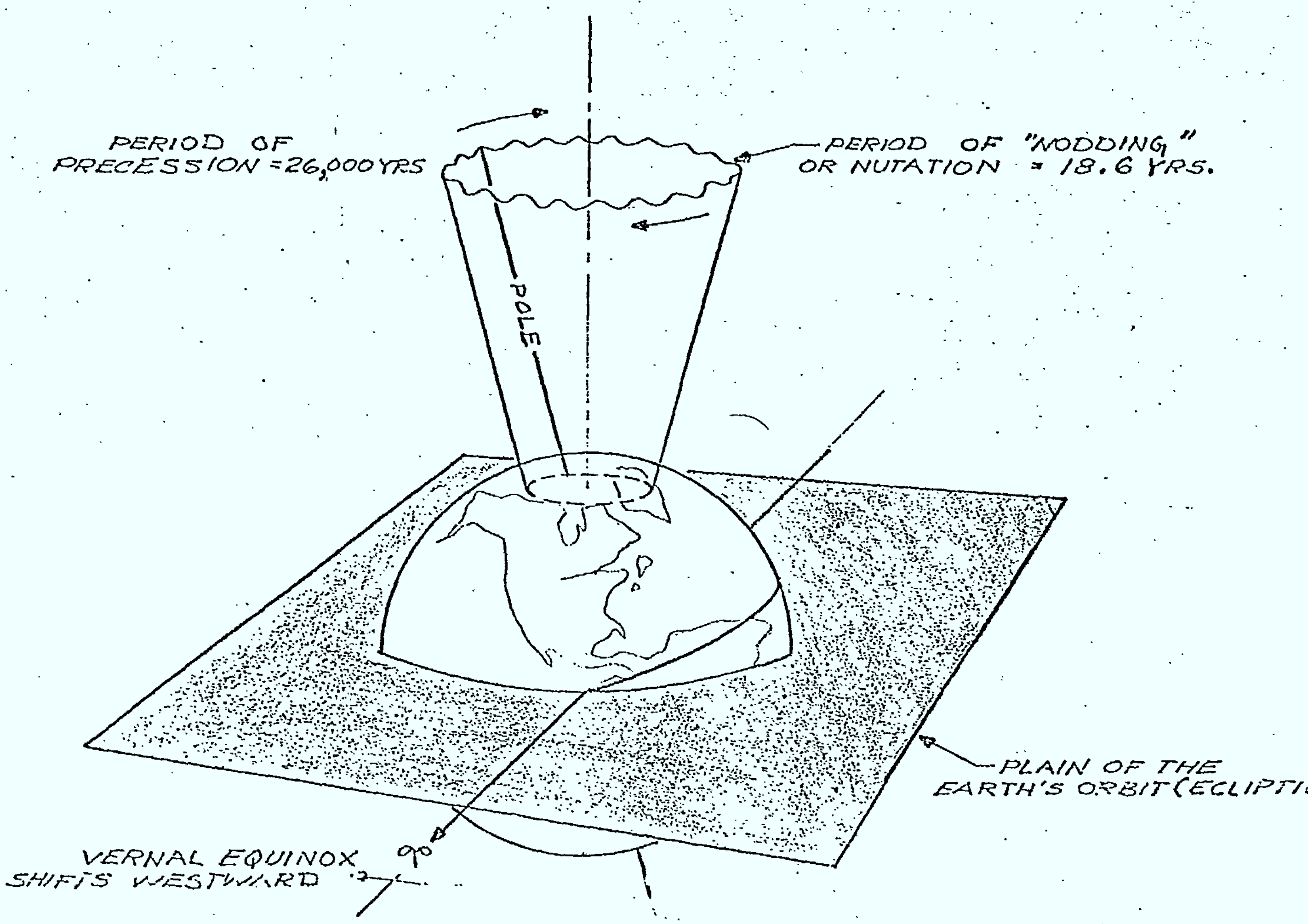
2.1.1 Introduction

The motion of a satellite is best described in a coordinate frame which is not rotating, or is rotating so slowly that the rotation may be neglected. For orbit determination and prediction, the plane of the earth's equator and the plane of the earth's orbit, or ecliptic, are used as fundamental quantities.

While the plane of the ecliptic is almost fixed relative to the stars, the equatorial plane is not. Due to the asphericity of the earth, the sun produces a torque on the earth which results in a wobbling or precessional motion similar to that of a simple top. Because the earth's equator is tilted $23\frac{1}{2}^{\circ}$ to the plane of the ecliptic, the polar axis sweeps out a cone-shaped surface in space with a semi-vertex angle of $23\frac{1}{2}^{\circ}$. As the earth's axis precesses, the line-of-intersection of the equator and the ecliptic swings westward slowly. The period of the precession is about 26,000 years, so the equinox direction shifts westward about 50 arc-seconds per year.

The moon also produces a torque on the earth's equatorial bulge. However, the moon's orbital plane precesses due to solar perturbation with a period of about 18.6 years, so the lunar-caused precession has this same period. The effect of the moon is to superimpose a slight nodding motion called "nutation", with a period of 18.6 years, on the slow westward precession caused by the sun.

The mean equator is the position of the equator when precession alone is taken into account. The true equator is the actual equator, accounting for both precession and nutation. Similarily the mean equinox is the intersection of the mean equator and the ecliptic, and the true equinox is the intersection of the true equator and the ecliptic.



PRECESSION OF THE EQUINOXES

The mean equator and equinox is frequently used as a reference for specifying the motion of the sun and the planets, because simple equations may be used which are valid over many years. The true equator and equinox is used for specifying the orientation of the earth.

When extreme accuracy is required, it is necessary to consider the motion of the ecliptic (less than 50 arc seconds per century), but all rotations of the ecliptic are neglected herein. Also, polar wandering over a distance of 50 m (or <2 arc seconds angular change) is neglected herein.

The following axes X, Y, Z are also referred to as g1, g2, g3; as in Cg1, Cg2, Cg3 the components of the C vector.

2.1.2 Mean Equator and Mean Equinox

Has centre at earth centre of gravity and coordinate axes

X - direction of mean vernal equinox

Y - forms right-handed system with X and Z axes

Z - north normal to mean equator.

2.1.3 True Equator and True Equinox

Has centre at earth centre of gravity and coordinate axes

X - direction of true vernal equinox

Y - forms right-handed system with X and Z axes

Z - north normal to true equator.

2.1.4 Ecliptic and Mean Equinox

Has centre at earth centre of gravity and coordinate axes

X - direction of mean vernal equinox

Y - forms right-handed system with X and Z axes

Z - northward normal to ecliptic.

2.2 Other Reference Frames

2.2.1 Introduction

In this section, we define additional reference frames which are used throughout this document.

2.2.2 Orbit Frame

Has centre at earth centre of gravity and coordinate axes

X - direction of perigee

Y - in orbit plane 90^{o} from X axis such that the spacecraft moves from X to Y

Z - forms a right handed coordinate system with the X and Y axes, and is in the direction of the orbit angular momentum.

If the orbit is circular, the perigee is undefined. Then it is arbitrarily set to coincide with the ascending node; that is, the argument of the periapsis is zero. If the inclination is zero, the longitude of ascending node is undefined, and it is arbitrarily set to coincide with the vernal equinox.

2.2.3 Orbit Frame Referenced to Ascending Node

Has origin at earth centre of gravity and coordinate axes

X - direction of ascending node, or crossing of the equator in a northward direction.

Y - in orbit plane 90^{o} from X axis such that the satellite moves from X to Y

Z - forms a right-handed coordinate system with the X and Y axes, and is in the direction of the orbit angular momentum.

If the inclination is zero, the longitude of the ascending node is undefined and the X axis is arbitrarily set to coincide with the direction of the vernal equinox.

2.2.4 Earth Fixed Cartesian

Has origin at earth centre of gravity and coordinate axes

X - direction of prime meridian intersection with equator

Y - on equator forming right-handed system with X and Z axes

Z - normal to the equator, North.

Note that this frame rotates with the earth

2.2.5 Earth Fixed Spherical

Has origin at earth centre and coordinates

r - radial distance to the point being measured

L - latitude, positive North of equator

λ - longitude, East of prime meridian.

2.2.6 Observation Frame

Has origin at the site and coordinates

El - elevation - angular measurement from the horizon to the range vector.

Az - azimuth - an angle measured in the plane of the local horizon. It is measured from the projection of the north direction eastward to the projection of the range vector.

ρ - radial distance to the point being measured.

Note the range vector goes from the site to the point being measured.

2.2.7 Topocentric Local Tangent

Has origin at the site and coordinate axes

E - projection of the East direction in the local horizon plane

N - projection of the North direction in the local horizon plane

U - local vertical.

2.2.8 Orbit Frame f_1 f_2 f_3

Has origin at centre of earth and principal axes

f_1 - in the orbit plane such that the angle between f_1 and the ascending node is equal to the longitude of the ascending node in inertial coordinates

f_2 - in the orbit plane 90^o from the f_1 axis such that the satellite moves from f_1 to f_2.

f_3 - forms a right-handed coordinate system with f_1 and f_2 and is in the direction of the orbit angular momentum.

2.2.9 Orbit Frame e_1 e_2 e_3

Has origin at centre of earth and principal axes

e_1 - directed toward the satellite

e_2 - in the orbit plane 90^o from the e_1 axis such that the satellite moves from e_1 toward e_2

e_3 - normal to the orbit plane, completing the right-handed coordinate system.

This frame is also called the radial, tangential, normal frame.

2.3 Sets of Orbital Elements

2.3.1 Inertial Set

This set has elements defined by satellite position and velocity.

$$\left.\begin{matrix} r_x \\ r_y \\ r_z \end{matrix}\right\} \quad \text{satellite position in True Equator and True Equinox Frame}$$

$$\left.\begin{matrix} v_x \\ v_y \\ v_z \end{matrix}\right\} \quad \text{satellite velocity in True Equator and True Equinox Frame}$$

Note: This set of orbital elements is referred to as the inertial set throughout this work, although as noted later, it is not an inertial frame in the strictest definition.

2.3.2 Keplerian or Classical Set

This set has elements defined as follows:

a - semi major axis

e - eccentricity

i - inclination

Ω - longitude of ascending node

w - argument of periapsis

ϕ - true anomaly

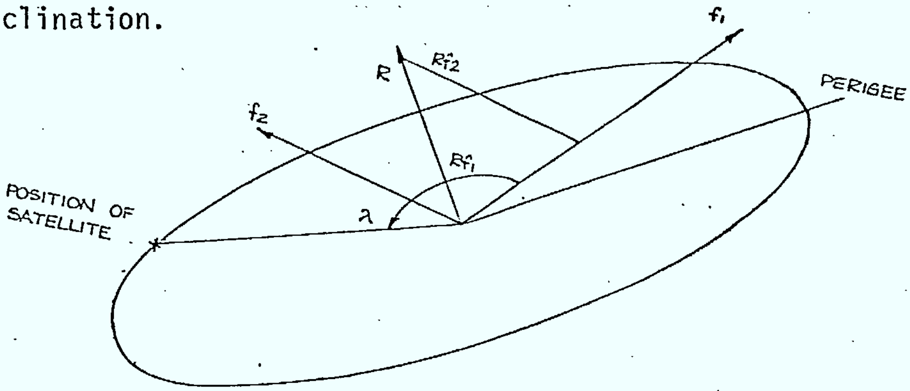
2.3.3 Velocity Space Elements (U6)

This set of orbital elements is (Cg_1, Cg_2, Cg_3, Rf_1, Rf_2, λ) called the velocity space set of elements because the first five elements have the dimension of velocity.

The first three elements, Cg_1, Cg_2 and Cg_3 are inertial components of a vector which is normal to the orbit plane. The magnitude of the vector, C, is such that

$$C = \frac{\mu m}{\text{orbit angular momentum}} = \frac{\mu}{|\vec{r} \times \vec{v}|} = \frac{\mu}{h}$$

The next three parameters are components in a coordinate frame which is obtained by rotating the inertial frame about the ascending node by the orbital inclination.



This rotation causes the inertial X axis to be rotated to the f_1 axis. The f_3 axis is normal to the orbit plane and the f_2 axis completes the right handed set.

The vector R is in the orbit plane and in the direction of the velocity vector at perigee. It may be expressed in terms of two components, R_{f1} and R_{f2}. The magnitude of R is given by

$$R = \left[2\varepsilon + C^2\right]^{\frac{1}{2}}$$

where ε = orbital energy per unit mass of the body

The angle λ is measured in the orbit plane from the f_1 axis to the position of the satellite.

There are several alternate expressions which may be obtained from the above, or which may be used to derive the above expressions.

Some of these alternate expressions are listed below

$$\overline{v} = \overline{C} \times \frac{\overline{r}}{|r|} + \overline{R}$$

$$\overline{R} = \tfrac{1}{2} (\overline{V} \text{ perigee} - \overline{V} \text{ apogee})$$

It may be shown that for an unperturbed orbit, the five parameters Cg_1, Cg_2, Cg_3, R_{f1} and R_{f2} are constant. The parameter λ steadily increases with time although only in a circular orbit is the rate of increase constant.

There are two singularities with the set of orbital elements. The first occurs when C=0, that is for an orbit straight up or down passing through the center of the earth. In this case, the orbit plane is undefined and parameters R_{f_1}, R_{f_2} and λ are undefined. The second occurs for an orbit with 180^0 inclination (purely retrograde). For this orbit, the ascending mode is undefined, and a 180^0 rotation about an undefined axis leads to an ambiguity in the location of the f_1, f_2, f_3 axes.

2.3.4 Velocity Space Elements Using Quaternions (U7)

An alternate form of velocity space element definition is given in reference 11.

There are seven orbital elements as follows:

C $= \sqrt{Cg_1^2 + Cg_2^2 + Cg_3^2}$ magnitude of C vector from section 2.3.3

$\left.\begin{array}{l} Rf_1 \\ Rf_2 \end{array}\right\}$ Defined in section 2.3.3

$\left.\begin{array}{l} e_{o1} \\ e_{o2} \\ e_{o3} \\ e_{o4} \end{array}\right\}$ A set of quaternions which specify rotation from inertial coordinates to the e_1 e_2 e_3 frame defined in section 2.2.9. See reference 14

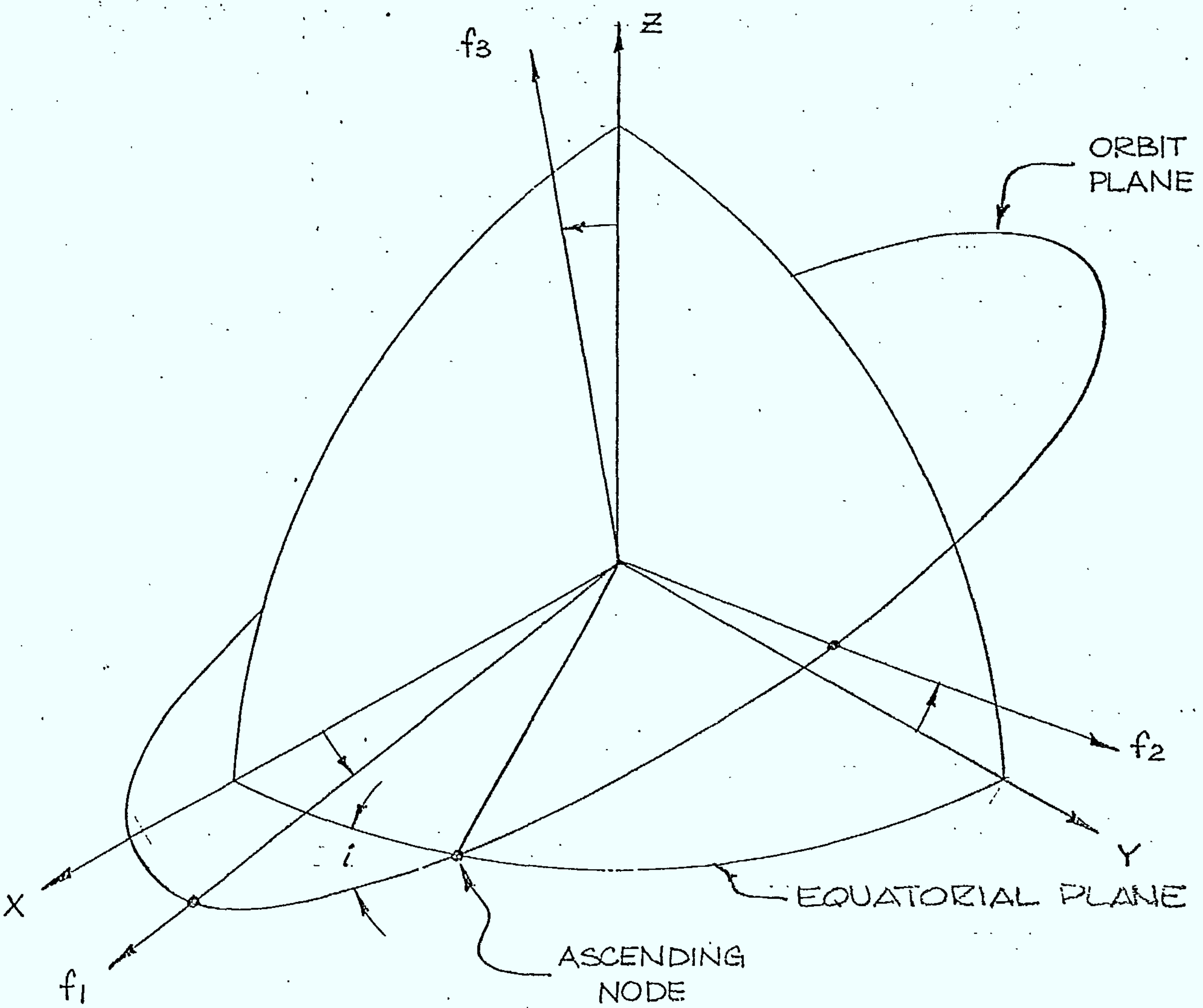


FIGURE 2.3/1 DEFINITION OF f_1, f_2, f_3 COORDINATES

2.4 Transformations

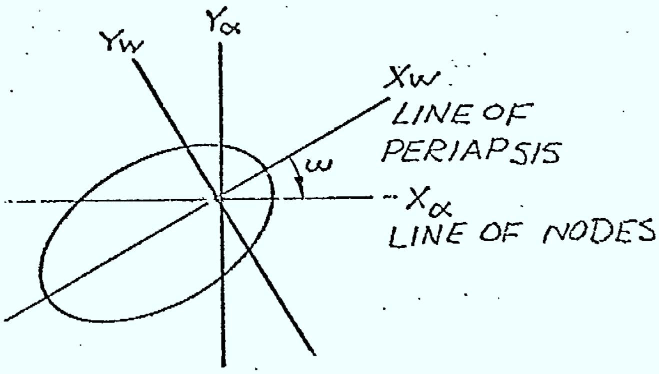
2.4.1 Transformation from Orbit Frame to A Fundamental Reference Frame

The orbit frame is defined in section 2.2.2 and the fundamental reference frames in section 2.1.

We assume that the following angles are known:

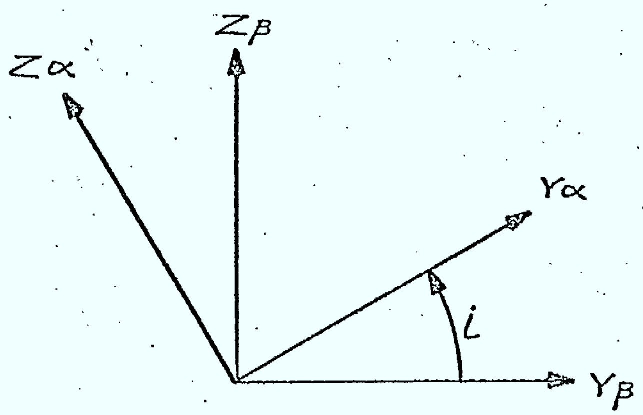
- w - argument of the perigee
- i - inclination
- Ω - longitude of the ascending node

First, rotate the frame about z_w (normal to orbit plane) by (-w) so that x points along the line of nodes. Call the new frame $x_\alpha\ y_\alpha\ z_\alpha$ and the transformation matrix C.



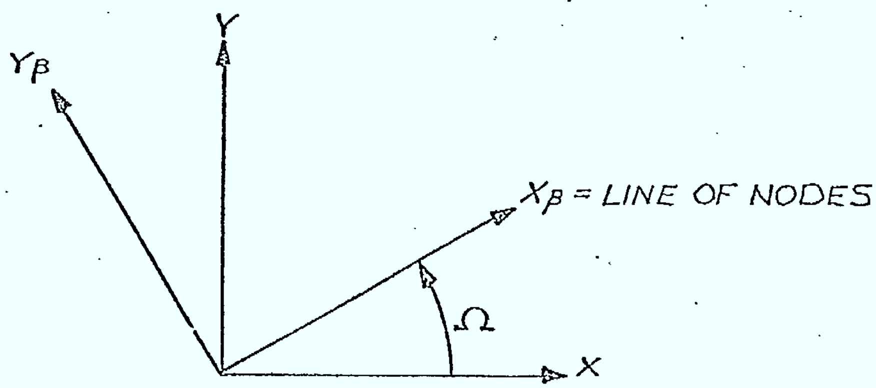
$$C = \begin{bmatrix} \cos w & -\sin w & 0 \\ \sin w & \cos w & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next, rotate the frame about x_α by $(-i)$ to bring the orbit plane into the equatorial plane; call the new coordinates $x_\beta y_\beta z_\beta$.



$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix}$$

Finally rotate the frame about z_β by $(-\Omega)$ so that the x axis is brought into the equinox.



$$A = \begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the transformation matrix from the orbital frame coordinates to the fundamental reference frame is:

$$D = ABC =$$

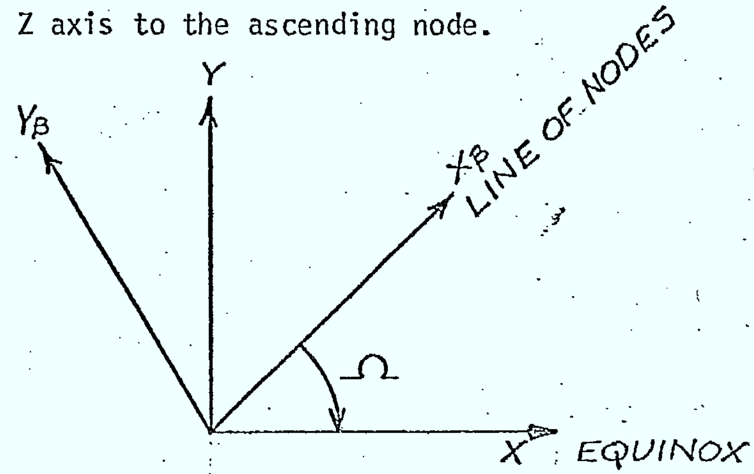
$$\begin{bmatrix} \cos w \cos\Omega - \sin w \cos i \sin\Omega & -\sin w \cos\Omega - \cos w \cos i \sin\Omega & \sin i \sin\Omega \\ \cos w \sin\Omega + \sin w \cos i \cos\Omega & -\sin w \sin\Omega + \cos w \cos i \cos\Omega & -\sin i \cos\Omega \\ \sin w \sin i & \cos w \sin i & \cos i \end{bmatrix}$$

Note that the angles i, w, Ω are measured from one of the fundamental reference frames given in section 2.1. For example, if i, w and Ω are measured from the ecliptic and mean equinox, the above transformation will go from orbital frame coordinates to the ecliptic and mean equinox frame.

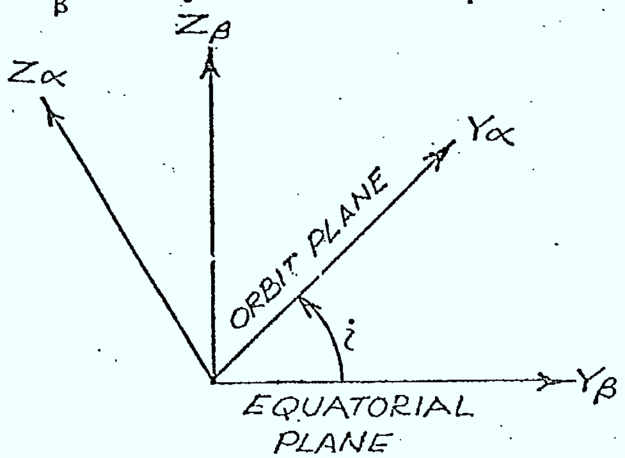
2.4.2 Transformation from a Fundamental Reference Frame to Orbit Frame Referenced to Ascending Node

The fundamental reference frames are defined in Section 2.1 and the orbit frame referenced to ascending node is defined in section 2.2.3.

First, rotate about the Z axis to the ascending node.



Next, rotate about the X_β axis by i to the orbit plane.



We note that these transformations are the inverse of the last two transformations in the previous section. The transformation matrix is given by

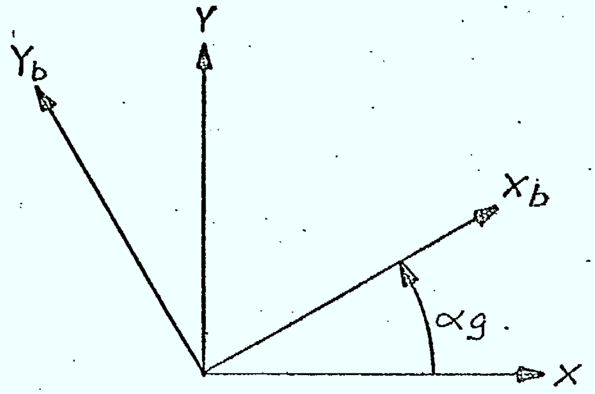
$$E = B^{-1}A^{-1} =$$

$$\begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega\cos i & \cos i\cos\Omega & \sin i \\ \sin i\sin\Omega & -\sin i\cos\Omega & \cos i \end{bmatrix}$$

2.4.3 Transformation from True Equator and True Equinox Frame to Earth Fixed Cartesian Frame

The True Equator and True Equinox Frame is defined in section 2.1 and the earth fixed cartesian frame is defined in section 2.2.4.

This transformation accounts for the earth's rotation.



Rotate about the Z axis by the Greenwich sidereal time, α_g

$$\begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} \cos \alpha_g & \sin \alpha_g & 0 \\ -\sin \alpha_g & \cos \alpha_g & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

The above matrix is called Λ

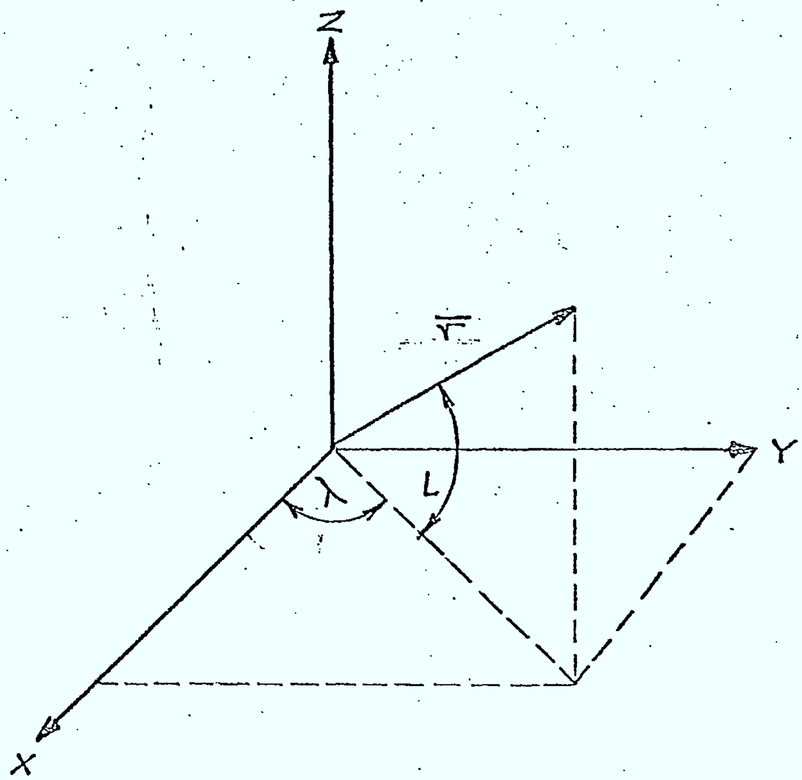
The value of α_g is found from section 7.4.

To transform velocities, it is necessary to account for the rotation of the earth

$$\begin{bmatrix} V_{xb} \\ V_{yb} \\ V_{zb} \end{bmatrix} = \begin{bmatrix} \cos\alpha_g & \sin\alpha_g & 0 \\ -\sin\alpha_g & \cos\alpha_g & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} +$$

$$\dot{\alpha}_g \begin{bmatrix} -\sin\alpha_g & \cos\alpha_g & 0 \\ -\cos\alpha_g & -\sin\alpha_g & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

2.4.4 Transformation from Cartesian to Spherical Coordinates



The transformation is defined by

$$L = \tan^{-1} \frac{r_z}{\sqrt{r_x^2 + r_y^2}} \quad \text{or} \quad L = \sin^{-1} \frac{r_z}{\sqrt{r_x^2 + r_y^2 + r_z^2}}$$

$$\lambda = \tan^{-1} \frac{r_y}{r_x}$$

$$r = \sqrt{r_x^2 + r_y^2 + r_z^2}$$

Note that the arctan function is a four quadrant function in the expression for longitude (λ).

2.4.5 Transformation from Spherical to Cartesian Coordinates

This is the inverse of the transformation described in section 2.4.4

$$\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = r \begin{bmatrix} \cos L \cos \lambda \\ \cos L \sin \lambda \\ \sin L \end{bmatrix}$$

2.4.6 Transformation from True Equator and True Equinox Frame to Topocentric Local Tangent Frame

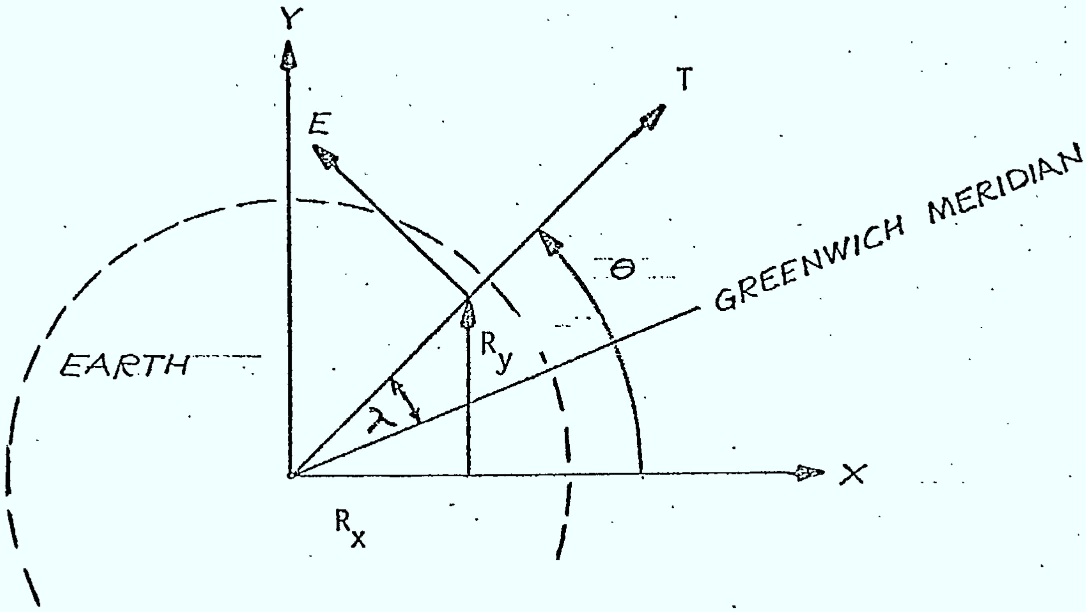
The True Equator and True Equinox frame is defined in section 2.1 and the topocentric local tangent frame is defined in section 2.2.7.

Translate the inertial frame to a new frame centred at the site.

$$\begin{bmatrix} \rho_x \\ \rho_y \\ \rho_z \end{bmatrix} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} - \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix}$$

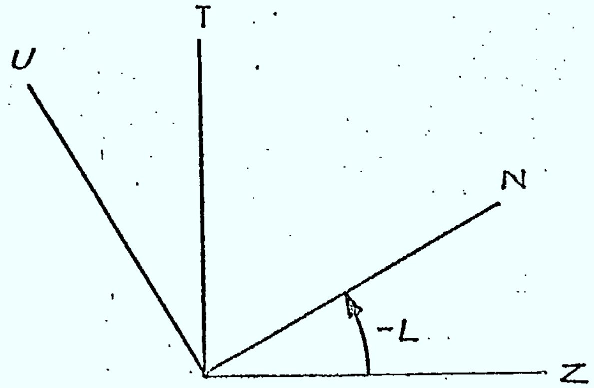
Where R_x, R_y, R_z are the coordinates of the site in the True Equator and True Equinox Frame.

Rotate about the z^1 axis by inertial longitude $\theta = \lambda + \alpha_g$ (longitude + Greenwich sidereal time)



$$
\begin{bmatrix} \rho_t \\ \rho_E \\ \rho_z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho_x \\ \rho_y \\ \rho_z \end{bmatrix}
$$

Rotate about Y axis by negative of latitude.



$$\begin{bmatrix} \rho_U \\ \rho_E \\ \rho_N \end{bmatrix} = \begin{bmatrix} \cos L & 0 & +\sin L \\ 0 & 1 & 0 \\ -\sin L & 0 & \cos L \end{bmatrix} \begin{bmatrix} \rho_T \\ \rho_E \\ \rho_Z \end{bmatrix}$$

Multiplying matrices gives:

$$\begin{bmatrix} \rho_U \\ \rho_E \\ \rho_N \end{bmatrix} = \begin{bmatrix} \cos L \cos \Theta & \cos L \sin \Theta & \sin L \\ -\sin \Theta & \cos \Theta & 0 \\ -\sin L \cos \Theta & -\sin L \sin \Theta & \cos L \end{bmatrix} \begin{bmatrix} \rho_x \\ \rho_y \\ \rho_z \end{bmatrix}$$

or, after rearranging

$$\begin{bmatrix} \rho_E \\ \rho_N \\ \rho_U \end{bmatrix} = \begin{bmatrix} -\sin \Theta & \cos \Theta & 0 \\ -\sin L \cos \Theta & -\sin L \sin \Theta & \cos L \\ \cos L \cos \Theta & \cos L \sin \Theta & \sin L \end{bmatrix} \left[\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} - \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \right]$$

$$= \left[E_{TOPO} \right] \left[\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} - \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \right],$$

2.4.7 Transformation from Topocentric Local Tangent to Observation Frame

This is an example of conversion from Cartesian to spherical coordinates which was given in Section 2.4.4. The equations are:

$$El = \sin^{-1} \frac{U}{\sqrt{E^2 + N^2 + U^2}}$$

$$Az = \tan^{-1} \frac{E}{N}$$

$$\rho = \sqrt{E^2 + N^2 + U^2}$$

2.4.8 Transformation from A Fundamental Reference Frame to Orbit Frame $f_1 f_2 f_3$

This transformation is a rotation of the inertial x y plane into the orbit plane. The axis of rotation is the intersection of the two planes, or the ascending node. The amount of rotation is the orbit inclination. See figure 2.3/1

To develop an analytical expression for the rotation matrix, it is more convenient to consider rotations about principal axes. Then the rotation is replaced by

i) A rotation about the inertial z axis to bring the X axis to the ascending node.

$$\begin{bmatrix} X' \\ Y' \\ Z \end{bmatrix} = \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

ii) A rotation about the X' axis by the orbit inclination to bring the Z axis normal to the orbit plane.

$$\begin{bmatrix} X' \\ Y'' \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix} \begin{bmatrix} X' \\ Y' \\ Z \end{bmatrix}$$

iii) A rotation about the normal to the orbit plane to bring the X axis back from the ascending node.

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X' \\ Y'' \\ f_3 \end{bmatrix}$$

The total transformation is:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \cos^2\Omega + \sin^2\Omega \cos i & \cos\Omega \sin\Omega \left[1-\cos i\right] & -\sin\Omega \sin i \\ \sin\Omega \cos\Omega \left[1-\cos i\right] & \sin^2\Omega + \cos^2\Omega \cos i & \cos\Omega \sin i \\ \sin\Omega \sin i & -\cos\Omega \sin i & \cos i \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

where the rotation matrix above is called $[\alpha]$

It is frequently necessary to express this matrix in terms of the velocity space parameters Cg_1, Cg_2 and Cg_3. We use the relationships

$$\cos i = \frac{Cg_3}{C}$$

$$\sin i = \frac{\sqrt{Cg_1^2 + Cg_2^2}}{C}$$

$$\cos \Omega = \frac{-Cg_2}{\sqrt{Cg_1^2 + Cg_2^2}}$$

$$\sin \Omega = \frac{Cg_1}{\sqrt{Cg_1^2 + Cg_2^2}}$$

Where $C = \sqrt{Cg_1^2 + Cg_2^2 + Cg_3^2}$

Then the matrix becomes

$$[\alpha] = \begin{bmatrix} \left(\frac{Cg_2^2}{Cg_1^2 + Cg_2^2} + \frac{Cg_1^2 Cg_3}{C(Cg_1^2 + Cg_2^2)}\right) & \left(\frac{-Cg_1 Cg_2}{Cg_1^2 + Cg_2^2} + \frac{Cg_1 Cg_2 Cg_3}{C(Cg_1^2 + Cg_2^2)}\right) & -\frac{Cg_1}{C} \\ \left(\frac{-Cg_1 Cg_2}{Cg_1^2 + Cg_2^2} + \frac{Cg_1 Cg_2 Cg_3}{C(Cg_1^2 + Cg_2^2)}\right) & \left(\frac{Cg_1^2}{Cg_1^2 + Cg_2^2} + \frac{Cg_2^2 Cg_3}{C(Cg_1^2 + Cg_2^2)}\right) & -\frac{Cg_2}{C} \\ \frac{Cg_1}{C} & \frac{Cg_2}{C} & \frac{Cg_3}{C} \end{bmatrix}$$

This form of the matrix has a singularity if both Cg_1 and Cg_2 are zero. To avoid this problem we substitute

$$Cg_1^2 + Cg_2^2 = C^2 - Cg_3^2$$

The previous matrix can be rearranged to give

$$[\alpha] = \frac{1}{C}\begin{bmatrix} Cg_3 + \dfrac{Cg_2^2}{C + Cg_3} & \dfrac{-Cg_1\,Cg_2}{C + Cg_3} & -Cg_1 \\ \dfrac{-Cg_1\,Cg_2}{C + Cg_3} & Cg_3 + \dfrac{Cg_1^2}{C + Cg_3} & -Cg_2 \\ Cg_1 & Cg_2 & Cg_3 \end{bmatrix}$$

or, alternatively

$$[\alpha] = \begin{bmatrix} 1 - \dfrac{Cg_1^2}{C(C + Cg_3)} & \dfrac{-Cg_1Cg_2}{C(C + Cg_3)} & -\dfrac{Cg_1}{C} \\ \dfrac{-Cg_1Cg_2}{C(C + Cg_3)} & 1 - \dfrac{Cg_2^2}{C(C + Cg_3)} & -\dfrac{Cg_2}{C} \\ \dfrac{Cg_1}{C} & \dfrac{Cg_2}{C} & \dfrac{Cg_3}{C} \end{bmatrix}$$

This form only has a singularity if $C = -Cg_3$, or the orbit is purely retrograde.

2.4.9 Transformation From Orbit Frame $f_1f_2f_3$ to Orbit Frame $e_1e_2e_3$

This is simply a rotation by λ about the f_3 axis to bring the f_1 axis to the direction of the satellite. The angle λ is one of the orbital elements defined in section 2.2.3.

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

This rotation matrix is called $[\lambda]$.

2.4.10 Transformation From Fundamental Reference Frame to Orbit Frame $e_1e_2e_3$

This is simply a product of the transformations in sections 2.4.8 and 2.4.9. The transformation matrix, which we call E is given by

$$E_{11} = \frac{1}{C}\left[\left(Cg_3 + \frac{Cg_2^2}{C + Cg_3}\right)\cos\lambda - \left(\frac{Cg_1Cg_2}{C + Cg_3}\right)\sin\lambda\right]$$

$$E_{12} = \frac{1}{C}\left[-\left(\frac{Cg_1Cg_2}{C + Cg_3}\right)\cos\lambda + \left(Cg_3 + \frac{Cg_1^2}{C + Cg_2}\right)\sin\lambda\right]$$

$$E_{13} = \frac{1}{C}\left[-Cg_1\cos\lambda - Cg_2\sin\lambda\right]$$

$$E_{21} = \frac{1}{C}\left[-\left(Cg_3 + \frac{Cg_2^2}{C + Cg_3}\right)\sin\lambda - \left(\frac{Cg_1\,Cg_2}{C + Cg_3}\right)\cos\lambda\right]$$

$$E_{22} = \frac{1}{C}\left[\left(\frac{Cg_1\,Cg_2}{C + Cg_3}\right)\sin\lambda + \left(Cg_3 + \frac{Cg_1^2}{C + Cg_3}\right)\cos\lambda\right]$$

$$E_{23} = \frac{1}{C}\left[Cg_1\sin\lambda - Cg_2\cos\lambda\right]$$

$$E_{31} = \frac{Cg_1}{C}$$

$$E_{32} = \frac{Cg_2}{C}$$

$$E_{33} = \frac{Cg_3}{C}$$

where

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} E \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

and $E = [\lambda] \; [\alpha]$

In inertial coordinates, the direction of the unit vectors e_1, e_2, e_3 are given by

$$\vec{e}_1 = \frac{\vec{r}}{r}$$

$$\vec{e}_3 = \frac{\vec{r} \times \vec{v}}{rv} = \frac{\vec{h}}{h}$$

$$\vec{e}_2 = \frac{\vec{r} \times \vec{v}}{rv} \times \frac{\vec{r}}{r}$$

Or, e_2 may be written

$$\vec{e}_2 = \frac{\vec{v}}{v} - \frac{(\vec{r}.\vec{v})\vec{r}}{vr^2} \equiv \vec{t}$$

Then the rotation matrix may be written

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \frac{r_x}{r} & \frac{r_y}{r} & \frac{r_z}{r} \\ t_x & t_y & t_z \\ \frac{h_x}{h} & \frac{h_y}{h} & \frac{h_z}{h} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

In U7 elements, the rotation matrix is shown in reference 14 to be

$$E_{11} = 1 - 2(e_{02}^2 + e_{03}^2)$$

$$E_{12} = 2(e_{01}e_{02} + e_{03}e_{04})$$

$$E_{13} = 2(e_{01}e_{03} - e_{02}e_{04})$$

$$E_{21} = 2(e_{01}e_{02} - e_{03}e_{04})$$

$$E_{22} = 1 - 2(e_{01}^2 + e_{03}^2)$$

$$E_{23} = 2(e_{02}e_{03} + e_{01}e_{04})$$

$$E_{31} = 2(e_{01}e_{03} + e_{02}e_{04})$$

$$E_{32} = 2(e_{02}e_{03} - e_{01}e_{04})$$

$$E_{33} = 1 - 2(e_{01}^2 + e_{02}^2)$$

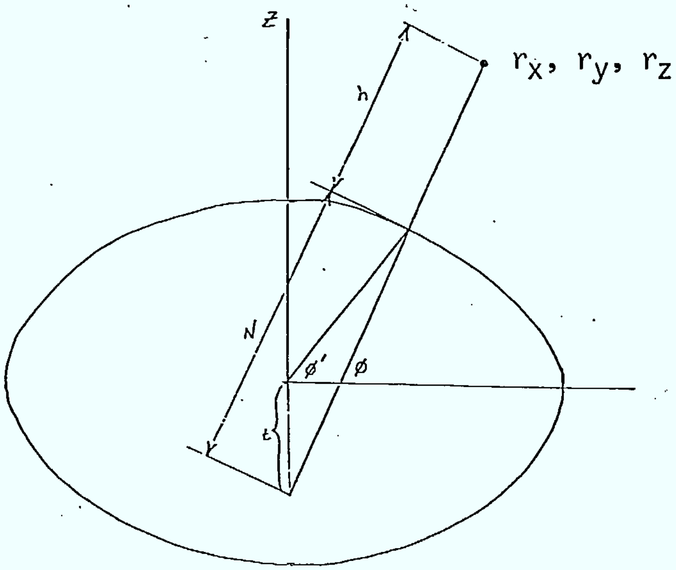
2.4.11 Transformation From Inertial Coordinates to Latitude, Longitude, Height

Given r_x, r_y, r_z, the inertial coordinates expressing satellite position referenced to the true equator and true equinox, the longitude (α) of the satellite is given by:

$$\alpha = \tan^{-1}\left(\frac{r_y}{r_x}\right) - \alpha_g$$

where α_g is the longitude of Greenwich (Section 7.4).

For latitude and height, the ellipsoidal shape of the earth must be taken into account. If this is done, the latitude and height are the geodetic latitude and height (the familiar values used in maps, for example). The figure illustrates, in an exaggerated way, the geometry.



By means of an iterative procedure, a value for t is determined, thus determining ϕ and h (the geodetic latitude and height). The procedure followed is:

let P be the desired precision

let K be the maximum no. of iterations

initialize: $t \leftarrow 0$

$i \leftarrow 0$

loop: $r_z' \leftarrow r_z + t$

$R \leftarrow \sqrt{r_x^2 + r_y^2 + r_z'^2}$ (R is N+h in the figure)

$\phi \leftarrow \sin^{-1}\left(\frac{r_z'}{R}\right)$ (estimated latitude)

$N \leftarrow \frac{Re}{\sqrt{1-e^2 \sin^2 \phi}}$ (Re is earth radius at the equator; e is the eccentricity)

$t \leftarrow N e^2 \sin\phi$

$i \leftarrow i + 1$

if $i \neq 1$, then if $\phi - \phi_{old} \leq P$, then go to exit

$\phi_{old} \leftarrow \phi$

if $i<K$, then repeat loop

exit: ϕ is the geodetic latitude at this point

$h \leftarrow R-N$

2.5 Conversions Between Sets of Orbital Elements

2.5.1 Keplerian to Inertial

This transformation uses eccentric anomaly, E, to define the position in the orbit plane. If mean anomaly is given, the eccentric anomaly may be found from section 2.5.6. If true anomaly is given, the eccentric anomaly is found from

$$\tan {}^{E}/_{2} = \sqrt{\frac{1 - e}{1 + e}} \quad \tan {}^{\phi}/_{2}$$

The position in the orbit frame is obtained from Appendix A.

$$x_w = a\,(\cos E - e)$$

$$y_w = a\sqrt{1 - e^2}\ \sin E$$

$$r = a\,(1 - e \cos E)$$

The velocity is found by differentiating the expressions for x_w and y_w.

$$\dot{x}_w = -a(\sin E)\,\dot{E}$$

$$\dot{y}_w = a\sqrt{1 - e^2}\ (\cos E)\,\dot{E}$$

The quantity $\dot{E}$ can be found by differentiating Kepler's eqn:

$$\dot{M} = \dot{E}\,(1 - e \cos E)$$

Using the fact $\dot{M} = \sqrt{\frac{\mu}{a^3}}$ and the expression for r above

$$\dot{E} = \frac{1}{r}\sqrt{\frac{\mu}{a}}$$

Therefore, the components of the velocity vector in the orbit plane frame are

$$\dot{x}_W = -\sqrt{\frac{\mu a}{r}} \sin E$$

$$\dot{y}_W = \sqrt{\frac{\mu a(1-e^2)}{r}} \cos E$$

The magnitude of the velocity is therefore

$$v = \sqrt{\frac{\mu a}{r}} \quad \sqrt{1-e^2 \cos^2 E}$$

Using the position and velocity in the orbit plane, and the transformation matrix from section 2.4.1 we obtain

$$\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = D \begin{bmatrix} x_W \\ y_W \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = D \begin{bmatrix} \dot{x}_W \\ \dot{y}_W \\ 0 \end{bmatrix}$$

Note: For the suns orbit about the earth, $i=0$ and $\Omega=0$, so position may be found from

$$\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = C \begin{bmatrix} x_W \\ y_W \\ 0 \end{bmatrix}$$

2.5.2 Inertial Coordinates to Keplerian Elements

The transformation starts with the following elements

$$\left.\begin{array}{l} r_x \\ r_y \\ r_z \end{array}\right\} \quad \text{Components of satellite position in Equator and Equinox Frame}$$

$$\left.\begin{array}{l} v_x \\ v_y \\ v_z \end{array}\right\} \quad \text{Components of satellite velocity in Equator and Equinox Frame}$$

Calculation of semimajor axis starts with the equations for radial distance and velocity which were developed in Appendix A and section 2.5.1.

$$r = a(1-e \cos E)$$

$$V = \sqrt{\frac{\mu a}{r}} \sqrt{1-e^2\cos^2 E}$$

Substituting for e cos E in the second equation gives

$$V = \sqrt{\frac{\mu a}{r}} \sqrt{1 - (1 - \frac{r}{a})^2}$$

or, after rearranging

$$a = \frac{1}{\frac{2}{r} - \frac{V^2}{\mu}}$$

The above equations may be rearranged to give

$$e \cos E = \frac{rv^2}{\mu} - 1$$

A similar expression in e sin E may be obtained by using the dot product of r and v vectors. From the definition of a dot product

$$\vec{r} \cdot \vec{v} = r\,v\,\cos\alpha$$

where α is the angle between r and v.

Substituting for v from section 2.5.2 and cos α from appendix A gives

$$\vec{r} \cdot \vec{v} = r\,\frac{\sqrt{\mu a}}{r}\,\sqrt{1-e^2\cos^2 E}\;\frac{e \sin E}{\sqrt{1-e^2\cos^2 E}}$$

or, after rearranging

$$e \sin E = \frac{\vec{r} \cdot \vec{v}}{\sqrt{\mu a}} \qquad \text{or,}$$

$$e \sin E = \frac{r_x v_x + r_y v_y + r_z v_z}{\sqrt{\mu a}}$$

The orbital angular momentum is $\vec{r} \times \vec{v}$, and has components and magnitude as follows

$$h_x = r_y v_z - r_z v_y$$

$$h_y = r_z v_x - r_x v_z$$

$$h_z = r_x v_y - r_y v_x$$

$$h = \sqrt{h_x^2 + h_y^2 + h_z^2}$$

The orbit inclination and longitude of ascending node may be found from geometry using the property that the angular momentum is normal to the orbit plane.

$$i = \cos^{-1} \frac{h_z}{h}$$

$$\Omega = \tan^{-1} \frac{h_x}{-h_y} \qquad \text{if } i \neq 0$$

$$\text{or} \quad \Omega = 0 \text{ by definition if } i = 0$$

In order to calculate other angles, it is necessary to calculate the angle from the ascending node to the position, that is, $w + \phi$. First we calculate the position of the satellite in the orbit frame referenced to the ascending node, using the transformation from section 2.4.2.

$$x_p = r_x \cos\Omega + r_y \sin\Omega$$

$$y_p = -r_x \sin\Omega \cos i + r_y \cos i \cos\Omega + r_z \sin i$$

$$z_p = r_x \sin i \sin\Omega - r_y \sin i \cos\Omega + r_z \cos i$$

Substituting for i and Ω from above gives

$$x_p = \frac{h_x r_y - h_y r_x}{\sqrt{h_x^2 + h_y^2}}$$

$$y_p = \frac{\sqrt{h_x^2 + h_y^2}}{h} r_z - \frac{h_z}{h\sqrt{h_x^2 + h_y^2}} (h_x r_x + h_y r_y)$$

$$= \frac{h\, r_z}{\sqrt{h_x^2 + h_y^2}}$$

$z_p = 0$, if $i \neq 0$

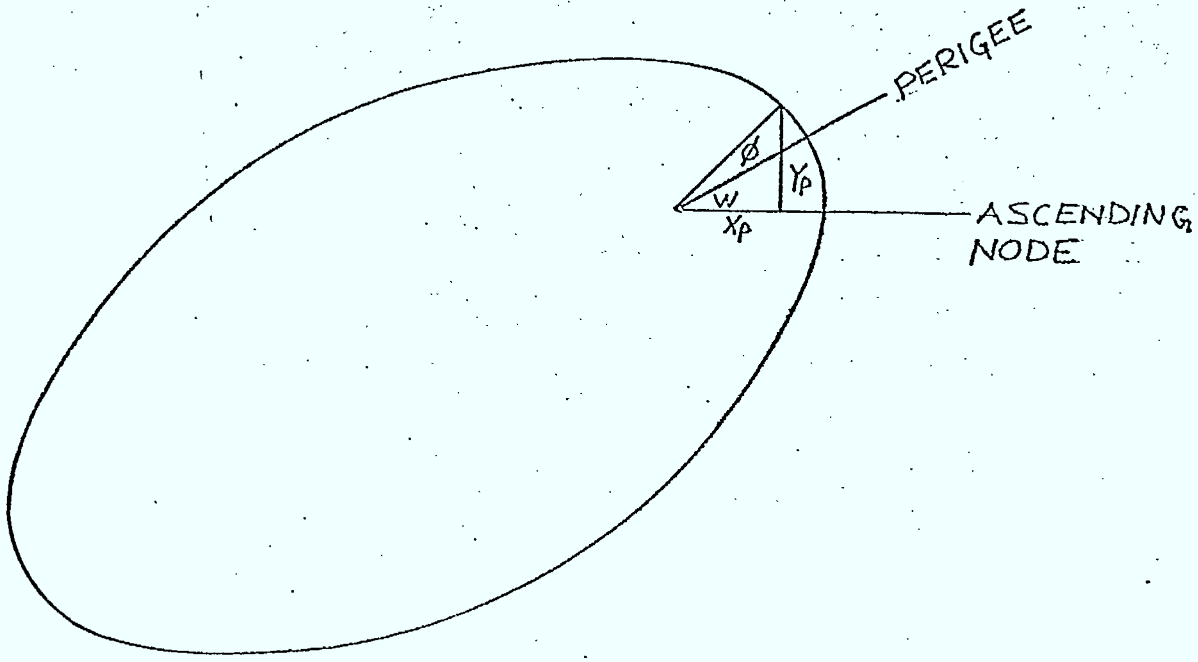
If $i = 0$ then

$x_p = r_x$ and

$y_p = r_y$.

The value $w + \phi$ can be found from the diagram

$$w + \phi = \tan^{-1} \frac{y_p}{x_p} = \tan^{-1} \frac{h\, r_z}{(h_x r_y - h_y r_x)}$$



The eccentric anomaly E is given by

$$E = \tan^{-1} \frac{e \sin E}{e \cos E} \qquad \text{if } e \neq 0$$

$$E = \tan^{-1} \frac{y_p}{x_p} \qquad \text{if } e = 0$$

The remaining elements e, ϕ and w are

$$e = \sqrt{(e \sin E)^2 + (e \cos E)^2}$$

$$\phi = 2 \tan^{-1}\left[\sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2}\right]$$

$$w = (w + \phi) - \phi$$

2.5.3 Conversion from I.C. to Velocity Space

The calculation of Cg_1, Cg_2, and Cg_3, uses the definition

$$\vec{C} = \frac{\mu \vec{h}}{h^2}$$

$$h_x = r_y v_z - r_z v_y$$

$$h_y = r_z v_x - r_x v_z$$

$$h_z = r_x v_y - r_y v_x$$

$$Cg_1 = \frac{C h_x}{h}$$

$$Cg_2 = \frac{C h_y}{h}$$

$$Cg_3 = \frac{C h_z}{h}$$

where

$$h = \sqrt{h_x^2 + h_y^2 + h_z^2}$$

$$C = \mu/h$$

Then λ is found in the f_1, f_2, f_3 frame $\cos \lambda = \frac{r_{f_2}}{r}$

Then r_{f_2} is calculated from r_x, r_y, r_z using the conversions from section 2.4.8. This gives

$$\cos\lambda = \frac{1}{r}\left[r_x - \frac{r_x Cg_1^2}{C(C+Cg_3)} - \frac{r_y Cg_1 Cg_2}{C(C+Cg_3)} - \frac{r_z Cg_1}{C}\right]$$

This may be rearranged to give

$$\cos\lambda = \frac{1}{r}\left[r_x - \frac{r_z Cg_1}{C+Cg_3}\right] - \frac{Cg_1}{rC(C+Cg_3)}\left[r_x Cg_1 + r_y Cg_2 + r_z Cg_3\right]$$

The right-hand term is $\vec{r} \cdot \vec{C}$ and it is zero because the two vectors, $\vec{r}$ and $\vec{C}$ are orthagonal. Then

$$\cos\lambda = \frac{1}{r}\left[r_x - \frac{r_z Cg_1}{C+Cg_3}\right]$$

Similarly, we find

$$\sin\lambda = \frac{1}{r}\left[r_y - \frac{r_z Cg_2}{C+Cg_3}\right]$$

From Section 2.3.3 we have

$$\vec{C} = \frac{\mu}{|\vec{r} \times \vec{v}|}, \text{ or}$$

$$C = \frac{\mu}{r\, v_{e_2}}$$

$$v_{e_2} = \frac{\mu}{rC} = \frac{h}{r}$$

and $Ve_1 = \dfrac{\vec{r} \cdot \vec{v}}{r}$

$$= \frac{r_x v_x + r_y v_y + r_z v_z}{r}$$

Then from Section 2.3.3

$$\vec{v} = \vec{C} \times \frac{\vec{r}}{|r|} + \vec{R}$$

or

$$Ve_1 = C + Re_1$$

$$Ve_2 = Re_2$$

Substituting

$$Re_1 = Rf_1 \cos \lambda + Rf_2 \sin \lambda$$

$$Re_2 = Rf_1 \sin \lambda + Rf_2 \cos \lambda$$

and solving for Rf_1 and Rf_2 gives

$$Rf_1 = Ve_1 \cos \lambda - (Ve_1 - C) \sin \lambda$$

$$Rf_2 = Ve_1 \sin \lambda + (Ve_1 - C) \cos \lambda$$

2.5.4 Transformation from Velocity Space to Inertial

For this transformation, we first develop expressions for position and velocity in the orbit frame $e_1e_2e_3$. Then the transformation from section 2.4.10 is used to convert to inertial coordinates.

The velocity in the $e_1e_2e_3$ frame is obtained from

$$\vec{v} = \vec{C} \times \frac{\vec{r}}{|\vec{r}|} + \vec{R}$$

where

$$\begin{bmatrix} R_{e1} \\ R_{e2} \end{bmatrix} = \begin{bmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{bmatrix} \begin{bmatrix} R_{f1} \\ R_{f2} \end{bmatrix}$$

and $\vec{C} \times \frac{\vec{r}}{|\vec{r}|} = C\ e_2$ since

vector C is in e_3 direction and r is in e_1 direction.

Then

$$v_{e1} = R_{f1} \cos\lambda + R_{f2} \sin\lambda$$

$$V_{e2} = C - R_{f1} \sin\lambda + R_{f2} \cos\lambda$$

the radius vector is obtained from

$$C = \frac{\mu}{|\vec{r} \times \vec{v}|} \qquad \text{(from section 2.3.3)}$$

$$C = \frac{\mu}{r\ v_{e2}} \qquad \text{or}$$

$$r = \frac{\mu}{Cv_{e2}}$$

where $C = \sqrt{Cq_1^{\ 2} + Cq_2^{\ 2} + Cq_3^{\ 2}}$

Now having obtained position and velocity in the $e_1e_2e_3$ frame, we transform to X Y Z.

$$\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} E^T \end{bmatrix} \begin{bmatrix} r_{e1} \\ 0 \\ 0 \end{bmatrix} \qquad \text{or}$$

$$r_x = \frac{\mu}{Cv_{e2}} \left[\frac{1}{C} \left[\left[Cg_3 + \frac{Cg_2{}^2}{C + Cg_3} \right] \cos\lambda - \frac{Cg_1 \, Cg_2}{C + Cg_3} \sin\lambda \right] \right]$$

$$r_y = \frac{\mu}{Cv_{e2}} \left[\frac{1}{C} \left[\left[Cg_3 + \frac{Cg_1{}^2}{C + Cg_3} \right] \sin\lambda - \frac{Cg_1 \, Cg_2}{C + Cg_3} \cos\lambda \right] \right]$$

$$r_z = \frac{\mu}{Cv_{e2}} \left[\frac{C_{g1}}{C} \sin\lambda - \frac{Cg_2}{C} \cos\lambda \right]$$

Similarly the velocity is given by

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} E^T \end{bmatrix} \begin{bmatrix} v_{e1} \\ v_{e2} \\ 0 \end{bmatrix}$$

2.5.5 Transformation from Velocity Space to Keplerian

Semi-major axis

From reference 1 page 28, we obtain

$$\varepsilon = -\frac{\mu}{2a}$$

Using the definition of R, or

$$R = \sqrt{2\varepsilon + C^2}$$

we may solve for a in terms of R and C. This gives

$$a = \frac{\mu}{C^2 - R^2}$$

Eccentricity

From reference 1 page 29, we obtain

$$e = \sqrt{1 + \frac{2\varepsilon h^2}{\mu^2}}$$

Substituting $C = \frac{\mu}{h}$ and $R = \sqrt{2\varepsilon + C^2}$

and simplifying gives

$$e = \frac{R}{C}$$

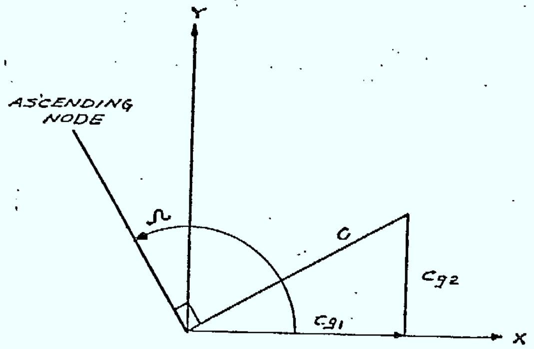
Inclination

The orbit inclination is the angle between the orbit normal and the inertial z axis. Since the vector C is in the direction of the orbit normal.

$$\cos i = \frac{C.z}{|C|} = \frac{Cg_3}{C}$$

Longitude of ascending node

This may be found from the projection of the orbit normal in the X Y plane as shown.



From the geometry and noting that the ascending node is 90^0 from the projection of the C vector we obtain

$$\tan(\Omega - 90^0) = \frac{Cg_2}{Cg_1}$$

or

$$\Omega = \tan^{-1}\left[\frac{Cg_1}{-Cg_2}\right]$$

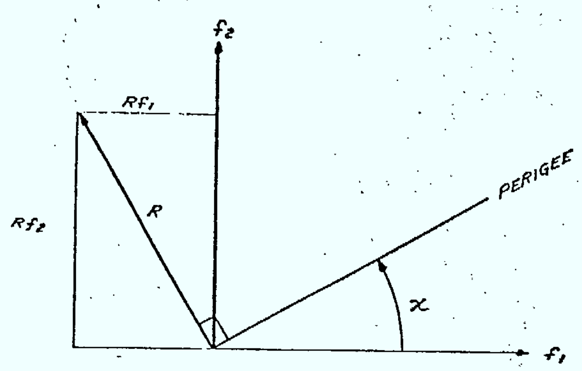
and

$$\mathrm{sir}\ \Omega = \frac{Cg_1}{\sqrt{Cg_1^{\ 2} + Cg_2^{\ 2}}}$$

$$\cos\Omega = \frac{-Cg_2}{\sqrt{Cg_1^{\ 2} + Cg_2^{\ 2}}}$$

Argument of perigee and true anomaly

The perigee in the orbit frame $f_1f_2f_3$ is 90^0 before the R vector as shown.



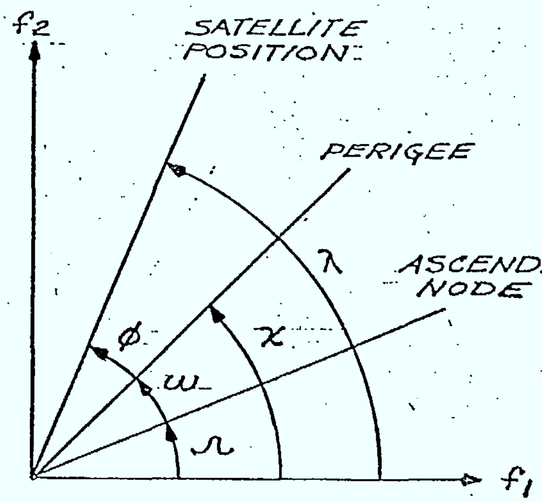
First, we find the angle χ

$$\sin\chi = \frac{-R_{f1}}{\sqrt{R_{f1}^2 + R_{f2}^2}} = -\frac{R_{f_1}}{R}$$

$$\cos\chi = \frac{R_{f2}}{\sqrt{R_{f1}^2 + R_{f2}^2}}$$

$$\tan\chi = \frac{-R_{f1}}{R_{f2}}$$

Then the remaining angles may be found from the following diagram.



argument of perigee

$$\omega = \chi - \Omega$$

true anomaly

$$\phi = \lambda - \chi$$

2.5.6 Mean Anomaly to True Anomaly

As a first step, we find the eccentric anomaly, E, from the equation

$$M = E - e \sin E$$

Since the equation cannot be solved analytically, we use the following iterative technique

M is in radians; assume $0 \le e < 1$
P is the desired precision
Kmax is the maximum number of iterations

$E_0 \leftarrow M + e \sin M$
if $e < .3$ then $E_0 \leftarrow E_0 + \frac{e^2}{2} \sin 2M$

$K \leftarrow 0$

loop: $K \leftarrow K+1$

$$E_k \leftarrow E_{k-1} - \frac{E_{k-1} - e \sin E_{k-1} - M}{1 - e \cos E_{k-1}}$$

if $\left| E_k - E_{k-1} \right| < P$ then done

if $k >$ Kmax then nonconvergent
repeat

The resulting E_k is a close approximation to E in radians.

Having calculated eccentric anomaly, the true anomaly, ϕ, is found using the following equation from reference 4.

$$\tan(\phi/2) = \sqrt{\frac{1 + e}{1 - e}} \quad \tan(E/2)$$

2.5.7 True Anomaly to Mean Anomaly

As a first step, the eccentric anomaly, E, is found using the following relationship from reference 4

$$\tan\left(\frac{E}{2}\right) = \sqrt{\frac{1 - e}{1 + e}} \quad \tan\left(\frac{\phi}{2}\right)$$

The mean anomaly, M, in radians is found from the eccentric anomaly in radians using Keplers eouation

$$M = E - e \sin E$$

2.5.8 Transformation from Keplerian to Velocity Space

This is the inverse of the transformation in Section 2.5.5. From Section 2.5.5 we get

$$a = \frac{\mu}{C^2 - R^2}$$

and $e = \frac{R}{C}$

Solving for R and C gives

$$C = \sqrt{\frac{\mu}{a\,(1-e)}}$$

$$R = e\sqrt{\frac{\mu}{a\,(1-e)}}$$

Then using the geometry from Section 2.5.5 gives

$$Cg_3 = C \cos i$$

$$Cg_1 = \sqrt{C^2 - Cg_3^2}\ \sin \Omega = C \sin i \sin \Omega$$

$$Cg_2 = \sqrt{C^2 - Cg_3^2}\ \cos \Omega = C \sin i \cos \Omega$$

$$R_{f_1} = -R \sin \chi = -R \sin (\omega + \Omega)$$

$$R_{f_2} = R \cos \chi = R \cos (\omega + \Omega)$$

$$\lambda = \omega + \Omega + \phi$$

2.5.9 Inertial Coordinates to U7

The calculation of elements C, R_{f1} and R_{f2} is given in section 2.5.3 as well as intermediate terms λ, h_x, h_y, h_z, h.

Then

$$e_{01} = \frac{1}{\sqrt{2h\,(h + h_z)}} \left(h_x \sin\frac{\lambda}{2} - h_y \cos\frac{\lambda}{2}\right)$$

$$e_{02} = \frac{1}{\sqrt{2h\,(h + h_z)}} \left(h_x \cos\frac{\lambda}{2} + h_y \sin\frac{\lambda}{2}\right)$$

$$e_{03} = \sqrt{\frac{h + h_z}{2h}} \quad \sin\frac{\lambda}{2}$$

$$e_{04} = \sqrt{\frac{h + h_z}{2h}} \quad \cos\frac{\lambda}{2}$$

2.5.10 U7 to Inertial Coordinates

These are given in reference 14 and are first found in the e_1 e_2 e_3 frame

$$r = r_{e1} = \frac{\mu}{C\ Ve2}$$

$$V_{e1} = R_{f1} \cos\lambda + R_{f2} \sin\lambda$$

$$V_{e2} = C - R_{f1} \sin\lambda + R_{f2} \cos\lambda$$

$$V_{e3} = 0$$

Then, the elements are rotated to the inertial frame using the transformation in section 2.4.10.

$$\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = E^T \begin{bmatrix} r \\ o \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = E^T \begin{bmatrix} v_{e1} \\ v_{e2} \\ 0 \end{bmatrix}$$

where T denotes transpose.

2.6 Transformations between inertial reference frames

2.6.1 Mean Equator and Equinox to True Equator and Equinox

To calculate this transformation, the following are obtained from section 7

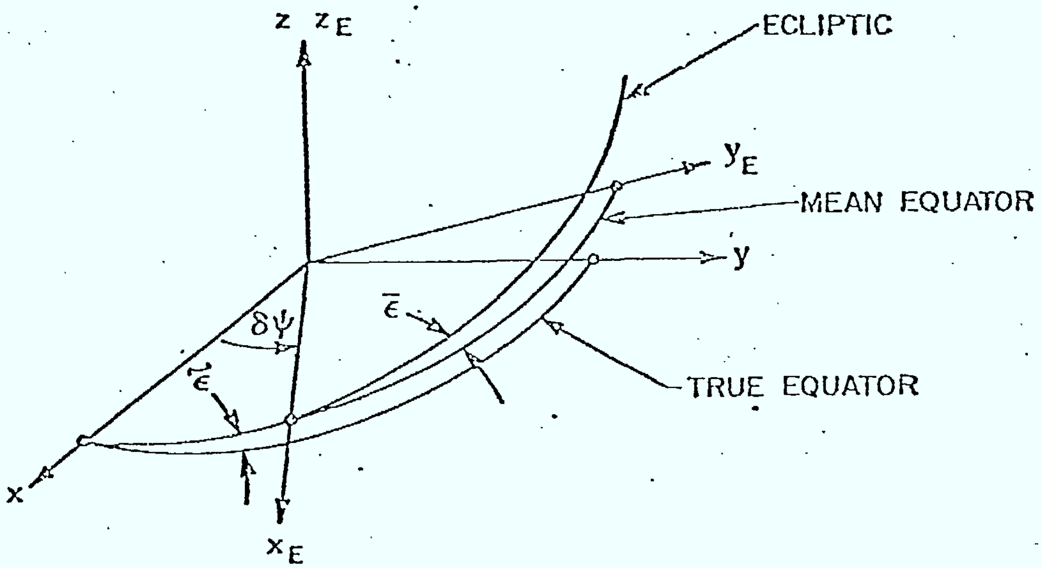
$\bar{\epsilon}$ = mean obliquity of the ecliptic
$\delta\epsilon$ = true obliquity less mean obliquity
$\delta\psi$ = longitude of true equinox less longitude of mean equinox
$\tilde{\epsilon}$ = $\bar{\epsilon} + \delta\epsilon$ = true obliquity

the rotation from the mean equator and equinox of epoch to the true equator and equinox is given by the following three rotations: (see sketch)

$R_x(\bar{\epsilon})$ ~ the rotation about the x_e axis into the ecliptic of epoch

$R_z(\delta\psi)$ ~ the negative rotation about the ecliptic pole, through the nutation in longitude to the true vernal equinox of epoch

$R_x(\tilde{\epsilon})$ ~ the rotation about the new x-axis through the true obliquity to the true equator of epoch.



The total rotation matrix may be expressed as

$$N = R_x(\tilde{\epsilon})\, R_z(\delta\psi)\, R_x(\bar{\epsilon}) = \{n_{ij}\}.$$

Denoting the true of epoch coordinates by $\bar{r}$, we have

$$\bar{r} = N\, \bar{r}_E$$

where the elements of N are

$$n_{11} = \cos\delta\psi$$

$$n_{12} = -\sin\delta\psi\cos\bar{\epsilon}$$

$$n_{13} = -\sin\delta\psi\sin\bar{\epsilon}$$

$$n_{21} = \sin\delta\psi\cos\tilde{\epsilon}$$

$$n_{22} = \cos\delta\psi\cos\tilde{\epsilon}\cos\bar{\epsilon} + \sin\tilde{\epsilon}\sin\bar{\epsilon}$$

$$n_{23} = \cos\delta\psi\cos\tilde{\epsilon}\sin\bar{\epsilon} - \sin\tilde{\epsilon}\cos\bar{\epsilon}$$

$$n_{31} = \sin\delta\psi\sin\tilde{\epsilon}$$

$$n_{32} = \cos\delta\psi\sin\tilde{\epsilon}\cos\bar{\epsilon} - \cos\tilde{\epsilon}\sin\bar{\epsilon}$$

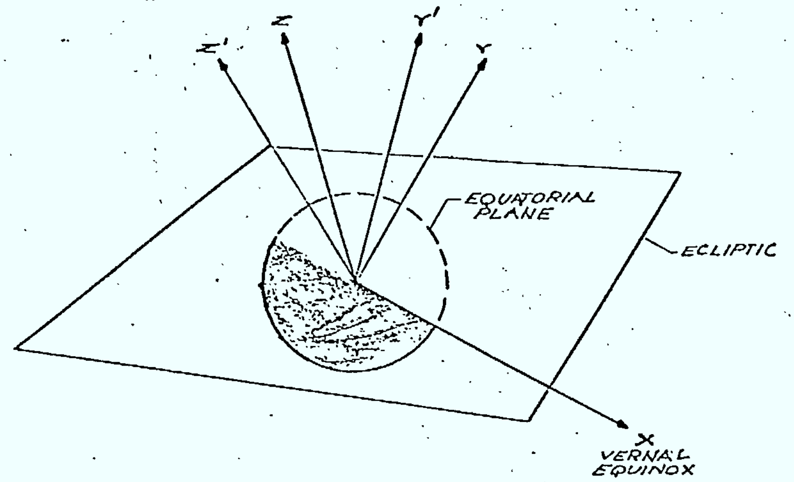
$$n_{33} = \cos\delta\psi\sin\tilde{\epsilon}\sin\bar{\epsilon} + \cos\tilde{\epsilon}\cos\bar{\epsilon}.$$

The time derivative of N is assumed to be negligible. Therefore the veloci coordinates are transformed as follows

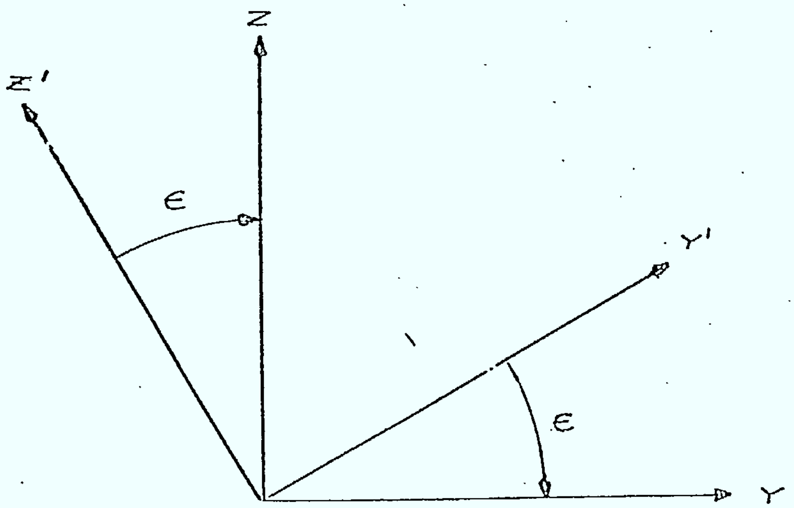
$$\dot{\bar{r}} = N\, \dot{\bar{r}}_E.$$

2.6.2 Ecliptic and Mean Equinox to Mean Equator and Mean Equinox

This is a rotation by the mean obliquity of the ecliptic, about the vernal equinox.



Let position in mean equator and mean equinox be r_x, r_y, r_z and position in ecliptic and mean equinox be r_x^1, r_y^1, r_z^1.



$$\begin{bmatrix} r_x^1 \\ r_y^1 \\ r_z^1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\epsilon & -\sin\epsilon \\ 0 & \sin\epsilon & \cos\epsilon \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

$$= \left[\epsilon\right] \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

The same transformation may be used for the velocity since the rate of change of ϵ is negligible.

A similar transformation is used to go from ecliptic and true equinox to true equator and true equinox. In this case, the amount of rotation is the true obliquity of the ecliptic.

2.6.3 True Equator and True Equinox to True Equator and True Equinox of Different Time

This consists of three rotations as follows:

Rotate about the X axis by the obliquity of the ecliptic at time t_i to bring the XY plane to the ecliptic.

Rotate about the Z axis by the precession of the equinoxes plus the difference in nutation in longitude from time t_i to t_f.

Rotate about the X axis by the obliquity of the ecliptic at time t_f to bring the XY plane to the equator.

The total rotation equation using first order approximations with $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ is

$$\begin{bmatrix} X_f \\ Y_f \\ Z_f \end{bmatrix} = \begin{bmatrix} 1 & -\beta\cos\varepsilon_i & -\beta\sin\varepsilon_i \\ \beta\cos\varepsilon_f & 1 & (\varepsilon_i - \varepsilon_f) \\ \beta\sin\varepsilon_f & (\varepsilon_f - \varepsilon_i) & 1 \end{bmatrix} \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix}$$

where ε is the true obliquity of the ecliptic (radians)

β is the precession of the equinoxes plus the difference in nutation from time t_i to t_f (radians)

$\beta = 7.738 \times 10^{-12} \; (t_f - t_i) + \delta\psi_f - \delta\psi_i$

2.7 Transformation of Error Covariance Matrices

The error covariance may be defined as

$$Px = \varepsilon \left[(\tilde{X})\ (\tilde{X})^T \right]$$

where

Px is the error covariance matrix with X as state variables

$\varepsilon\ (X)$ is the expectation of X

$\tilde{X}$ is the difference between the estimate and the true value of X

$\tilde{X}^T$ is the transpose of $\tilde{X}$

It may be useful to find the error covariance for a different state vector. An example of this occurs when we wish to transform the state and error covariance from Cartesian orbital elements to velocity space orbital elements. Then the error covariance matrix in the new orbital elements is

$$P_z = \varepsilon \left[(\tilde{z})\ (\tilde{z})^T \right]$$

An expression for the new orbital elements of the form

$\tilde{z} = A\ \tilde{x}$ is required.

Then substituting for $\tilde{x}$

$$P_z = \varepsilon \left[(A\ \tilde{x})\ (A\ \tilde{x})^T \right]$$

$$= \varepsilon \left[A\ \tilde{x}\ \tilde{x}^T\ A^T \right]$$

$$= A \left[\varepsilon\ (\tilde{x}\ \tilde{x}^T) \right] A^T$$

$$= A\ P_x\ A^T$$

3.0 EQUATIONS OF MOTION

3.1 Inertial Frames

It is necessary to integrate the equations of motion in an inertial frame, in order to avoid introduction of fictitious forces such as centrifugal force and coreolis force into the equations of motion. However, the frames used herein are not truly inertial, so it is necessary to list the approximations used.

A truly inertial frame should have no translation as well as no rotation. However, it is more convenient to consider the centre of the earth as the origin of the co-ordinate system. We must then calculate all accelerations relative to the acceleration of the center of the earth, in particular the solar and lunar perturbations. The inertial reference frames used for integration of the equations of motion are based on the true equator and equinox of date. This is a convenient frame for calculation of gravity gradient forces. In any other frame, it is necessary to transform satellite position to true equator and equinox frame and then transform the perturbing accelerations back to the working frame. However, there is an approximation involved since the precession and nutation of the equinoxes is neglected. If higher accuracy is required, the orbital elements are corrected for this precession and nutation at each integration step using the transformation in section 2.6.3.

3.2 Basic Forms

3.2.1 Inertial Coordinates

The general form of the equations of motion is:

$$\frac{dx}{dt} = f\,(x,\ t)$$

OR

$$\frac{dr_x}{dt} = v_x$$

$$\frac{dr_y}{dt} = v_y$$

$$\frac{dr_z}{dt} = v_z$$

$$\frac{dv_x}{dt} = -\frac{\mu}{r^3} r_x + a_x\,(r_x,\ r_y,\ r_z,\ v_x,\ v_y,\ v_z,\ t)$$

$$\frac{dv_y}{dt} = -\frac{\mu}{r^3} r_y + a_y\,(r_x,\ r_y,\ r_z,\ v_x,\ v_y,\ v_z,\ t)$$

$$\frac{dv_z}{dt} = -\frac{\mu}{r^3} r_z + a_z\,(r_z,\ r_y,\ r_z,\ v_x,\ v_y,\ v_z,\ t)$$

$$\text{where } r = \sqrt{r_x^2 + r_y^2 + r_z^2}$$

The expressions a_x, a_y, a_z, represent the perturbing accelerations. These are developed in more detail in the following subsections.

3.2.2 Velocity Space (U6)

The general form of the equations of motion is

$$\frac{dCg_1}{dt} = -\frac{C}{v_{e2}}\left(e_{21}\, a_{e3} + e_{31}\, a_{e2}\right)$$

$$\frac{dCg_2}{dt} = -\frac{C}{v_{e2}}\left(e_{22}\, a_{e3} + e_{32}\, a_{e2}\right)$$

$$\frac{dCg_3}{dt} = -\frac{C}{v_{e2}}\left(e_{23}\, a_{e3} + e_{33}\, a_{e2}\right)$$

$$\frac{dR_{f1}}{dt} = a_{e1} \cos\lambda - a_{e2}\left(1 + \frac{C}{v_{e2}}\right) \sin\lambda - \gamma \frac{a_{e3}}{v_{e2}} R_{f2}$$

$$\frac{dR_{f2}}{dt} = a_{e1} \sin\lambda + a_{e2}\left(1 + \frac{C}{v_{e2}}\right) \cos\lambda + \gamma \frac{a_{e3}}{v_{e2}} R_{f1}$$

$$\frac{d\lambda}{dt} = \frac{C v_{e2}^{2}}{\mu} + \gamma \frac{a_{e3}}{v_{e2}}$$

where

$e_{21}, e_{22} \ldots e_{33}$ are elements in the rotation matrix from inertial coordinates to the $e_1\ e_2\ e_3$ frame (section 2.4.10)

$$C = \sqrt{Cg_1^{\,2} + Cg_2^{\,2} + Cg_3^{\,2}}$$

$$v_{e2} = C + R_{f2} \cos\lambda - R_{f1} \sin\lambda$$

$$\gamma = -\frac{C_{g1} \cos\lambda + C_{g2} \sin\lambda}{C + C_{g3}}$$

$$\begin{bmatrix} a_{e1} \\ a_{e2} \\ a_{e3} \end{bmatrix} = \begin{bmatrix} E \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

and $\begin{bmatrix} E \end{bmatrix}$ is given in section 2.4.10.

These equations are developed in references 11 and 12.

Parameters Cg_1, Cg_2 ... are defined in section 2.3.3.

The terms a_x, a_y and a_z represent the perturbing accelerations. These are developed in more detail in the following subsections.

3.2.3 Velocity Space (U7)

The general form of the equations of motion is:

$$\frac{dc}{dt} = -\frac{c}{V_{e2}} \; a_{e2}$$

$$\frac{dR_{f1}}{dt} = \cos\lambda \; a_{e1} - \left(1 + \frac{c}{V_{e2}}\right) \sin\lambda \; a_{e2} - \gamma w_1 R_{f2}$$

$$\frac{dR_{f2}}{dt} = \sin\lambda \; a_{e1} + \left(1 + \frac{c}{V_{e2}}\right) \cos\lambda \; a_{e2} + \gamma \, w.R_{f1}$$

$$\frac{de_{02}}{dt} = \frac{w_3 \, e_{02} + w_1 \, e_{04}}{2}$$

$$\frac{de_{03}}{dt} = \frac{-w_3 \, e_{01} + w_1 \, e_{03}}{2}$$

$$\frac{de_{03}}{dt} = \frac{-w_1 \, e_{02} + w_3 \, e_{04}}{2}$$

$$\frac{de_{04}}{dt} = \frac{-w_1 e_{01} - w_3 e_{03}}{2}$$

where $\sin\lambda = \dfrac{2e_{03} \, e_{04}}{e_{03}^2 + e_{04}^2}$

$$\cos\lambda = \frac{e_{04}^2 - e_{03}^2}{e_{03}^2 + e_{04}^2}$$

$$w_1 = \frac{a_{e3}}{V_{e2}}$$

$$w_3 = \frac{C \, V_{e2}^2}{\mu}$$

$$V_{e2} = C - \sin\lambda\, R_{f1} + \cos\lambda\, R_{f2}$$

$$\gamma = \frac{e_{01}\, e_{03} - e_{02}\, e_{04}}{e_{03}^2 + e_{04}^2}$$

$$\begin{bmatrix} a_{e1} \\ a_{e2} \\ a_{e3} \end{bmatrix} = \begin{bmatrix} E \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

where E is given in section 2.4.10.

3.3 Gravity Gradient

The gravity field of the earth is very well approximated by describing the earth as homogeneous in spherical shells, resulting in only a central gravity term. More accurate approximations account for irregularities in the gravity field of the earth. This is generally done using a series of Legendre polynomials. The series is developed using spherical coordinates in order to make use of the spherical symmetry of the earth, thus giving a good approximation to the earth's gravity field in the first few terms of the series.

The potential field of the earth is

$$\psi(r, L, \lambda) = \frac{\mu}{r} + \frac{\mu}{r} \sum_{n=1}^{\infty} C_n^o \left(\frac{a_p}{r}\right)^n P_N^o (\sin L)$$

$$+ \frac{\mu}{r} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \left(\frac{a_p}{r}\right)^n P_n^m (\sin L) \left[S_n^m \sin m\lambda + C_n^m \cos m\lambda\right]$$

where

μ ∿ the gravitational parameter of the earth

a_p ∿ equatorial radius of the earth

P_n^m ∿ the associated Legendre function

S_n^m, C_n^m ∿ harmonic coefficents.

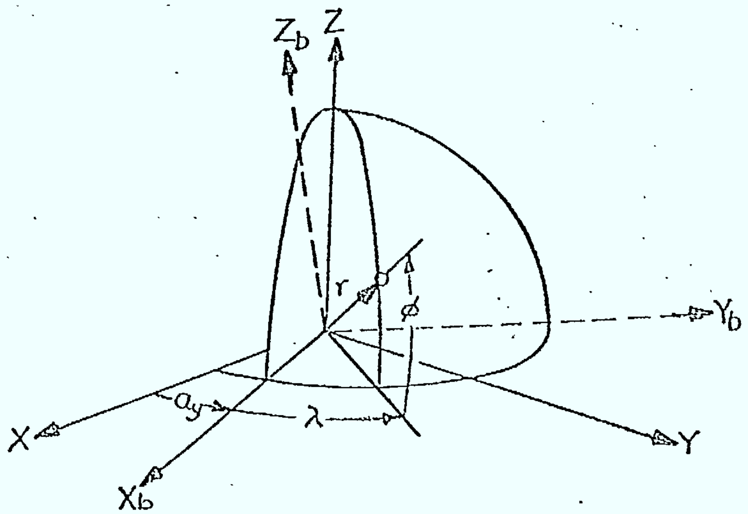
Note that the coordinates r, λ, L are used in this expression. These may be obtained as follows, starting from inertial position r_x, r_y, r_z, transform to earth fixed cartesian, and then to spherical coordinates :

$$\begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} \cos \alpha_g & \sin \alpha_g & 0 \\ -\sin \alpha_g & \cos \alpha_g & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

$$r = \sqrt{x_b^2 + y_b^2 + z_b^2}$$

$$\lambda = \tan^{-1} \frac{y_b}{x_b}$$

$$L = \tan^{-1} \frac{Z_b}{\sqrt{x_b^2 + y_b^2}}$$



The terms $\frac{\partial\psi}{\partial r}$, $\frac{\partial\psi}{\partial L}$, $\frac{\partial\psi}{\partial\lambda}$ for the nonspherical portion are given by:

$$\frac{\partial\psi}{\partial r} = -\frac{1}{r}\left(\frac{\mu}{r}\right)\sum_{n=2}^{N}\left(\frac{a_p}{r}\right)^n (n+1)\sum_{m=0}^{n}(C_n^m \cos m\lambda + S_n^m \sin m\lambda)P_n^m(\sin L)$$

$$\frac{\partial\psi}{\partial L} = \left(\frac{\mu}{r}\right)\sum_{n=2}^{N}\left(\frac{a_p}{r}\right)^n \sum_{m=0}^{n}(C_n^m \cos m\lambda + S_n^m \sin m\lambda)[P_n^{m+1}(\sin L) - m\tan L\, P_n^m(\sin L)]$$

$$\frac{\partial\psi}{\partial\lambda} = \left(\frac{\mu}{r}\right)\sum_{n=2}^{N}\left(\frac{a_p}{r}\right)^n \sum_{m=0}^{n} m(S_n^m \cos m\lambda - C_n^m \sin m\lambda)P_n^m(\sin L).$$

The Legendre functions and the terms cos $m\lambda$, sin $m\lambda$, are computed via recursion formulae:

$$P_n^0(\sin L) = [(2n-1)\sin L\, P_{n-1}^0(\sin L) - (n-1)P_{n-2}^0(\sin L)]/n$$

$$P_n^m(\sin L) = P_{n-2}^m(\sin L) + (2n-1)\cos L\, P_{n-1}^{m-1}(\sin L) \qquad m \neq 0,\ m < n$$

$$P_n^n(\sin L) = (2n-1)\cos L\, P_{n-1}^{n-1}(\sin L) \qquad m \neq 0,\ m = n$$

where

$$P_0^0(\sin L) = 1, \qquad P_1^0(\sin L) = \sin L, \qquad P_1^1(\sin L) = \cos L$$

$$\sin m\lambda = 2\cos\lambda\ \sin(m-1)\lambda - \sin(m-2)\lambda$$

$$\cos m\lambda = 2\cos\lambda\ \cos(m-1)\lambda - \cos(m-2)\lambda$$

The acceleration in body fixed coordinates is found using the chain rule

$$a_{xb} = \frac{\partial\psi}{\partial x_b} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x_b} + \frac{\partial\psi}{\partial L}\frac{\partial L}{\partial x_b} + \frac{\partial\psi}{\partial\lambda}\frac{\partial\lambda}{\partial x_b}$$

$$a_{yb} = \frac{\partial\psi}{\partial y_b} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial y_b} + \frac{\partial\psi}{\partial L}\frac{\partial L}{\partial y_b} + \frac{\partial\psi}{\partial\lambda}\frac{\partial\lambda}{\partial y_n}$$

$$a_{zb} = \frac{\partial\psi}{\partial z_b} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial z_b} + \frac{\partial\psi}{\partial L}\frac{\partial L}{\partial z_b} + \frac{\partial\psi}{\partial\lambda}\frac{\partial\lambda}{\partial z_b}$$

The expressions relating r, L, λ to x_b, y_b, z_b, are those in sections 2.4.4 and 2.4.5. For example

$$r = \sqrt{x_b^2 + y_b^2 + z_b^2}$$

$$\frac{\partial r}{\partial x_b} = \frac{x_b}{\sqrt{x_b^2 + y_b^2 + z_b^2}} = \frac{x_b}{r}$$

Calculating similar expressions for all terms gives:

$$\begin{bmatrix} a_{xb} \\ a_{yb} \\ a_{zb} \end{bmatrix} = \begin{bmatrix} \frac{x_b}{r} & -\frac{z_b\, x_b}{r^2\sqrt{x_b^2 + y_b^2}} & -\frac{y_b}{x_b^2 + y_b^2} \\ \frac{y_b}{r} & -\frac{z_b\, y_b}{r^2\sqrt{x_b^2 + y_b^2}} & \frac{x_b}{x_b^2 + y_b^2} \\ \frac{z_b}{r} & \frac{\sqrt{x_b^2 + y_b^2}}{r^2} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial\psi}{\partial r} \\ \frac{\partial\psi}{\partial L} \\ \frac{\partial\psi}{\partial\lambda} \end{bmatrix}$$

These accelerations are then converted to inertial coordinates using the transformation from section 2.4.3.

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \Lambda^T \begin{bmatrix} a_{xb} \\ a_{yb} \\ a_{zb} \end{bmatrix}$$

The terms a_x, a_y, a_z are part of the perturbing acceleration which appear in the equations of motion in section 3.0.

For velocity space orbital elements, two different approaches may be used, either

. convert to inertial coordinates and use the above directly

OR

. develop partial derivaties in a similar manner to go directly to a_{e_1}, a_{e_2}, a_{e_3} components.

3.4 Atmospheric Drag

The acceleration due to atmospheric drag is in a direction opposite to the velocity of the satellite with respect to the atmosphere. In vector form:

$$a = \frac{AC_D}{m} \rho \frac{V_{rel} \left| V_{rel} \right|}{2} \qquad \text{Equation 3.4.1}$$

where a and V_{rel} are vector quantities.

The atmosphere is assumed to rotate with the earth, and the components of velocity are:

$$V_{ax} = - r_y w_z$$

$$V_{ay} = r_x w_z$$

where w_z is the earth's rotation rate of $.7292116 \times 10^{-4}$ rad./sec..

The relative velocity of the satellite with respect to the atmosphere is:

$$V_{rel\ x} = V_x - V_{ax}$$

$$V_{rel\ y} = V_y - V_{ay}$$

$$V_{rel\ z} = V_z$$

$$\left| V_{rel} \right| = \sqrt{V_{rel\ x}^2 + V_{rel\ y}^2 + V_{rel\ z}^2}$$

The density of the atmosphere is assumed to be a function only of height above the earth's surface. In fact, there are significant variations in density due to solar influences. These variations are more significant at higher altitudes, and at 1000 km. result in more than an order of magnitude differences between maximum and minimum density. The density used in the simple model is a mean density, and as a result of this approximation, the calculation of atmospheric drag gives a first order approximation only.

The density of the atmosphere is obtained by exponential interpolation between the points in Table 3.1 using the following formula:

$$\rho = \rho_h * e^{\left[\frac{h_h - h}{h_h - h_l} \ln \frac{\rho_l}{\rho_h}\right]}$$

where:

h_h = height at the nearest higher point in the table

h_l = height at the nearest lower point in the table

ρ_h = density at the nearest higher point in the table

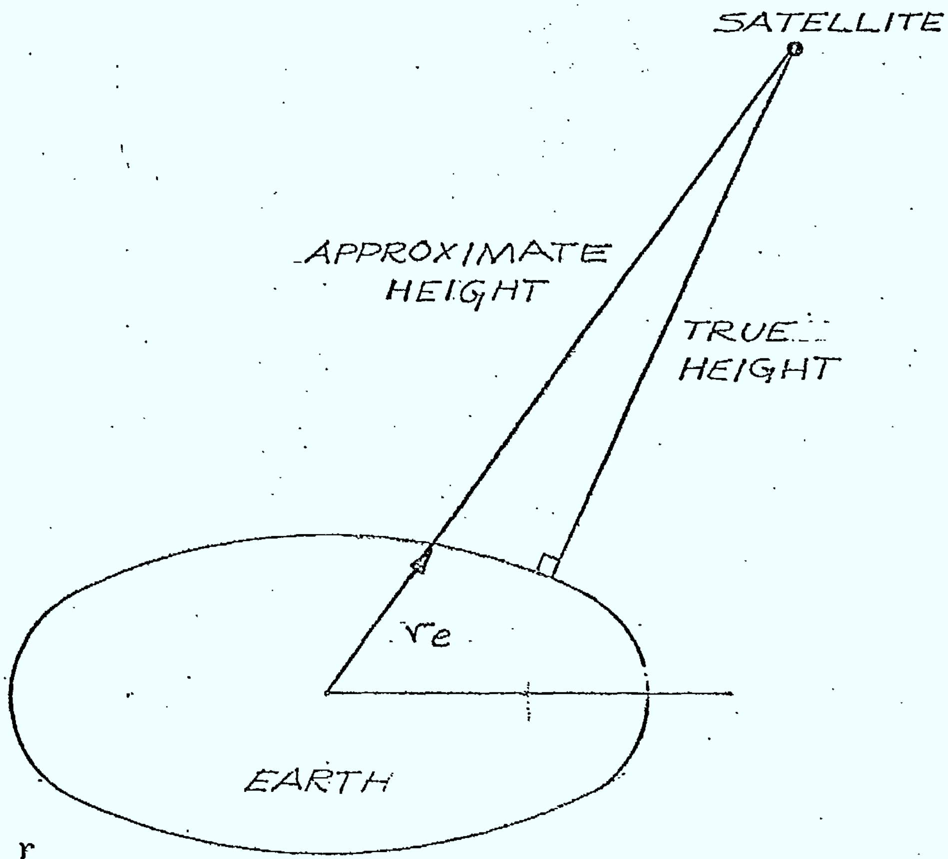
ρ_l = density at the nearest lower point in the table

Height (kilometers)	Density (kg/km^3)
100	448.
140	3.34
180	.589
220	.202
260	.0819
300	.0368
340	.0179
380	.00921
420	.00488
460	.00270
500	.00158
540	.00096
580	.00059
620	.00037
660	0
above 660	0

From Reference 3

TABLE 3.4/1 Atmospheric Density

The height is the height above sea level, for the earth's elipsoid of revolution. This is approximated as shown in the figure.



$$h \approx r - r_e$$

$$= \quad r \quad - \quad \frac{a_p}{\sqrt{1 + \left(\frac{e^2}{1 - e^2}\right) \frac{r_z^2}{r^2}}}$$

where $r = \sqrt{r_x^2 + r_y^2 + r_z^2}$

The expression for drag contains the term $\frac{A\ C_D}{M}$. This term must be supplied for a particular satellite. The units for input are:

A = area in km^2

C_D = drag coefficient - dimensionless

M = mass in kilograms

which gives units of km^2/kilogram

Then acceleration in kilometers/second^2 is calculated from equation 3.4.1. In component form, that is

$$a_x = -\frac{A\ C_D}{M}\ \rho\ \frac{Vrel_x\ |Vrel|}{2}$$

$$a_y = -\frac{A\ C_D}{M}\ \rho\ \frac{Vrel_y\ |Vrel|}{2}$$

$$a_z = -\frac{A\ C_D}{M}\ \rho\ \frac{Vrel_z\ |Vrel|}{2}$$

3.5 Lunar and Solar Gravity

The acceleration of a satellite by the sun is given by

$$a = \mu_s \frac{r_{sun} - r}{(r_{sun} - r)^3}$$

where r is the position vector of the satellite

r_{sun} is the position vector of the sun

μ_s is the gravitational constant of the sun

We wish to find the acceleration of the satellite relative to our "inertial" coordinate system with origin at the center of the earth. Therefore we must subtract the acceleration of the earth by the sun, to give the net acceleration.

$$a_s = \mu_s \left[\frac{r_{sun} - r}{|r_{sun} - r|^3} - \frac{r_{sun}}{|r_{sun}|^3} \right]$$

Similarly, the net acceleration of the satellite by the moon is

$$a_m = \mu_m \left[\frac{r_m - r}{|r_m - r|^3} - \frac{r_m}{r_m^3} \right]$$

The constants μ_s and μ_m are

$$\mu_s = 132.71545 \times 10^9 \quad km^3/sec^2$$
$$\mu_m = 4.902778 \times 10^3 \quad km^3/sec^2$$

The position of the sun, r_s and the moon, r_m are found in sections 7.2 and 7.3.

3.6 Solar Radiation Pressure

The acceleration of the satellite due to solar radiation pressure is given by

$$a = -\left(K\,\frac{A}{m}\right)\left(P\left(\frac{a_{sun}}{r_{sun}}\right)^2\right)\frac{\vec{r}_{sun}}{|\vec{r}_{sun}|}$$

where

K	is the reflectivity coefficient $K = 1$ for perfect absorption $K = 2$ for perfect reflection
A	is the effective area of the satellite
m	is the mass of the satellite
P	is the solar radiation force per unit area
a_{sun}	is the semi-major axis of the suns orbit or mean distance from the earth to the sun

The quantity $K\,\frac{A}{m}$ must be supplied for each satellite. The value of $\vec{r}_{sun}$ is found from section 7.2. The constants are

$$P = 4.5 \times 10^{-6}\ \frac{n}{m^2}\ \left[\text{or } 4.5 \times 10^{-3}\ \frac{kg}{sec^2\ km}\right]$$

$$a_{sun} = 149.6 \times 10^{6}\ \text{km}$$

When the satellite is in eclipse (50% or greater) the radiation pressure is set to zero. See section 8 for eclipse calculations.

3.7 Impulsive Maneuvers

Impulsive maneuvers are specified as instantaneous velocity increments in the orbit frame e1, e2, e3 described in section 2.2.9. These velocity increments are transformed to the inertial frame using the mathematics in section 2.4.10. Then the velocity increments are added to the velocity of the satellite. If U6 elements are used for the equations of motion, they are transformed to inertial coordinates before adding the velocity increments and then transformed back to U6 elements.

4.0 SATELLITE OBSERVABLES

4.1 Introduction

The satellite observables of interest herein are those which are used by a tracking station to determine a satellite orbit. Depending on the type of tracking stations, there may be many types of observables. The ones treated herein are azimuth, elevation, range and range-rate.

The observables are expressed as functions of the estimated parameters, or

$$y = \mathbf{h}\,(\mathbf{p}) = h\,(x,b)$$

The observables are developed for inertial position and velocity as orbital elements. If the velocity space set of orbital elements is used, the observable functions become

$$y = h\,(\tilde{x},b) = h'\,(F\,(x),\,b)$$

where F(x) is the transformation from velocity space elements to inertial coordinates (developed in section 2.4) and h' is the set of functions developed to transform from inertial coordinates to observables (section 4).

The inertial position and velocity used in calculating observables is based on true equator and equinox.

4.2 Azimuth, Elevation and Range

These observables are calculated as

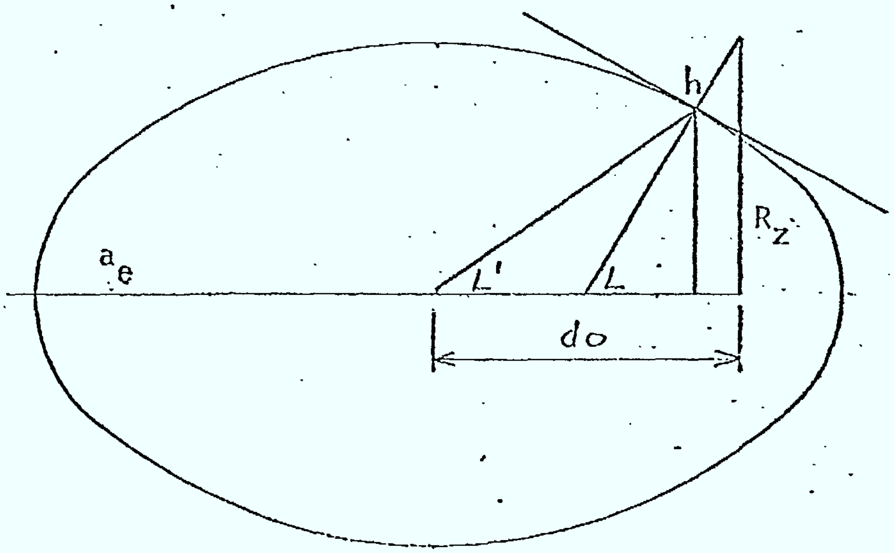
$$\begin{bmatrix} A_z \\ El \\ \rho \end{bmatrix} = F_1\left(F_2\left(\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}\right)\right) + \begin{bmatrix} \Delta Az \\ \Delta El \\ \Delta \rho \end{bmatrix}$$

where

F_2 is a set of functions to transform from inertial position to topocentric local tangent (ENU) frame

F_1 is a set of functions to transform from topocentric (ENU) frame to azimuth, elevation and range. These are stated in section 2.4.7.

A first step in calculating the observables is to calculate the inertial position of the site. To do this, we first find the site position in the earth fixed Cartesian frame. The earth is assumed to be an ellipsoid of revolution as shown in the figure



From the geometry of ellipse (see Appendix A) we obtain.

$$d_o = \left[\frac{a_e}{\sqrt{1 - e^2 \sin^2 L}} + h \right] \cos L$$

(equation 4.2/1)

$$R_z = \left[\frac{a_e (1-e^2)}{\sqrt{1 - e^2 \sin^2 L}} + h \right] \sin L$$

(equation 4.2/2)

The inertial position of the site is found by rotating about the Z axis by the inertial longitude of the site, Lo. The component R_z has been calculated above and the other components are

$R_x = d_o \cos Lo$ and (equation 4.2/3)

$R_y = d_o \sin Lo$. (equation 4.2/4)

The vector from the site to the satellite in inertial coordinates is

$$\begin{bmatrix} \rho_x \\ \rho_y \\ \rho_z \end{bmatrix} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} - \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix}$$

(equation 4.2/5)

This vector is transformed into topocentric local tangent frame using the transformation from section 2.3.6.

$$\begin{bmatrix} \rho_E \\ \rho_N \\ \rho_U \end{bmatrix} = \begin{bmatrix} E_{TOPO} \end{bmatrix} \left[\begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} - \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix} \right]$$

(equation 4.2/6)

Having found the E, N, U components, the transformation in section 2.3.7 is used to find the observables. That is,

$$El = \sin^{-1}\left(\frac{\rho_U}{\sqrt{\rho_E^2 + \rho_N^2 + \rho_U^2}} \right) + \Delta El$$

$$Az = \tan^{-1}\left(\frac{\rho E}{\rho_N} \right) + \Delta Az$$

$$\rho = \sqrt{\rho_E^2 + \rho_N^2 + \rho_U^2} + \Delta\rho$$

An alternate calculation of range is found from equation 4.2/5.

$$\rho = \sqrt{\rho_x^2 + \rho_y^2 + \rho_z^2} + \Delta\rho$$

4.3 Range Rate

The relative velocity of the satellite with respect to the site is

$$\dot{\vec{\rho}} = \vec{V} - \vec{V}_{SITE}$$

The velocity of the site is due to the rotation of the earth, and so the components of relative velocity are

$$\begin{bmatrix} \dot{\rho}_x \\ \dot{\rho}_y \\ \dot{\rho}_z \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} - \begin{bmatrix} -w\ R_y \\ w\ R_x \\ 0 \end{bmatrix}$$

The range rate which we call $\dot{\rho}$ is the component of $\dot{\vec{\rho}}$ in the direction of $\vec{\rho}$. Or

$$\dot{\rho} = \frac{1}{\rho}\ \dot{\vec{\rho}} \cdot \vec{\rho} + \Delta\dot{\rho}$$

$$= \frac{1}{\rho} \begin{bmatrix} v_x - w\ R_y \\ v_y - w\ R_x \\ v_z \end{bmatrix} \cdot \begin{bmatrix} r_x - R_x \\ r_y - R_y \\ r_z - R_z \end{bmatrix}$$

Where the expression for $\vec{\rho}$ is obtained from section 4.2.

Expanding and simplifying the above expression gives

$$\dot{\rho} = \frac{1}{\rho} (\rho_x\ v_x + \rho_y\ v_y + \rho_z\ v_z + w_z r_x R_y - w_z\ R_x\ r_y) + \Delta\dot{\rho}$$

or an alternate form

$$\dot{\rho} = \frac{1}{\rho} (\rho_x\ v_x + \rho_y\ v_y + \rho_z\ v_z + \rho_x\ w\ r_y - \rho_y\ w\ r_x) + \Delta\dot{\rho}$$

equation 4.3/1

5.0 ORBIT ESTIMATION

5.1 Introduction

The orbit estimator uses a Kalman filter, and processes each measurement as soon as it is available. The measurement is not stored, and the only information of the history is contained in the estimate and the error covariance matrix. This is a standard technique with a Kalman filter and it greatly reduces the storage requirements compared to least squares data processing. There are, however, several approximations and limitations of the Kalman filter, which are discussed in Section 5.2. The estimator is described in Section 5.3.

5.2 Discussion of Kalman filtering

Kalman filtering is the method used to find an estimate of the orbital elements. The terms "Kalman filter" and "estimator" are used interchangably.

There are many papers and books on Kalman filtering in general, and on Kalman filtering applied to orbit determination. The reader may refer to reference 5 for detailed information and for the theory of Kalman filtering. The following discussion just gives the assumptions necessary to apply Kalman filtering to orbit determination.

Kalman filtering will provide an optimal estimate provided that certain conditions are met. For orbit determination, these conditions are never satisfied and so the estimate is not optimum. The Kalman filtering theory applies to linear equations, and satellite orbital equations are nonlinear. The equations are linearized about the estimated orbital elements. This introduces errors due to linearization because the estimated orbital elements are not the same as the true orbital elements. The Kalman filtering theory can account for measurement noise and state noise, but assumes Gaussian white noise, a condition which is never satisfied in practise. The Kalman filter requires an initial state error covariance matrix. An optimum value of this error covariance matrix can only be obtained if the true orbital elements are known, and since the true orbital elements are not known it is necessary to use a sub-optimum value. These problems can be partly overcome by including a state noise matrix in the equations. The terms in the matrix, must be carefully chosen to give good results.

5.3 Estimator Analytics

5.3.1 Introduction

The equations listed in this section are a summary of the estimator analytics. Vector and matrix notation is used as in reference 5. For example, the notation

$$\hat{x}\ (k+1/k)$$

represents the best estimate of the state (or orbital elements) at time t_{k+1} using information up to and including time t_K. The state is defined by:

$$x \equiv \begin{bmatrix} r_x \\ r_y \\ r_z \\ v_x \\ v_y \\ v_z \end{bmatrix} \quad \text{or} \quad x \equiv \begin{bmatrix} C_{g1} \\ C_{g2} \\ C_{g3} \\ R_{f1} \\ R_{f2} \\ \lambda \end{bmatrix} \quad \text{or} \quad x \equiv \begin{bmatrix} C \\ R_{f1} \\ R_{f2} \\ e_{01} \\ e_{02} \\ e_{03} \\ e_{04} \end{bmatrix}$$

As mentioned in the introduction (section 1.0) the estimator uses a process of prediction and correction. The prediction equations in section 5.3.2 and the estimator equations in section 5.3.3 comprise the orbit estimator.

5.3.2 Orbital Element Prediction

The orbit prediction equations are used to update the orbital elements and error covariance matrix to a new time. They are

$$\hat{x}\ (k+1/k) = \hat{x}\ (k/k) + \int_{t_k}^{t_{k+1}} f\ (\hat{x}\ (t/t_k),\ t)\ dt$$

$$P\ (k + 1/k) = \Phi P\ (k/k)\ \Phi^T + Q$$

where

The function f represents the equations of motion which are given in Section 3.0, i.e.

$$\dot{x} = f\ (x,t)$$

$\Phi \equiv \Phi\ (k + 1,\ k)$ is the state transition matrix from time t_k to time t_{k+1} obtained from a relinearization of the nonlinear equations of motion about state $\hat{x}(k/k)$. and is developed in section 6.1.

P is the error covariance matrix

Q is the state noise covariance matrix which is used to account for limitations of the numerical computations and inaccuracies in the equations of motion.

5.3.3 Orbital Element Correction

The orbital element correction equations are used each time a new measurement is available. They are

$$K(k+1/k) = P(k+1/k)\, H^T \,(HP(k+1/k)\, H^T + R)^{-1}$$

$$\hat{x}(k+1/k+1) = \hat{x}(k+1/k) + K(k+1/k)(y(k+1) - h(\hat{x}(k+1/k), t_{k+1}))$$

$$P(k+1/k+1) = ((I-KH)\, P(k+1/k)\, (I-KH)^T + KRK^T)$$

or $$P(k+1/k+1) = (I-KH)\, P(k+1/k)$$

where

$y\,(k+1)$ is the measurement at time t_{k+1}

h is the nonlinear function mapping an orbital element into the corresponding measurements

$H = H(k+1)$ is the measurement matrix at time t_{k+1} obtained from linearization of the h function about the state reference $\hat{x}(k+1/k)$

K is the Kalman gain matrix

6.0 LINEARIZATIONS

6.1 State Transition Matrix

6.1.1 Introduction

The state transition matrix, Φ is used to propagate the error covariance matrix, P, forward in time. In order to generate the state transition matrix for use in the extended Kalman filter, it is necessary to linearize the (nonlinear) equations of motion.

We start with the nonlinear equations of motion which are given explicitly in section 3.0. In general, these are

$$\dot{x} = f\,(x,t)$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1\,(x_1, x_2 \ldots x_n, t) \\ f_2\,(x_1, x_2 \ldots x_n, t) \\ f_n\,(x_1, x_2 \ldots x_n, t) \end{bmatrix}$$

These equations are linearized by expanding in a Taylor series about the reference state Xref and neglecting second order terms, giving

$$\begin{bmatrix} \dot{x}_1 \\ | \\ \dot{x}_n \end{bmatrix} \simeq \begin{bmatrix} \dot{x}_{1ref} + \dfrac{\partial \dot{x}_1}{\partial x_1}\,(x_1 - x_{1ref}) \;\ldots\; \dfrac{\partial \dot{x}_1}{\partial x_n}\,(x_n - x_{nref}) \\ | \qquad\qquad\qquad\qquad | \\ \dot{x}_{nref} + \dfrac{\partial \dot{x}_n}{\partial x_1}\,(x_1 - x_{1ref}) \;\ldots\; \dfrac{\partial \dot{x}_n}{\partial x_n}\,(x_n - x_{nref}) \end{bmatrix}$$

If we let $\delta x_1 = x_1 - x_{1ref}$

$$\delta \dot{x}_1 = \dot{x}_1 - \dot{x}_{1ref}$$

etc.

$$\begin{bmatrix} \delta \dot{x}_1 \\ | \\ \delta \dot{x}_n \end{bmatrix} \approx \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} \delta x_1 & \cdots & \frac{\partial \dot{x}_1}{\partial x_n} \delta x_n \\ | & & | \\ \frac{\partial \dot{x}_n}{\partial x_1} \delta x_1 & \cdots & \frac{\partial \dot{x}_n}{\partial x_n} \delta x_n \end{bmatrix}$$

Or, in vector form

$$\delta \dot{x} = [J] \delta x$$

where

J stands for the matrix of partial derivatives evaluated at the reference state.

$$J \equiv \left[\frac{\partial \dot{X}}{\partial X}\right]$$

Then the state transition matrix ϕ is found from the equation

$$\frac{d\Phi}{dt} = J\Phi$$

The state transition matrix has the property

$$\delta x\ (t + \Delta t) = \Phi\ (t + \Delta t,\ t)\ \delta\ x\ (t)$$

So far, we have only considered orbital elements. If biases are also being estimated then state transition for biases must be included.

This is very simple since biases are constant

$$\delta\ b\ (t + \Delta t) = \delta\ b\ (t)$$

Then the total I matrix may be partitioned as follows

$$\Phi_{pxp} = \begin{bmatrix} \Phi_{(x \text{ by } x)} & 0 \\ 0 & I_{(b \text{ by } b)} \end{bmatrix}$$

where I is the identity matrix.

It may be shown that Φ is used to propogate the error covarince matrix forward in time (see section 2.7).

Φ is evaluated using the equation listed above

$$\frac{d\Phi}{dt} = J\Phi$$

where $\Phi\ (t,t) = I$

The approach used in finding Φ is to assume that J is constant over the interval of interest. Then

$$\Phi\ (t + \Delta t, t) = e^{J\,\Delta t}$$

and the exponential is approximated as

$$\Phi = e^{J\Delta t} = I + \frac{J\Delta t}{1!} + \frac{(J\Delta t)^2}{2!} + \frac{(J\Delta t)^3}{3!} + \cdots$$

where

I is an identity matrix of the required order.

$$(J\Delta t)^2 = \Delta t^2\ [J]\ [J]$$

The above discussion provides a summary of the technique used. For more background information, see reference 5, sections 8.1 to 8.3 and reference 7 section 6-3 and 6-4.

The individual elements in J are found by taking partial derivatives of the equations of motion given in section 3. The equations contain perturbing accelerations which are small to first order and partial derivatives of these terms are second order and are neglected.

For example in inertial coordinates, one term in the equations of motion is

$$\dot{v}_x = \frac{\mu}{r^3} r_x + a_x (r_x, r_y, r_z, v_x, v_y, v_z, t)$$

where $r = \sqrt{r_x^{\,2} + r_y^{\,2} + r_z^{\,2}}$

Then

$$\frac{\partial \dot{v}_x}{\partial r_x} = \frac{\mu}{r^3} - \frac{3\mu r_x}{r^4} \frac{\partial r}{\partial r_x} + \frac{\partial a_x}{\partial r_x}$$

$$= \frac{\mu}{r^3} - \frac{3\mu r_x^{\,2}}{r^5}$$

Other terms are calculated in a similar manner. See reference 15 for a complete development of all terms in the J matrix.

In U6 elements, the Jacobian J has a specially simple form

$$J = \begin{bmatrix} 0 & & \cdots & & & 0 \\ \vdots & & & & & \vdots \\ 0 & & \cdots & & & 0 \\ j_{61} & j_{62} & j_{63} & j_{64} & j_{65} & j_{66} \end{bmatrix}$$

For this special case, it may be shown that

$$\Phi = e^{J\Delta t} = I + \frac{\Delta t}{j_{66}} \left(e^{j_{66}\Delta t} - 1\right) \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ j_{61} & j_{62} & j_{63} & j_{64} & j_{65} & j_{66} \end{bmatrix}$$

and a matrix exponential calculation is not required, which gives a significant saving in computation time.

6.2 Linearized Measurement Matrix

6.2.1 Introduction

A linearized measurement matrix, H, is required for orbital element correction. It maps differences between estimated parameters and a reference into a difference between the observables and a reference.

Starting from the nonlinear measurement equations in Section 4 of the form

$$y = h\,(x,b) = h(p)$$

we linearize and obtain

$$\tilde{y} = H\,[\tilde{p}] \qquad \text{where } H = \frac{\partial y}{\partial p}$$

The matrix H contains (# observations) rows by (# parameters) columns. The matrix H may be partitioned as follows

$$\tilde{y} = \begin{bmatrix} H_x & H_b \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{b} \end{bmatrix}$$

where

$$Hx = \frac{\partial y}{\partial x} \quad \text{and } H_b = \frac{\partial y}{\partial b}$$

Terms in H_x are evaluated in sections 6.2.2 and 6.2.3. Terms in H_b are evaluated in section 6.2.4.

6.2.2 Inertial Coordinates

The linearized measurement equation is

$$\begin{bmatrix} \widetilde{El} \\ \widetilde{Az} \\ \tilde{\rho} \\ \tilde{\dot{\rho}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \widetilde{El}}{\partial \tilde{r}_x} & \cdots & \frac{\partial \widetilde{El}}{\partial \tilde{V}_z} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial \tilde{\dot{\rho}}}{\partial \tilde{r}_x} & \cdots & \frac{\partial \tilde{\dot{\rho}}}{\partial \tilde{V}_z} \end{bmatrix} \begin{bmatrix} \tilde{r}_x \\ \tilde{r}_y \\ \tilde{r}_z \\ \tilde{V}_x \\ \tilde{V}_y \\ \tilde{V}_z \end{bmatrix}$$

where $\widetilde{E\ell} = E\ell - E\ell_{ref}$

etc.

The first three rows of the measurement matrix refer to elevation, azimuth and range, and the last row refers to range rate. These will be developed separately below.

Elevation, Azimuth and Range

The measurement matrix for elevation, azimuth and range is developed in two stages. The first stage relates the measurements to East North Up coordinates, and the second stage relates the East North Up coordinates to state variables.

The equations relating measurables to East, North Up coordinates are given in section 2.4.7 and are repeated below.

$$El = \sin^{-1} \sqrt{\rho_E^2 + \rho_N^2 + \rho_U^2}$$

$$Az = \tan^{-1} \frac{\rho_E}{\rho_N}$$

$$\rho = \sqrt{\rho_E^2 + \rho_N^2 + \rho_U^2}$$

These equations are linearized by expanding in a Taylor series and neglecting second order terms

$$\begin{bmatrix} \widetilde{E\ell} \\ \widetilde{Az} \\ \tilde{\rho} \end{bmatrix} = [A] \begin{bmatrix} \tilde{\rho}_E \\ \tilde{\rho}_N \\ \tilde{\rho}_U \end{bmatrix}$$

Where $[A] \equiv \frac{\partial y}{\partial x}$

The terms in A may be evaluated as

$$A = \begin{bmatrix} \frac{-\rho_E \rho_U}{\rho^2 \sqrt{\rho_E^2 + \rho_N^2}} & \frac{-\rho_N \rho_U}{\sqrt{\rho_E^2 + \rho_N^2}} & \frac{\sqrt{\rho_E^2 + \rho_N^2}}{\rho^2} \\ \frac{\rho_N}{\rho_E^2 + \rho_N^2} & \frac{-\rho_E}{\rho_E^2 + \rho_N^2} & 0 \\ \frac{\rho_E}{\rho} & \frac{\rho_N}{\rho} & \frac{\rho_U}{\rho} \end{bmatrix}$$

The equation relating East North Up coordinates to state variables are given in section 2.4.6, and repeated below

$$\begin{bmatrix} \rho_E \\ \rho_N \\ \rho_U \end{bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta \ \sin L & -\sin\theta \ \sin L & \cos L \\ \cos\theta \ \cos L & \sin\theta \ \cos L & \sin L \end{bmatrix} \begin{bmatrix} r_x - R_x \\ r_y - R_y \\ r_z - R_z \end{bmatrix}$$

$$= \begin{bmatrix} E_{TOPO} \end{bmatrix} \begin{bmatrix} r_x - R_x \\ r_y - R_y \\ r_z - R_z \end{bmatrix}$$

We expand the above expressions in a Taylor series about the reference. In this expansion, the position of the site is assumed to be fixed and independent of the state of the satellite so $R_x = R_{xref}$. Then the expansion gives

$$\begin{bmatrix} \tilde{\rho}_E \\ \tilde{\rho}_N \\ \tilde{\rho}_U \end{bmatrix} = \begin{bmatrix} E_{TOPO} \end{bmatrix} \begin{bmatrix} \tilde{r}_x \\ \tilde{r}_y \\ \tilde{r}_z \end{bmatrix}$$

Combining with the previous equation gives

$$\begin{bmatrix} \tilde{E}\ell \\ \tilde{A}_z \\ \tilde{\rho} \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \rho_E \\ \rho_N \\ \rho_U \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} E_{TOPO} \end{bmatrix} \begin{bmatrix} \tilde{r}_x \\ \tilde{r}_y \\ \tilde{r}_z \end{bmatrix}$$

The matrix product $\left[A\right]\left[E_{TOPO}\right]$ gives the upper left 3x3 elements of the H matrix. The upper right 3x3 elements are all zero because azimuth, elevation and range do not depend on the spacecraft velocity.

An alternate calculation for partials of range may be used, coming directly from the definition of range in inertial coordinates.

$$\rho = (r_x - R_x)^2 + (r_y - R_y)^2 + (r_z - R_z)^2$$

Then

$$\frac{\partial\rho}{\partial r_x} = \frac{r_x - R_x}{\rho} = \frac{\rho_x}{\rho}$$

$$\frac{\partial\rho}{\partial r_y} = \frac{r_y - R_y}{\rho} = \frac{\rho_y}{\rho}$$

$$\frac{\partial\rho}{\partial r_z} = \frac{r_z - R_z}{\rho} = \frac{\rho_z}{\rho}$$

Range-Rate

The fourth row of the H matrix is derived from the expression for range-rate from section 4.3.

$$\dot{\rho} = \frac{1}{\rho}\left(\rho_x v_x + \rho_y v_y + \rho_z v_z + w r_x R_y - w r_y R_x\right)$$

where

$$\rho_x = r_x - R_x$$

$$\rho_y = r_y - R_y$$

$$\rho_z = r_z - R_z$$

Then expanding in a Taylor series about the reference gives

$$\dot{\rho} = \dot{\rho}_{ref} + \frac{\partial\dot{\rho}}{\partial r_x}(r_x - r_{ref}) + \ldots \frac{\partial\dot{\rho}}{\partial v_z}(v_z - v_{zref})$$

$$\tilde{\dot{\rho}} = \left[C\right]\begin{bmatrix} \tilde{r}_x \\ \vdots \\ \tilde{v}_z \end{bmatrix}$$

The elements of C are evaluated next. Intermediate results are

$$\frac{\partial\rho_x}{\partial r_x} = 1 \qquad \frac{\partial\rho_y}{\partial r_y} = 1 \qquad \frac{\partial\rho_z}{\partial r_z} = 1$$

$$\frac{\partial}{\partial r_x}\left(\frac{1}{\rho}\right) = \frac{-\rho_x}{\rho^3} \qquad \frac{\partial}{\partial r_y}\left(\frac{1}{\rho}\right) = \frac{-\rho_y}{\rho^3} \qquad \frac{\partial}{\partial r_z}\left(\frac{1}{\rho}\right) = \frac{-\rho_z}{\rho^3}$$

$$C_{11} = -\frac{\rho_x}{\rho^2}\dot{\rho} + \frac{1}{\rho}(v_x + w R_y)$$

$$C_{12} = -\frac{\rho_y}{\rho^2}\dot{\rho} + \frac{1}{\rho}(v_y - w R_x)$$

$$C_{13} = -\frac{\rho_z}{\rho^2}\dot{\rho} + \frac{v_z}{\rho}$$

$$C_{14} = \frac{\rho_x}{\rho}$$

$$C_{15} = \frac{\rho_y}{\rho}$$

$$C_{16} = \frac{\rho_z}{\rho}$$

6.2.3 Velocity Space (U6)

The position and velocity in terms of velocity space elements may be written

$$r_x = F_1 \, (Cg_1, Cg_2 \ldots \lambda)$$

$$r_y = F_2 \, (Cg_1, Cg_2 \ldots \lambda)$$

etc.

These equations may be linearized by expanding in a Taylor series about the reference, and neglecting second order terms gives

$$r_x = r_{xref} + \frac{\partial r_x}{\partial Cg_1}(Cg_1 - Cg_{1ref}) + \ldots \frac{\partial r_x}{\partial \lambda}(\lambda - \lambda_{ref})$$

etc.

These may be written in matrix form as

$$\begin{bmatrix} \widetilde{r}_x \\ \widetilde{r}_y \\ \vdots \\ \widetilde{v}_z \end{bmatrix} = \begin{bmatrix} F \end{bmatrix} \begin{bmatrix} \widetilde{Cg}_1 \\ \widetilde{Cg}_2 \\ \vdots \\ \widetilde{\lambda} \end{bmatrix}$$

In section 6.2.2, an expression was developed relating difference in observables to difference in position and velocity of the form

$$\begin{bmatrix} \widetilde{E\ell} \\ \widetilde{A}_z \\ \widetilde{\rho} \\ \widetilde{\dot{\rho}} \end{bmatrix} = \begin{bmatrix} H' \end{bmatrix} \begin{bmatrix} \widetilde{r}_x \\ \widetilde{r}_y \\ \cdot \\ \cdot \\ \widetilde{v}_z \end{bmatrix}$$

Note that the matrix H' above was called H in section 6.2.2. The reason for the change is that the name H is reserved for the measurement matrix. The above matrix is the measurement matrix for inertial coordinates, but not for velocity space elements.

Combining the two previous equations gives:

$$\begin{bmatrix} \widetilde{E\ell} \\ \widetilde{A}_z \\ \widetilde{\rho} \\ \widetilde{\dot{\rho}} \end{bmatrix} = \begin{bmatrix} H' \end{bmatrix} \begin{bmatrix} F \end{bmatrix} \begin{bmatrix} \widetilde{Cg}_1 \\ \widetilde{Cg}_2 \\ \cdot \\ \cdot \\ \widetilde{\lambda} \end{bmatrix} = \begin{bmatrix} H \end{bmatrix} \begin{bmatrix} \widetilde{Cg}_1 \\ \widetilde{Cg}_2 \\ \cdot \\ \cdot \\ \widetilde{\lambda} \end{bmatrix}$$

The measurement matrix for the velocity space set is the product $\left[H'\right]\left[F\right]$. The matrix $\left[H'\right]$ has already been found in section 6.2.2.

The matrix F is developed in references 8 and 9. The results are summarized below.

The matrix F may be divided into two parts.

$$\left[F\right] = \begin{bmatrix} \dfrac{\partial r}{\partial x^T} \\[2em] \dfrac{\partial v}{\partial x^T} \end{bmatrix}$$

where $\dfrac{\partial r}{\partial x^T}$ and $\dfrac{\partial v}{\partial x^T}$ are defined by

$$\frac{\partial r}{\partial x^T} = \begin{bmatrix} \dfrac{\partial r_x}{\partial Cg_1} & \dfrac{\partial r_x}{\partial Cg_2} & \cdots & \dfrac{\partial r_x}{\partial \lambda} \\[2em] \dfrac{\partial r_y}{\partial Cg_1} & & & \\[2em] \dfrac{\partial r_z}{\partial Cg_1} & & \cdots\cdots & \dfrac{\partial r_z}{\partial \lambda} \end{bmatrix}$$

$$\frac{\partial v}{\partial x^T} = \begin{bmatrix} \dfrac{\partial v_x}{\partial Cg_1} & \dfrac{\partial v_x}{\partial Cg_2} & \cdots\cdots & \dfrac{\partial v_x}{\partial \lambda} \\[2em] \dfrac{\partial v_y}{\partial Cg_1} & & & \dfrac{\partial vy}{\partial \lambda} \\[2em] \dfrac{\partial v_z}{\partial Cg_1} & & \cdots\cdots & \dfrac{\partial vz}{\partial \lambda} \end{bmatrix}$$

Then in references 8 and 9, it is shown that $\frac{\partial r}{\partial x^T}$ and $\frac{\partial v}{\partial x^T}$ are given by

$$\frac{\partial r}{\partial x^T} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \end{bmatrix} \begin{bmatrix} \frac{\partial r}{\partial Cg_1} & \frac{\partial r}{\partial Cg_2} & \cdots & \frac{\partial r}{\partial \lambda} \end{bmatrix} + r \cos \lambda \, A + r \sin \lambda \, B$$

$$\frac{\partial v}{\partial x^T} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \end{bmatrix} \frac{\partial v_{e1}}{\partial x^T} + \begin{bmatrix} \varepsilon_{21} \\ \varepsilon_{22} \\ \varepsilon_{23} \end{bmatrix} \frac{\partial v_{e2}}{\partial x^T} + (v_{e1} \cos\lambda - v_{e2} \sin\lambda) \, A$$

$$+ (v_{e1} \sin\lambda + v_{e2} \cos\lambda) \, B$$

where

$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

is the matrix E developed in section 2.4.10 to convert from inertial coordinates to orbit frame $e_1 e_2 e_3$.

and $$\frac{\partial r}{\partial Cg_1} = - \frac{\mu}{C v_{e2}^2} \frac{Cg_1}{C} \left(1 + \frac{v_{e2}}{C} \right)$$

and $$\frac{\partial r}{\partial Cg_2} = - \frac{\mu}{C v_{e2}^2} \frac{Cg_2}{C} \left(1 + \frac{v_{e2}}{C} \right)$$

and $$\frac{\partial r}{\partial Cg_3} = -\frac{\mu}{Cv_{e2}^{2}} \frac{Cg_3}{C} \left(1 + \frac{v_{e2}}{C}\right)$$

and $$\frac{\partial r}{\partial R_{f1}} = \frac{\mu}{Cv_{e2}^{2}} \sin \lambda$$

and $$\frac{\partial r}{\partial R_{f2}} = -\frac{\mu}{Cv_{e2}^{2}} \cos \lambda$$

and $$\frac{\partial r}{\partial \lambda} = \frac{\mu V_{e1}}{Cv_{e2}^{2}}$$

and

$$A = \begin{bmatrix} \frac{\partial \alpha_{21}}{\partial Cg_1} & \frac{\partial \alpha_{21}}{\partial Cg_2} & \frac{\partial \alpha_{21}}{\partial Cg_3} & 0 & 0 & -\alpha_{11} \\ \frac{\partial \alpha_{22}}{\partial Cg_1} & \frac{\partial \alpha_{22}}{\partial Cg_2} & \frac{\partial \alpha_{22}}{\partial Cg_3} & 0 & 0 & -\alpha_{12} \\ \frac{\partial \alpha_{23}}{\partial Cg_1} & \frac{\partial \alpha_{23}}{\partial Cg_2} & \frac{\partial \alpha_{23}}{\partial Cg_3} & 0 & 0 & -\alpha_{13} \end{bmatrix}$$

and

$$B = \begin{bmatrix} \frac{\partial \alpha_{11}}{\partial Cg_1} & \frac{\partial \alpha_{11}}{\partial Cg_2} & \frac{\partial \alpha_{11}}{\partial Cg_3} & 0 & 0 & \alpha_{21} \\ \frac{\partial \alpha_{12}}{\partial Cg_1} & \frac{\partial \alpha_{12}}{\partial Cg_2} & \frac{\partial \alpha_{12}}{\partial Cg_3} & 0 & 0 & \alpha_{22} \\ \frac{\partial \alpha_{13}}{\partial Cg_1} & \frac{\partial \alpha_{13}}{\partial Cg_2} & \frac{\partial \alpha_{13}}{\partial Cg_3} & 0 & 0 & \alpha_{23} \end{bmatrix}$$

and
$$\frac{\partial v_{e1}}{\partial x_T} \equiv \left[\frac{\partial v_{e1}}{\partial Cg_1} \quad \frac{\partial v_{e1}}{\partial Cg_2} \quad \frac{\partial v_{e1}}{\partial Cg_3} \quad \frac{\partial v_{e1}}{\partial R_{f1}} \quad \frac{\partial v_{e1}}{\partial R_{f2}} \quad \frac{\partial v_{e1}}{\partial \lambda} \right]$$

and
$$\frac{\partial v_{e2}}{\partial x_T} \equiv \left[\frac{\partial v_{e2}}{\partial Cg_1} \quad \frac{\partial v_{e2}}{\partial Cg_2} \quad \frac{\partial v_{e2}}{\partial Cg_3} \quad \frac{\partial v_{e2}}{\partial R_{f1}} \quad \frac{\partial v_{e2}}{\partial R_{f2}} \quad \frac{\partial v_{e2}}{\partial \lambda} \right]$$

and
$$\left. \begin{aligned} v_{e1} &= R_{f1} \cos\lambda + R_{f2} \sin\lambda \\ v_{e2} &= C - R_{f1} \sin\lambda + R_{f2} \cos\lambda \end{aligned} \right\} \quad \text{see section 2.5.4}$$

In addition, the following are used above

$$\frac{\partial \alpha_{11}}{\partial Cg_1} = 2\, Cg_1\, S + Cg_1^{\,3}\, u$$

$$\frac{\partial \alpha_{11}}{\partial Cg_2} = Cg_1^{\,2}\, Cg_2\, u$$

$$\frac{\partial \alpha_{11}}{\partial Cg_3} = \frac{Cg_1^{\,2}}{C^3}$$

$$\frac{\partial \alpha_{12}}{\partial Cg_1} = \frac{\partial\ 21}{\partial Cg_1} = Cg_2\, S + Cg_1^{\,2}\, Cg_2\, u$$

$$\frac{\partial \alpha_{12}}{\partial Cg_2} = \frac{\partial \alpha_{21}}{\partial Cg_2} = Cg_1\, S + Cg_1\, Cg_2^{\,2}\, u$$

$$\frac{\partial \alpha_{12}}{\partial Cg_3} = \frac{\partial \alpha_{21}}{\partial Cg_3} = \frac{Cg_1\, Cg_2}{C^3}$$

$$\frac{\partial \alpha_{22}}{\partial Cg_1} = Cg_2^2 \; Cg_1 \; u$$

$$\frac{\partial \alpha_{22}}{\partial Cg_2} = 2 \; Cg_2 + Cg_2^3 \quad u$$

$$\frac{\partial \alpha_{22}}{\partial Cg_3} = \frac{Cg_2^2}{C^3}$$

$$\frac{\partial \alpha_{13}}{\partial Cg_1} = \frac{Cg_1^2}{C^3} - \frac{1}{C}$$

$$\frac{\partial \alpha_{13}}{\partial Cg_2} = \frac{Cg_1 \; Cg_2}{C^3}$$

$$\frac{\partial \alpha_{13}}{\partial Cg_3} = \frac{Cg_1 \; Cg_3}{C^3}$$

$$\frac{\partial \alpha_{23}}{\partial Cg_1} = \frac{Cg_2 \; Cg_1}{C^3}$$

$$\frac{\partial \alpha_{23}}{\partial Cg_2} = \frac{Cg_2^2}{C^3} - \frac{1}{C}$$

$$\frac{\partial \alpha_{23}}{\partial Cg_3} = \frac{Cg_2 \; Cg_3}{C^3}$$

$$C = \sqrt{Cg_1^2 + Cg_2^2 + Cg_3^2}$$

and
$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$$

is the matrix $\begin{bmatrix} \alpha \end{bmatrix}$ developed in section 2.4.8 to transform from inertial coordinates to the orbit frame $f_1 f_2 f_3$.

The following relationships are used above

$$s \equiv \frac{-1}{C(C + Cg_3)}$$

$$u \equiv \frac{2 + \frac{Cg_3}{C}}{C^2 \, (C + Cg_3)^2}$$

6.2.4 Velocity Space (U7)

This development follows that in section 6.3.3. The equations for $\frac{\partial r}{\partial x^T}$ and $\frac{\partial v}{\partial x^T}$ and obtained from references 8 and 9 and are

$$\frac{\partial r}{\partial x^T} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \end{bmatrix} \begin{bmatrix} \frac{\partial r}{\partial c}, & \frac{\partial r}{\partial R_f}, & \frac{\partial r}{\partial R_{f2}}, & \frac{\partial r}{\partial e_{01}}, & \frac{\partial r}{\partial e_{02}}, & \frac{\partial r}{\partial e_{03}}, & \frac{\partial r}{\partial e_{04}} \end{bmatrix}$$

$$+ 2r \begin{bmatrix} 0000 & -2e_{02} & -2e_{03} & 0 \\ 000 & e_{02} & e_{01} & e_{04} & e_{03} \\ 000 & e_{03} & -e_{04} & e_{01} & -e_{02} \end{bmatrix}$$

where ε_{11}, ε_{12}, ε_{13} are elements of the matrix E developed in section 2.4.

$$\frac{\partial r}{\partial c} = \frac{-\mu}{cV_{e2}^2} \left(1 + \frac{V_{e2}}{c}\right)$$

$$\frac{\partial r}{\partial R_{f1}} = \frac{+\mu}{cV_{e2}^2} \sin \lambda$$

$$\frac{\partial r}{\partial R_{f2}} = \frac{-\mu}{cV_{e2}^2} \cos \lambda$$

$$\frac{\partial r}{\partial e_{01}} = 0$$

$$\frac{\partial r}{\partial e_{02}} = 0$$

$$\frac{\partial r}{\partial e_{03}} = \frac{2\mu\, e_{04}}{cV_{e2}^{\;2}\,(e_{03}^{\;2} + e_{04}^{\;2})} \left(R_{f1}\cos\lambda + R_{f2}\sin\lambda\right)$$

$$\frac{\partial r}{\partial e_{04}} = \frac{-2\mu\, e_{03}}{cV_{e2}^{\;2}\,(e_{03}^{\;2} + e_{04}^{\;2})} \left(+R_{f1}\cos\lambda + R_{f2}\sin\lambda\right)$$

$$r = \frac{\mu}{cV_{e2}}$$

$$\sin\lambda = \frac{2e_{03}\, e_{04}}{e_{03}^{\;2} + e_{04}^{\;2}}$$

$$\cos\lambda = \frac{e_{04}^{\;2} - e_{03}^{\;2}}{e_{03}^{\;2} + e_{04}^{\;2}}$$

$$\frac{\partial v}{\partial x^T} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \end{bmatrix} \begin{bmatrix} \frac{\partial V_{e1}}{\partial c} & \frac{\partial V_{e1}}{\partial R_{f1}} & \frac{\partial V_{e1}}{\partial R_{f2}} & \frac{\partial V_{e1}}{\partial e_{01}} & \frac{\partial V_{e1}}{\partial e_{02}} & \frac{\partial V_{e1}}{\partial e_{03}} & \frac{\partial V_{e1}}{\partial e_{04}} \end{bmatrix}$$

$$+ \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{13} \end{bmatrix} \begin{bmatrix} \frac{\partial V_{e2}}{\partial c} & \frac{\partial V_{e2}}{\partial R_{f1}} & \frac{\partial V_{e2}}{\partial R_{f2}} & \frac{\partial V_{e2}}{\partial e_{01}} & \frac{\partial V_{e2}}{\partial e_{02}} & \frac{\partial V_{e2}}{\partial e_{03}} & \frac{\partial V_{e2}}{\partial e_{04}} \end{bmatrix}$$

$$+ \quad 2V_{e1} \begin{bmatrix} 0 & 0 & 0 & 0 & -2e_{02} & -2e_{03} & 0 \\ 0 & 0 & 0 & e_{02} & e_{01} & e_{04} & e_{03} \\ 0 & 0 & 0 & e_{03} & -e_{04} & e_{01} & -e_{02} \end{bmatrix}$$

$$+ \quad 2V_{e2} \begin{bmatrix} 0 & 0 & 0 & e_{02} & e_{01} & -e_{04} & -e_{03} \\ 0 & 0 & 0 & -2e_{01} & 0 & -2e_{03} & 0 \\ 0 & 0 & 0 & e_{04} & e_{03} & e_{02} & e_{01} \end{bmatrix}$$

where

$$\frac{\partial V_{e1}}{\partial c} = 0$$

$$\frac{\partial V_{e1}}{\partial R_{f1}} = \cos \lambda$$

$$\frac{\partial V_{e1}}{\partial R_{f2}} = \sin \lambda$$

$$\frac{\partial V_{e1}}{\partial e_{01}} = 0$$

$$\frac{\partial V_{e1}}{\partial e_{02}} = 0$$

$$\frac{\partial V_{e1}}{\partial e_{03}} = \frac{2\, e_{04}}{e_{03}^{\,2} + e_{04}^{\,2}} \left(-R_{f1} \sin \lambda + R_{f2} \cos \lambda\right)$$

$$\frac{\partial V_{e1}}{\partial e_{04}} = \frac{2\, e_{03}}{e_{03}^{\,2} + e_{04}^{\,2}} \left(R_{f1} \sin \lambda - R_{f2} \cos \lambda\right)$$

$$\frac{\partial V_{e2}}{\partial c} = 1$$

$$\frac{\partial V_{e2}}{\partial R_{f2}} = -\sin \lambda$$

$$\frac{\partial V_{e2}}{\partial R_{f2}} = \cos \lambda$$

$$\frac{\partial V_{e2}}{\partial e_{01}} = 0$$

$$\frac{\partial V_{e2}}{\partial e_{02}} = 0$$

$$\frac{\partial V_{e2}}{\partial e_{03}} = \frac{-2\ e_{04}}{e_{03}^{\ 2} + e_{04}^{\ 2}} \quad (+R_{f1} \cos \lambda + R_{f2} \sin \lambda)$$

$$\frac{\partial V_{e2}}{\partial e_{04}} = \frac{2\ e_{03}}{e_{03}^{\ 2} + e_{04}^{\ 2}} \quad (R_{f1} \cos \lambda + R_{f2} \sin \lambda)$$

6.2.5 Biases

The terms in the measurement matrix which are from biases in the measurements are

$$\frac{\partial Az}{\partial \Delta Az} = 1 \qquad \frac{\partial Az}{\partial \Delta E\ell} = 0 \qquad \frac{\partial Az}{\partial \Delta \rho} = 0 \qquad \frac{\partial Az}{\partial \Delta \dot{\rho}} = 0$$

$$\frac{\partial E\ell}{\partial \Delta Az} = 0 \qquad \frac{\partial E\ell}{\partial \Delta E\ell} = 1 \qquad \text{etc.}$$

6.3 State Transformations

We wish to calculate the linearized transformation matrix A which is used in section 2.7. The method for calculating the matrix is given below, but the elements of the matrices are not developed.

For example, consider the transformation from inertial coordinates to U7 elements. Then the state vectors x and z are:

$$x = \begin{bmatrix} c \\ R_{f1} \\ R_{f2} \\ e_{01} \\ e_{02} \\ e_{03} \\ e_{04} \end{bmatrix} \qquad z = \begin{bmatrix} r_x \\ r_y \\ r_z \\ v_x \\ v_y \\ v_z \end{bmatrix}$$

The linearized transformation matrix A satisfies the equation

$$\tilde{x} = A\,\tilde{z}$$

where

$$A = \begin{bmatrix} \frac{\partial c}{\partial r_x} & \frac{\partial c}{\partial r_y} & \cdots & \frac{\partial c}{\partial v_z} \\ \vdots & & & \vdots \\ \frac{\partial e_{01}}{\partial r_x} & & \cdots & \frac{\partial e_{01}}{\partial v_z} \end{bmatrix} \equiv \begin{bmatrix} \frac{\partial x}{\partial z} \end{bmatrix}$$

The terms in A are found by taking partial derivatives of the nonlinear transformation equations in sections 2.5.3 and 2.5.9. Some of the equations are

$$h_x = r_y v_z - r_z v_y$$

$$h_y = r_z v_x - r_x v_z$$

$$h_z = r_x v_y - r_y v_x$$

$$h = \sqrt{h_x^2 + h_y^2 + h_z^2}$$

$$c = \frac{u}{h}$$

Then the first term is evaluated using the chain rule as

$$\frac{\partial c}{\partial r_x} = \frac{\partial c}{\partial h} \left(\frac{\partial h}{\partial h_y} \frac{\partial hy}{\partial r_x} + \frac{\partial h}{\partial h_z} \frac{\partial hz}{\partial r_x} \right)$$

$$= \frac{-\mu}{h^2} \quad \frac{h_y}{h} (-v_z) + \frac{h_z}{h} (v_y)$$

$$= \frac{\mu}{h^3} (h_y v_z - h_z v_y)$$

and the other terms may be evaluated in a similar fashion.

7.0 EPHEMERIS

7.1 Time

The orbit determination and prediction programs use a form of Universal Time as the working unit of time. It is worth noting two approximations that result

i) The first is because no correction is made for seasonal variations in the earth's rate of rotation. This may result in an error of up to 0.025 seconds of time, or for a satellite with semimajor axis of 7000 km a position error of up to 185 meters.

ii) The second results from the difference in the length of the units of time. The year in universal time differs from the year in Ephemeris time by about one second. In a period of one day of prediction, this amounts to an error of less than 0.003 seconds of time or less than 20 meters in position for a satellite with semimajor axis of 7000 km.

The universal time is in the form of Modified Julian Date where Julian Date 2,400,000.5 is 0 in MJD. The value for other MJD's of interest are

1900 January 0 at 12 h	=	MJD	15019.5
1900 January 1 at 0 h	=	MJD	15020.0
1950 January 1 at 0 h	=	MJD	33282.0
1978 January 1 at 0 h	=	MJD	43509.0

A form of Ephemeris Time is used for the calculation of the position of the sun and moon, since the equations for the elements of the position of the sun and the moon use ephemeris time. The calculation of ephemeris time from universal time is done using a linear fit which is probably accurate to within about 1 second from 1965 - 1985. The linear fit is performed using the following data from references 9 and 10.

Date	Δt
Jan 1 1979 at 0^h UT	49.3
Jan 1 1969 at 0^h UT	39.20
Jan 1 1900 at 12^h UT	-4.5

This data is fit to an equation of the form

$$d_E = a + b * MJD$$

where d_E is the time in ephemeris days since 1900 January 0 at 12^h ET (1899 Dec 31 at 12^h)

The above table may be rewritten

MJD (UT)		MJD (ET)
43874.0	Jan 1 1979	43874.0005706
40222.0	Jan 1 1969	40222.0004537

Then with d_E being measured from MJD 15019.5 ET, an expression for d_E is

$$d_E = -15019.5008338 + 1.00000003201 *MJD_{UT}$$

and $T_E = \dfrac{d_E}{36525}$

where T_E is in Julian centuries.

7.2 Position of the Sun

As a first step in finding the mean motion of the sun, we find its inertial position and velocity based on the ecliptic and mean equinox (see section 2.1.4).

The orbital elements are obtained from reference 5 and they are

Semi-major axis

$$a = 149.6 \times 10^6 \text{ km}$$

Eccentricity

$$e = 0.01675\,104 - 0.00004\,180\,T - 0.00000\,0126\,T^2$$

Inclination

$i \equiv 0$ since the sun's orbit is in the ecliptic

Longitude of Ascending Node

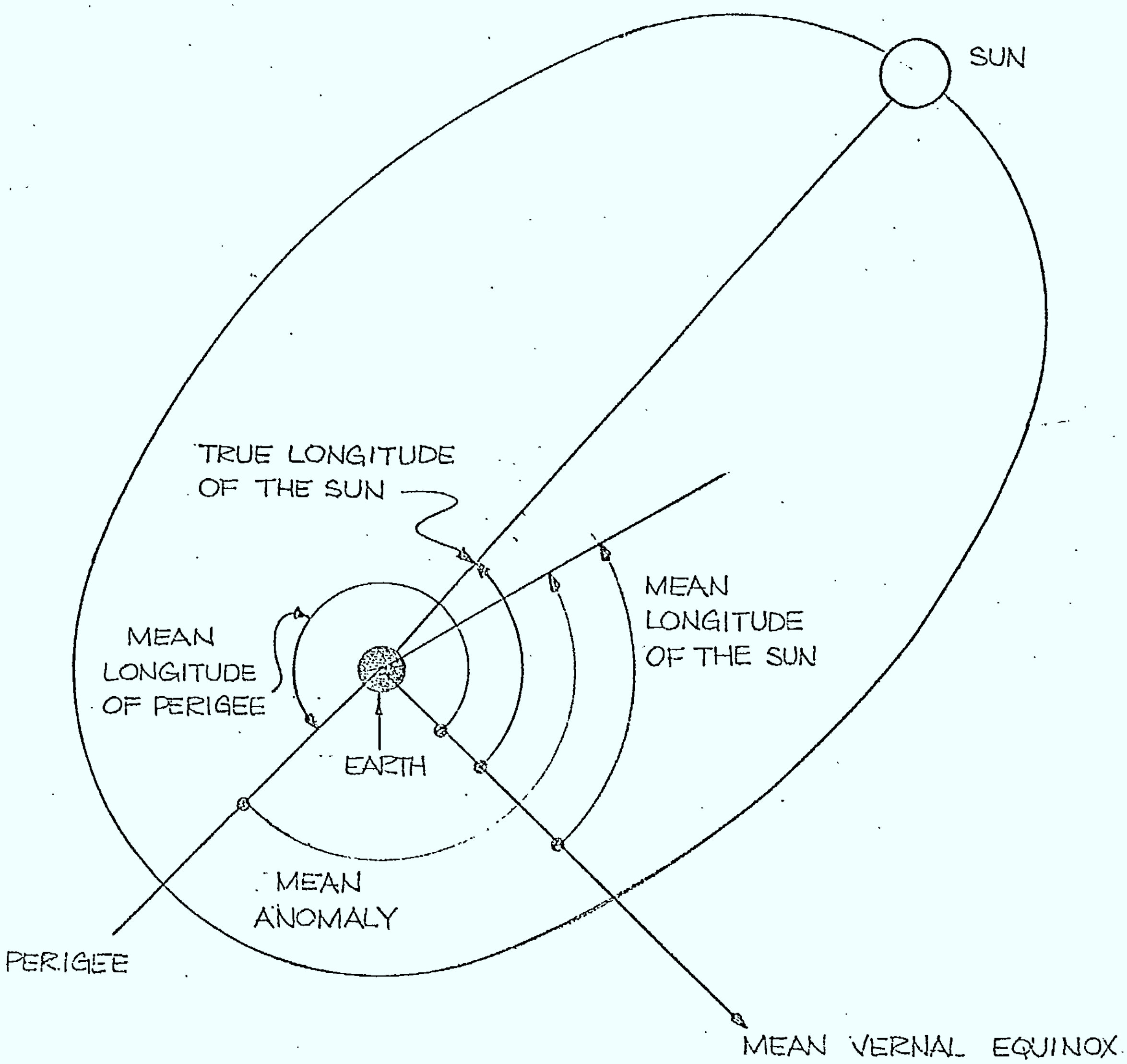
Ω - not required since $i = 0$

Mean longitude of perigee, mean equinox of date

$$w = 281\overset{\circ}{.}22083 + 0\overset{\circ}{.}00004\,70684\,d + 0\overset{\circ}{.}000453\,T^2 + 0.00000\,3\,T^3$$

Mean anomaly

$$M = 358\overset{\circ}{.}47583 + 0\overset{\circ}{.}98560\,02670\,d - 0\overset{\circ}{.}000150\,T^2 - 0\overset{\circ}{.}000003\,T^3$$



VIEW LOOKING SOUTHWARD ON THE ECLIPTIC

NOTE:

ALTHOUGH THE EARTH REVOLVES ABOUT THE SUN, WE ARE INTERESTED IN THE POSITION OF THE SUN WITH RESPECT TO THE EARTH, AND SO IT IS MORE CONVENIENT TO DESCRIBE IT AS MOTION OF THE SUN ABOUT THE EARTH.

FIGURE 7.2/1 MOTION OF THE SUN

where

T : is time in Julian centuries of 36525 ephemeris days from 1900 January 0 at 12 hrs ephemeris time

d is time in ephemeris days from 1900 January 0 at 12 hours ephemeris time

As the orbit determination program works in universal time, the value of T and d must be calculated using the formulae from section 7.1.

The true anomaly is found from the mean anomaly and eccentricity using the conversion in section 2.5.6. Then position and velocity are found in the inertial frame based on ecliptic and mean equinox using the transformation from section 2.5.1. The mean obliquity of the ecliptic which is required for this transformation is obtained from reference 5.

$$\epsilon = 23^{\circ}\!.452294 - 0^{\circ}\!.01301\,25\ T - 0.00000\,164\ T^2 + 0^{\circ}\!.00000\,0503\ T^3$$

The position and velocity are transformed into the inertial frame based on true equator and equinox using the rotations from sections 2.6.2 and 2.6.1.

7.3 Position of the Moon

The position of the moon is found following the procedure from reference 13, which is based on the Brown lunar theory. In order to reduce the amount of computation, major simplifications have been made. These simplifications are:

- All periodic terms with coefficients less than 180 seconds of arc have been eliminated in the latitude and longitude developments.

- All periodic terms with coefficients less than 3 seconds of arc have been eliminated from the lunar parallax development.

- All additive terms in the fundamental arguments have been neglected.

The fundamental arguments are evaluated from a power series as functions of ephemeris time in days from 1900 January 0.5 ET; or Julian day 2415020.0. These are:

- the geocentric mean longitude of the moon

- the mean anomaly of the moon

- the mean anomaly of the sun

- the mean distance of the moon from the ascending node

- the mean elongation of the moon from the sun

Then the lunar longitude in the ecliptic referred to the true equinox of date is calculated as the sum of:

- the geocentric mean longitude of the moon,
- the nutation in longitude, and
- a number of sine terms involving sums and differences of the other fundamental arguments.

The lunar latitude with respect to the ecliptic is calculated as

$$= A \sin S + B \sin 3S + C \sin 5S$$

where

- A, B, and C are constants
- S is the sum of the mean distance of the moon from the ascending node plus a number of sine terms involving sums and differences of the fundamental arguments.

The lunar parallax, π, is calculated as 3422.7 seconds of arc plus a number of cosine terms involving sums and differences of the fundamental arguments. The lunar radius is calculated as

$$R = \frac{6378.16}{\sin \pi}$$

The inertial coordinates of the moon with respect to the true ecliptic of date are found by converting to inertial coordinates using the conversion from section 2.4.5. These are transformed to true equator and true equinox using the same transformation as in 2.6.2, a rotation by the true obliquity of the ecliptic.

7.4 Position of the Earth

The position of the earth is obtained using universal time from the definition of universal time (ref 10 pg. 73)

"12 hours + the Greenwich hour angle of a point on the equator whose right ascension measured from the mean equinox of date is

$$Ru = 18^h\ 38^m\ 45^s\!.836 + 8640184^s\!.542\ Tu + 0^s\!.0929\ Tu^2$$

where Tu is the number of Julian centuries of 36525 days of universal time elapsed since 12 h universal time 1900 January 0"

From this definition we obtain the longitude of Greenwich, or Greenwich mean sidereal time as

$$\alpha_{gm} = 12^h + UT + Ru$$

In terms of the Modified Julian Date (section 7.1) the expression for α_g may be derived from the above as

$$\alpha_{gm} = 100.075542 + 360.985647346\ (t-33282) + 0.29 \times 10^{-12}\ (t-33282)^2$$

where α_{gm} is in degrees
t is the MJD

The longitude of Greenwich referred to the true equator and equinox of date is

$$\alpha_g = \alpha_{gm} + \delta\psi \cos(\varepsilon + \delta\varepsilon)$$

where $\delta\psi$ and $\delta\varepsilon$ are obtained from section 7.5 and ε is given in section 7.2.

7.5 Nutation

The expressions for nutation are obtained from reference 10 pages 41 to 45. Only the largest terms are included. The largest term neglected has an amplitude of 0.13 seconds of arc. This corresponds to a position error of less than 5 meters for a semimajor axis of 7000 km.

The equations for nutation in longitude and obliquity of the equinox are

$$\delta\Psi = -17''.2327 \sin\Omega - 1.2729 \sin(2F - 2D + 2\Omega) + 0.2088 \sin 2\Omega - 0.2037 \sin 2(\Omega + F)$$

$$\delta\varepsilon = 9''.2100 \cos\Omega + 0''.5522 \cos(2F - 2D + 2\Omega)$$

where the above are in seconds of arc, and

$$F = 11^{\circ}.250889 + 13^{\circ}.2293504490\, d - 2^{\circ}.407 \times 10^{-12}\, d^2$$

$$D = 350^{\circ}.737486 + 12^{\circ}.1907491914\, d - 1^{\circ}.076 \times 10^{-12}\, d^2$$

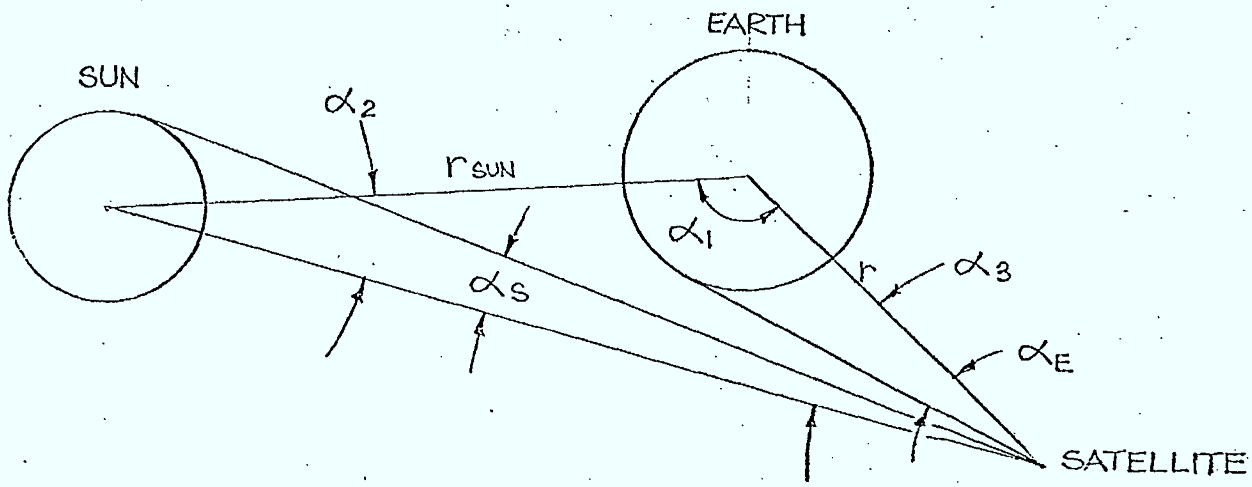
$$\Omega = 259^{\circ}.183275 - 0^{\circ}.0529539222\, d + 1^{\circ}.557 \times 10^{-12}\, d^2$$

8.0 EVENTS

8.1 Eclipses

An eclipse occurs when the earth or the moon comes between the satellite and the sun.

8.1.1 Eclipse of the Sun by the Earth



α_2 is the angle between the earth and the satellite as viewed from the sun. At the time of an eclipse, it is the earth half angle as viewed from the sun, or tan $\alpha_2 \approx 6378/149.6 \times 10^6$

$\alpha_2 \approx .0024$ degrees, or less than 9 seconds of arc

α_2 is assumed to be zero and neglected in eclipse calculations

α_1 is the angle between the satellite and the sun as viewed from the center of the earth. It is found from

$$\cos \alpha_1 = \frac{r.r_{sun}}{|r||r_{SUN}|}$$

where r_{SUN} is found in section 7.2.

α_E is the half-angle subtended by the earth as viewed from the satellite. The mean radius of the earth which used in this calculation is 6367.48 km

$$\alpha_E = \tan^{-1}\left[\frac{6367.48}{r}\right]$$

α_S is the semi-diameter of the sun as viewed from the satellite. This is approximated by the semi-diameter of the sun as viewed from the earth, or 15' 59"6, or 0.2666 degrees

α_3 is the angle between the earth and the sun as viewed from the satellite.

$$\alpha_3 = 180^0 - (\alpha_1 + \alpha_2)$$

$$\alpha_3 = 180^0 - \alpha_1$$

then the conditions for various events are

Transition from sunlight to penumbra

$$\alpha_3 = \alpha_E + \alpha_S$$

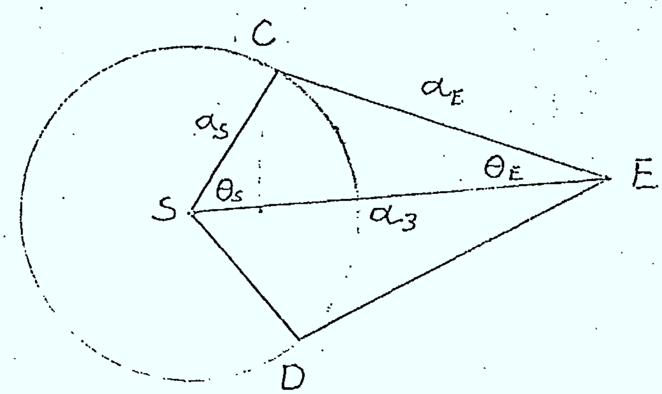
50% Eclipse

$$\alpha_3 = \alpha_E$$

Transition from penumbra to umbra

$$\alpha_3 = \alpha_E - \alpha_S$$

The calculation of percentage eclipse uses a plane trigonometry calculation as shown



The overlapping area is found as the area of the two intersecting arcs, CSD and CED, less the area of the two triangles CSE and DSE.

$$\cos \Theta_S = \frac{\alpha_S^2 + \alpha_3^2 - \alpha_E^2}{2 \alpha_S \alpha_3}$$

$$\cos \Theta_E = \frac{\alpha_E^2 + \alpha_3^2 - \alpha_S^2}{2 \alpha_E \alpha_3}$$

$$\sin \theta_s = \sqrt{1 - \cos^2 \theta_s}$$

$$A = \alpha_s^2 \theta_s + \alpha_E^2 \theta_E - \alpha_s \alpha_3 \sin \theta_s$$

Then the percentage obscured is given by A divided by the area of the sun

$$\% \text{ obscured} = \frac{100 \left[\alpha_s^2 \theta_s + \alpha_E^2 \theta_E - \alpha_s \alpha_3 \sin \theta_s\right]}{\pi \alpha_s^2}$$

8.2 Ascending Node Crossings

8.2.1 Inertial Coordinates

The ascending node is crossed when $r_z = 0$ going from negative r_z to positive r_z.

8.2.2 Velocity Space

The ascending node is crossed when

λ = longitude of ascending node

or

$$\lambda = \tan^{-1}\left[\frac{Cg_1}{-Cg_2}\right]$$

8.3 Crossings of an Identified Latitude

Latitude checking of a satellite is accomplished by monitoring previous and current latitude differences from a specified latitude coordinate. If these differences are of opposite sign, the satellite has crossed the desired latitude within the last integration step. When this event occurs, the differences are then utilized in a linear interpolation to estimate required time step size (forward or backward) to propagate the satellite to within an allowable deadband of the desired latitude.

In order to reduce the possibility of missing a desired latitude crossing because of the nature of the checking algorithm (a check is made only at discrete instants of time - after each numerical integration through a time step), the maximum time step size for integration should be relatively small, say no larger than 1/180th of the satellite orbital period, i.e.,

$$s < s_{max} = \frac{2\pi \sqrt{a^3/\mu}}{180}$$

where a is the orbit semi-major axis, in km., and μ is the earth central body term.

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10.0 NOTATION AND LIST OF SYMBOLS

Vectors are not normally distinguished from scalars and it is up to the reader to identify vectors from the context, except for a few instances where vectors are shown with arrows above them.

List of Symbols

a or a_p		semi-major axis
a		acceleration
A		cross sectional area of satellite; occulted area of sun during eclipse
Az		azimuth
b		number of biases
C		parameter in the velocity space set of orbital elements
C_D		drag coefficient
C_n^n		earth harmonic coefficient (constant)
d or d_E		days since the epock 1900 Jan. 0.5 ET
e		eccentricity; as $\vec{e}$, a vector with magnitude equal to the eccentricity directed towards peridpsis.
	or	number of orbital elements (6 or 7)
E		eccentric anomaly
	or	East direction or distance
E_{TOPO}		matrix for conversion from inertial to topocentric ENU coordinates
El		elevation
h		orbital angular momentum per unit mass
	or	nonlinear measurement function
	or	height above geospheroid
H		measurement matrix
i		inclination
I		identity matrix
J		Jacobian matrix of the equations of motion

K	Kalman gain matrix
L	Geodetic Latitude
	or mean longitude of the moon
L^1	Geocentric Latitude
m	mass
M	Mean Anomaly
N	North direction or distance
P	error covariance matrix
	or solar radiation force per unit area
P_n^m	Legendre polynomial
Q	state noise covariance matrix
r	position (without subscript: distance)
R	position (without subscript: distance)
	or measurement noise covariance matrix
	or parameter in the velocity space set
S_n^m	harmonic coefficient (constant)
t	time
T or T_E	time in Julian centuries since the epoch 1900 Jan. 0.5 ET
U	Up direction or distance
v	velocity
w	argument of peridpsis
x	state vector (vector of orbital elements)
y	observable vector
α_g	Greenwich sidereal time (longitude of Greenwich)
ε	orbit energy per unit mass
ϵ	obliquity of the ecliptic
Θ	local sidereal time
λ	longitude
	or parameter in the velocity space set of orbital elements
μ	gravitational constant (without subscript: of the earth)
ρ	range
	or density of the atmosphere
ϕ	true anomaly
Φ	state transition matrix
ψ	gravitational potential function; in $\delta\psi$, nutation in longitude
Ω	longitude of ascending node

Subscripts

ref — reference, for example $x_{ref} = \hat{x}\,(k+1/k)$

X, Y, Z — inertial coordinate axes. For example the components of inertial position are r_x, r_y, and r_z.

e_1, e_2, e_3 — coordinate axes defined in section 2.2.9

f_1, f_2, f_3 — coordinate axes defined in section 2.2.8

g_1, g_2, g_3 — inertial coordinate axes - the same as X,Y,Z.

E — relating to the earth
m — relating to the moon
s — relating to the sun
(of μ, etc.)

Superscripts

T — as in X^T indicates the transpose operation

· — as in $\dot{x}$ shows time derivative

→ — as in $\vec{v}$ indicates a vector

^ — as in $\hat{x}$ indicates an estimate of a quantity. Parenthesized indices following an estimate may give the time of the estimate and the time at which the last measurement was taken. Thus $\hat{x}(k + 1/K)$ means the estimate of $\hat{x}$ at time t_k with information up to and including time t_{k+1}.

~ — as in $\tilde{X}$ denotes a residual, commonly with respect to a reference.

APPENDIX A - Geometry of the ellipse

This appendix summarizes the equations which are used elsewhere in the document. It does not include proofs or development of the equations.

Figure A.1 shows the geometry of the ellipse. Point O is the center of the ellipse, F is the focus, P is a point on the ellipse, and a is the semi major axis. A circle is drawn around the ellipse and the point P' is on the circle with P'P parallel to the y axis. The lengths x_w and y_w are the x and y components of FP. The line CP is normal to the slope of the ellipse at point P.

The relationships which are used are listed below:

- the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(from analytic geometry)

- the ratio of the y coordinate of the ellipse to the y coordinate of the circle is b:a (ref 1)

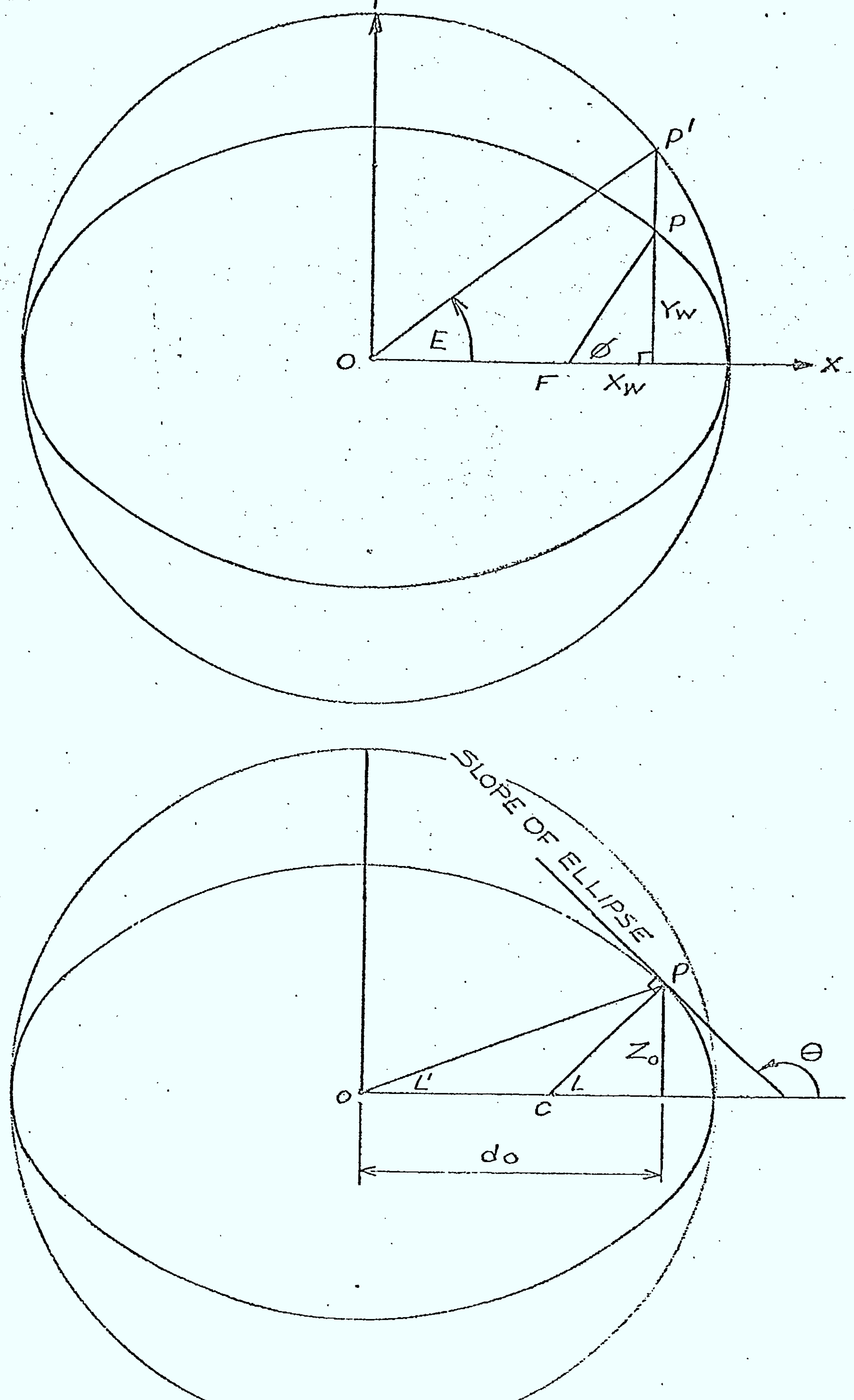
- $OF = ae$

- $b = a\sqrt{1 - e^2}$ (ref 1)

- $x_w = a(\cos E - e)$ (ref 1)

- $y_w = \sqrt{1 - e^2}\; a \sin E$ (ref 1)

- $r = a(1 - e \cos E)$ (ref 1)



GEOMETRY OF AN ELLIPSE
FIGURE A.1

$$\cdot\ \cos(\theta - \phi) = \frac{e \sin E}{\sqrt{1 - e^2 \cos^2 E}}$$

$$\cdot\ \tan \frac{\phi}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan E/2 \qquad \text{(ref 4)}$$

$$\cdot\ y_w = \frac{a \ (1 - e^2)}{\sqrt{1 - e^2 \sin^2 L}} \sin L \qquad \text{(ref 1)}$$

$$\cdot\ d_o = \frac{a}{\sqrt{1 - e^2 \sin^2 L}} \cos L \qquad \text{(ref 1)}$$

$$\cdot\ r = \frac{a}{\sqrt{1 + \frac{e^2}{1 - e^2} \sin^2 L}}$$

APPENDIX B - Integration Routines

The formulae for integration routines used in the orbit prediction software are summarized below. Each formula is given in the form used for a single differential equation, which is the simplest form. In actual practice, the integration routines are used on several simultaneous equations.

The basic equation is

$$\dot{x}(t) = f(t,x) \qquad \text{where } x(t_o) = x_o$$

Runge Kutta (fourth order)

$$x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = \Delta t\, f(t_n, x_n)$$

$$k_2 = \Delta t\, f\left(t_n + \frac{\Delta t}{2}, x_n + \frac{1}{2}k_1\right)$$

$$k_3 = \Delta t\, f\left(t_n + \frac{\Delta t}{2}, x_n + \frac{1}{2}k_2\right)$$

$$k_4 = \Delta t\, f(t_n + \Delta t, x_n + k_3)$$