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ABSTRACT

The signal-to-noise ratio (SNR) is calculated for a radar system consisting of a nonlinear amplifier, a mixer, a matched filter, and a two-pulse canceller. The principal result of this report is an expression for the SNR loss which is due to the nonlinear amplifier. The amplifier is assumed to consist of a memoryless limiter followed by a bandpass filter. For the purposes of calculation three limiter models were chosen: a hard limiter, an error function limiter, and a ν-th power device. Quite simple expressions are obtained for the SNR loss for these three models at input SNR less than -15dB. For normal radar system parameters, the losses range from 1dB or less up to 8dB. The highest loss occurs for the ideal hard limiter.

In order to obtain these results, previously known mathematical methods had to be extended. A significant feature of the extension is that it permits the calculation of the autocorrelation function of the amplifier output even when the input signal is completely deterministic, whereas most previous treatments assumed a random phase in the signal component.

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1. INTRODUCTION

In this report we present an analytical performance analysis of a scanning, air traffic control radar, operating in a strong stationary clutter environment. The particular problem to which we direct our attention is the effect of saturation in the IF strip on the operation of the moving target indicator (MTI). The saturating IF strip is modelled by various soft and hard bandpass limiters. The MTI is assumed to be a two-pulse canceller.

Although the effect of bandpass limiters on signal-to-noise ratio (SNR) has been studied extensively (see for example [1-3]) these results are not directly applicable to a MTI receiver. The studies [4] and [5] involve receivers similar to the one discussed here, but examine the case where the input consists of clutter alone. There have been digital computer simulation studies [6-8] of problems similar to the one discussed here. However, simulation of nonlinear IF amplifiers involves approximations because of time constraints. Hence an analytical study was undertaken. The principal result of the study is a relatively simple expression for the degradation in output SNR which is due to the saturation. The expression is fairly accurate when the input SNR is less than -15dB.

A simplified block diagram of the system under consideration is shown in Figure 1. Section 2 of this report is devoted to a small-signal analysis of the nonlinear portion of the system. This analysis is similar to and inspired by, the analyses of Jain [9] and Blachman [10]. However, because of the presence of the MTI, the autocorrelation function of the amplifier output is needed and consequently the analysis must be carried farther than Blachman's. In Section 3 the results of Section 2 are applied to evaluate the overall system performance and to compare the system with one having no saturation in the IF amplifiers. In Section 4 some simple approximate expressions for SNR loss are obtained, and in Section 5 the application of these results to two particular radar systems is discussed.

2. NONLINEAR AMPLIFIER ANALYSIS

2.1 Modelling the nonlinearity.

It will be assumed that the nonlinear bandpass amplifier can be represented by a memoryless device with characteristic $f(u)$ followed by a bandpass filter. More precisely, if the input to the device is

$$u(t) = V(t)\cos(\omega t+\phi(t)) \qquad (1)$$

it is assumed that the output of the memoryless nonlinearity is $f(u(t))$. If, in addition, $V(t)$ and $\phi(t)$ vary slowly compared with ωt , it is assumed that the output of the amplifier is

$$u_0(t) = g(V(t))\cos(\omega t+\phi(t)) \qquad (2)$$

where

$$g(V) = \frac{2}{\pi}\int_0^{\pi} f(V\cos\alpha)\cos\alpha d\alpha \quad . \qquad (3)$$

That is, it is assumed that the bandpass filter passes the fundamental term in the Fourier series for $f(u)$. Blachman [10,11] discusses this representation of nonlinear amplifiers more fully and provides a table [11] of values of g for many f .

It will be convenient for our purposes to use a Fourier integral representation for f in many of our calculations. Since we wish to model symmetric limiters we will assume that f is an odd function and that f can be represented as a sine transform. That is, we assume that

$$f(u) = \int_0^{\infty} F(z)\sin uz\, dz \qquad (4)$$

for some suitable function F . (The restriction to sine transforms is more a matter of convenience than importance. The methods used here could be extended to other representations by contour integrals with a Fourier kernel.) Then it is easily shown from (3) and (4) that

$$g(V) = 2\int_0^{\infty} F(z)J_1(Vz)dz \qquad (5)$$

where J_1 is the Bessel function of order 1. Three commonly used models of limiters, together with the corresponding F and g are displayed in Table 1. In Table 1, $a > 0$, $c > 0$, and $0 < \nu < 1$. The parameters have been chosen so that c and ν measure the "softness" of the limiters, in the sense that $f_2 \to f_1$ and $f_3 \to f_1$ as $c \to 0$ and $\nu \to 0$ respectively. In addition, $-a < f_2(u) < a$ for all u, with $f_2(u) \to \pm a$ as $u \to \pm\infty$. The functions f_1, f_2, f_3 are plotted in Figure 2 for various values of the parameters ν and $\lambda = (1+c^2)^{-2}$ (see Section 4.2 for significance of λ).

2.2 Statistical analysis

It will be assumed that the input to the nonlinear amplifier is

$$u(t) = s(t) + n(t) \tag{6}$$

where $s(t)$ is the signal term and $n(t)$, the clutter or noise, is a zero mean, stationary, narrowband, Gaussian random process which is independent of $s(t)$. Let

$$s(t) = P(t)\cos(\omega t+\theta(t)) \tag{7}$$

where $P(t) \geq 0$ and $\theta(t)$ vary slowly compared with ωt. Let the narrow band representation of $n(t)$ be

TABLE 1 -- Limiter Characteristics and Outputs

Limiter Characteristic f	Inverse Transform F	Output Amplitude g
$f_1(u) = \begin{cases} a & u>0 \\ -a & u<0 \end{cases}$	$F_1(z) = \frac{2a}{\pi z}$	$g_1(V) = \frac{4a}{\pi}$
$f_2(u) = \frac{a}{c}\sqrt{\frac{2}{\pi}}\int_0^u e^{-t^2/2c^2}\,dt$	$F_2(z) = \frac{2a}{\pi z}\,e^{-c^2 z^2/2}$	$g_2(V) = \frac{aV}{c}\sqrt{\frac{2}{\pi}}\;{}_1F_1\left(\frac{1}{2};2;-\frac{V^2}{2c^2}\right)$
$f_3(u) = \begin{cases} au^{\nu} & u\geq 0 \\ -a(-u)^{\nu} & u\leq 0 \end{cases}$	$F_3(z) = \frac{\nu a \csc(\nu\pi/2)}{\Gamma(1-\nu)\,z^{\nu+1}}$	$g_3(V) = \frac{2a}{\sqrt{\pi}}\,\frac{\Gamma\left(\frac{\nu+2}{2}\right)}{\Gamma\left(\frac{\nu+3}{2}\right)}\;V^{\nu}$

$$n(t) = A(t)\cos(\omega t + \alpha(t)) \tag{8}$$

where $A(t) \geq 0$ has a Rayleigh distribution [12] and $\alpha(t)$ is uniformly distributed on $[0,2\pi]$. For convenience of analysis it will be further assumed that the spectral density of $n(t)$ is symmetric about ω . Then [12, p. 298] the autocorrelation function of $n(t)$ can be written

$$E[n(t)n(t+\tau)] = \sigma^2 r(\tau)\cos\omega\tau \tag{9}$$

where $r(0) = 1$, and where E denotes the expectation operator.

Now u can be put in the form (1) where V and ϕ are determined by the complex sum

$$Ve^{j\phi} = Pe^{j\theta} + Ae^{j\alpha} \quad .$$

It follows from Graf's addition theorem [13, p. 361] that

$$J_1(Vz)e^{j\phi} = \sum_{m=-\infty}^{\infty} J_{m+1}(Az)J_{-m}(Pz)e^{j[(m+1)\alpha - m\theta]} \quad ,$$

where J_n denotes the Bessel function of order n . Combining this result with (2) and (5) we get the output

$$u_0(t) = 2\sum_{m=-\infty}^{\infty} \cos[\omega t + (m+1)\alpha - m\theta]\left(\int_0^{\infty} F(z)J_{m+1}(Az)J_{-m}(Pz)dz\right) \quad . \tag{10}$$

This is the total output of the nonlinear amplifier. The output signal is

$$s_0(t) = E[u_0(t)]$$

where the expectation is over the noise variables A and α. Since $A(t)$ and $\alpha(t)$ are independent and since $\alpha(t)$ is uniformly distributed, only the term $m = -1$ in (10) contributes to the average, the other cosines averaging to zero. Also [13, p. 393],

$$E[J_0(Az)] = \int_0^\infty \frac{A}{\sigma^2} e^{-A^2/2\sigma^2} J_0(Az)dA = e^{-\sigma^2 z^2/2} \quad . \tag{11}$$

Thus the output signal is

$$s_0(t) = 2\left(\int_0^\infty F(z)J_1(P(t)z)e^{-\sigma^2 z^2/2}\, dz\right)\cos(\omega t+\theta(t)) \; . \tag{12}$$

The noise output of the nonlinear amplifier is then

$$n_0(t) = u_0(t) - s_0(t) \tag{13}$$

2.3 Small-signal analysis

If $P(t)$ is always small compared with σ^2, it is evident from (12) that $J_1(Pz)$ can be replaced by $Pz/2$, the first term in its Taylor expansion about zero. Thus, for $P \ll \sigma^2$, we have the approximate output signal

$$\hat{s}_0(t) = QP(t)\cos(\omega t+\theta(t)) \quad , \tag{14}$$

where

$$Q = \int_0^\infty zF(z)e^{-\sigma^2 z^2/2}\, dz \tag{15a}$$

is the signal gain due to the nonlinear amplifier.

The gain, Q , can also be calculated from the equivalent expression

$$Q = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^\infty f'(u)e^{-u^2/2\sigma^2}\, du \quad . \tag{15b}$$

A simple, but formal, demonstration of the equivalence of (15a) and (15b) is obtained by differentiating both sides of (4) with respect to u and then using the integral formula

$$\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^\infty \cos uz\, e^{-u^2/2\sigma^2}\, du = e^{-\sigma^2 z^2/2} \quad .$$

One can also derive (15b) from (15a) more carefully by using Parseval's relation and a theorem about Fourier transforms of derivatives. Even if $f(u)$ is discontinuous, (15b) is valid provided that $f'(u)$ is regarded as a generalized function derivative.

Although it is not immediately apparent from (14) the approximation $\hat{s}_0(t)$ is the same as one obtained by Blachman [10] by another method. From the identity [13, p. 18]

$$zJ_1'(Az) + J_1(Az)/A = zJ_0(Az)$$

and (5) we have

$$g'(A) + g(A)/A = 2\int_0^\infty F(z)zJ_0(Az)dz \quad .$$

Combining this with (11) we get

$$\hat{s}_0(t) = \frac{1}{2} E[g'(A) + \frac{g(A)}{A}]\, s(t) \quad ,$$

which is Blachman's [10] expression.

Also, if $P \ll \sigma^2$, we can approximate the output noise by putting $P = 0$ in (13) . The approximate noise is then

$$\hat{n}_0(t) = g(A(t))\cos(\omega t+\alpha(t)) \quad , \qquad (16)$$

where g is given by (3) or (5).

The validity of these small-signal approximations will be examined more carefully in Appendix 1.

2.4 Autocorrelation of output noise

In order to discuss the noise output of the two-pulse canceller, we will require the autocorrelation function

$$\hat{r}_0(\tau) = E[g(A(t))g(A(t+\tau))\cos\alpha(t)\cos\alpha(t+\tau)] \quad . \qquad (17)$$

It is shown in Appendix 2 that

$$E[J_1(z_1A(t))J_1(z_2A(t+\tau))\cos\alpha(t)\cos\alpha(t+\tau)] = \frac{1}{2} I_1(r(\tau)\sigma^2 z_1 z_2)e^{-\frac{\sigma^2}{2}(z_1^2+z_2^2)} , \qquad (18)$$

where σ^2 and $r(\tau)$ are defined in (9) and I_1 is a modified Bessel function. It now follows from (5), (17) and (18) that

$$\hat{r}_0(\tau) = 2\int_0^\infty \int_0^\infty F(z_1)F(z_2)e^{-\frac{\sigma^2}{2}(z_1^2+z_2^2)} I_1(r(\tau)\sigma^2 z_1 z_2)dz_1dz_2 . \qquad (19)$$

In Table 2 we show the results of evaluating the integrals in (15) and (19) for the three limiter models of Table 1. The quantities σ^2 and $r(\tau)$ are related to the input noise as in (9). The integrals leading to Table 2 were evaluated using standard results on special functions in [13-15]. The double integral (19) can be evaluated either by expanding I_1 in a power series or by using suitable pairs of integral formulas for special functions. The error function limiter results can be obtained by direct integration or by the replacements of σ^2 by $\sigma^2 + c^2$ and $r(\tau)$ by $\sigma^2 r(\tau)/(\sigma^2+c^2)$ in the results for the hard limiter [16, eq. (50)]. The symbols Γ and ${}_2F_1$ denote the gamma function and the hypergeometric function in the notation of [14].

TABLE 2 -- Gain and Output Autocorrelation

Limiter Characteristic	Gain, Q	Output Autocorrelation, $\hat{r}_0(\tau)$
f_1 (Hard limiter)	$\frac{a}{\sigma}\sqrt{\frac{2}{\pi}}$	$\frac{2a^2}{\pi}\, r(\tau)\ {}_2F_1(\frac{1}{2},\frac{1}{2};2;r^2(\tau))$
f_2 (Error function limiter)	$\frac{a}{\sqrt{\sigma^2+c^2}}\sqrt{\frac{2}{\pi}}$	$\frac{2a^2\ \sigma^2}{\pi(\sigma^2+c^2)}\, r(\tau)\ {}_2F_1(\frac{1}{2},\frac{1}{2};2;\left[\frac{r(\tau)\sigma^2}{\sigma^2+c^2}\right]^2)$
f_3 (ν-th power device)	$\frac{a}{\sigma}\sqrt{\frac{2}{\pi}}\,(2\sigma^2)^{\nu/2}\ \Gamma(1+\frac{\nu}{2})$	$\frac{2a^2}{\pi}\, r(\tau)(2\sigma^2)^{\nu}\ \Gamma^2(1+\frac{\nu}{2})\,{}_2F_1(\frac{1-\nu}{2},\frac{1-\nu}{2};2;r^2(\tau))$

Most of the results in Table 2 are more or less well-known. For example, the value of $\hat{r}_0(\tau)$ for the hard limiter agrees, apart from notation, with the expression derived by Price [17]. The results for the ν-the power device agree with eq. (13.85) and (13.86) of Middleton [18].

3. DEMODULATION AND MTI FILTERING

A model of a scanning air traffic control radar is depicted in Figure 3. Two adjacent positions of the antenna and one range ring are shown. As the antenna rotates $\Delta\theta$ radians, N_B transmissions of $s(t)$ are received and processed by a mixer and matched filter, where N_B is the number of hits per beamwidth. To distinguish moving targets from clutter, the returns for two adjacent antenna positions are subtracted on a component-by-component basis. This operation is performed by the two-pulse canceller of Figure 1.

We will use the small-signal approximations $\hat{s}_0(t)$ and $\hat{n}_0(t)$ as the signal and noise at the output of the nonlinear amplifier. It will be assumed that the action of the mixer is to remove the ωt terms in (14) and (16) (i.e. mathematically, the mixer multiplies by $2\cos\omega t$ and filters out high-frequency components). Thus, after mixing, the input to the matched filter is

$$QP(t)\cos\theta(t) + g(A(t))\cos\alpha(t) .$$

The matched filter is matched to the modulation $P(t)\cos\theta_1(t)$, where $\theta(t) = \theta_1(t) + \theta_2(t)$ and θ_2 models Doppler shifts and any other random phase effects. The modulation will typically have a bandwidth in the megahertz range. The noise, on the other hand, typically has a bandwidth of about 100 hertz. This band is spread somewhat by the nonlinear amplification, but not to the mehahertz range. Thus it will be assumed that the noise passes through the matched filter unchanged, while the signal is modified in such a way that $P(t)$ is replaced by $P_1(t)$. The precise form of $P_1(t)$ is not needed for our development; hence we do not consider the structure of the matched filter in detail. To summarize, after the signal plus noise passes through the nonlinear amplifier, mixer, the matched filter, the output is

$$w(t) = QP_1(t)\cos\theta_2(t) + g(A(t))\cos\alpha(t) .$$

If the signal is a reflection from a moving target, and if the random phase effects are small, then

$$\theta_2(t) = \omega_d t \qquad (20)$$

where ω_d is the Doppler frequency. We now model the MTI by a two-pulse canceller, which forms the difference

$$w(t) - w(t-T) ,$$

where T is the reciprocal of the pulse repetition frequency. In the best possible case $\omega_d T = n\pi$, where n is an odd integer, i.e. $\cos\theta_2(t) = -\cos\theta_2(t-T)$, and $P_1(t) = P_1(t-T)$. For this case, the peak signal output of the two-pulse canceller is

$$S_0 = 2Q\bar{P} \quad , \tag{21}$$

where $\bar{P}$ is the peak value of $P_1(t)$.

The noise output of the canceller is

$$N(t) = g(A(t))\cos\alpha(t) - g(A(t-T))\cos\alpha(t-T) \quad . \tag{22}$$

From (17) and (22), the mean noise power at the output is

$$E[N^2(t)] = 2[\hat{r}_0(0)-\hat{r}_0(T)]$$

and thus the output signal-to-noise power ratio is

$$(SNR)_0 = \frac{4Q^2\bar{P}^2}{2[\hat{r}_0(0)-\hat{r}_0(T)]} \quad . \tag{23}$$

If, instead of the nonlinear amplifier, there had been a linear amplifier, it is not hard to see that the corresponding signal-to-noise ratio would be

$$(SNR)_\ell = \frac{4\bar{P}^2}{2\sigma^2[1-r(T)]} \quad . \tag{24}$$

That is, we replace Q by one and $\hat{r}_0(T)$ by $\sigma^2 r(T)$. Thus, if we define a SNR loss by

$$L = \frac{(SNR)_\ell}{(SNR)_0} \quad ,$$

we have

$$L = \frac{\hat{r}_0(0)-\hat{r}_\theta(T)}{\sigma^2[1-r(T)]Q^2} \quad . \tag{25}$$

Table 2 exhibits the values of Q and r_0 for three simple limiter models. The general expressions for Q , $\hat{r}_0$, σ^2 , r are in (15), (19) and (9).

We have only considered the SNR at the peak of the matched filter output. However, in the small signal approximation, the noise does not depend on the signal. Further, the second moment of $N(t)$ in (22) is independent of t . Thus the SNR at other points of the matched filter output is just attenuated by the ratio $P_1{}^2(t)/\overline{P}^2$. Consequently the peak to side-lobe ratio is the same as that for the linear case.

4. COMPUTATIONAL ASPECTS AND SIMPLIFIED EXPRESSIONS

An expression for SNR loss was the main object of this investigation. Thus, in principle, (25) meets this requirement. However, both numerator and denominator of (25) contain differences of almost equal quantities. In typical applications, $r(T) > 0.99$. Moreover, as will be seen from Table 2, $\hat{r}_0(T)$ is a function of $r(T)$ which tends to $\hat{r}_0(0)$ as $r(T)$ tends to one. If we set

$$\beta = r(T)$$

we can try to express L as a series of powers of $1 - \beta$ and to express $1 - \beta$ in terms of the radar system parameters. Regrettably, the matter is not that simple. In two of the examples of Table 2, we see $\hat{r}_0$ as a hypergeometric function of the variable β^2 . Since the series defining the hypergeometric function has radius of convergence equal to one, this function is not analytic at $\beta = 1$ and an attempt to find a Taylor series in powers of $1 - \beta$ will fail. (This failure could also have been predicted from (19) and an examination of the asymptotic behavior of $I_1(r\sigma^2 z_1 z_2)$ when $r > 1$). It is still possible to get simple approximations for L , but we shall have to treat each of the three examples separately.

4.1 Hard limiter

In order to simplify the loss expression for the hard limiter, $f_1(u)$ in Tables 1 and 2 , we use equations (15.1.20) and (15.3.11) of [14] to get

$$ {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};2;\beta^2) = {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};2;1) + \frac{(1-\beta^2)}{\pi}\,[\ell n(1-\beta^2)+3-4\ell n2] + O((1-\beta^2)^2\,\ell n(1-\beta^2)) \quad . $$

Ommitting terms of order $(1-\beta^2)^2 \ln(1-\beta^2)$ in this, we get

$$L \doteq \frac{4}{\pi} - \frac{\beta(1+\beta)}{\pi} [\ln(1-\beta^2)+3-4\ln 2] \quad .$$

To this order of approximation we also have $\beta \doteq 1$ and $1 + \beta \doteq 2$ so that

$$L \doteq \frac{6\ln 2-2-2\ln(1-\beta)}{\pi} \tag{26}$$

for the hard limiter.

The fact that $L \to \infty$ as $\beta \to 1$ (or $T \to 0$) may be disconcerting. An examination of (23) and (24) shows that the SNR $\to \infty$ with both linear and nonlinear amplifiers as $\beta \to 1$. It is just that the approach to ∞ is faster (order $(1-\beta)^{-1}$) in the linear case than in the nonlinear case (order $[(1-\beta)\log(1-\beta)]^{-1}$) .

4.2 Error function limiter

For $f(u) = f_2(u)$ with $c > 0$, define

$$\lambda = \left(\frac{\sigma^2}{\sigma^2+c^2}\right)^2 \quad .$$

When $2 - \lambda^{-1/2} < \beta \leq 1$, we can use a Taylor's series expansion of L in powers of $1 - \beta$. The reason for the limitation on the values of β is that ${}_2F_1(\frac{1}{2},\frac{1}{2};2;\lambda\beta^2)$ fails to be analytic when $\lambda\beta^2 = 1$. Thus the radius of convergence of the Taylor's series is $\lambda^{-1/2} - 1$ and the lower end of the interval of convergence is $1 - (\lambda^{-1/2} - 1) = 2 - \lambda^{-1/2}$. The resulting first-order approximation is

$$L \doteq {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};2;\lambda) + \frac{\lambda}{4}\,{}_2F_1(\tfrac{3}{2},\tfrac{3}{2};3,\lambda)$$

$$- \frac{3}{16}\lambda(1-\beta)[2\,{}_2F_1(\tfrac{3}{2},\tfrac{3}{2};3;\lambda) + \lambda\,{}_2F_1(\tfrac{5}{2},\tfrac{5}{2};4;\lambda)] \qquad (27a)$$

Since the complete elliptic integrals K and E (see [14], Chap. 17 for definitions and relations with the hypergeometric function) are more extensively tabulated, the following alternative expression may be more useful:

$$L \doteq \frac{4}{\pi}(2-\beta)\frac{[K(\lambda)-E(\lambda)]}{\lambda} - \frac{2}{\pi}\frac{1-\beta}{1-\lambda}E(\lambda) \quad . \qquad (27b)$$

The result can be derived from differentiation formulas for elliptic integrals [15, vol. 2].

4.3 ν-th power device.

To deal with $f_3(u)$ we use equations (15.1.20) and (15.3.6) of [14] and retain powers of $1 - \beta$ of order ν or less. The result is

$$L \doteq \frac{\Gamma(1+\nu)}{\Gamma^2(\frac{3+\nu}{2})}\left[\frac{1+\nu^2}{2\nu} - \frac{2^{1+\nu}\Gamma(-1-\nu)\Gamma^2(\frac{3+\nu}{2})}{\Gamma(1+\nu)\Gamma^2(\frac{1-\nu}{2})}(1-\beta)^{\nu}\right] . \qquad (28)$$

In the limit as $\nu \to 0$, the right side of (28) approaches the right side of (26).

5. EXPRESSION IN TERMS OF RADAR PARAMETERS

For some purposes it is more convenient to express $\beta = r(T)$ in terms of radar system parameters. We consider two possible clutter spectra and examine the SNR loss for each of these. First, we adopt a model which has been examined by McAuley [19]. In his development, the low pass component of the clutter spectrum has autocorrelation function

$$\frac{g^2}{2\pi} \omega_s \int_{-\infty}^{\infty} h(t+\tau)h(t)dt$$

where ω_s is the angular velocity of the antenna and $h(t)$ is its two-way beam pattern. A good approximation to many beam patterns is

$$h(t) = \text{sinc}^2\left(\frac{.88\pi\ f_r t}{N_B}\right)$$

where $\text{sinc}\ x = \sin x/x$, $f_r = T^{-1}$ is the pulse repetition frequency, and N_B is the number of hits per 3dB beamwidth (see Figure 3).* Standard calculations give

$$r(\tau) = \frac{3N_B^{\ 2}}{2f_r^{\ 2}\tau^2}\left[1 - \text{sinc}\ \frac{1.76\pi f_r \tau}{N_B}\right] .$$

Thus

$$\beta = r(f_r^{-1}) = \frac{3N_B^{\ 2}}{2}\left[1 - \text{sinc}\ \frac{1.76\pi}{N_B}\right] .$$

* In McAuley's clutter model, $\Delta\theta$ has been altered to be equal to the 3dB beamwidth.

Since $N_B \geq 10$ in most applications we use the approximation

$$\beta \doteq 1 - 1.53 N_B^{-2} \qquad (29)$$

when dealing with McAuley's [19] clutter spectrum.

Another model for clutter which has been used by Grasso and Guarguaglino [5] is a Gaussian autocorrelation,

$$R(\tau) = \sigma^2 \exp(-2\pi^2 \sigma_c^2 \tau^2) \quad ,$$

where the value $\sigma_c = 0.265 f_r/N_B$ is given by Barton [20, eq.(7.51)] This, in turn, gives

$$\beta \doteq 1 - 1.38 N_B^{-2} \qquad (30)$$

where N_B is the number of hits per beamwidth as before.

In Figures 4 and 5 we plot the SNR loss as a function of N_B for these two clutter spectrum models. For each clutter spectrum we plot the loss for the hard limiter as calculated from (26), for the error function limiter with $\lambda = 0.5, 0.75$ and 0.9 as calculated from (27), and for the ν-th power device for $\nu = 0.5, 0.25$ and 0.125 as calculated from (28). The formulas for calculating loss in dB as a function of N_B are also displayed in Table 3. It appears that the loss increases as the limiter becomes harder ($\nu \to 0$ or $\lambda \to 1$). For soft limiters the loss is almost independent of N_B, and for all limiter models considered, the loss is almost independent of the spectrum model.

6. CONCLUSION

The saturation of the front end of an air traffic control radar circuit has been modelled using ν-th power and error function devices. Simple formulas for the loss in signal-to-noise ratio due to the nonlinear amplifier have been obtained. These formulas have been used to evaluate the loss numerically for two clutter spectra of practical importance. The losses were found to range from 1dB to 8dB for interesting values of the parameters.

We have also developed a significant extension of Blachman's [10] method of dealing with nonlinear amplifiers. The extension, which may be useful in other applications, permits calcuation of a correlation function and facilitates estimates of error due to the small-signal approximation. It also holds some promise for being useful when the SNR is large.

TABLE 3

Amplifier	Loss(dB) General Formula	Loss(dB) McAuley Spectrum	Loss(dB) Gaussian Spectrum
Hard Limiter f_1	$10\log_{10}[0.687-0.637\ln(1-\beta)]$	$10\log_{10}[0.418+2.93\log_{10}N_B]$	$10\log_{10}[0.481+2.93\log_{10}N_B]$
Error Function			
f_2, $\lambda=0.90$	$3.19-10.31(1-\beta)$	$3.19-15.78/N_B^2$	$3.19-14.23/N_B^2$
$\lambda=0.75$	$2.06-4.00(1-\beta)$	$2.06-6.12/N_B^2$	$2.06-5.52\ N_B^2$
$\lambda=0.50$	$1.08-1.48(1-\beta)$	$1.08-2.26/N_B^2$	$1.08-2.04/N_B^2$
ν-th power device			
f_3, $\nu=0.125$	$6.84-3.71(1-\beta)^{1/8}$	$6.84-3.91/N_B^{1/4}$	$6.84-3.86/N_B^{1/4}$
$\nu=0.25$	$3.80-3.01(1-\beta)^{1/4}$	$3.80-3.34/N_B^{1/2}$	$3.80-3.26/N_B^{1/2}$
$\nu=0.5$	$1.18-1.68(1-\beta)^{1/2}$	$1.18-2.07/N_B$	$1.18-1.97/N_B$

APPENDIX 1 -- VALIDITY OF SMALL-SIGNAL APPROXIMATIONS

In this appendix we examine the small-signal approximations of Section 2.3 in order to estimate the range of validity of these approximations. These approximations involve the replacement of s_0 and n_0 of (12) and (13) by $\hat{s}_0$ and $\hat{n}_0$ of (14)-(16).

Consider first

$$s_0 - \hat{s}_0 = H(P)\cos(\omega t+\theta) \tag{A.1}$$

where

$$H(P) = \int_0^\infty F(z)[2J_1(Pz)-Pz]e^{-\sigma^2 z^2/2}\,dz \quad . \tag{A.2}$$

It is shown by Watson [13, p. 17] that

$$|2J_1(Pz)-Pz| \le |Pz|[e^{P^2z^2/8}-1] \quad . \tag{A.3}$$

This bound is crude, but sufficient for our purposes. From (A.2) and (A.3) ,

$$|H(P)| \le |P|\int_0^\infty ze^{-\sigma^2z^2/2}[e^{P^2z^2/8}-1]|F(z)|dz \quad .$$

Thus the relative error in the signal approximation is bounded by

$$\frac{|s_0-\hat{s}_0|}{|\hat{s}_0|} \le Q^{-1}\int_0^\infty ze^{-\sigma^2z^2/2}[e^{P^2z^2/8}-1]\,F(z)\,dz \quad , \tag{A.4}$$

where Q is given by (15), and in Table 2. The value of the integral in (A.4) can be obtained easily for the limiter characteristics of Table 1. The resulting relative errors are as follows:

Hard Limiter:

$$\frac{|s_0-\hat{s}_0|}{|\hat{s}_0|} \leq \left(1 - \frac{P^2}{4\sigma^2}\right)^{-1/2} - 1 \quad .$$

Error function limiter:

$$\frac{|s_0-\hat{s}_0|}{|\hat{s}_0|} \leq \left[1 - \frac{P^2}{4(\sigma^2+c^2)}\right]^{-1/2} - 1 \quad .$$

ν-th power device:

$$\frac{|s_0-\hat{s}_0|}{|\hat{s}_0|} \leq \left[1 - \frac{P^2}{4\sigma^2}\right]^{(\nu-1)/2} - 1 \quad .$$

An examination of these bounds shows that the hard limiter is the "worst case", in the sense that for given SNR , $P^2/2\sigma^2$, the bound is largest for the hard limiter. A little calculation then shows that the relative error in the signal amplitude is less than 3% if the SNR is below -10dB and less than 1% if the SNR is below -14dB .

The noise is somewhat more difficult to deal with. We will estimate the error in the noise power which is due to our approximation. To simplify the writing we will do our calculations after the mixer has shifted the amplifier output to low frequency. The noise is then, from (2), (12) and (13)

$$n_L = u_L - s_L$$

where

$$u_L = g(V)\cos\phi \quad ,$$

and

$$s_L = E[u_L] = 2\left(\int_0^\infty F(z)J_1(Pz)e^{-\sigma^2 z^2/2}\, dz\right)\cos\theta \quad . \tag{A.5}$$

Thus

$$E(n_L{}^2) = E(u_L{}^2) - s_L{}^2 \quad , \tag{A.6}$$

where the expectation operator E denotes averaging over the noise statistics. Now

$$Ve^{j\phi} = Pe^{j\theta} + X + jY$$

where X and Y are independent Gaussian random variables with mean 0 and variance σ^2 , and $X + jY = Ae^{j\alpha}$.
Thus

$$V^2 = (P\cos\theta+X)^2 + (P\sin\theta+Y)^2$$

and

$$\cos\phi = (P\cos\theta+X)/V \quad .$$

Hence

$$E[u_L{}^2] = \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^2(V)\cos^2\phi \exp[-\frac{(X^2+Y^2)}{2\sigma^2}]dXdY \quad ,$$

or with the change of variables

$$X + P\cos\theta = r\cos\beta \quad , \quad Y + P\sin\theta = r\sin\beta \ ,$$

$$E[u_L{}^2] = \frac{1}{2\pi\sigma^2} \int_0^{\infty}\int_0^{2\pi} g^2(r)\cos^2\beta \exp\{-\frac{[r^2+P^2-2rP\cos(\beta-\theta)]}{2\sigma^2}\} r d\beta dr$$

The integration with respect to β can be carried out with the aid of eq. (9.6.18) of [14] to give

$$E[u_L{}^2] = \frac{e^{-P^2/2\sigma^2}}{\sigma^2} \int_0^{\infty} re^{-r^2/2\sigma^2} g^2(r)[\cos^2\theta I_2(\frac{rP}{\sigma^2})+\frac{\sigma^2}{rP} I_1(\frac{rP}{\sigma^2})]dr \tag{A.6}$$

To check our approximations we put $g = g_3$ from Table 1. That is, we consider the ν-th power device. We get, with the aid of [13] and [14]

$$E[u_L{}^2] = \frac{2^{\nu+1} a^2 \sigma^{2\nu} \Gamma^2(\frac{\nu+2}{2})\Gamma(\nu+1)}{\pi\Gamma^2(\frac{\nu+3}{2})} [{}_1F_1(1-\nu,2,-\frac{P^2}{2\sigma^2})$$

$$+ (\nu+1) \frac{P^2}{2\sigma^2} \cos^2\theta \ {}_1F_1(1-\nu,3,-\frac{P^2}{2\sigma^2})] \ .$$

For the same device we get, upon evaluating (A.5) ,

$$s_L = \left(\frac{\nu\pi}{2} \csc \frac{\nu\pi}{2}\right) \frac{2^{1+\nu/2}\ \sigma^\nu\ a}{\sqrt{\pi}\ \Gamma(1-\nu/2)} \left(\frac{P}{\sqrt{2}\,\sigma}\right)\ {}_1F_1\left(\frac{1-\nu}{2}, 2; \frac{-P^2}{2\sigma^2}\right) \cos\theta \quad ,$$

where the first bracketed expression is equal to 1 for $\nu = 0$. From these expressions we get $E(n_L{}^2)$ from (A.6). To get an indication of the dependence on the SNR, we expand in powers of $x = P^2/2\sigma^2$. The result is

$$E[n_L{}^2] = \frac{2^{\nu+1}\Gamma^2(\frac{\nu+2}{2})\Gamma(\nu+1)a^2\ \sigma^{2\nu}}{\pi\ \Gamma^2(\frac{\nu+3}{2})}\ [1-Cx+O(x^2)] \quad , \qquad (A.7)$$

where

$$C = \frac{1-\nu}{2} - (\nu+1)\cos^2\theta + \frac{2(\frac{\nu\pi}{2} \csc \frac{\nu\pi}{2})^2\ \Gamma^2(\frac{\nu+3}{2})}{\Gamma^2(\frac{2-\nu}{2})\ \Gamma^2(\frac{2+\nu}{2})\ \Gamma(1+\nu)} \cos^2\theta \qquad (A.8)$$

The effect of our approximation in Section 2.3 is to take $x = 0$ in the noise terms. As will be seen from (A.7), this leads to a relative error approximately equal to Cx , for small x .

If we put $\nu = 0$ (for the hard limiter) and $\cos^2\theta = \frac{1}{2}$ (its average value if θ is uniformly distributed) we get $C = \frac{\pi}{4}$ and the relative error is $\pi x/4$. For $x = .03$ (-15dB) , $\pi x/4 \doteq 2.4\%$ and for $x = .01$ (-20dB) , $\pi x/4 \doteq 0.8\%$. Thus at least in this case our approximations are adequate for SNR less than -15dB .

Calculations with other values of ν lead to similar conclusions. For example, if $\nu = 0.25$ and $\cos^2\theta = \frac{1}{2}$

we get $C = .637$. Thus if $= 0.25$ and $x = .03$, $Cx = 1.9\%$ and hence the error is even smaller here.

It thus appears safe to use the small signal approximation for SNR less than -15dB , since neither the signal power nor the noise power is changed by more than about 2.5% by making this approximation.

APPENDIX 2 -- AN ORTHOGONALITY RELATION

We derive here a general orthogonality relation (equation (A.9) below) of which (18) is the special case for $m = 1$. The general relation is almost as easy to derive as (18) and can be used in conjunction with (10) to treat cases where the small-signal approximation does not hold.

First we introduce the notation. Let $A_1 = A(t)$, $A_2 = A(t+\tau)$, $\alpha_1 = \alpha(t)$, $\alpha_2 = \alpha(t+\tau)$ where A and α are the amplitude and phase of the narrowband Gaussian noise process $n(t)$ with autocorrelation given by (9). Let θ_1 and θ_2 be any numbers. (In use with (10) , $\theta_1 = \theta(t)$, $\theta_2 = \theta(t+\tau)$.) Let m and n be integers. Then we show that

$$E\{J_m(A_1z_1)J_n(A_2z_2)\cos[m\alpha_1-(m-1)\theta_1]\cos[n\alpha_2-(n-1)\theta_2]\}$$

$$= \frac{1}{2} e^{-\frac{1}{2}\sigma^2(z_1^2+z_2^2)} I_n(\sigma^2 r(\tau)z_1z_2)\{\delta_{m,n} \cos(n-1)(\theta_2-\theta_1)$$

$$+ (-1)^n \delta_{-m,n} \cos[(n-1)\theta_2-(n+1)\theta_1]\} \quad . \qquad (A.9)$$

To form the expectation in (A.9) we multiply by the density function $f(A_1,A_2,\alpha_1,\alpha_2)$ and integrate over these four variables. The density function is given by (see [12], eq. (8-102) with a simplification because we assume a noise spectrum symmetric about ω here)

$$f(A_1,A_2,\alpha_1,\alpha_2) =$$

$$\frac{A_1\ A_2}{4\pi^2\ \sigma^4(1-r^2)} \exp\{\frac{-1}{2\sigma^2(1-r^2)}\ [A_1^2+A_2^2-2A_1A_2 r\cos(\alpha_2-\alpha_1)]\} \ . \qquad (A.10)$$

The integration with respect to α_2 gives

$$\frac{1}{2\pi}\int_0^{2\pi} \cos[n\alpha_2-(n-1)\theta_2]\ \exp\ \frac{A_1A_2 r\cos(\alpha_2-\alpha_1)}{\sigma^2(1-r^2)}\ d\alpha_2$$

$$= I_n\left(\frac{A_1A_2\ r}{\sigma^2(1-r^2)}\right)\ \cos[n\alpha_1-(n-1)\theta_2] \quad , \qquad (A.11)$$

where eq. (9.6.19) of [14] and elementary trigonometric identities have been used. The integration with respect to α_1 then becomes

$$\frac{1}{2\pi}\int_0^{2\pi} \cos[m\alpha_1-(m-1)\theta_1]\cos[n\alpha_1-(n-1)\theta_2]d\alpha_1$$

$$= \frac{1}{2}\{\delta_{m,n} \cos(n-1)(\theta_2-\theta_1) + \delta_{-m,n} \cos[(n-1)\theta_2-(n+1)\theta_1] , \quad (A.12)$$

since m and n are integers.

Using (A.10-A.12) and the relation $J_{-n} = (-1)^n J_n$ [13], the expectation of (A.9) becomes

$$\frac{1}{2}\{\delta_{m,n} \cos(n-1)(\theta_2-\theta_1) + (-1)^n \delta_{-m,n} \cos[(n-1)\theta_2-(n+1)\theta_1]\}d_n \quad (A.13)$$

where

$$d_n = \int_0^\infty \int_0^\infty \frac{A_1A_2\, e^{-(A_1{}^2+A_2{}^2)/2\sigma^2(1-r^2)}}{\sigma^4(1-r^2)} \cdot$$

$$\cdot I_n\left(\frac{A_1A_2\, r}{\sigma^2(1-r^2)}\right) J_n(A_1z_1)J_n(A_2z_2)dA_1dA_2 . \quad (A.14)$$

Using $I_n(z) = (-j)^n J_n(jz)$ [13, p. 77] and Weber's second exponential integral [13, p. 395] the integration with respect to A_2 gives

$$d_n = \frac{1}{\sigma^2} \exp[-\frac{\sigma^2}{2}(1-r^2)z_2{}^2] \int_0^\infty A_1J_n(A_1z_1)J_n(rz_2A_1)e^{-A_1{}^2/2\sigma^2} dA_1$$

and a second use of Weber's integral yields

$$d_n = e^{-\frac{\sigma^2}{2}(z_1{}^2+z_2{}^2)}\ I_n(\sigma^2 r z_1 z_2) \quad . \tag{A.15}$$

Finally (A.13) and (A.15) immediately yield (A.9).

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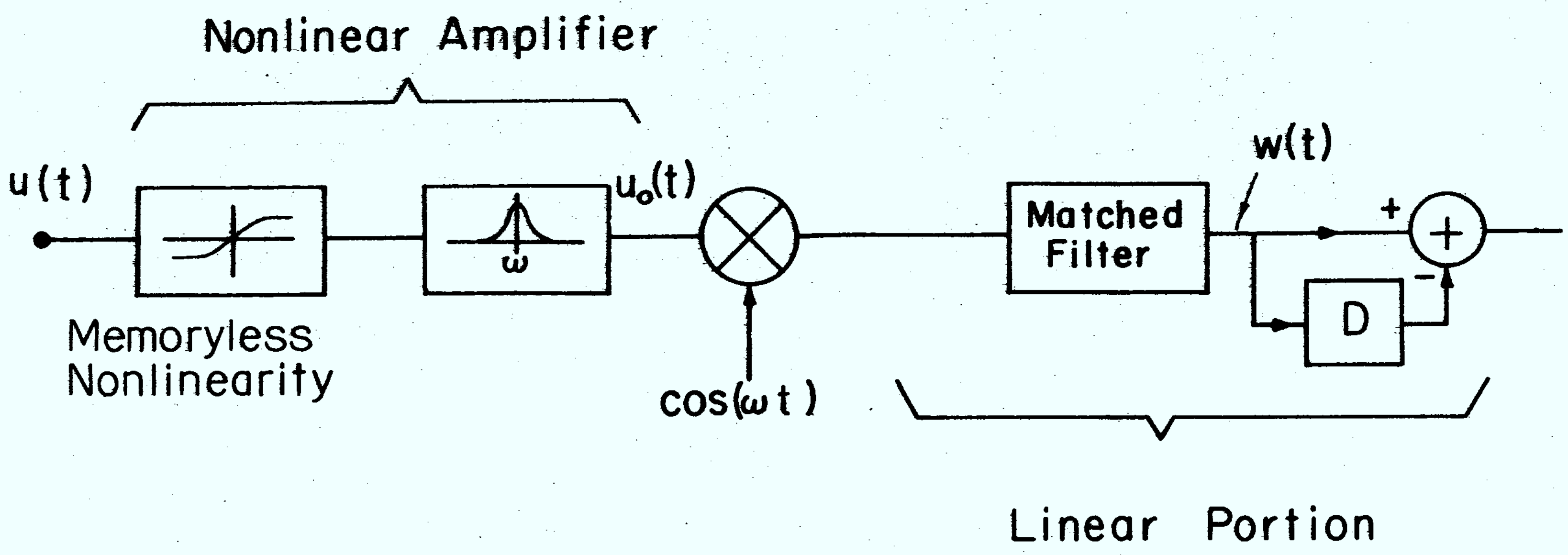

FIGURE 1 — BLOCK DIAGRAM OF THE SIGNAL PROCESSING SYSTEM

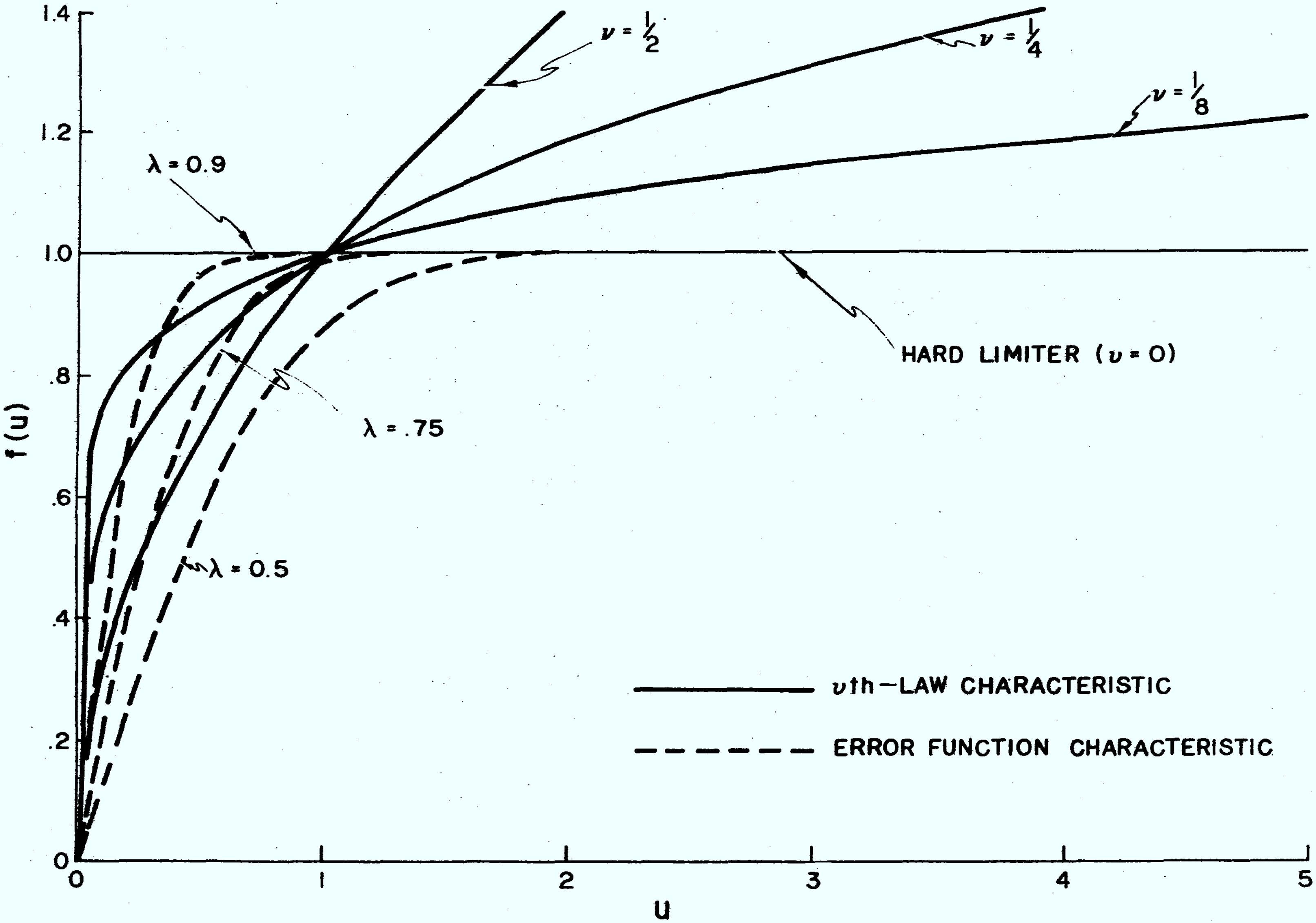

FIGURE 2: COMPARISON OF LIMITER CHARACTERISTICS.

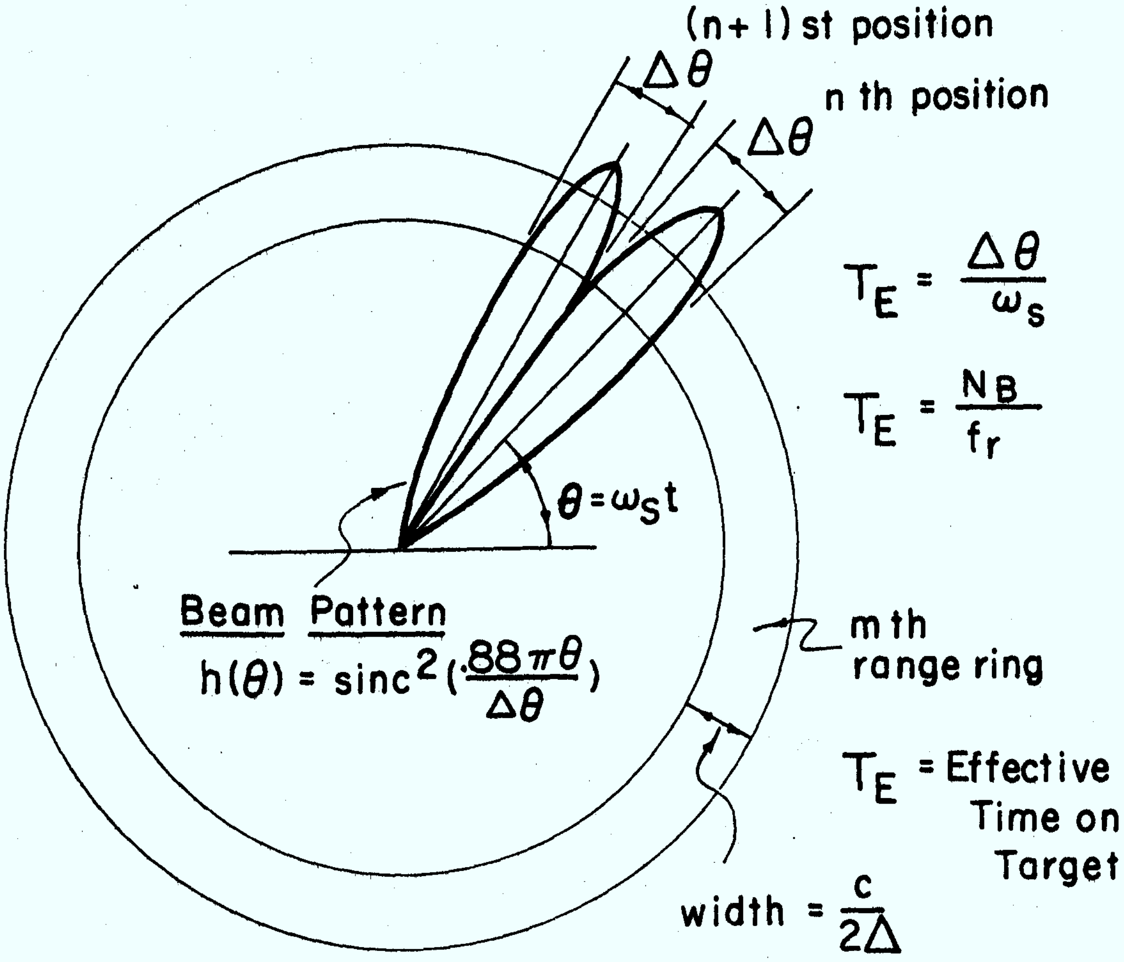

FIGURE 3: - SCANNING RADAR MODEL

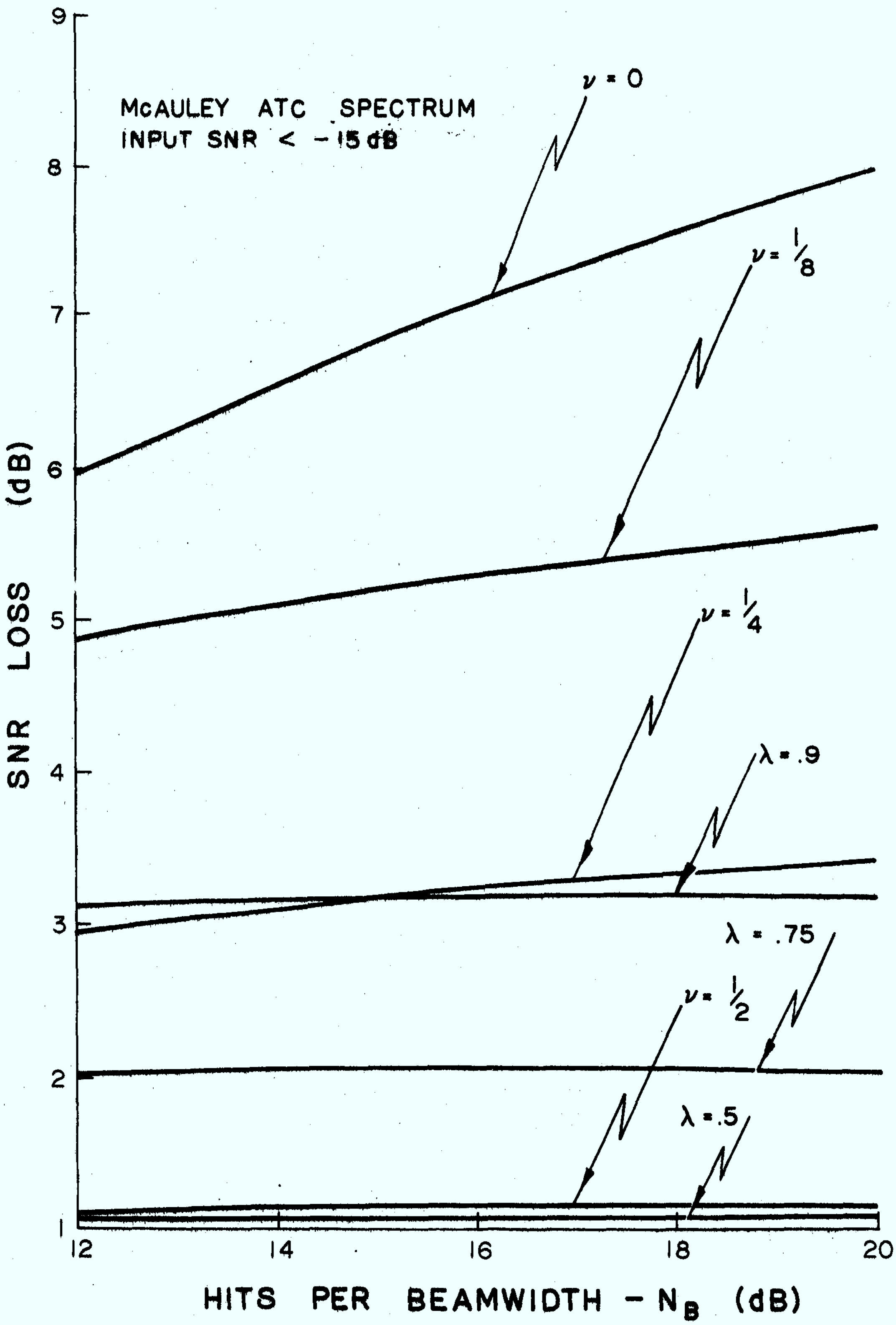

FIGURE 4: SNR LOSS VS N_B FOR McAULEY ATC SPECTRUM.

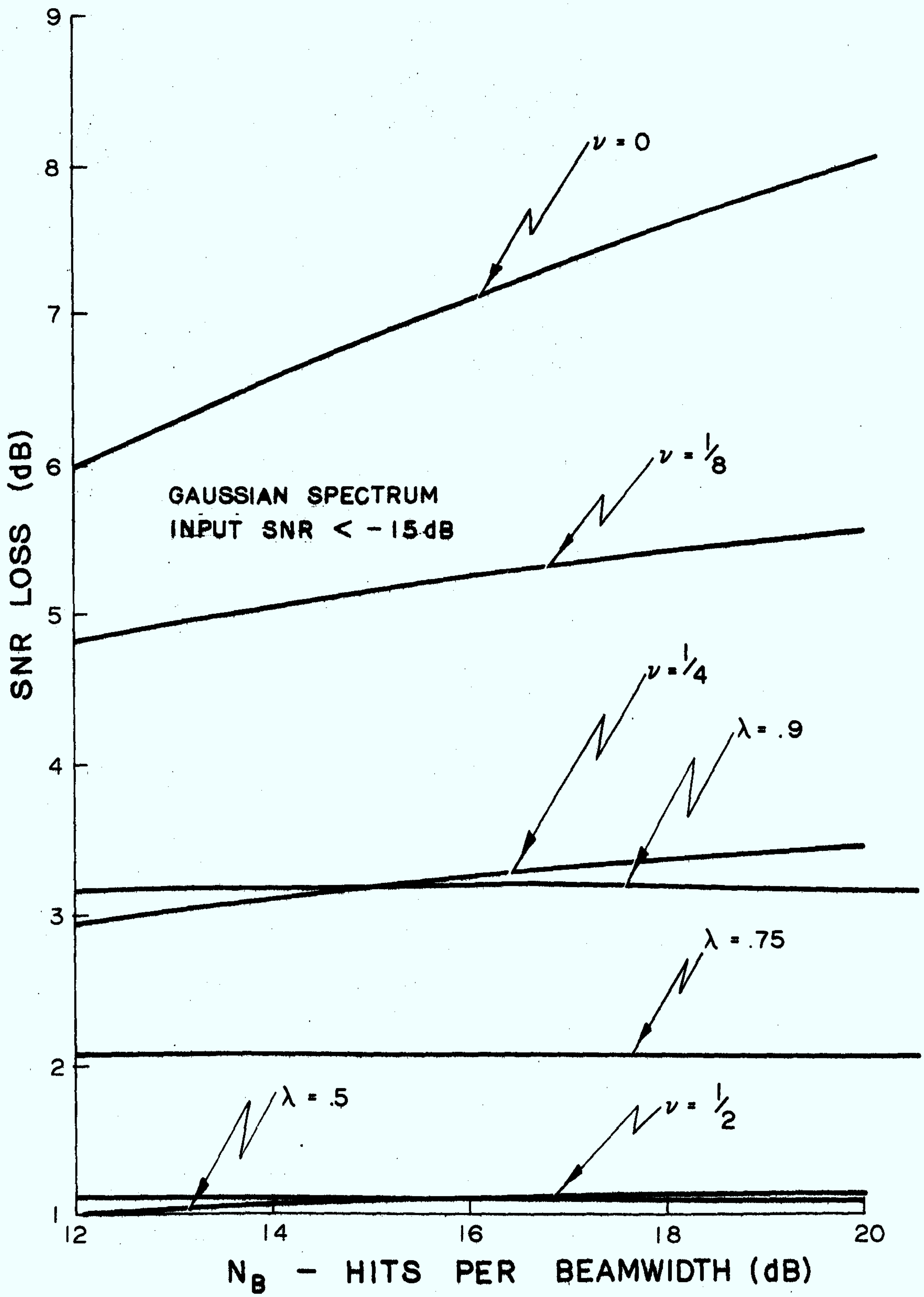

FIGURE 5: SNR LOSS VS N_B FOR GAUSSIAN SPECTRUM.

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