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Interpolation of Complex-Valued Response Amplitude Operators

Approaches and Effects

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IMPORTANT INFORMATIVE STATEMENTS

Abstract

Response amplitude operator (RAO) data sets are used in many applications, such as for calculating RMS ship motions in arbitrary sea conditions. In the process of performing these calculations, there can be cases where the complex-numbers of a RAO data set must be interpolated. This report presents the results of a study examining different approaches to interpolating complex RAO data, such as using its real & imaginary parts, or by using its magnitudes & phase angles. Performing interpolation at different steps within a calculation procedure for RMS motions is also examined to evaluate the effects on output.

Significance for defence and security

Calculating RMS ship motions using RAO data is a routine activity in support of naval activities such as planning, operations, and design evaluation. Identifying approaches to reduce even minor sources of interpolation error in these calculations is part of a continuous effort at DRDC to develop and deliver the best possible data to the Royal Canadian Navy.

Résumé

Les ensembles de données d’amplitude des réactions de l’opérateur (response amplitude operator, ou RAO) servent à de nombreuses applications dont le calcul des mouvements des navires de la MRC en conditions en mer arbitraires. Cependant, pour ces calculs, il arrive de devoir interpoler les nombres complexes de l’ensemble de données RAO. Le rapport scientifique présente les résultats d’une étude de diverses stratégies d’interpolation des données RAO complexes, comme séparer les parties réelles et imaginaires des nombres ou utiliser plutôt leur magnitude et leur angle de phase. L’étude s’est aussi penchée sur l’interpolation à diverses étapes du calcul des mouvements des navires de la MRC afin d’en cerner les répercussions sur les résultats obtenus.

Importance pour la défense et la sécurité

On calcule souvent les mouvements d’un navire à l’aide des données RAO pour appuyer les activités navales comme la planification, les opérations et l’évaluation de la conception d’un navire. Trouver des stratégies de réduction des erreurs d’interpolation (même légères) dans ces calculs s’intègre dans les efforts continus de RDDC de développer et livrer à la Marine royale canadienne des données de la qualité la plus élevée qu’il est possible d’atteindre.

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1 Introduction

Response amplitude operators (RAOs) are used in marine hydrodynamics for the prediction of RMS ship motions for arbitrary combinations of ship speed, heading, and sea conditions. They are frequency response transfer functions describing the relationships between wave excitation forces (input) and modes of ship response (output). Response amplitude operators (RAOs) are most commonly generated using frequency-domain potential flow based codes which output the RAO over discrete ranges of ship speeds, wave frequencies, and relative directions.

There can be cases where RAO data may be needed at values of ship speeds, wave frequencies, and/or relative directions not included in the discrete sets defining the RAO. This necessitates some form of interpolation in order to perform the computation. Various approaches to interpolation can be taken, and at different steps of the calculation procedure. The choice of approach can affect the accuracy of the final result.

This report will examine various interpolation approaches of RAO for the calculation of RMS motions using an example case of a tanker ship. These include different approaches to interpolating RAO and those that can be applied to avoid RAO interpolation altogether.

2 Complex Response Amplitude Operators

Response amplitude operators (RAOs) are transfer functions used to calculate ship motions in the frequency domain, such as RMS amplitude and zero-crossing period. RAO can be derived by various means but are often produced by seakeeping prediction programs. The RAOs used for the test cases in this report were generated using the in-house ship motions prediction suite ShipMo3D (Version 3.1) [1–3].

Note that the conventions used for defining RAO data sets are not standardized and data sets from sources other than ShipMo3D may differ from those discussed in this report. Common differences include: how the coordinate system is defined, alternative definitions for relative direction (see Annex A), the of use encounter frequency instead of incident wave frequency (see Annex C), and the choice of units (particularly for angular motions, see Annex B).

An RAO data set consists of number pairs that define the ship response (e.g., roll) to a specific wave direction and period. Each pair include an amplitude, r, usually expressed as a ratio (e.g., heave response per unit wave height), and a phase, α, which defines the timing of the vessel motion relative to the wave. For example, the highest point of the ship heave motion may occur after the crest of the wave passes the ship.

These number pairs (amplitude and phase) are often expressed as complex numbers of the form, $z = xi + y$. Eqns. (1) to (5) can be used to convert between amplitude & phase to z with ‘real’ & ‘imaginary’ components, x and y.

$$z = xi + y = r\,e^{i\alpha} \tag{1}$$

$$r = |z| = \sqrt{x^2 + y^2} \tag{2}$$

$$\alpha = \arctan(y, x) \tag{3}$$

$$x = r\cos\alpha \tag{4}$$

$$y = r\sin\alpha \tag{5}$$

where,

z is a complex number
x is the real component of z
y is the imaginary component of z
r is the magnitude of z
α is the phase angle of z

The relationship between the polar form of the RAO number pair (amplitude & phase) and the rectangular form (real & imaginary components) is best illustrated graphically using the complex plane where the horizontal axis is used for the real component and the vertical axis for the imaginary component. Figure 1 shows the complex plane with an example point using both polar and rectangular forms of the same position.

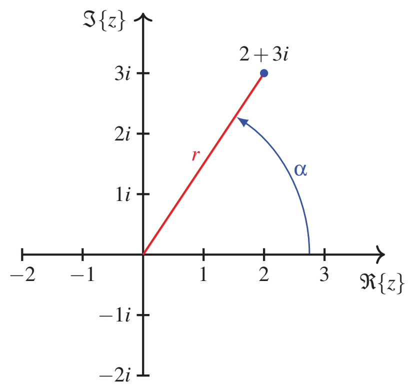

Figure 1: *Example coordinates in the complex plane.*

Another way of thinking of positions in the complex plane relate to the dynamics of a mass attached to a spring. At any given moment in a mass-spring system, the mass is either gaining or losing momentum in response to be pushed/pulled by the spring, and the spring is applying more or less force as it is stretched/compressed.

Energy is cycled between kinetic energy of the mass (how fast its moving) and the potential energy of the spring (how much it is stretched or compressed). This transfer can be shown in the complex plane as Figure 2. For the case of ship motions, the mass is the ship and the spring is the combination of gravitational and hydrodynamic forces from the waves.

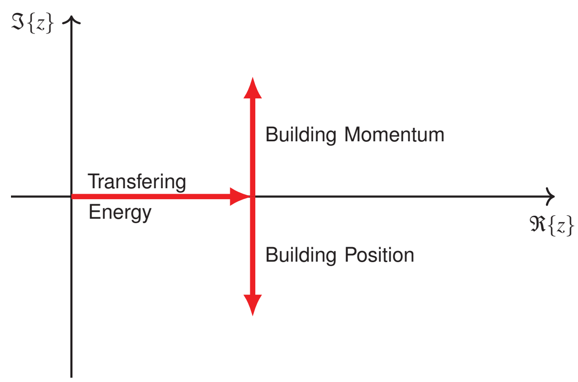

Figure 2: *Energy transfer in the complex plane.*

2.1 Complex Number Interpolation

Interpolation is a way of finding new data within a set of known data in a way that best estimates the 'true' value of the underlying function or process defining that data. This underlying function is assumed to be both continuous and smooth (its derivatives are also continuous). There are many methods for performing interpolation ranging from simple linear to sophisticated high-order piecewise—each with its own set of strengths and drawbacks. For the sake of simplicity, only basic linear interpolation will be used here.

As complex numbers consist of two independent components, these components are interpolated separately and then recombined. The complication is that interpolation may give different results depending on whether the rectangular or polar components are used. This is illustrated in Figure 3 which shows two points C1 and C2 in the complex plane. All possible values between these two points are shown as coloured paths for the interpolation of rectangular components (green) and again for the polar components (red). The rectangular approach generates a straight line path between the two points whereas the polar approach creates a curved path.[1]

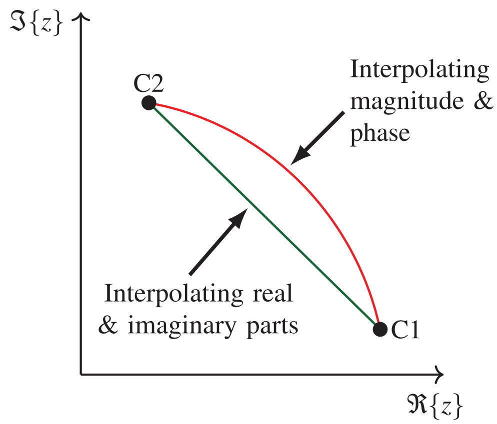

Figure 3: *Example of the two paths formed when linearly interpolating all values between points C1 and C2 using interpolation of rectangular and polar components.*

To further complicate the issue, interpolation of the phase angle may or may not account for the property of angles wrapping around the negative real axis when α is simultaneously equal to both 180° and −180°. Approaches that first 'unwrap' the phase angle (see Annex D) will generate a different path than those that do not. This is demonstrated in Figure 4 where C1 and C2 are on either side of the negative real axis; C1 has a positive phase angle and C2 has a negative phase angle. As in Figure 3, interpolation of rectangular components creates a straight line path (green) between the two points. Interpolation of polar components without unwrapping α creates the long curved path (purple) from C1 passing through the positive real axis ($\alpha = 0°$) to C2. Interpolation of the phase angle after unwrapping produces the shorter curved path (red) which passes through the negative real axis.

Using basic linear interpolation, Figure 4 shows three very different paths which are all 'between' points C1 and C2. So which is one is correct? The answer depends on the nature of the data being interpolated. If the underlying processes producing the data set follow approximately straight line paths in the complex plane, then rectangular component interpolation is best. If the data tends to follow more curved paths akin to polar component interpolation, then that approach would be best.

[1] In the special case when both C1 and C2 lie on a straight line that also passes through the origin (i.e., a radial line), both approaches are equivalent.

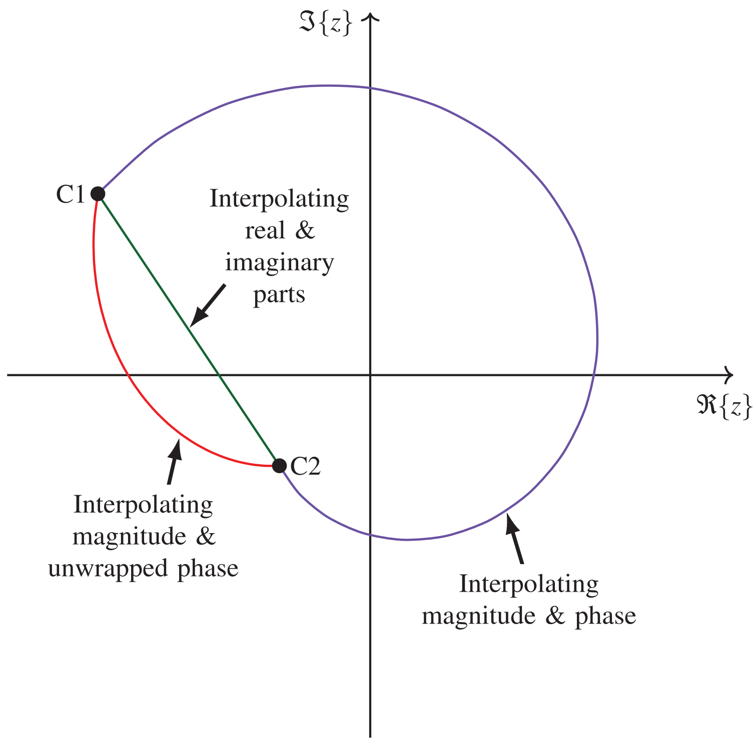

Figure 4: *Example of the paths formed when linearly interpolating all values between points C1 and C2 using different approaches.*

Note that polar component interpolation should almost always be performed after first unwrapping the phase angle (e.g., the red path in Figure 4). However, most commonly used library functions used for interpolation (also for statistics like mean and standard deviation) are not written to handle wrapped angles, so unwrapping is an extra step that is sometimes forgotten. The purple path shown in Figure 4 is intended to show the potential significance of this step. When discussing polar component interpolation in other sections, only the unwrapped approach is applied.

3 Interpolating RAO

Seakeeping prediction programs provide a discrete set of RAO data defined over a range of ship speeds, wave frequencies, and relative wave headings. Ideally, an RAO set will (for each parameter of speed, frequency, and direction) cover the full range of expected values, have equally sized increments over the range that are small enough to fully resolve the character of the RAO data set. However, even if all of these conditions are met there may be times where data is needed at specific values for speed, heading, and/or frequency that are not explicitly defined in the RAO. In these cases, interpolation may be necessary.

It is common for RAO data to be expressed in terms of complex numbers which, as discussed in Section 2.1, can give rise to different answers from interpolation, depending on the whether using rectangular or polar coordinates. This section will examine the results that arise from using these different approaches.

As an example case, the following two sets of RAO for a tanker ship roll motions were generated using a seakeeping program: a coarse set with dimensions 14x37x72 and a fine set with dimensions 105x181x360, see Table 1. The evaluation involved applying different approaches of interpolation to the coarse data set to fill in points so as to match the fine set. The interpolated results were then compared against the fine set to see how well each approach performed. The process was conducted separately for each dimension of the RAO: wave direction, frequency, then ship speed. Following that, an additional interpolation approach is introduced for the specific case when RAO are used to calculate RMS motions.

While the RAO data sets were generated using ShipMo3D, the interpolation tests were performed in MATLAB® (Version R2019a). Example code for performing complex-number interpolation in MATLAB is given in Annex E.3.

Table 1: *Generated RAO data sets.*

RAO Set	Parameter	V_s	ω	β_s
Fine Set	Count	105	181	360
	Increment	0.25 knots	0.01 rad/s	1°
	Range	0–26	0.2–2.0	0–359
Coarse Set	Count	14	37	72
	Increment	2 knots	0.05 rad/s	5°
	Range	0–26	0.2–2.0	0–355

3.1 Interpolating Relative Direction

As as first example, a subset of RAO was first extracted from both the fine and coarse sets corresponding to a ship speed of 20 knots and a wave frequency of 0.4 rad/s. This reduces the independent variable for interpolation to just relative direction, β_s. These sets are plotted together in Figure 5. The blue curve represents the fine set and the coarse set is given by black dots which, expectedly, all fall on the blue curve. The top two charts show the real and imaginary parts of the RAO, while the bottom two show the magnitude and phase angle components.

The 72 values (every 5°) in the coarse subset were then interpolated to 360 values (every 1°) using rectangular component interpolation and polar component interpolation. The results are shown in Figure 6. The differences between the two approaches are mostly seen at the peaks when β_s is approximately 90° and 270° as was as when the phase angle crosses 0° at $\beta_s = 180°$.

Another way of comparing the results is to plot them in the complex plane as shown in Figure 7. In this example it is evident that the polar component interpolation is closer to matching the curvature of the fine set then rectangular component interpolation.

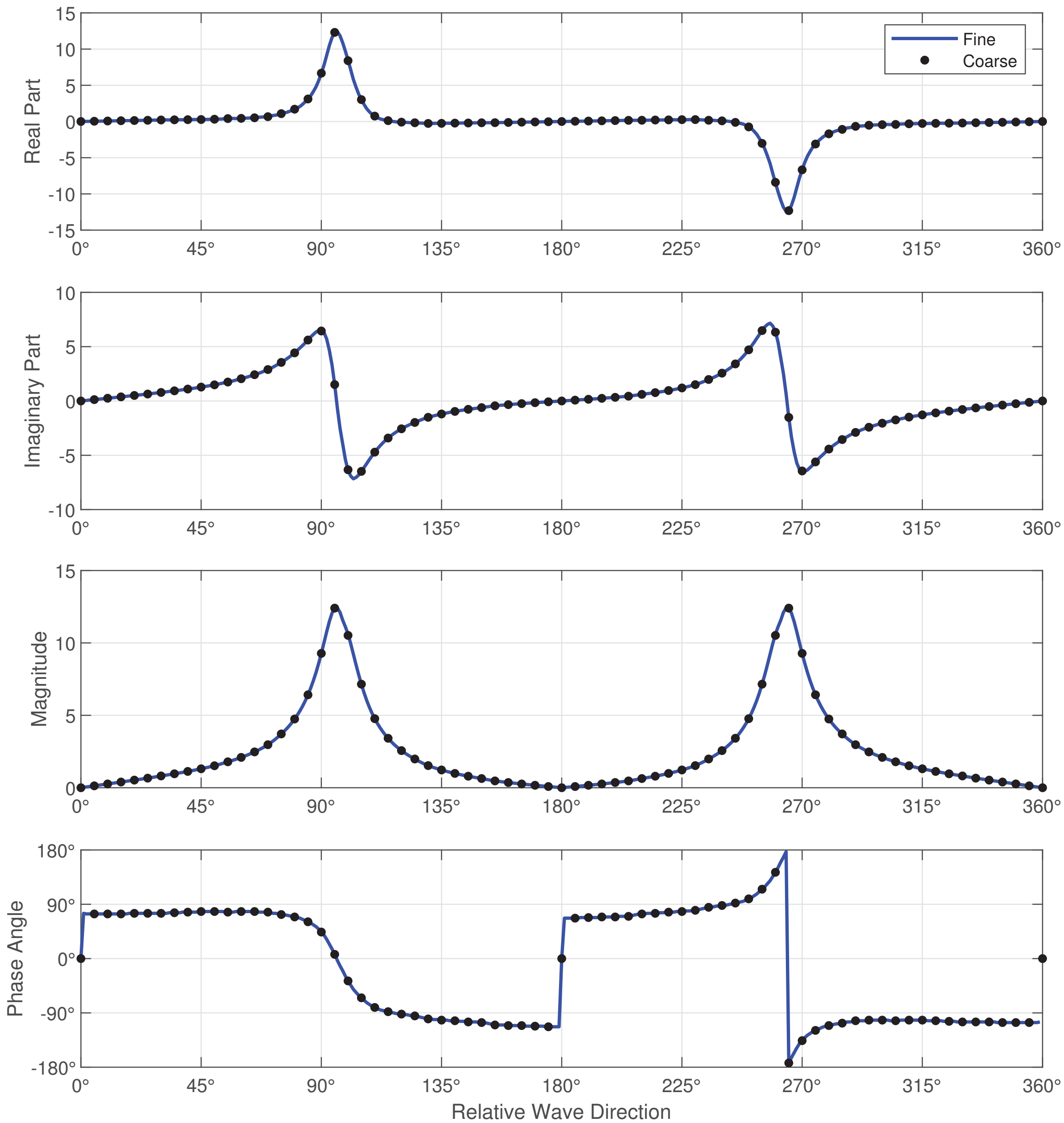

Figure 5: *Roll motion RAO for* $\omega = 0.4\,\mathrm{rad/s}$ *and* $V_S = 20\,\mathrm{knots}$.

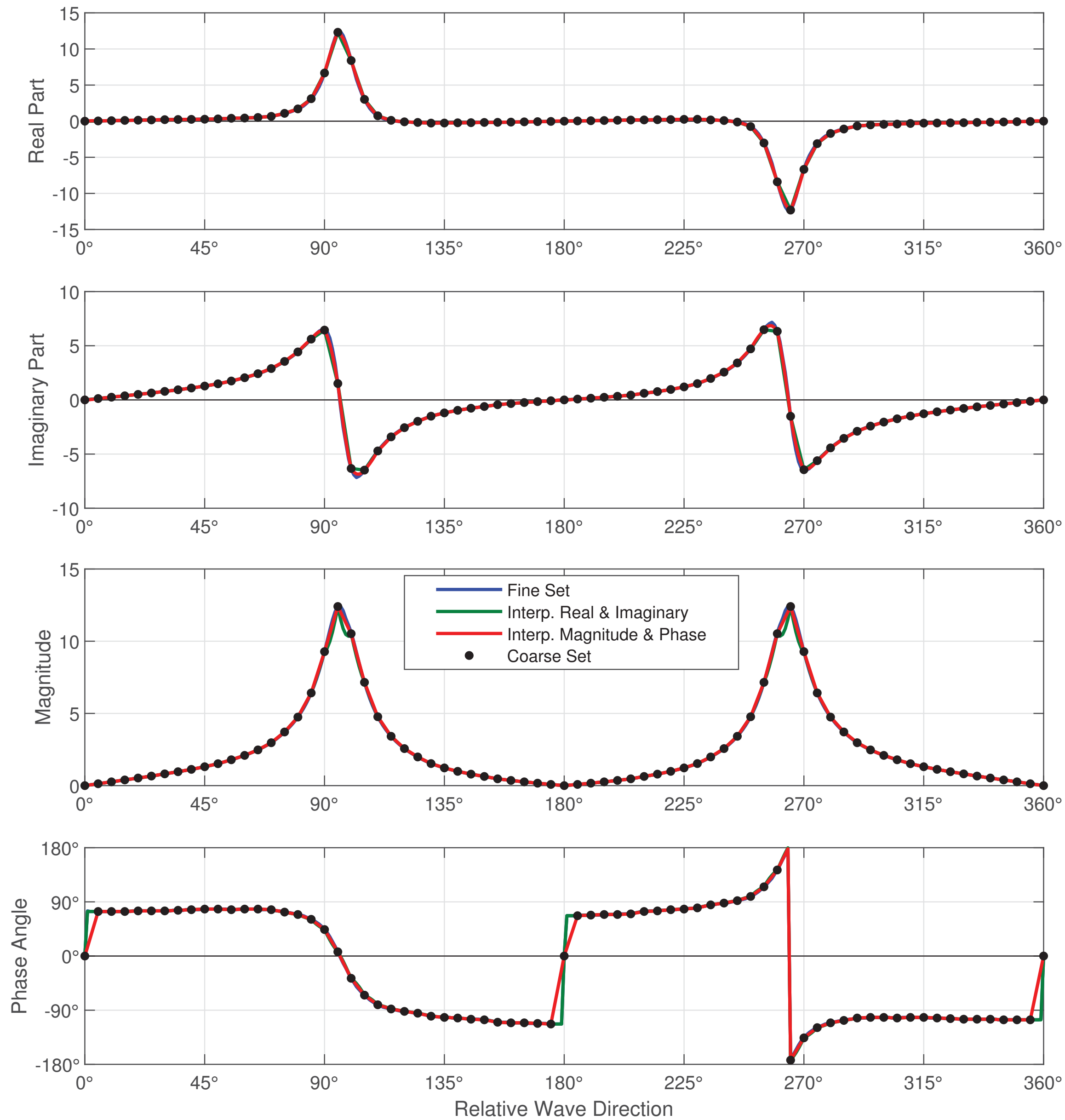

Figure 6: *Interpolated roll motion RAO for* $\omega = 0.4\,\mathrm{rad/s}$ *and* $V_S = 20\,\mathrm{knots}$.

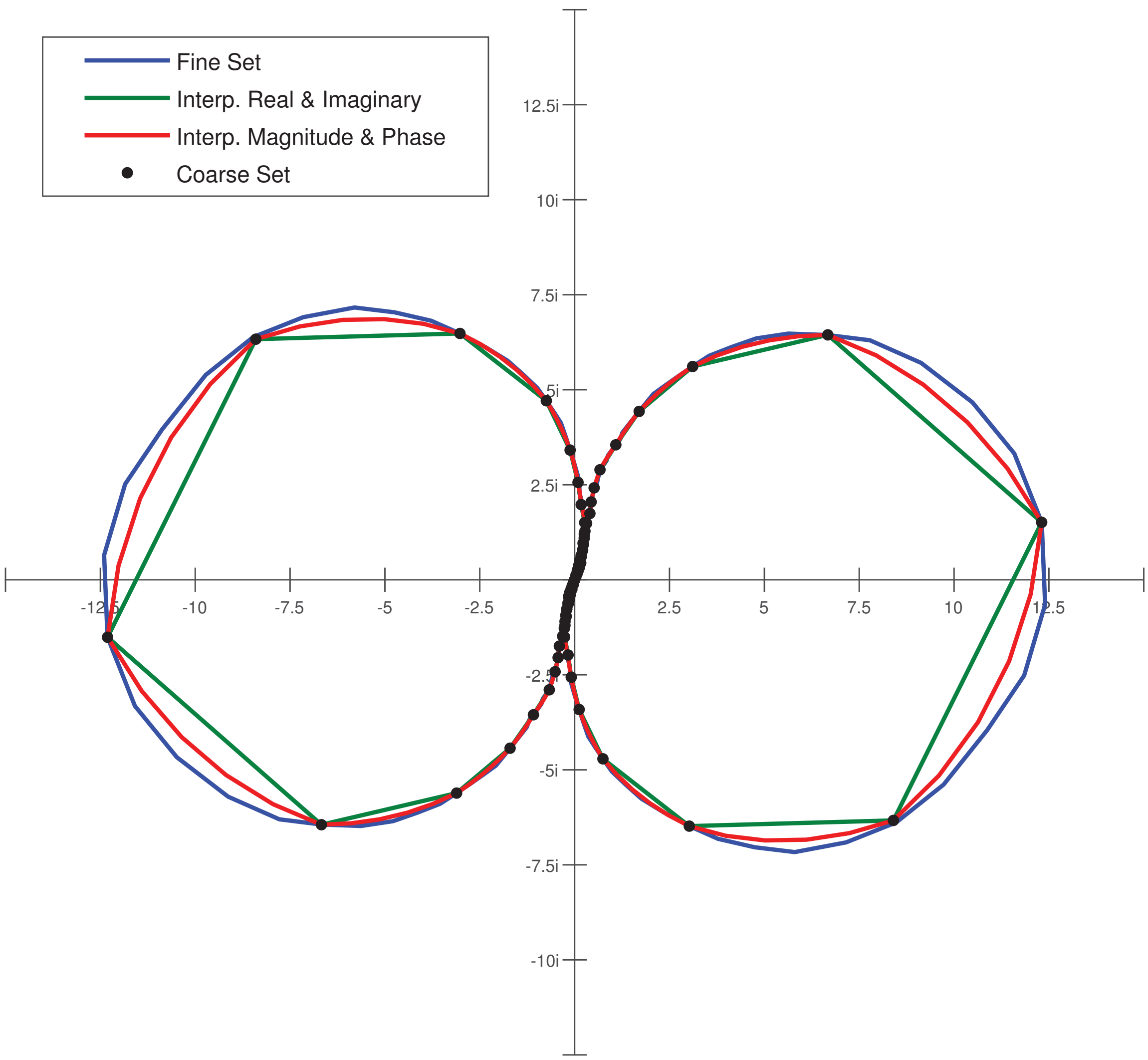

Figure 7: *Roll motion RAO for* $\omega = 0.4\,\mathrm{rad/s}$ *and* $V_S = 20\,\mathrm{knots}$ *in the complex plane.*

3.2 Interpolating Wave Frequency

In this second example, a subset of RAO was extracted from both the fine and coarse sets corresponding to a ship speed of 20 knots and a relative direction of 35° (stern quartering seas). This reduces the independent variable for interpolation to just wave frequency, ω.

The 37 values (every 0.05 rad/s) in the coarse subset were then interpolated to 181 values (every 0.01 rad/s) using the rectangular and polar approaches. The results are shown in Figure 8. The differences between the two are evident in a few places but mostly for the frequencies when $\omega > 1\,\mathrm{rad/s}$. The fine set had a sharp peak near $\omega = 1.48$ that fell between the available points in the coarse set defined at $\omega = 1.45$ and 1.5. The narrowness of this peak would make it very difficult for any interpolation scheme (even higher order methods) to capture it based on the coarse data set.

The results are shown in the complex plane in Figure 9. Unlike with relative direction, the results for frequency are less clear as the path traced by the fine data set (blue) is more erratic. Some portions appear better matched by the rectangular component interpolation, while others by the polar approach. It may be that best approach for wave frequency is to start with a fine resolution when initially creating the RAO so as to avoid the need to fill in gaps later on.

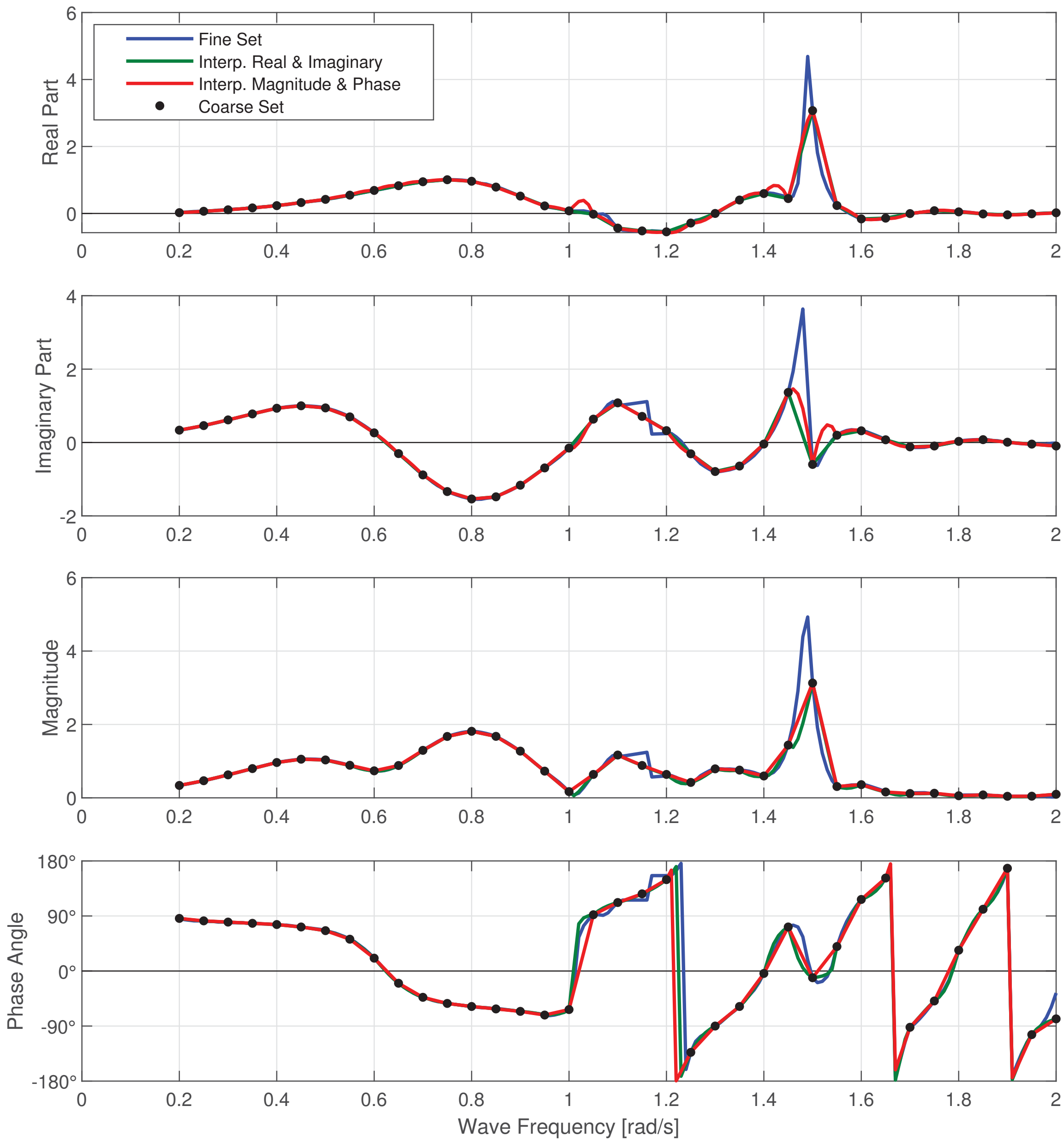

Figure 8: *Roll motion RAO for* $\beta_S = 35^\circ$ *and* $V_S = 20\,\mathrm{knots}$.

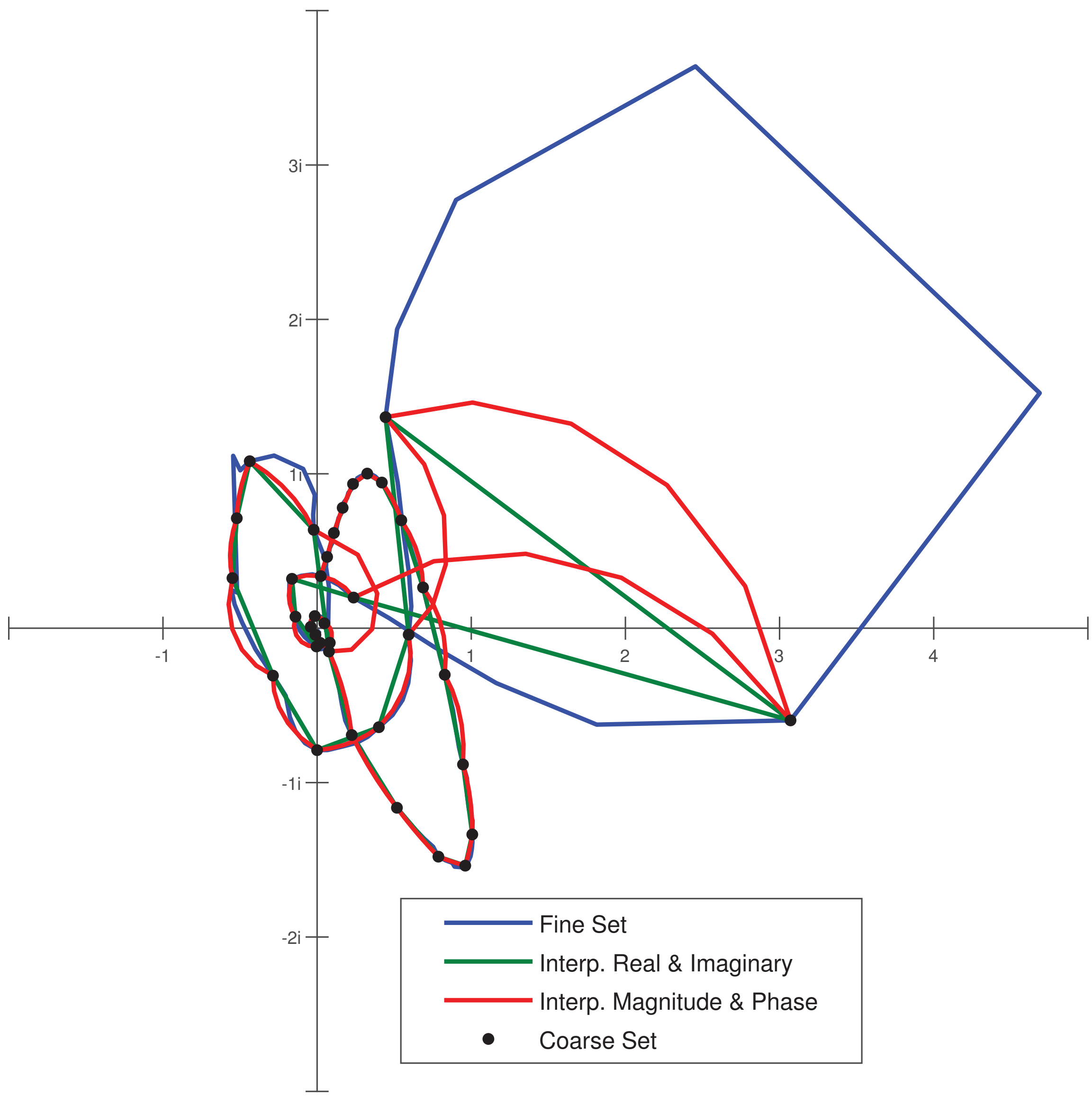

Figure 9: *Roll motion RAO for* $\beta_S = 35°$ *and* $V_S = 20\,\text{knots}$.

3.3 Interpolating Ship Speed

In this third example, a subset of RAO was extracted from both the fine and coarse sets corresponding to a wave frequency of 0.4 rad/s and a relative direction of 45° (stern quartering seas). This reduces the independent variable for interpolation to just ship speed, V_S.

The 14 values (every 2 knots) in the coarse subset were then interpolated to 105 values (every 0.25 knots) using the polar and rectangular forms. The results are shown in Figure 10. The differences between the two approaches were minor.

The results are shown on the complex plane in Figure 11 and again there does not appear to be major differences between the two interpolation methods. There is, however, a mismatch between some of the coarse data points not falling exactly on the blue curve of the fine data set; e.g., at the top left of the curve (corresponding to $V_S = 0$).

This is because the fine and coarse sets were both generated by ShipMo3D, as opposed to extracting the coarse points from the fine set (in which case there would never be a mismatch like that shown). When ShipMo3D was executing the fine set it was using more relative directions and frequencies which can result in slightly different results for some conditions, primarily due to how it calculates linearized roll damping.

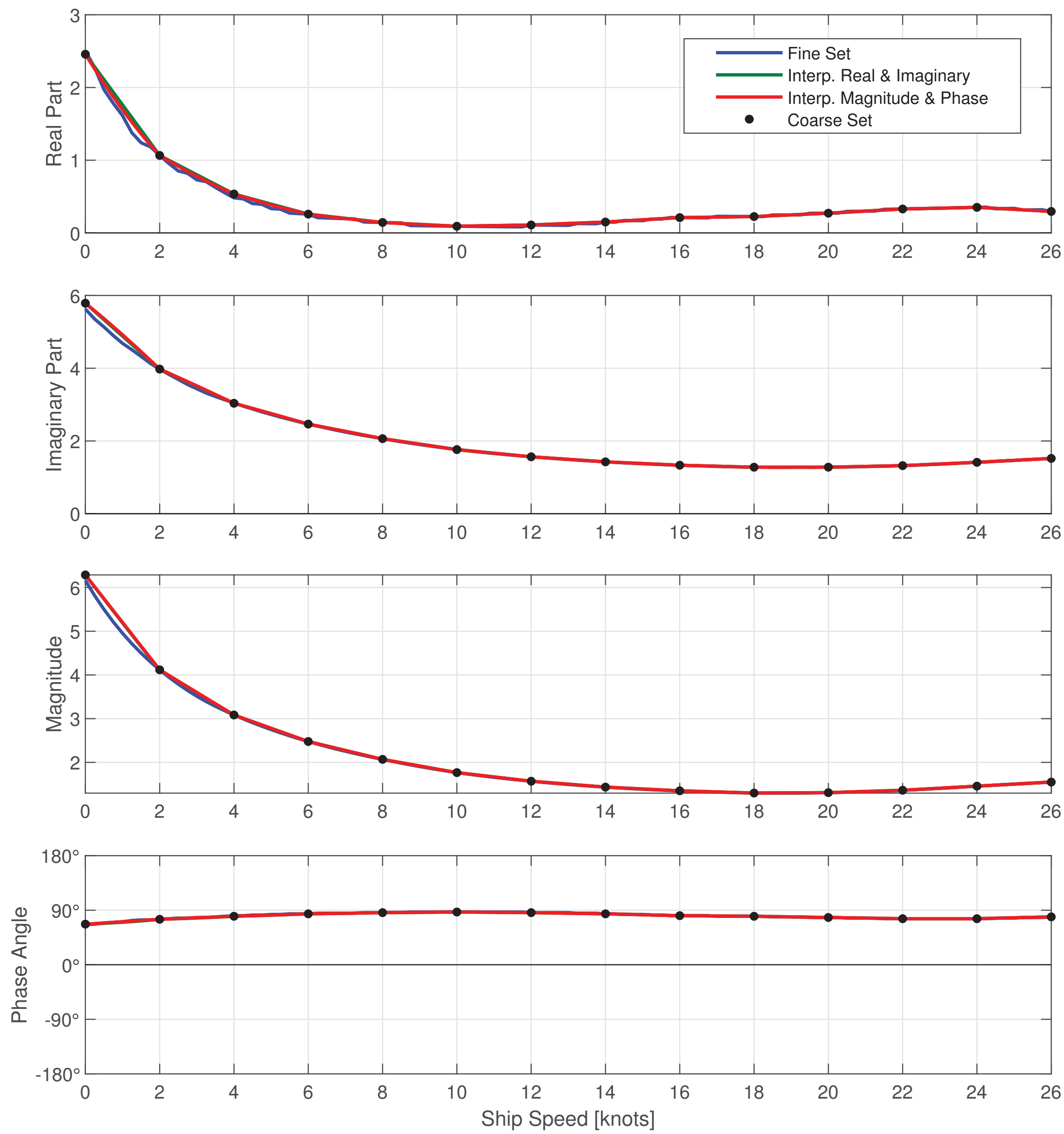

Figure 10: *Roll motion RAO for* $\omega = 0.4\,\mathrm{rad/s}$ *and* $\beta_S = 45^\circ$.

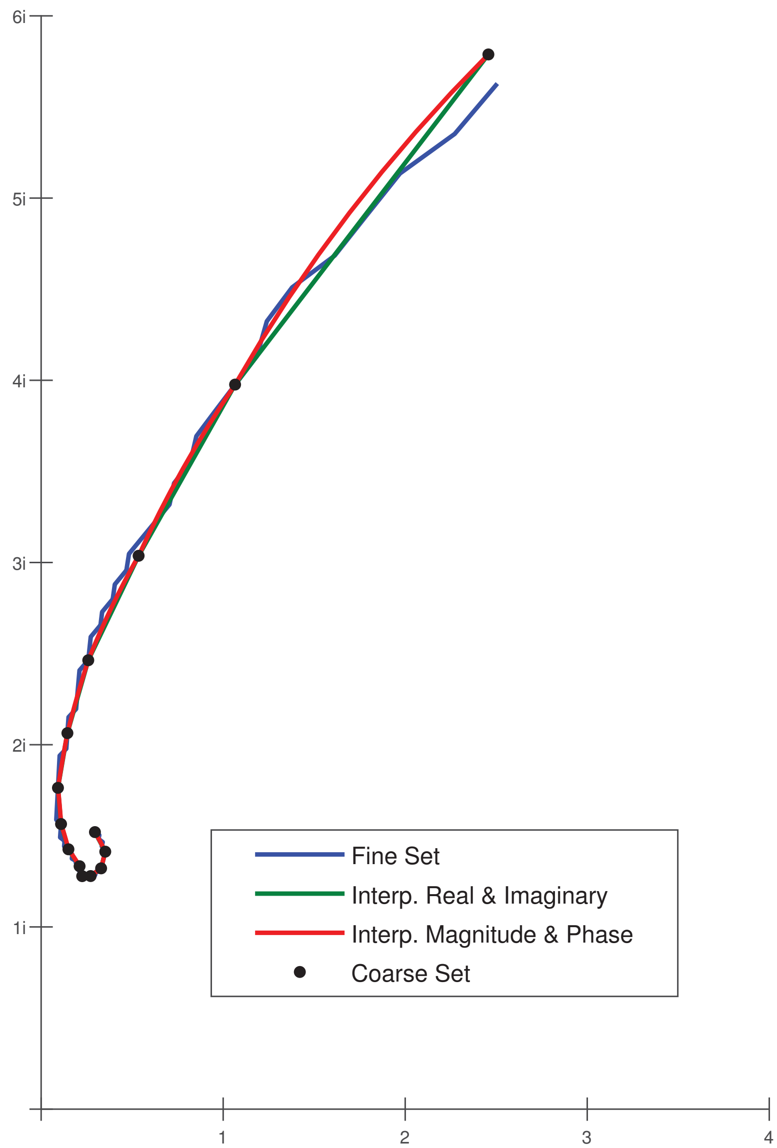

Figure 11: *Roll motion RAO for* $\omega = 0.4\,\mathrm{rad/s}$ *and* $\beta_S = 45°$.

3.4 Interpolating Magnitude Squared

One of the primary motivations for this study was using RAO to calculate RMS ship motions. The procedure (see Section 4.1) for computing these are based on the square of the magnitudes of the RAO values. Therefore one additional interpolation scheme is explored which disregards the phase angles and uses magnitude-squared values only (simplifying the RAO array to real values). This approach is also included in the example code listed in Annex E.3. Note that the magnitude-squared values of the polar form are equivalent to the sum of the squares of the rectangular components.

For comparison purposes, the rectangular and polar interpolation approaches are used to interpolate the complex values of the RAO first, after which the square of the magnitude of the resulting values are calculated. For the magnitude-squared approach, the square of the magnitudes of the complex values in the RAO are calculated first, and then interpolated. The results are compared with the magnitude-squared values from the fine RAO set.

Figures 12 to 14 show results for the difference schemes for the same subset RAO examples used in Sections 3.1 to 3.3, respectively. On average, the magnitude-squared interpolation produced slightly better results than the rectangular and polar approaches for relative direction and frequency. The results for ship speed were mixed as shown Figures 14 and 15 for two different wave frequencies.

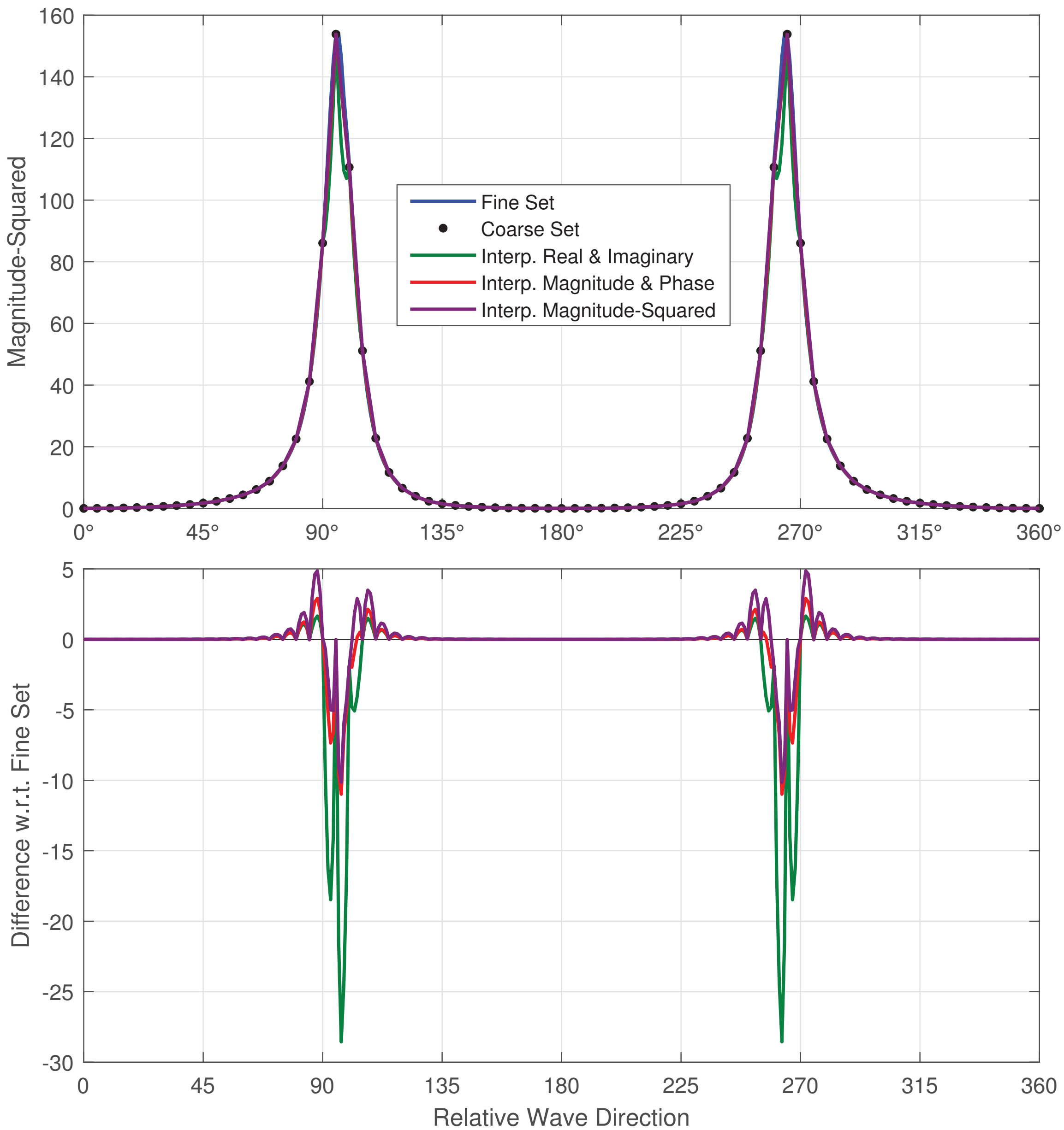

Figure 12: *Interpolated roll motion magnitude-squared RAO for* $\omega = 0.4\,\mathrm{rad/s}$ *and* $V_S = 20\,\mathrm{knots}$.

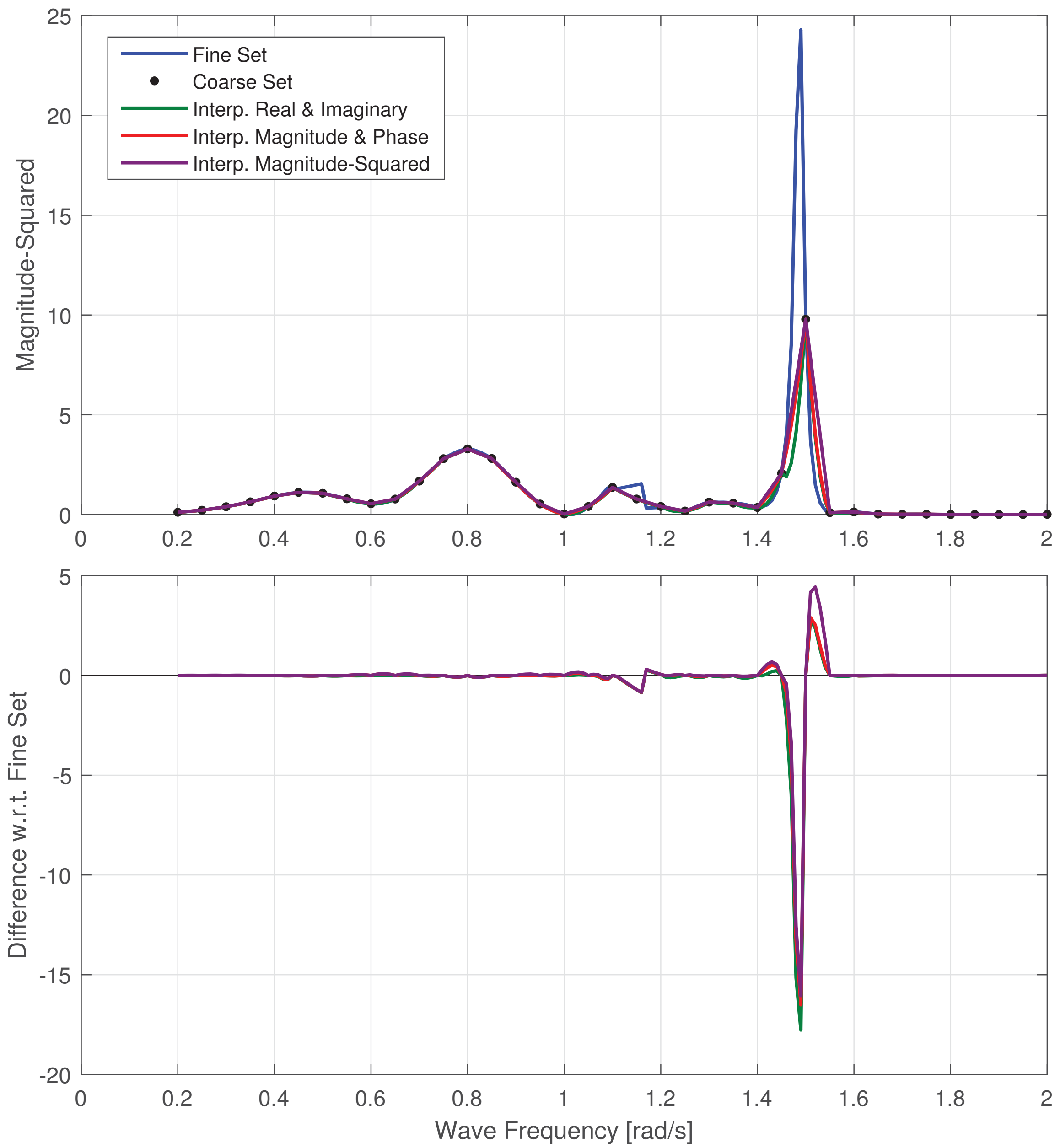

Figure 13: *Interpolated roll motion magnitude-squared RAO for* $\beta_S = 35°$ *and* $V_S = 20\,\text{knots}$.

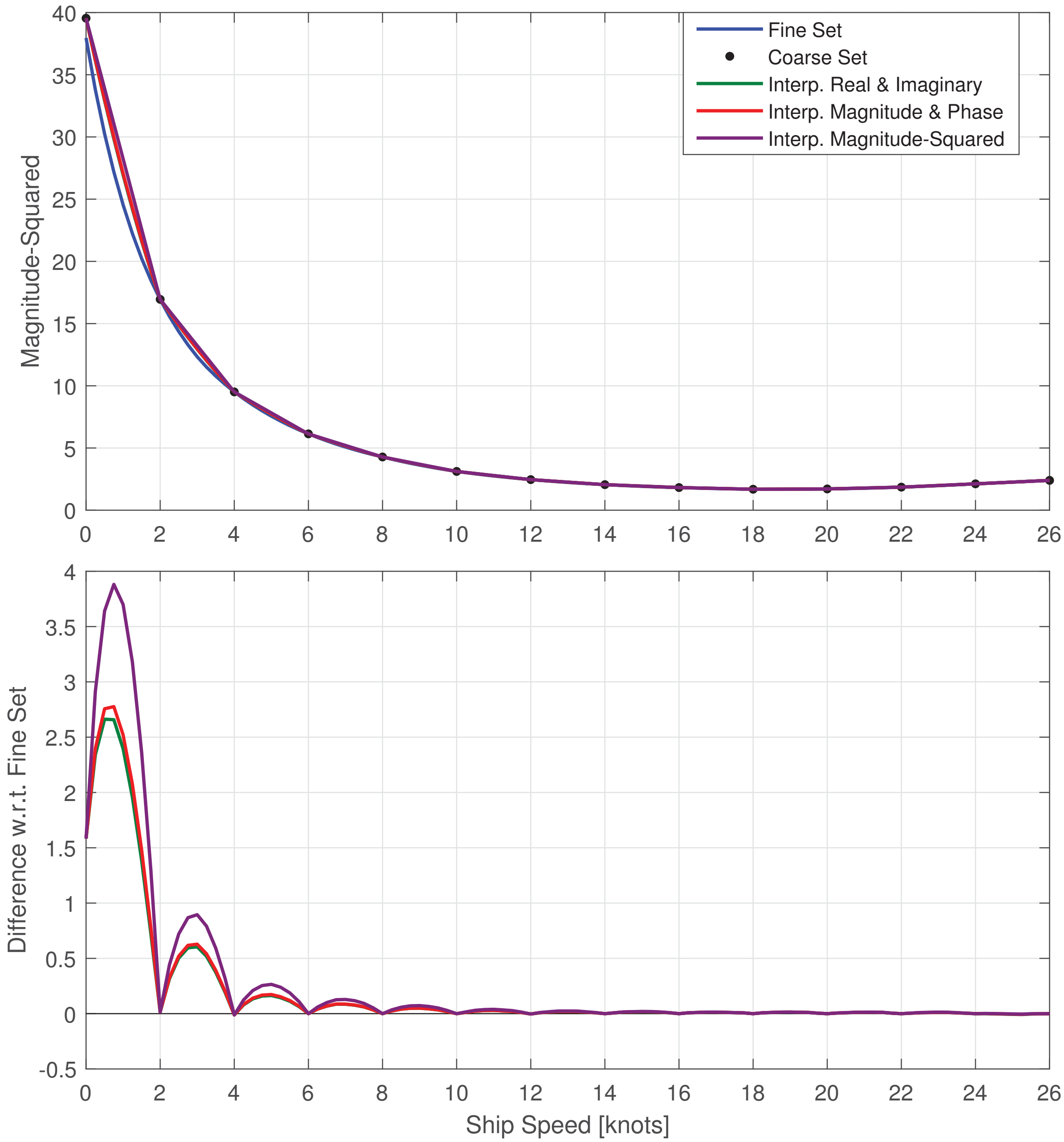

Figure 14: *Interpolated roll motion magnitude-squared RAO for* $\omega = 0.4\,\mathrm{rad/s}$ *and* $\beta_S = 45°$.

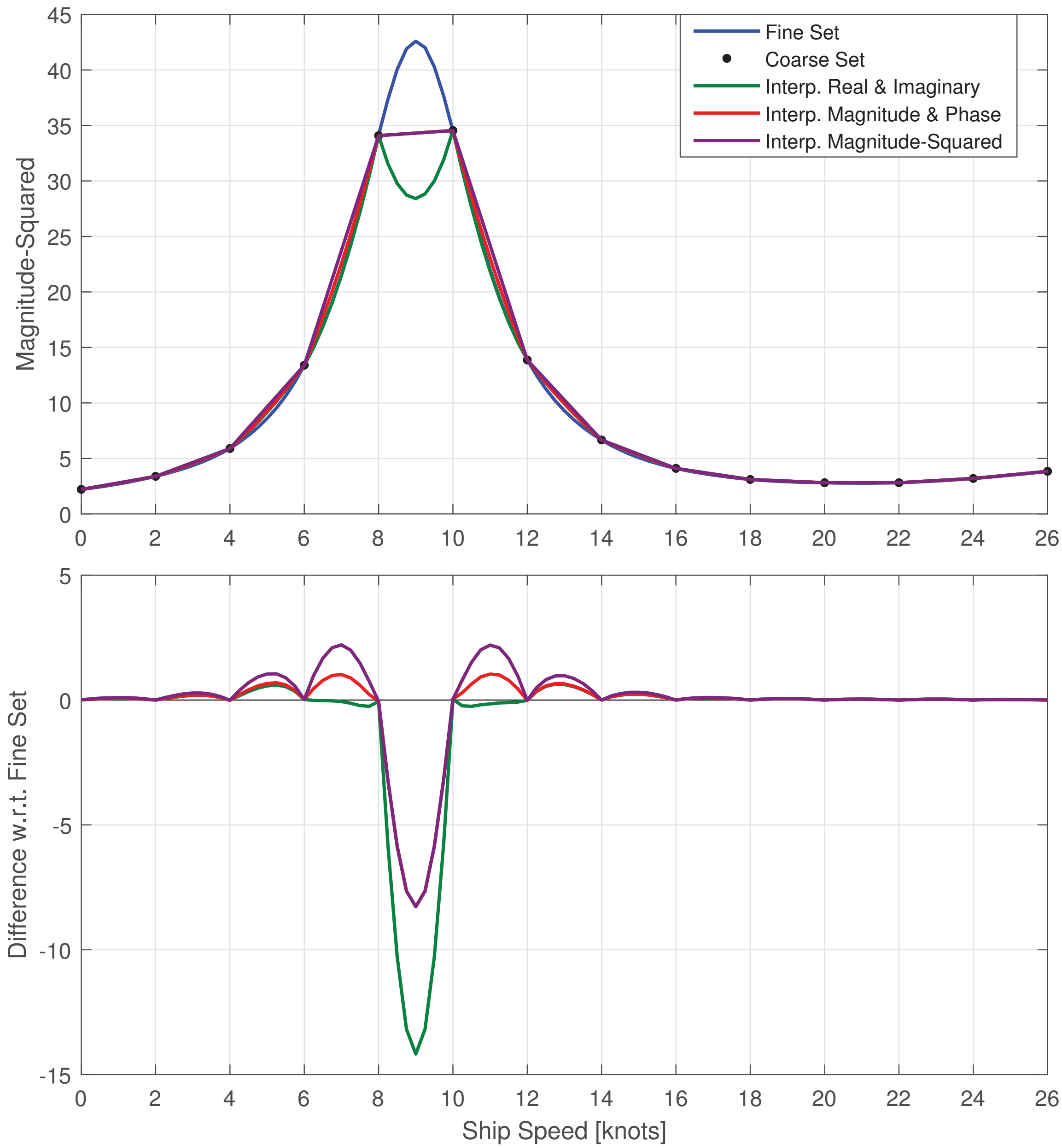

Figure 15: *Interpolated roll motion magnitude-squared RAO for* $\omega = 0.5\,\mathrm{rad/s}$ *and* $\beta_S = 45°$.

4 Interpolating Effects on RMS Motions

This set of tests will compare how using different interpolation approaches to the complex RAO from Section 3 can affect the resulting RMS motions. As in Section 3, parameters will be chosen such that interpolation is only needed along an isolated dimension (ship speed, freqency, direction).

Whereas Sections 3.1 to 3.3 examined how different complex interpolation approaches affect the resulting RAO. This section will look at how the differences in RAO due to interpolation could affect predicted ship motions.

4.1 Procedure for Calculating RMS Motions

A procedure for calculating an RMS ship motion from RAO in a given wave field, S_η, at an arbitrary ship speed, V_A and an arbitrary ship heading, χ_A is given by Equation (6).

The first step involves extracting the 2D (ω, β_S) RAO matrix from the larger 3D (V_S, ω, β_S) array at the chosen ship speed V_A. Next the directional wave spectrum S_η, which is defined using χ_W (the direction waves are coming from), must be converted to relative directions β_S for the chosen heading angle χ_A.

The motion energy spectrum S is then calculated in Equation (6c) by multiplying S_η by the square of the magnitudes of the complex values in the RAO. Note that when implementing this for discrete matrices that the operations (e.g., calculating the magnitude of the complex values, squaring, multiplying) are all done on an element-wise basis.

The motion energy spectrum S is then integrated over its range of frequencies and relative directions to get m_0, the zeroth spectral moment. The RMS ship motion is then simply the square-root of m_0 as given by Equation (6e).

$$\mathrm{RAO}_{VA}(\omega, \beta_S) = \mathrm{RAO}(V_S, \omega, \beta_S)\ @\ V_S = V_A \tag{6a}$$

$$S_\eta(\omega, \beta_S) = \mathrm{convert}\Big(S_\eta(\omega, \chi_W) \text{ to } \beta_S \text{ for } \chi_H = \chi_A\Big) \tag{6b}$$

$$S(\omega, \beta_S) = S_\eta(\omega, \beta_S)\ |\mathrm{RAO}_{VA}(\omega, \beta_S)|^2 \tag{6c}$$

$$m_0 = \int_{\beta_S}\int_{\omega} S(\omega, \beta_S)\, d\omega\, d\beta_S \tag{6d}$$

$$\mathrm{RMS} = \sqrt{m_0} \tag{6e}$$

For discrete matrices, there are several methods for performing numerical integration. For this study only the trapezoidal method (as implemented in the built-in MATLAB function `trapz`) was considered as given by the following sample code.

```
omega;    % Row vector containing RAO wave frequencies
betaS;    % Row vector containing RAO relative directions
RAO;      % 2D complex-value RAO matrix: RAO(omega,betaS)
waveSpec; % Wave spectrum matrix: waveSpec(omega,betaS)

% Calcualate motion enery spectrum
S = waveSpec .* abs(RAO).^2;

% Perform double numerical integration on S to get m0
m0 = trapz(omega,trapz(betaS,S,2),1);

% Calculate RMS motion
RMS = sqrt(m0);
```

4.1.1 Procedural Options for Interpolation

The procedure given by Equation (6) neglects certain situations such as: if the RAO and S_η are defined over different values of frequencies and/or angles, and if the speed V_A or the angle χ_A are not explicitly defined in the RAO.

If these occur, some form of interpolation will be required. However, there are different options on what form and at which step interpolation is performed. The following list outlines the various interpolation options available at different steps of the calculation. In an actual implementation of a given procedure some of these steps may be combined. However, for clarity here they will be listed and treated separately.

1. **Ship Speed**: The arbitrary ship speed, V_A, may fall between the speed values defined in the RAO; e.g., $V_A = 5$ knots when the RAO are only defined every 2 knots.
   a. Use nearest ship speed to V_A defined in the RAO.
   b. Calculate RMS motions using speeds defined in the RAO that are above and below V_A. Interpolate the resulting RMS motions to get motions at V_A.
   c. Calculate the RAO at V_A using interpolation.
2. **Wave Frequencies and Relative Directions**: The wave spectrum and RAO data may be defined over different frequency and angle ranges/increments. Extrapolation should be avoided, therefore any data that is outside the ranges of either the RAO and wave spectrum should be truncated.
   a. Resize/interpolate wave spectrum to match frequencies and angles of RAO.
   b. Resize/interpolate RAO to match frequencies and angles of wave spectrum.
3. **Relative Direction and Wave Direction**: The RAO is defined over a range of relative directions β_S (Annex A) while the wave spectrum is defined over a range of wave directions (coming from), χ_W.
   a. Convert wave spectrum from χ_W to β_S.
   b. Convert RAO from β_S to χ_W.

4. **Ship Heading**: The arbitrary ship heading, χ_{HA}, may fall between the angles defined in the RAO and wave spectrum; e.g., $\chi_{HA} = 13°$ when the RAO and wave spectrum are only defined every 5°.
   a. Use nearest angle to χ_{HA} defined in the RAO and wave spectrum.
   b. Calculate RMS motions using angles defined in the RAO and wave spectrum that are above and below χ_{HA}. Interpolate the resulting RMS motions to get motions at χ_{HA}.
   c. Convert RAO or wave spectrum (depending on choice for Item 3) to exactly χ_{HA} using interpolation.

The permutations of these choices can lead to many possible procedures for calculating RMS motions. These permutations increase when considering different approaches for handling complex number interpolation as discussed in Section 3. The following sections examine selected cases to test whether choices in implementation have an impact on the resulting output.

The tests will be conducted by using the coarse RAO set to calculate RMS data for cases that require some form of interpolation with different variations of the calculation procedure. These results will then be compared with the results using the fine RAO data set with a correspondingly fine resolution wave spectrum—i.e., a case where no interpolation is required at any step. All test cases will use the wave spectrum described in Section 4.2 (though they may involve different resolutions of frequency and wave direction).

4.2 Wave Spectrum Used for Tests

The wave spectrum used for the RMS procedure test cases was created using a ten-parameter model [4–8]. Using a spectrum based on analytic equations means that it can be resolved to any resolution of wave frequency and direction as needed for the given test case.

The spectrum was created as a bi-modal sea with a strong swell component and a weaker wind sea component, both originating from different directions. This was chosen to give a reasonably strong but non-symmetric response by the ship. A polar plot of the spectrum is shown in Figure 16 using the parameters from Table 2.

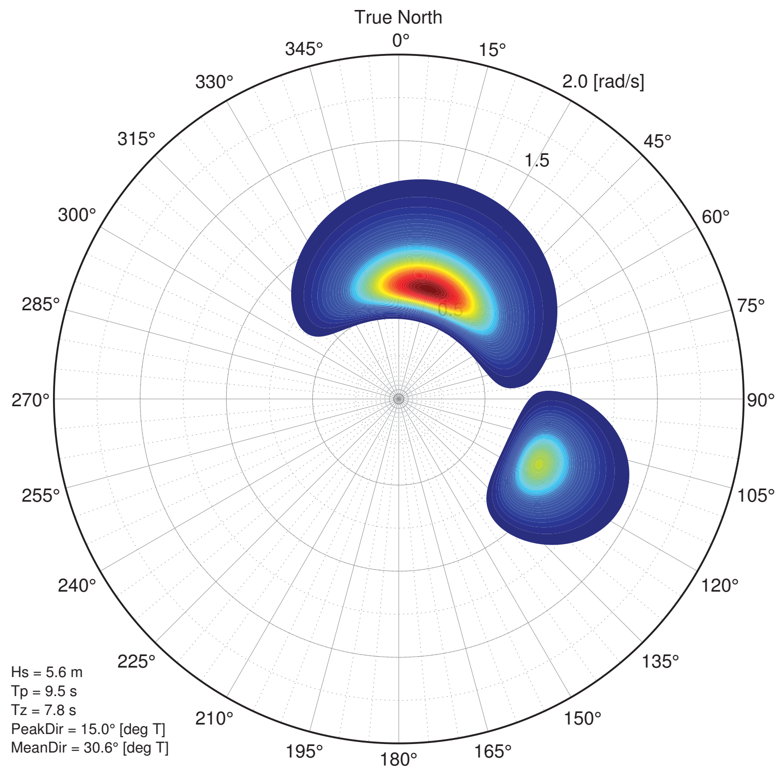

Figure 16: *Wave spectrum used for proceedure tests. Wave angles indicate the 'coming from' direction.*

Table 2: *Ten-parameter wave spectrum values.*

Parameter	Value	Units	Description
H_{s1}	5.0	[m]	Significant wave height of swell component
ω_{p1}	0.661	[rad/s]	Peak frequency of swell component
(T_{p1})	9.5	[s]	Peak wave period of swell component: $T = 2\pi/\omega$
λ_1	2	[-]	Spectral shape parameter of swell component
ν_{m1}	15	[deg T]	Mean compass direction (from) of swell component
P_1	10	[deg]	Directional spreading parameter of swell component
H_{s2}	2.5	[m]	Significant wave height of wind sea component
ω_{p2}	0.898	[rad/s]	Peak frequency of wind sea component
(T_{p2})	7.0	[s]	Peak wave period of swell component: $T = 2\pi/\omega$
λ_2	3	[-]	Spectral shape parameter of wind sea component
ν_{m2}	115	[deg T]	Mean compass direction (from) of wind sea
P_2	60	[deg]	Directional spreading parameter of wind sea component

4.3 Ship Speed

Figure 17 shows the resulting RMS roll motions for V_A = 13 knots, calculated over the range of ship headings, χ_A from 0° to 360° in 5° increments. The input wave spectrum was generated to match the frequencies and angles in the coarse set RAO. Therefore in this case, as 13 knots is not included in the coarse set RAO (see Table 1), some interpolation is needed.

The black curve in the top plot are the results using the fine RAO set, it is for reference as no interpolation was needed. The bottom chart show the differences between the other methods and these fine set results.

The green curve shows the results from creating an RAO at 13 knots using the rectangular components for interpolation. Likewise the red and purple curves shows the same for polar component and magnitude-squared interpolation, respectively. The cyan curve shows the results when no RAO interpolation is used. Instead the RMS motions at 12 knots and 14 knots are both calculated, and then interpolated to get the motions for 13 knots.

In this case the best results are from the magnitude-squared RAO interpolation, or the interpolation of RMS calculating from existing RAO data. This particular ship speed had some of the largest differences between the methods, but the trends were similar for other speeds.

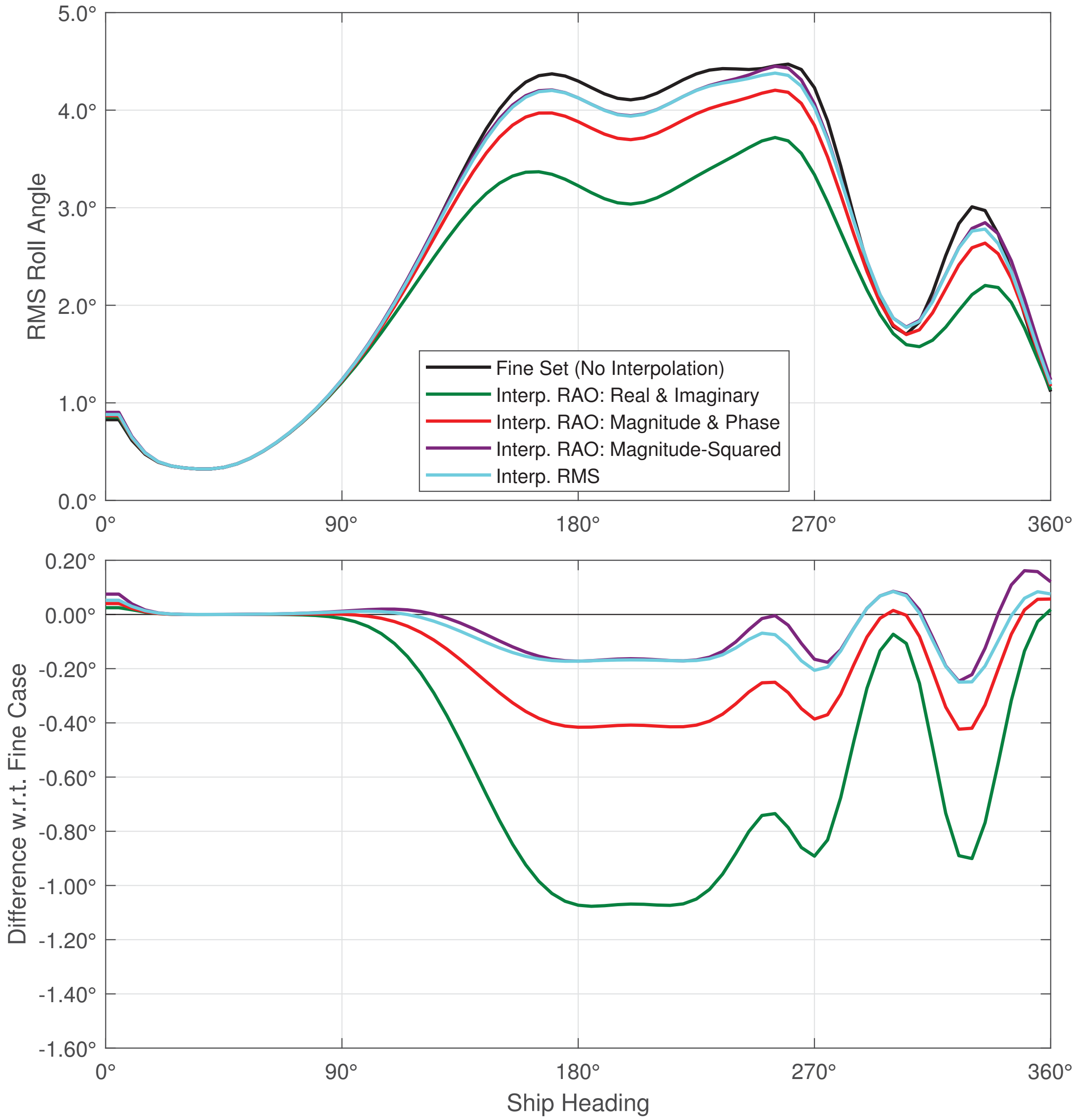

Figure 17: *RMS motions for different approaches to interpolating for ship speed = 13 knots.*

4.4 Wave Frequency

Figure 18 shows the resulting RMS roll motions for $V_A = 20$ knots, calculated over the range of ship headings, χ_A from 0° to 360° in 5° increments. The input wave spectrum was generated to match the angles in the coarse set RAO, but with wave frequencies matching the fine set RAO (see Table 1). RMS motions were calculating by interpolating the coarse set RAO using various approaches.

In this case, the differences between all methods tested were minor. Unlike for Section 4.3, the trends on which method performed best were not consistent across different speeds. When re-calculated for V_A = 14 knots as shown in Figure 19, the worse performer from Figure 18 switches to the best performer. The overall result is that the differences were minor and not consistently sensitive to the interpolation method used.

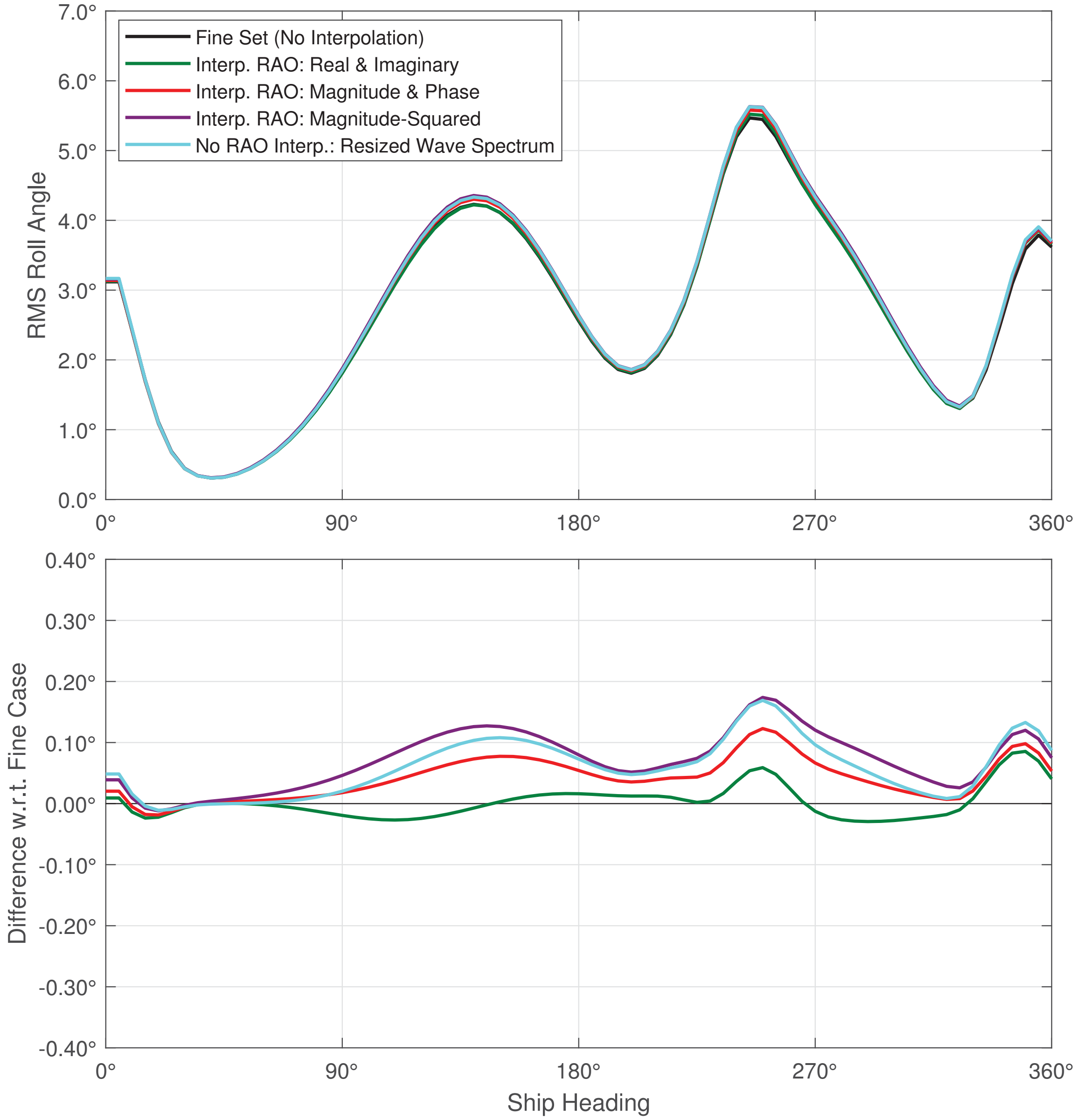

Figure 18: *RMS motions for different approaches to interpolating for wave frequency. V_A = 20 knots.*

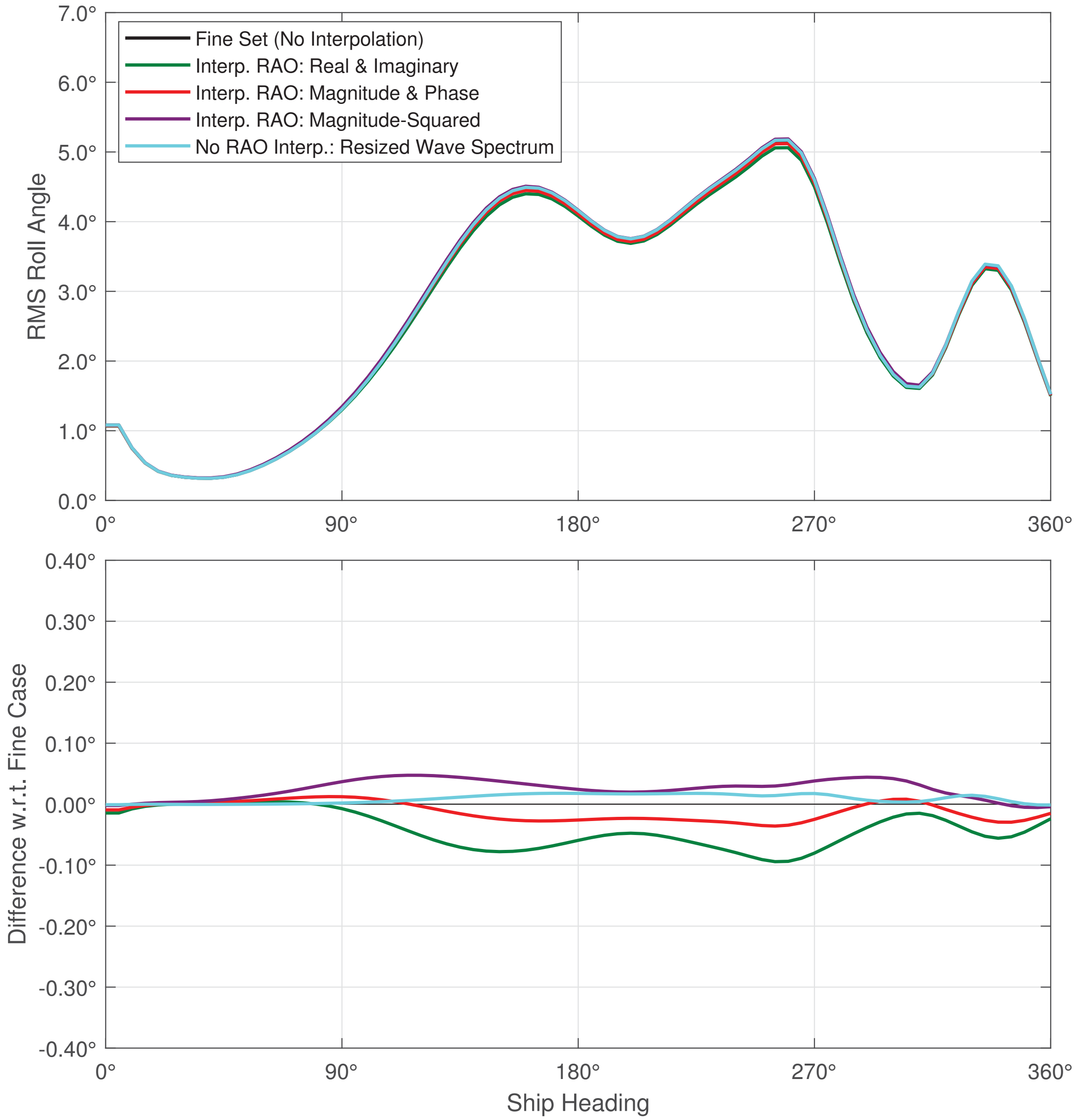

Figure 19: *RMS motions for different approaches to interpolating for wave frequency.* V_A *= 14 knots.*

4.5 Wave Direction

Figure 20 shows the resulting RMS roll motions for $V_A = 24$ knots, calculated over the range of ship headings, χ_A from 0° to 360° in 5° increments. The input wave spectrum was generated to match the frequencies in the coarse set RAO, but with wave directions matching the fine set RAO (see Table 1). RMS motions were calculating by interpolating the coarse set RAO using various approaches.

In this case, the polar component interpolation performed best (Figure 20), but for other speeds such as V_A = 14 knots in Figure 21, it was the magnitude-squared (purple curve) and the approach with no RAO interpolation (cyan curve) which performed best. For all speeds, the interpolation of rectangular components performed worst.

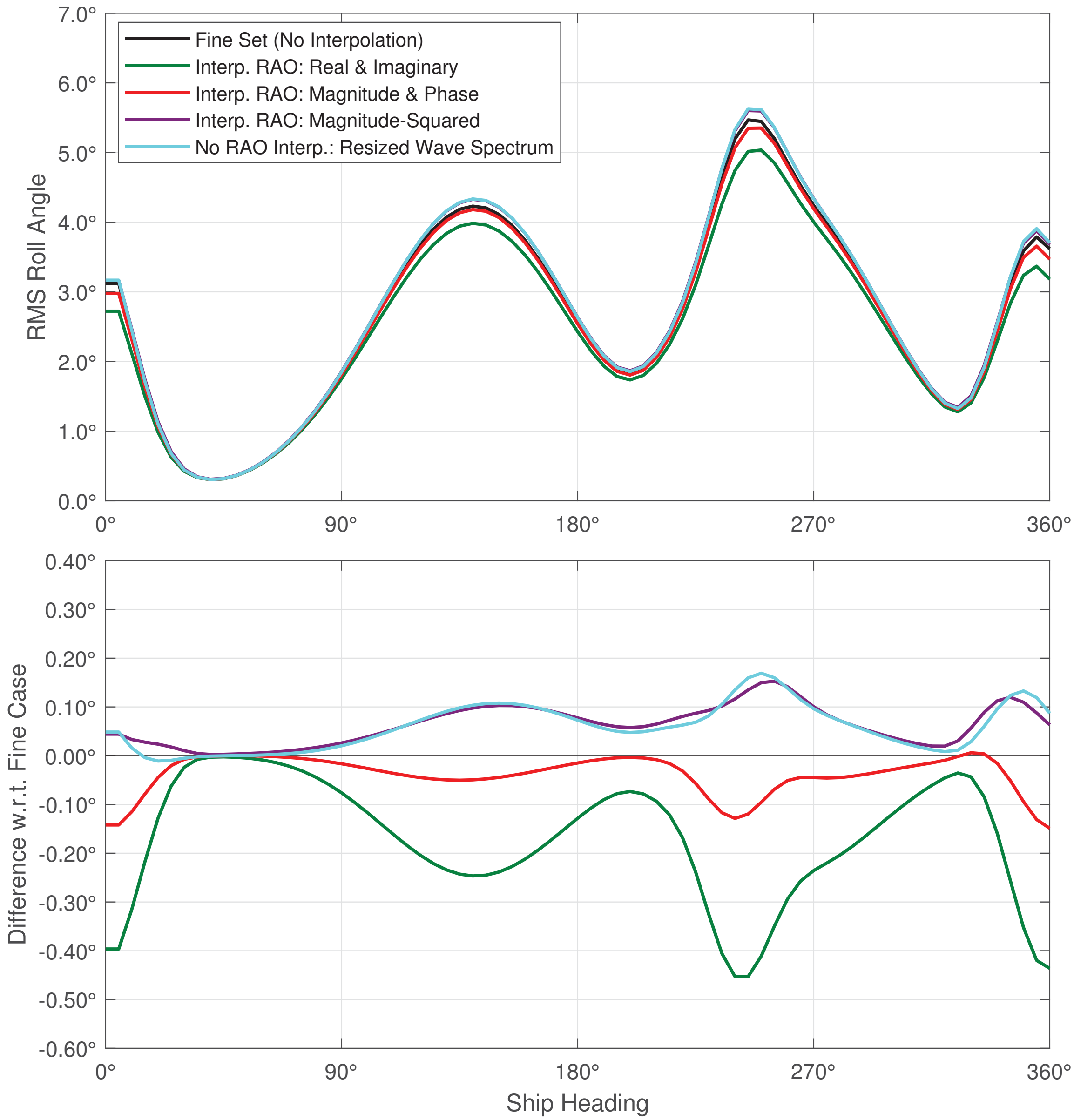

Figure 20: *RMS motions for different approaches to interpolating for wave direction.* V_A *= 24 knots.*

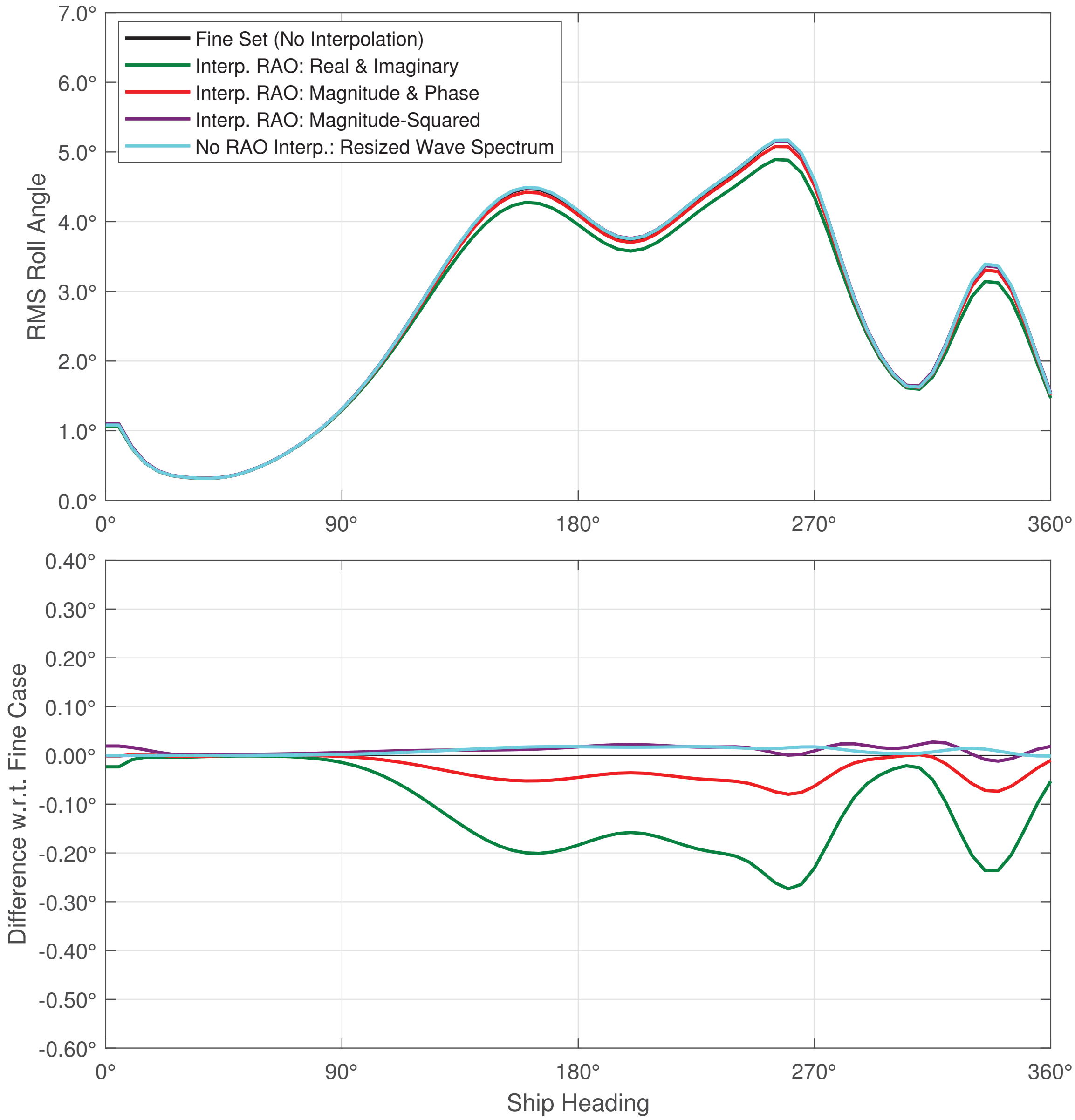

Figure 21: *RMS motions for different approaches to interpolating for wave direction.* V_A *= 14 knots.*

4.6 Ship Heading

This test set examines the effects from the various options from Item 4 from Section 4.1.1. This deals with the fact that the wave spectrum is (typically) defined in terms of wave directions using the ‘coming from’ convention, χ_W, w.r.t. compass north while the RAO are defined using relative directions, β_S (see Annex A). Examples of each with their respective angle conventions are shown in Figure 22.

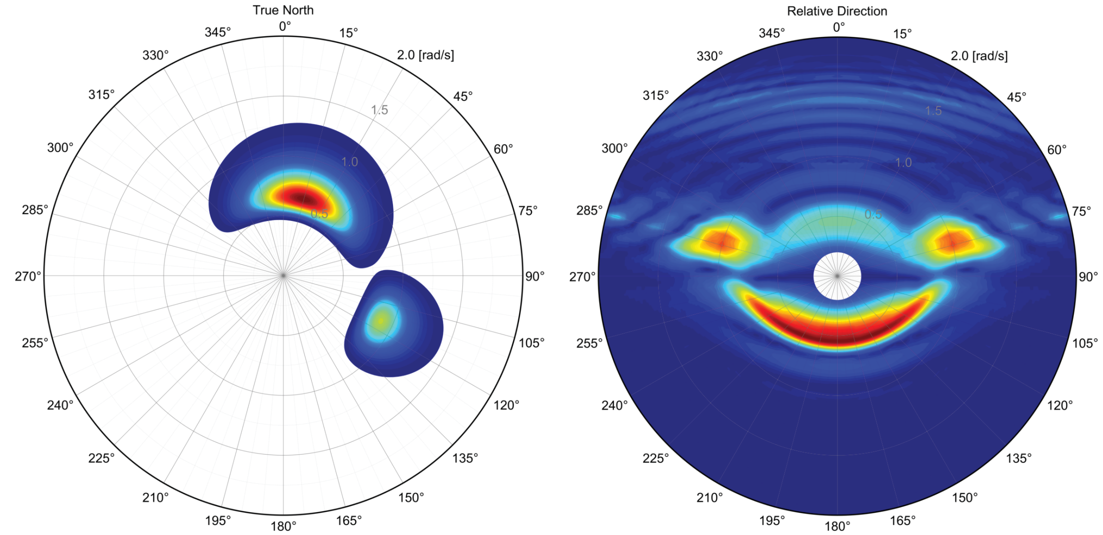

Figure 22: *Wave sprectrum with 'coming from' directions w.r.t. compass north (left) and RAO with relative directions (right).*

In order to calculate the motion energy spectrum, S, the wave spectrum and RAO must share the same angle convention; either χ_W or β_S. In Equation (6), the chosen option was to convert S_η to β_S, but in this set of tests, the alternative approach, given by Equation (7), of converting the RAO to χ_W will also be considered. Examples of the two approaches for the RAO and S_η are shown in Figure 23. Sample code for performing these operation is given in Annex E.

$$\mathrm{RAO_{VA}}(\omega,\beta_S) = \mathrm{RAO}(V_S,\omega,\beta_S) \ @ \ V_S = V_A \tag{7a}$$

$$\mathrm{RAO_{VA}}(\omega,\chi_W) = \mathrm{convert}\Big(\mathrm{RAO_{VA}}(\omega,\beta_S) \text{ to } \chi_W \text{ for } \ \chi_H = \chi_A\Big) \tag{7b}$$

$$S(\omega,\chi_W) = S_\eta(\omega,\chi_W)\ |\mathrm{RAO_{VA}}(\omega,\chi_W)|^2 \tag{7c}$$

$$m_0 = \int_{\chi_W}\int_{\omega} S(\omega,\chi_W)\, d\omega\, d\chi_W \tag{7d}$$

$$\mathrm{RMS} = \sqrt{m_0} \tag{7e}$$

Converting data from one angle convention to another, as illustrated in Figure 23, is akin to rotating it. In cases where the selected heading angle, χ_A, exists in the set of angles defining the data set, then no interpolation is necessary—the operation only re-arranges the data within the array without modifying any of the values. For example, if the data is defined every 5° from 0° to 355° and χ_A = 30°, then no interpolation is needed. It is only when χ_A falls between the defined angles, such as χ_A = 33°, that interpolation is needed. This section examines how the difference options for handling this can affect the output RMS motions.

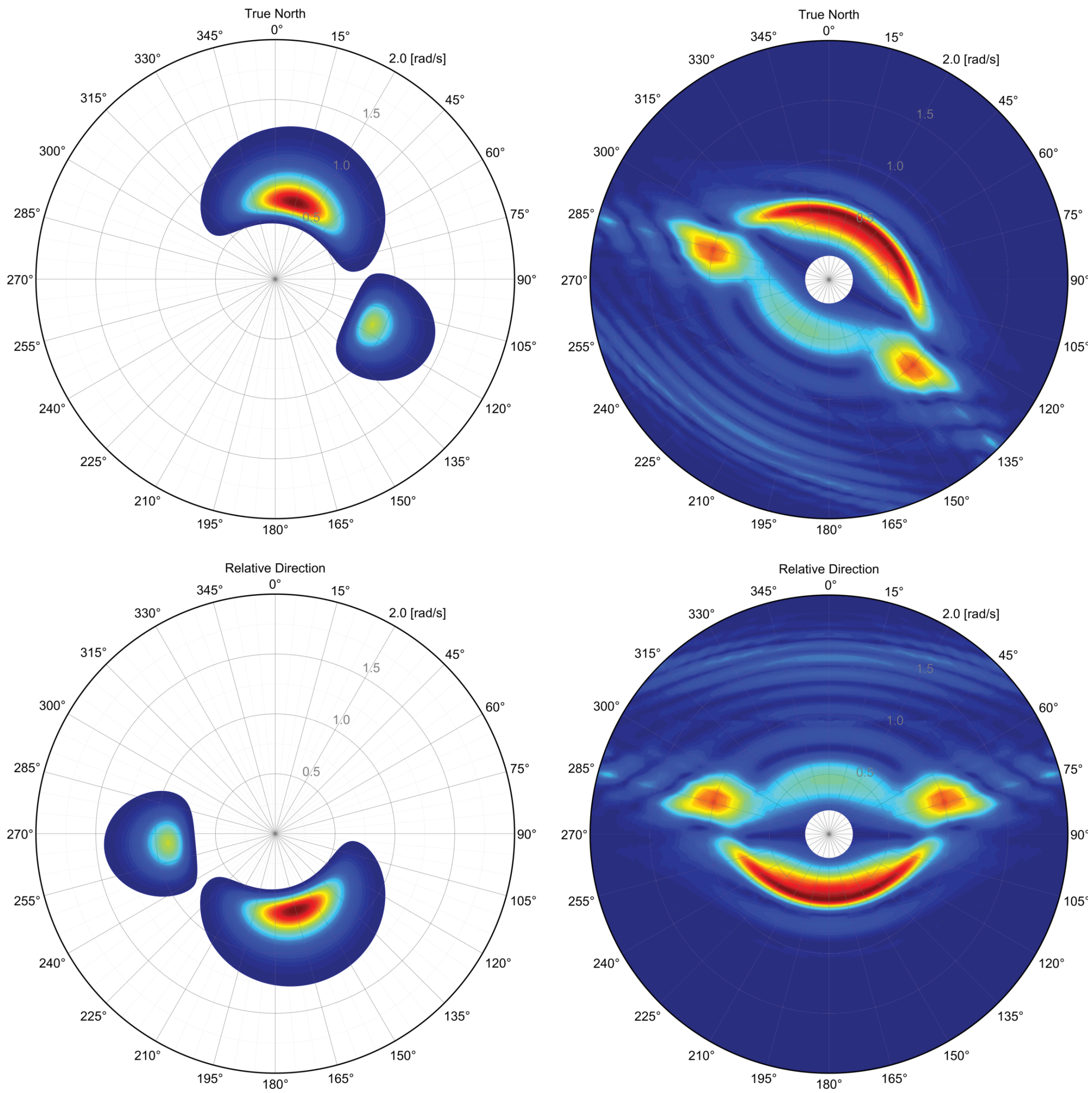

Figure 23: *Top set show both wave spectrum and RAO with directions w.r.t. compass north. Bottom set show both wave spectrum and RAO w.r.t. relative directions for a ship heading of 30°.*

The input wave spectrum was generated to match both the frequencies and directions as the coarse RAO, so no resizing interpolation is needed (Item 2. from Equation (6)). Ship speeds were also chosen that exist in the coarse RAO set, so no speed interpolation is needed (Item 1. in Equation (6)). However, in this case, the requested ship headings χ_A will range from 0° to 360° in increments of 3° instead of 5° as used in the previous cases. This will involved using options from Items 3. and 4. from Section 4.1.1.

The results are shown in Figure 24 for $V_A = 20$ knots. In this case, the results for converting the angle convention of the RAO using either rectangular or polar form interpolation resulted in oscillations of the output RMS motions. The results for using magnitude-squared interpolation on the RAO were nearly identical to both the results from converting the wave spectrum angle convention, and for interpolating the RMS results using neighbouring angles (i.e., no interpolation of either the wave spectrum or RAO).

The oscillations in the rectangular or polar form results were caused by how the peaks in the RAO (see Figure 5) were interpolated for certain angles. Note that for every 5^{th} point (e.g., $\chi_A = 15°, 30°, \ldots$) that the results from all the methods match exactly, as those angles are defined in the coarse RAO data set and therefore do not require interpolation.

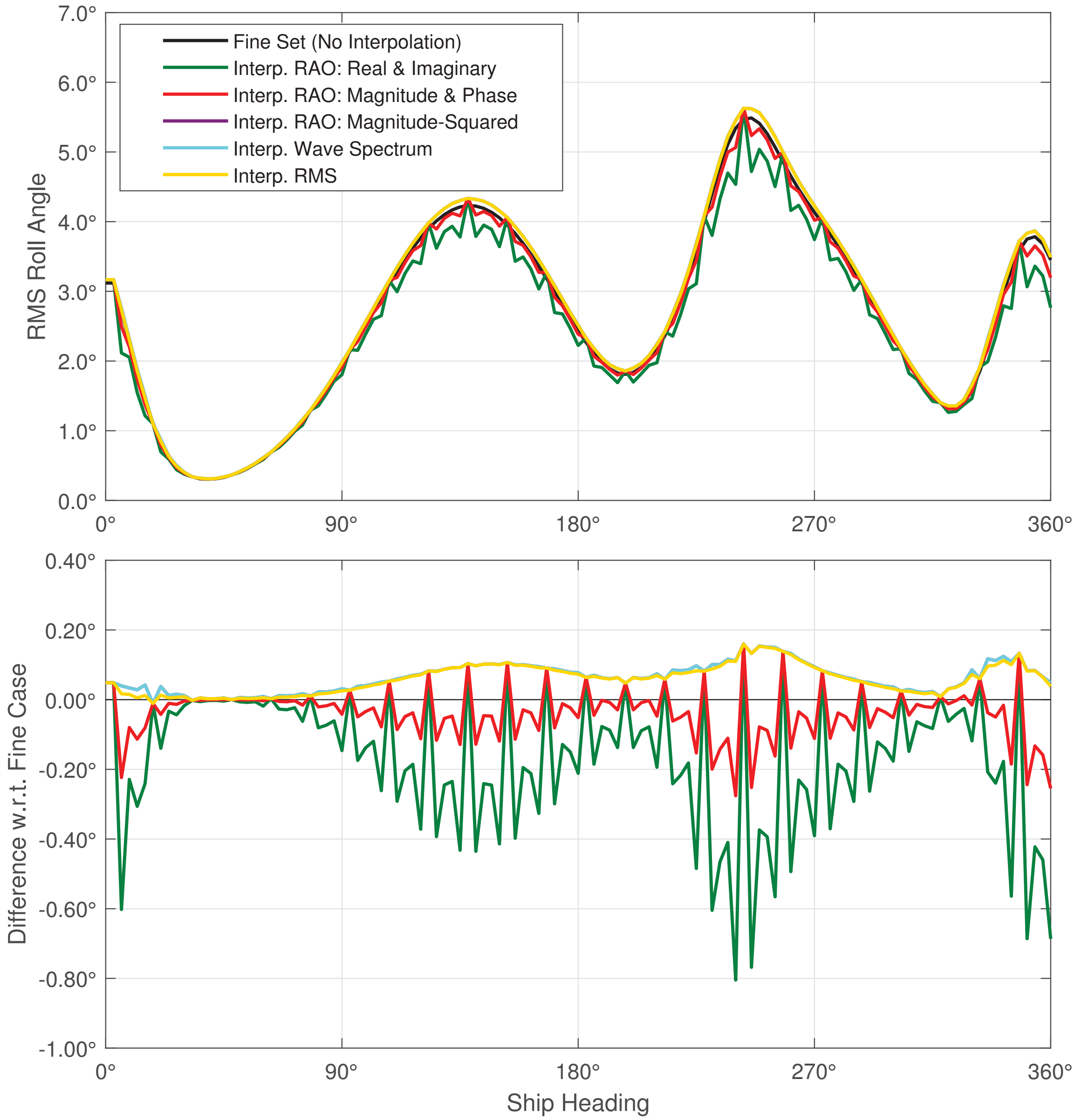

Figure 24: *RMS motions for different approaches to interpolating for ship heading.* $V_A = 20$ *knots.*

4.7 Higher Order Methods

Regardless of the approach taken for interpolation in the previous sections, a linear method was used for actually calculating the interpolated values. Linear interpolation is most commonly used as it is fast, easy to implement, and has the property that the interpolated value will never exceed (above or below) its neighbouring values. It's disadvantage is that it assumes a straight line between two points, which may not always be the true path of the data.

For comparison, an additional case was examined which uses the 'spline' option of the built-in MATLAB interpolation functions. This method fits a spline to the data allowing for curvature between points for the interpolation. The test case was chosen where interpolation was needed at each step, effectively combining the previous cases into one.

The wave spectrum was generated to match the frequencies and angles in the fine set RAO and the chosen speed of V_A of 13 knots is not included in the coarse set RAO (see Table 1). The results using a linear interpolation in the various approaches for this case are shown in Figure 25 and for spline interpolation in Figure 26.

Using the linear method, the results are consistent to those discussed in the preceding sections. The magnitude-squared and method with no RAO interpolation performed best, followed by the polar approach and the rectangular approach performing worst. With the spline method, the results from all methods improved by about 50%. However, the rankings of each method remained the same.

While computing the results using the spline method, negative values were created in the wave spectrum, and also when interpolating the RAO magnitudes as part of the polar approach. These are the sorts of artefacts that can result from higher order interpolation methods that are not bounded in the same way that a linear method is. In the current case, these negative values were changed to zeros.

Although not present in the current case, higher order interpolation methods can also sometimes produce wobbly or oscillating curves that can, depending on the input, be unrealistically steep. Although these methods can produce improved results, with poor input data quality, they can also generate very incorrect data as well. If used, higher order interpolation should be done carefully and with appropriate checks on the output.

A conference paper from 2011 [9], discussed a 'new' method for interpolation for RAO data sets. The approach, called harmonic interpolation, is where a data set is decomposed into cosine components using a Fast Fourier Transform (FFT) and then reconstructed with finer resolution.[2] The presented results were in excellent agreement, but like other interpolation approaches, the effectiveness of the method can depend on the input data. As an example, the results for RAO interpolation using this harmonic method is compared against a spline method in Figure 27. The phase angles in the harmonic results show clear oscillatory behaviour. This was also present in the other charts, but the amplitude was less pronounced.

[2] MATLAB has a built-in function `interpft` which can perform this procedure.

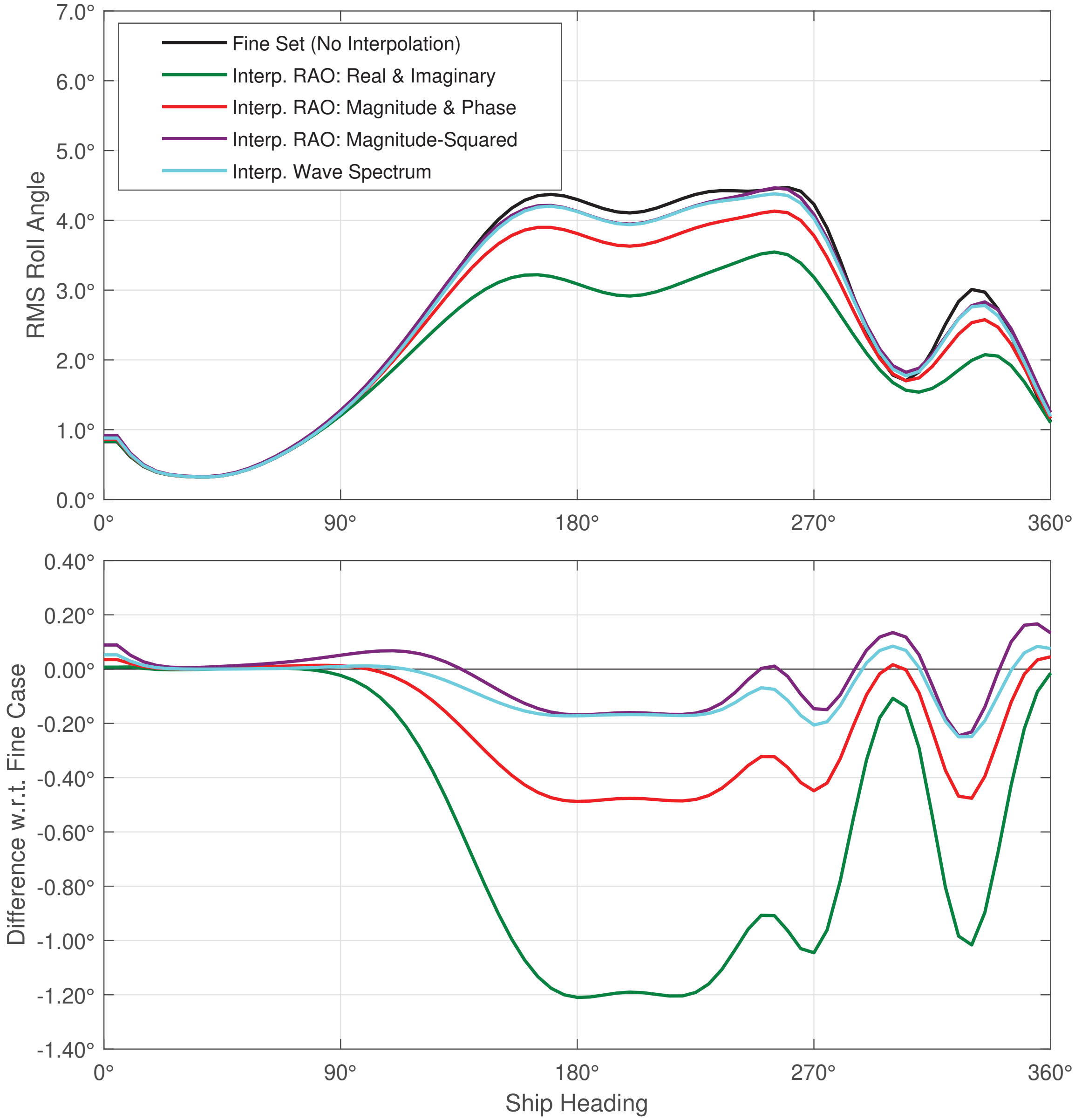

Figure 25: *Linear RMS motions for different approaches to interpolating for wave direction.* V_A *= 13 knots.*

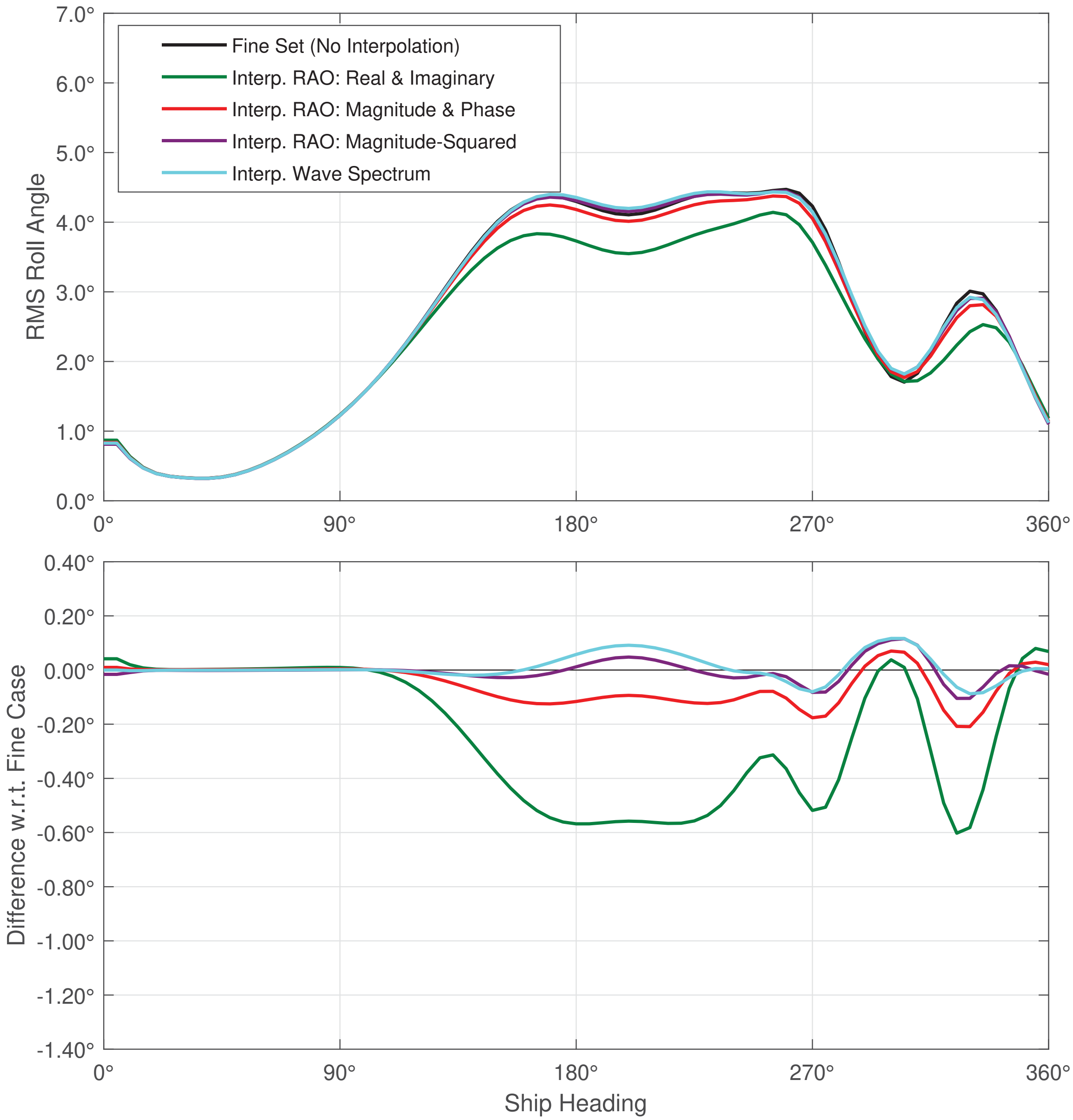

Figure 26: *Spline RMS motions for different approaches to interpolating for wave direction.* V_A *= 13 knots.*

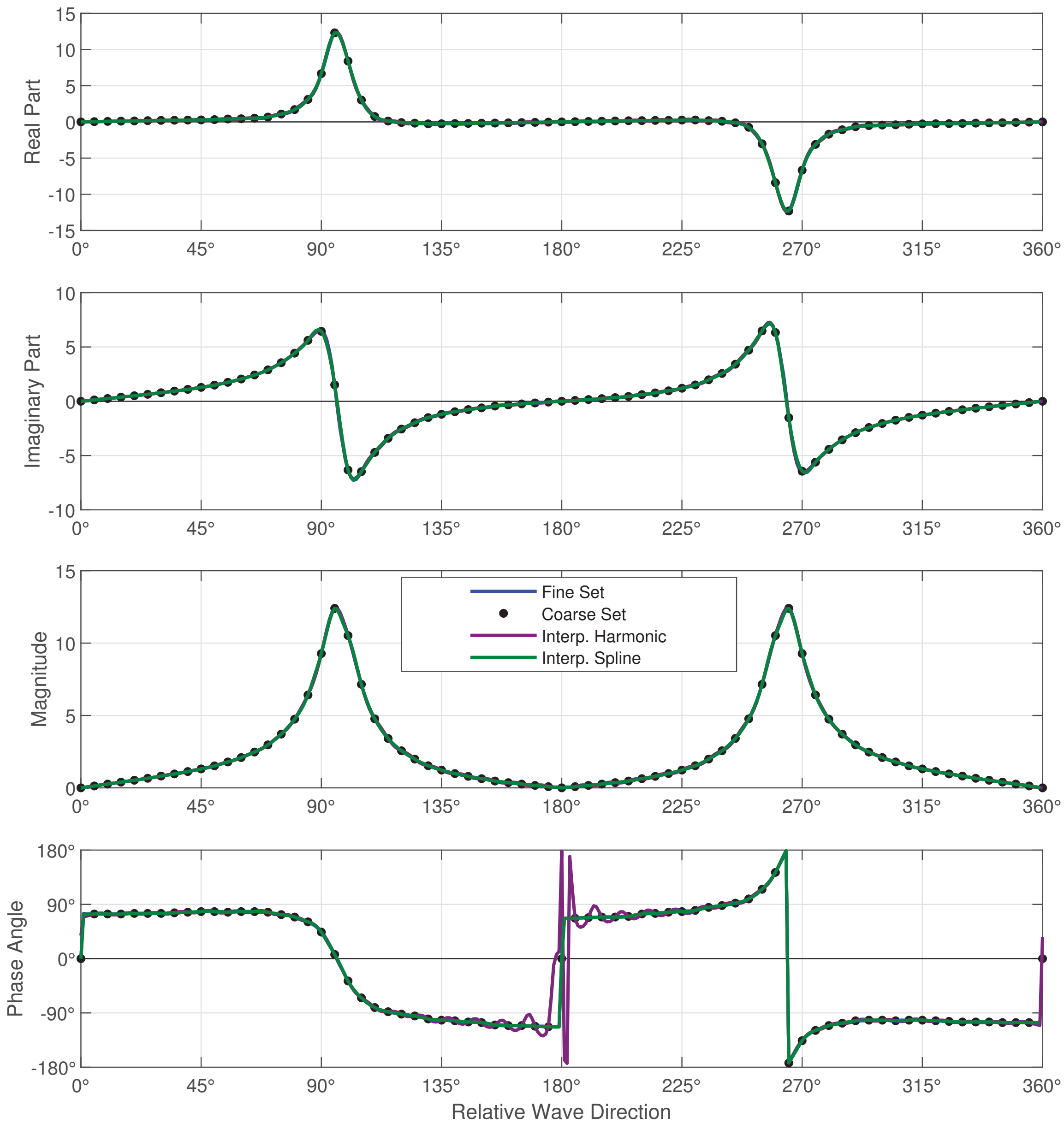

Figure 27: *RAO results using harmonic interpolation and spline interpolation.*

5 Conclusions

A study was conducted examining different interpolation approaches for complex RAO such as using its rectangular form, polar form, and magnitude-squared. Comparing interpolated results of coarse RAO data against fine RAO data showed that the polar form approach was best used when interpolating for relative direction. When interpolating along the frequency or ship speed dimensions of the RAO, the results from different approaches were mixed.

Next, the effects of RAO interpolation on the output when calculating RMS ship motions was examined, along with how the interpolation could be implemented at different steps in the calculation procedure. The following observations were made.

Interpolation of complex numbers in rectangular form for RAO data performed worst for most cases tested. Interpolation along the wave frequency dimension was the least sensitive to interpolation approach with all approaches performing similarly. Interpolation along the relative direction or ship speed dimensions was best performed by either using the polar form or with magnitude-squared.

One variation of the calculation procedure involved converting the RAO from relative directions to wave directions. This was found to produce oscillations in the output RMS ship motions when the selected ship headings, χ_A, did not match the angles available in the RAO. This was a result of how linear interpolation could emphasize/de-emphasize peaks in the RAO depending the selected angle. This was the case for the rectangular and polar approaches only, oscillations were not present for the magnitude-squared approach.

For all of the test cases discussed above, interpolation was always performed with a basic linear method. For comparison, an additional test case was performed which instead used a spline-based method. The results improved for all cases, but not the relative rankings; magnitude-squared faired best, followed by polar form, and then rectangular form.

It is possible to calculate RMS ship motions without any interpolation or modification of the RAO data set. In this approach the input wave spectrum is modified to match the frequency and angle distributions in the RAO, and the wave spectrum is converted to relative angles prior to calculation of the ship motion spectrum. If the selected ship speed, V_A, is not available in the RAO, then ship motions at the speeds above and below V_A can be computed, and then interpolated for motions at V_A. In all cases tested, this approach consistently provided as-good or better results than any of the approaches involving RAO interpolation.

5.1 Recommendations

Based on these results, the follow recommendations should be considered when using RAO for calculation of RMS ship motions.

- The RAO data set should be defined with speed, frequency, and angle ranges that cover the expected conditions along with increment sizes sufficient to resolve regions in the RAO that have large gradients.

- Never extrapolate.
- Depending on the specific application, avoiding interpolation of the RAO altogether may produce the best results and may also be more computationally efficient. The efficiency stems from the fact the wave spectrum is defined in real numbers only, avoiding the extra effort and potential issues involved with complex number interpolation. Secondly, if the input wave spectrum is first interpolated to match the RAO, it can then be used for every requested speed, and every ship motion (e.g., roll, heave, pitch). Whereas, interpolating the RAO would have to be done for every speed and motion.
- In cases where interpolation of a complex value RAO is absolutely required, then a magnitude-squared approach is best used for RMS calculations. Otherwise polar form interpolation is recommended over rectangular form.
- For selected ship heading angles that are not explicitly defined in the RAO (or wave spectrum), then the wave spectrum should be converted from χ_W to β_S for calculation of the motion energy spectrum instead of converting the RAO directions. This is to avoid unnecessary interpolation of the RAO. An alternate recommended approach is to calculate RMS motions at angles existing in the RAO set that are above and below the selected heading, and then interpolating between them.
- As with the point above, if the selected speed, V_A, is not defined in the RAO, then it may be better to interpolate the RMS motions at speeds above and below the selected speed, rather than interpolating the RAO.
- Linear interpolation is 'safest' to use as it ensures interpolated values will always be within the ranges of neighbouring values. However, higher-order methods are capable of improving results, as long as they are used carefully.

5.2 Future Work

A limitation of the current study was its focus on the RAO from a single ship. Potential follow-on work may include examination of interpolation of RAO for different ship types.

Another factor affecting ship motions is the ship loading condition which can alter the mass, inertia, and centre of gravity. For ships that may have large changes in loading due to its operational profile, then its RAO data set may include additional dimensions to account for them. For example, along with ship speed, wave frequency, and relative direction, an RAO set may also be defined for a range of values for vertical centre of gravity. Effects of interpolation along these extra dimensions could be examined.

Insufficient refinement or resolution of the data in areas with high gradients is a major source of error due for interpolation. A future study could involve examining interpolation error as a function of gradient for both RAO and wave spectrum data. Automated checks could potentially be developed into ship motions calculation procedures that could provide warnings if a given data set is found to be insufficiently resolved.

References

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Annex A Relative Wave Angle

The convention for reporting sea wave used by wave buoys and meteorological services is to report the direction the waves are **coming from**. These are usually given in the range 0°–359° with respect to true north (e.g., 90° means that waves are propagating from the east to the west). However, it is also common in fluid dynamics to describe waves in terms of the direction they are **going to** (a change of 180°). Relative wave angle, the difference between the ship and wave directions, can also therefore be defined using either the sea wave ‘coming from’ or ‘going to’ conventions.

Relative wave angles, β_{S}, are in defined in ShipMo3D as the angle between the ship direction[3] and the wave propagation direction, ν_p. This is expressed in terms of both the ‘coming from’ and ‘going to’ wave directions in Equation (A.1) and illustrated in Figure A.1. Using this approach head seas has a relative angle of $\beta_{\mathrm{S}} = 180°$. Nomenclature for other relative angles are given in Figure A.2.

$$\beta_{\mathrm{S}} = \nu_p - \psi_{\mathrm{H}} \tag{A.1a}$$

$$\nu_p = \nu + 180° \tag{A.1b}$$

$$\beta_{\mathrm{S}} = \nu + 180° - \chi_{\mathrm{H}} \tag{A.1c}$$

where,

ν_p is the direction waves are propagating to (w.r.t. true north).
ν is the direction waves are propagating from (w.r.t. true north).
β_{S} is the relative wave angle.
χ_{H} is the ship heading (w.r.t. true north).

[3] Ship heading, χ_{H}, and ship course, χ_{C} are assumed to be the same.

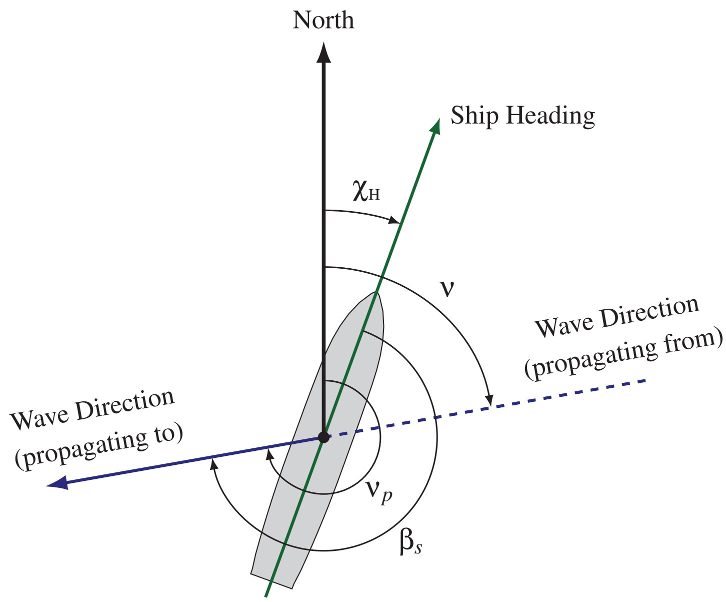

Figure A.1: *Wave direction angles.*

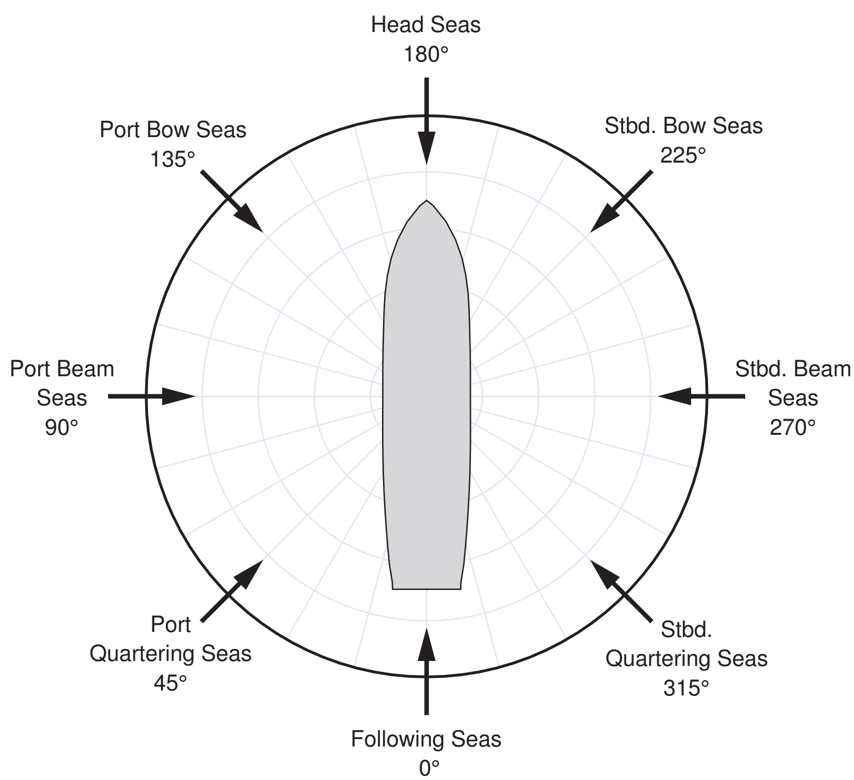

Figure A.2: *Relative wave directions (arrows indicate the direction of wave propagation).*

Annex B RAO Units

RAO data is generated by the ShipMo3D application `SM3DSeakeepRandom31.exe`. For each requested ship speed and relative direction, output RAO are listed in the form of amplitude and phase angle (degrees) for each requested wave frequency for the ship motions: surge, sway, heave, roll, pitch, and yaw. The RAO amplitudes for the displacement motions surge, sway, and heave are non-dimensionalized by wave amplitude (i.e., [m/m]. The RAO amplitudes for the angular motions roll, pitch, and yaw are given in radians divided by wave slope.

In order to use the same Matlab functions to calculate RMS motions for both displacements (surge, sway, heave) and angles (roll, pitch, yaw), the units of the RAO for the angular motions were changed from being non-dimensional to having units of [deg/m].

Whereas wave steepness is defined as the ratio of the wave height (H) over the wavelength (λ), the wave slope is defined as ka; the wave number (k) multiplied by the wave amplitude (a).

$$k = \frac{2\pi}{\lambda} \tag{B.1}$$

$$ka = \pi\frac{H}{\lambda} \tag{B.2}$$

For deep water waves, the wave number can be expressed in terms of wave frequency (ω) and gravity (g) as follows.

$$k = \frac{\omega^2}{g} \tag{B.3}$$

To be consistent with the displacement motion RAO data which is divided by wave amplitude, the RAO data for the angular motions as output by ShipMo3D (which as divided by ka) must therefore be multiplied by the wave number k. As it is more common to have motions like roll and pitch expressed in degrees rather than radians, a unit conversion is also applied. This is given by the following equation which shows the process of converting the native ShipMo3D angular RAO output to that used in the Matlab routines for the calculations in this report.

$$\mathrm{RAO}_{\mathrm{Matlab}} = \frac{\omega^2}{g}\frac{180^\circ}{\pi}\mathrm{RAO}_{\mathrm{ShipMo3D}} \tag{B.4}$$

where,

$\mathrm{RAO}_{\mathrm{ShipMo3D}}$	is the angular.
ν	is the direction waves are propagating from (w.r.t. true north).
β_S	is the relative wave angle.
χ_H	is the ship heading (w.r.t. true north).

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Annex C Encounter Frequency

Encounter frequency is the frequency of the regular wave system as perceived by a ship moving at a steady speed and course. For example, a ship moving into a wave system will run into crests more frequency than if it were moving in the same direction as the waves. The change in frequency is a function of the relative speeds of the ship and waves. In deep water, encounter frequencies, ω_e, can be calculated using Equation (C.1). Note that ship speed, V_S in this equation must be in [m/s] instead of knots.[4]

$$\omega_e = \left| \omega - \frac{V_S \cdot \omega^2}{g} \cos \beta_S \right| \tag{C.1}$$

where,

ω_e is the encounter frequency [rad/s]
ω is the wave frequency [rad/s]
V_S is the ship speed [m/s]
g is the gravitational acceleration [9.807 m/s^2]
β_S is the relative direction [deg R]

The following Matlab code can be used to calculate (deep water) encounter frequencies covering a range of wave frequencies, relative headings, and ship speeds. An example polar contour plot of ω_e for a ship speed of 20 knots ($U_{ms} = 10.3$ m/s) is given in Figure C.1. The radial axis is the wave frequency (ω) and the angular axis is the relative heading (see Annex A). Encounter frequency is plotted as colour contours which increase in head seas and are symmetrical for port and starboard relative headings.

```
betas;     % vector containing relative headings [deg]
omega;     % vector containing wave frequencies [rad/s]
shipSpeed; % vector containing ship speeds [knots]}
g = 9.807; % gravity [m/s^2]}

shipSpeedMS = shipSpeed*0.514444; % convert [knots] to [m/s]
[FREQ, SPD, BETAS] = meshgrid(omega, shipSpeedMS, betas);
encFreq = abs(FREQ - (FREQ.\^2) .* (SPD/g) .* cosd(BETAS));
% The result will be a matrix; encFreq(shipSpeeds, omega, betas)
```

[4] 1 knots ≈ 0.514444 m/s.

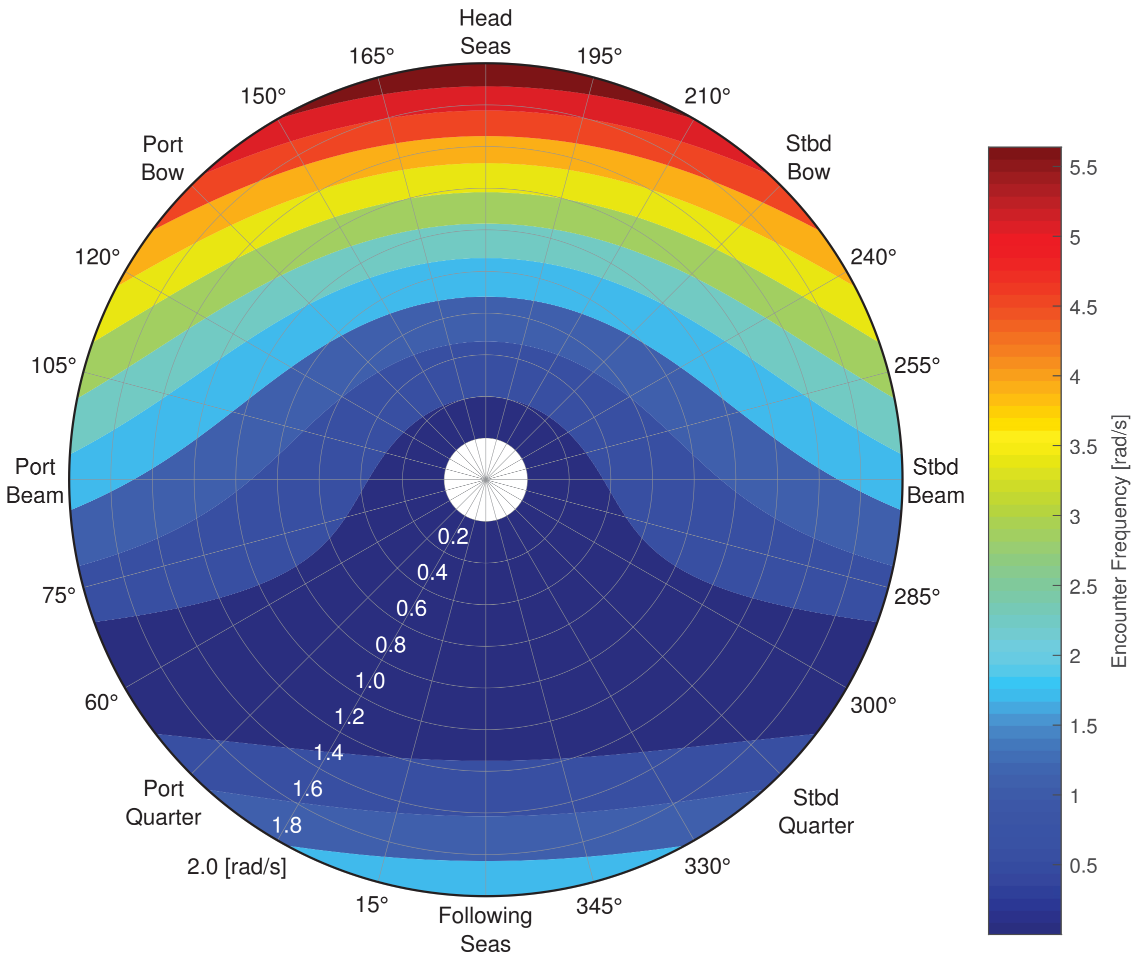

Figure C.1: *Example of encounter frequencies, ω_e, as a function of wave frequency, ω, and relative wave direction, β_s, for a ship speed of 20 knots.*

Annex D Circular Data

Circular data refers to angular data (degrees or radians) which ‘wrap’ around a discontinuity (either 360°/0° or −180°/180°) that causes problems with conventional analysis methods (filtering, differentiation, interpolation, etc.) which assume continuous data.

In some cases, the calculation of statistical metrics, such as standard deviation, for circular data should use alternate methods than for non-circular data. A collection of statistical procedures for circular data is available in the MATLAB ‘Directional Statistics Toolbox’ [10] developed based on methods of Fisher [11].

D.1 Unwrapping Circular Data

An approach to resolve apparent discontinuities due to angle wrapping, is to first unwrap the angles. For example, the angles $(355°, 0°, 5°)$ can be unwrapped to $(355°, 360°, 365°)$. With the discontinuities removed, standard approaches for processing the data could then be applied.

A basic procedure for unwrapping is given in Equation (D.1) for a set of wrapped angles, α (in degrees) of length n. The tolerance defining when a ‘wrap’ occurs is typically 180° (or π radians). An example of wrapped and unwrapped data is shown in Figure D.1 using ship heading angles recorded during a sea trial.

$$
\begin{aligned}
&\text{for } k = 2 \text{ to } n \\
&\quad \Delta\alpha_k = \alpha_k - \alpha_{(k-1)} \\
&\quad \text{if } (\Delta\alpha_k < -\text{tolerance}) \\
&\qquad \alpha_{(k \text{ to } n)} = \alpha_{(k \text{ to } n)} + 360° \\
&\quad \text{end} \\
&\quad \text{if } (\Delta\alpha_k > \text{tolerance}) \\
&\qquad \alpha_{(k \text{ to } n)} = \alpha_{(k \text{ to } n)} - 360° \\
&\quad \text{end} \\
&\text{end}
\end{aligned}
\tag{D.1}
$$

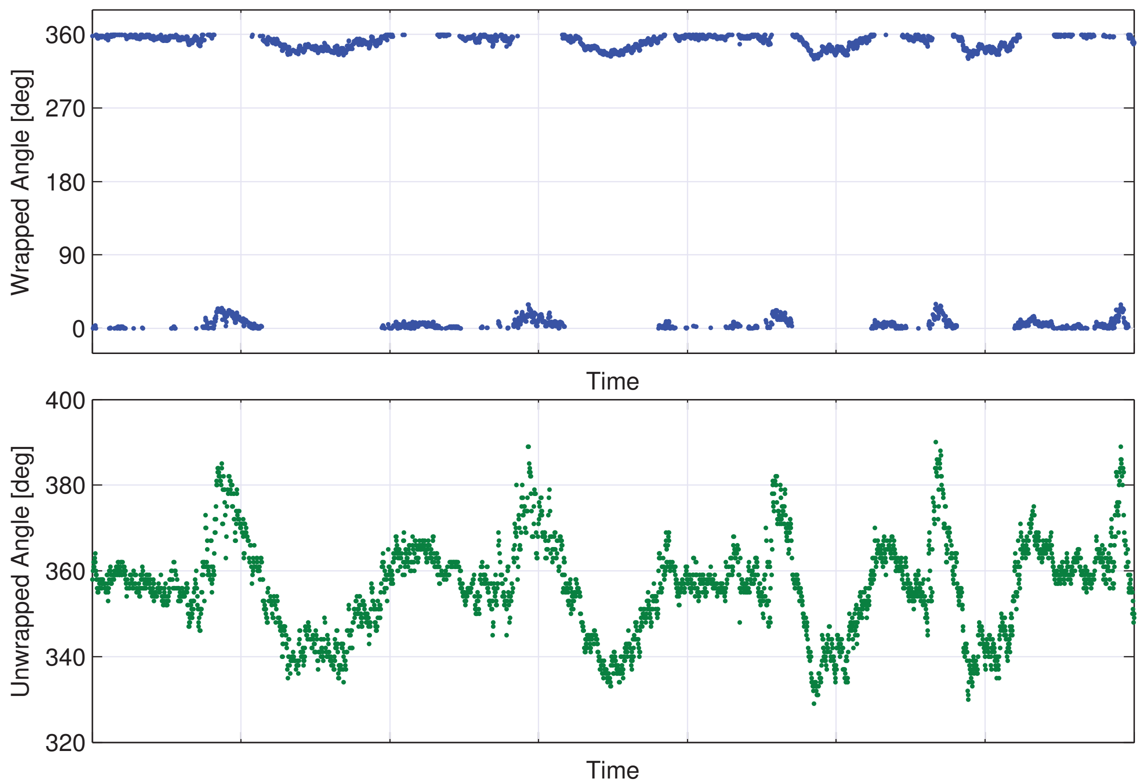

__Figure D.1:__ Example of wrapped and unwrapped circular data.

D.1.1 Unwrapping in MATLAB

Unwrapping of angles for the examples in this report was performed using the built-in MATLAB function `unwrap`. This function works on single vectors (in radians) so applying it to multi-dimensional arrays must be done in sequence as shown below. An example (for a 2D case) of unwrapping is shown in Figure D.2.

```
thetaDegrees;  % 3 dimensional array of wrapped angles in degrees
thetaRadians = thetaDegrees * pi/180;
thetaRadians = unwrap(thetaRadians,[],1); % unwrap along first dimension
thetaRadians = unwrap(thetaRadians,[],2); % unwrap along second dimension
thetaRadians = unwrap(thetaRadians,[],3); % unwrap along third dimension
thetaDegrees = thetaRadians * 180/pi;
```

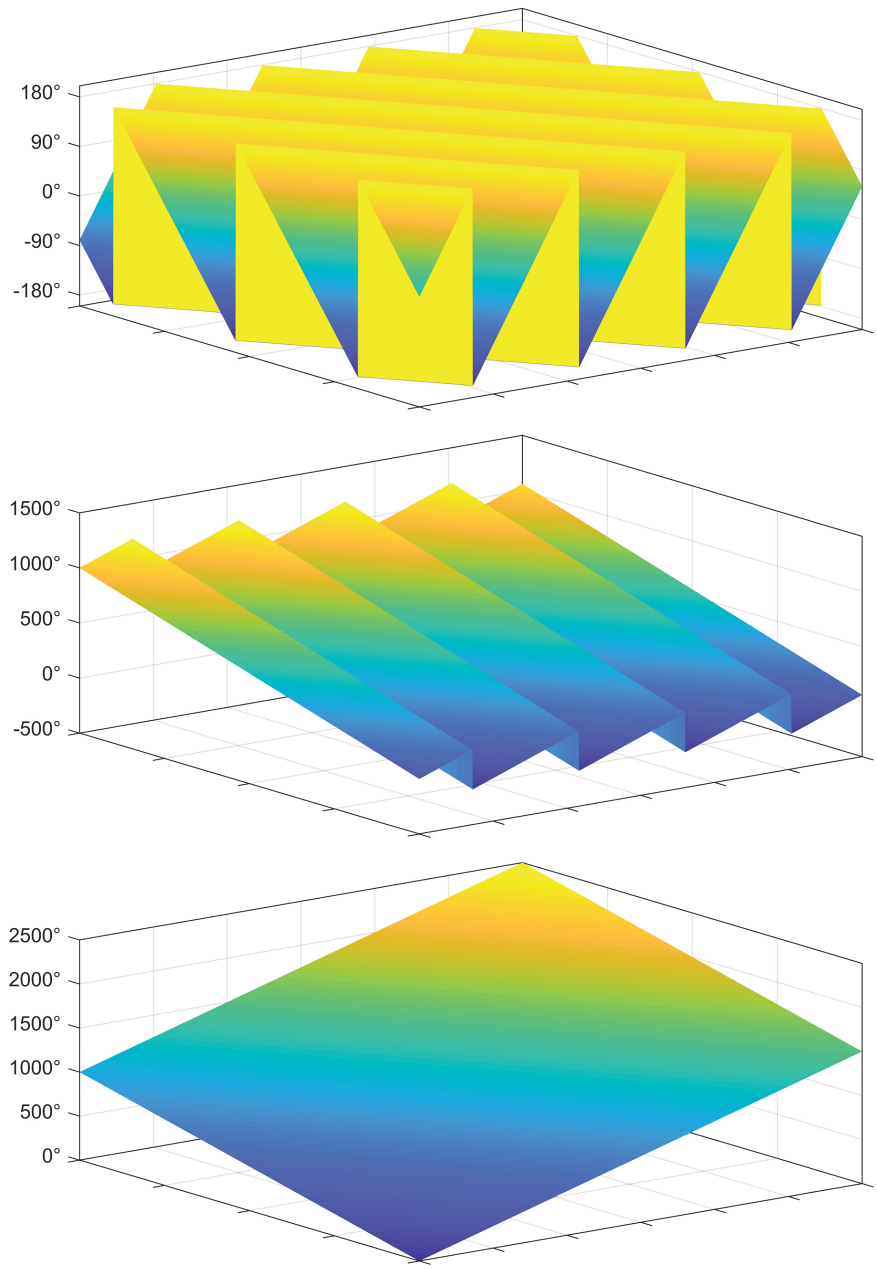

Figure D.2: *An example of a 2D matrix of angles being unwrapped. Top plot shows the matrix data wrapped to $-180°$ to $180°$. The middle plots shows the results after unwrapping along the first dimension. The bottom plot show the final results after unwrapping along the second dimension.*

D.2 Decomposing Circular Data

Another method which is sometimes used to handle wrapped data involves decomposing the angular data into sine and cosine components. The processing activity (filtering, interpolation, etc.) is then carried out seperately on each of these two components. The resulting sine and cosine components after the processing are then recombined using an inverse tangent function.[5]

This approach is equivalent to re-expressing complex numbers defined as polar form into rectangular form for processing, then returning them to the polar form. The difference with angular data is that the 'magnitude' is always of unit length. As an example, the procedure for angular interpolation by decomposition is given by Equation (D.2).

$$\begin{aligned} x &= \cos(\alpha) \\ y &= \sin(\alpha) \\ x_{\text{new}} &= \text{interp}(x) \\ y_{\text{new}} &= \text{interp}(y) \\ \alpha_{\text{new}} &= \arctan\left(y_{\text{new}}, x_{\text{new}}\right) \end{aligned} \tag{D.2}$$

where,

α is the initial set of wrapped angular data

x is the set of cosine components of α

y is the set of sine components of α

x_{new} is the set of cosine components after interpolation

x_{new} is the set of sine components after interpolation

α_{new} the set of wrapped angular data after interpolation

However, as discussed in Section 2.1, this method of decomposition can give different results than those from the unwrapping approach in Annex D.1. For example, the top plot in Figure D.3 shows a set of angular data that is interpolated by decomposition into rectangular components without first unwrapping the angles and by unwrapping. The example data increases linearly w.r.t. an independent variable shown by the black line. The data to be interpolated is shown with large blue circles. Interpolation by decomposition is shown with green crosses and by unwrapping with red dots. Visually, they all appear to match. However, the bottom chart shows the error resulting from the decomposition approach as compared with the actual true values. The error is small, alternating through $\pm 0.13°$. The results from the unwrapping interpolation were exact (in this example).

This oscillating error for the decomposition approach can be visualized geometrically by plotting the sine and cosine components (akin to plotting rectangular components of complex numbers) as shown

[5] Normally the inverse tangent function with two inputs (e.g., $\arctan(y,x)$), is used because it returns angles in the range $-180°$ to $180°$, as opposed to the function with only one input (e.g., $\arctan\left(\frac{y}{x}\right)$) which returns angles in the range $-90°$ to $90°$.

in Figure D.4. In this plot, the true angles trace out a semi-circle. Interpolation by decomposition connects the initial data (blue circles) with straight line segments. The error in the bottom of Figure D.3 is related to the gaps between these line segments the semi-circle.

The error associated with this method increases with the step size of the angle in the initial data set. In the current example, $\Delta\alpha = 30°$ which had an absolute error of $0.13°$. Smaller values of $\Delta\alpha$ will have smaller absolute errors, reaching zero at the limit of $\Delta\alpha \to 0°$.

Although the errors using decomposition are often small, the method offers no significant advantages over the unwrapping approach in Annex D.1 and in some implementations could even be slower as it may require more computational steps. The unwrapping method is therefore used for angular calculations in this report.

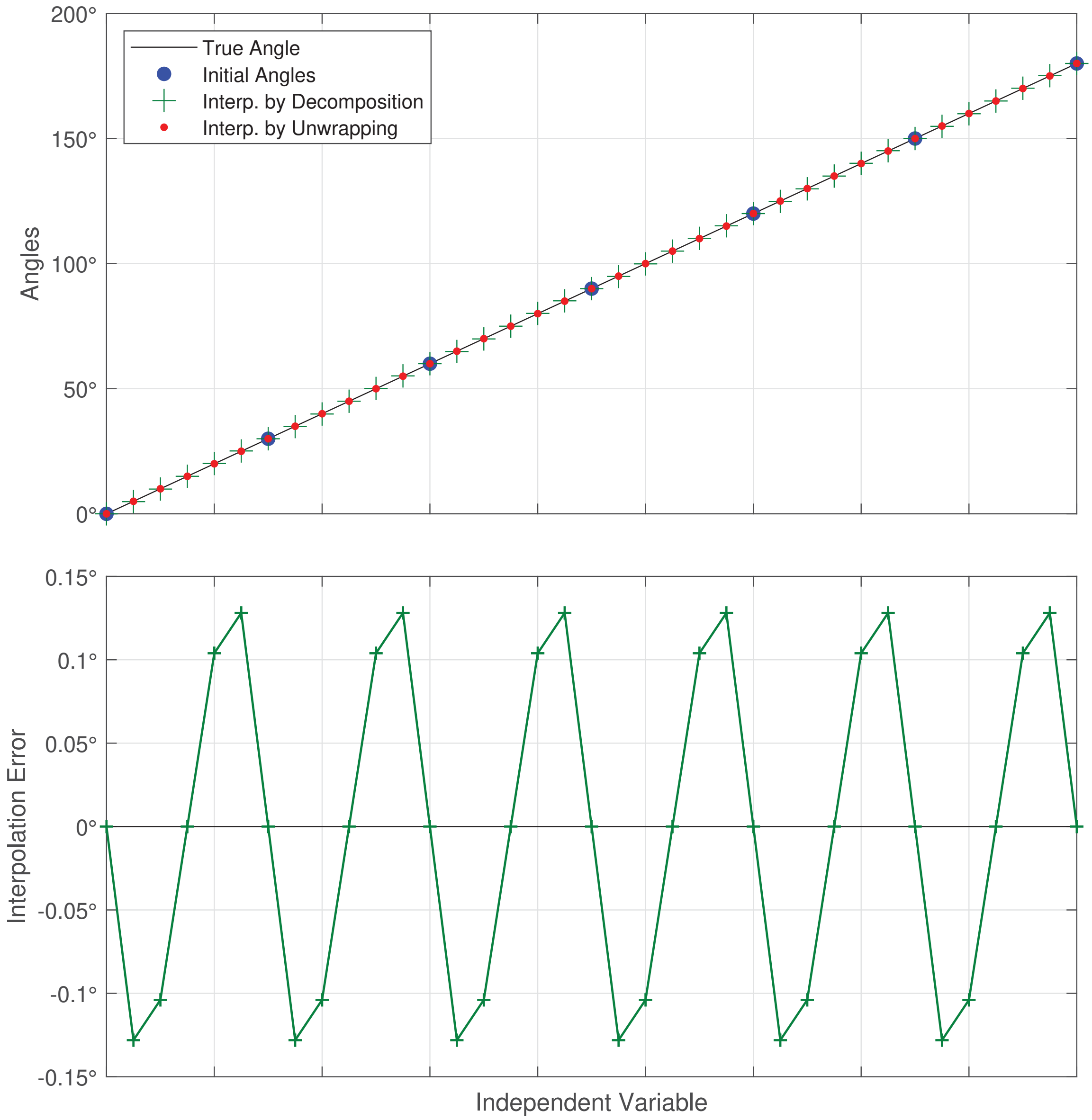

Figure D.3: *Example of angular data being interpolated along with the resulting error.*

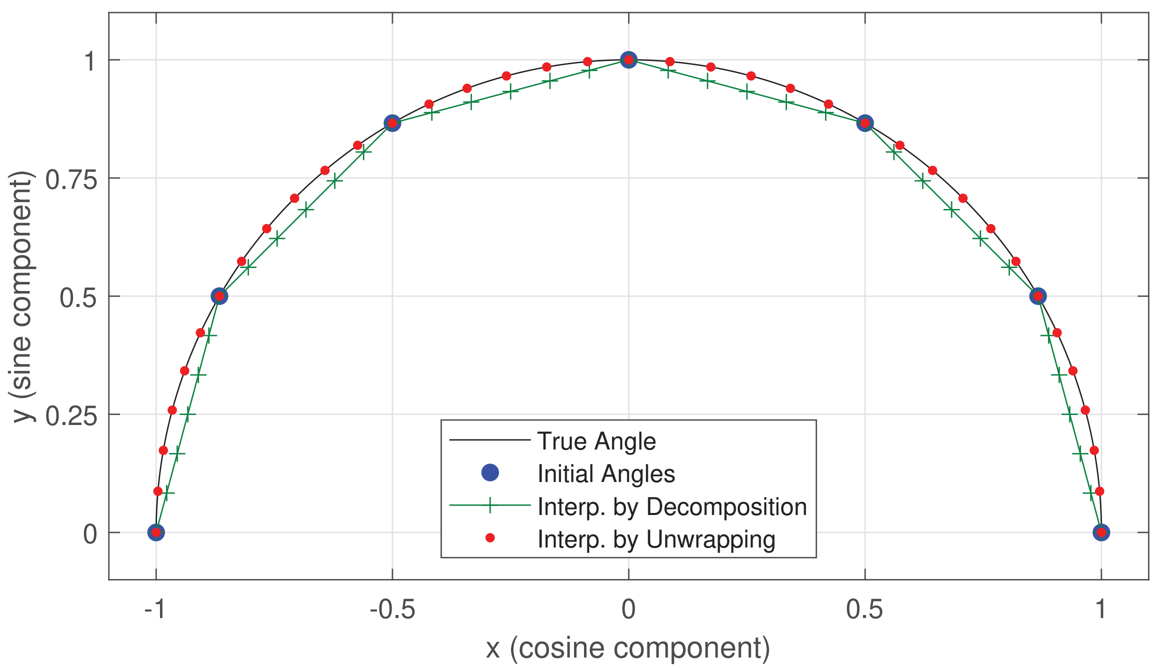

Figure D.4: *Angular data plotted w.r.t. its sine and cosine components.*

D.3 Open and Closed Forms

Wave spectrums and RAO data will have the same values at 0° and 360°. In some cases, the redundant data at 360° is left out; this will be referred to as an ‘open’ set, while in other cases, the 360° data is included to form a ‘closed’ set. Depending on the calculation being performed, one or other of these forms may be needed. An example of a wave spectrum with angles defined every 15° is plotted in both closed and open forms in Figure D.5. If for example, the spectrum were integrated to calculate wave statistics, then the closed form must be used, otherwise the integration would miss the sector spanning 345° to 360° as shown on the right side of Figure D.5.

e.g., Open circular data set: 0°, 15°, 30°, … 330°, 345°
Closed circular data set: 0°, 15°, 30°, … 330°, 345°, 360°

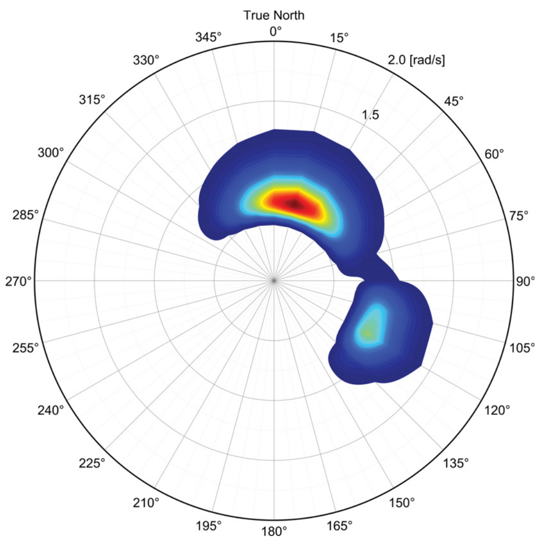

Figure D.5: *Wave spectrum data in ‘closed’ (left) and ‘open’ (right) forms.*

The following are samples of MATLAB code that can be used to ensure a wave spectrum or an RAO data set are in an open or closed form.

D.3.1 Opening Wave Spectrum Data

This sample of MATLAB code is used to ensure a wave spectrum data set is in an open state.

```
omega;     % Row vector containing wave frequencies
chiW;      % Row vector containing 'coming from' wave directions [degrees]
waveSpec;  % Wave spectrum matrix: waveSpec(omega,chiW)

if (chiW(end) - chiW(1)) == 360  % First check if already open

  chiW(end) = [];
  waveSpec(:,end) = [];

end
```

D.3.2 Closing Wave Spectrum Data

This sample of MATLAB code is used to ensure a wave spectrum data set is in an closed state.

```
omega;     % Row vector containing wave frequencies
chiW;      % Row vector containing 'coming from' wave directions [degrees]
waveSpec;  % Wave spectrum matrix: waveSpec(omega,chiW)

if (chiW(end) - chiW(1)) ~= 360  % First check if already closed

  chiW(end+1) = 360 + chiW(1);
  waveSpec = cat(2, waveSpec, waveSpec(:,1));

end
```

D.3.3 Opening RAO Data

This sample of MATLAB code is used to ensure a RAO data set is in an open state.

```
Vs;    % Row vector containing RAO ship speeds
omega; % Row vector containing RAO wave frequencies
betaS; % Row vector containing RAO relative directions
RAO;   % Complex-value RAO array: RAO(Vs,omega,betaS)

if (betaS(end) - betaS(1)) == 360  % First check if already open

  betaS(end) = [];
  RAO(:,:,end) = [];

end
```

D.3.4 Closing RAO Data

This sample of MATLAB code is used to ensure a RAO data set is in an closed state.

```
Vs;    % Row vector containing RAO ship speeds
omega; % Row vector containing RAO wave frequencies
betaS; % Row vector containing RAO relative directions
RAO;   % Complex-value RAO array: RAO(Vs,omega,betaS)

if (betaS(end) - betaS(1)) ~= 360  % First check if already closed

  betaS(end+1) = 360 + betaS(1);
  RAO = cat(3, RAO, RAO(:,:,1));

end
```

Annex E MATLAB Code Samples

E.1 Resizing Wave Spectrum Data

The following sample of MATLAB code can be used to resize a wave spectrum to different values of wave frequency and direction using linear interpolation. Note that the wave spectrum must first be set to a ‘closed’ form (see Annex D.3.2).

```
omega;      % Row vector containing wave frequencies
chiW;       % Row vector containing 'coming from' wave directions [degrees]
waveSpec;   % Wave spectrum matrix: waveSpec(omega,chiW)

omega_new; % Row vector containing new set of wave frequencies
chiW_new;  % Row vector containing new set of wave directions [degrees]

METHOD     = 'linear'; % Use linear interpolation
EXTRAPVAL  = NaN;      % Assign Not-A-Number (NAN) to out-of-range values

% First check that the wave spectrum is closed
if (chiW(end) - chiW(1)) ~= 360
  error('Wave spectrum must be in a closed form for interpolation');
end

waveSpec_new = interp2(chiW, omega, waveSpec, ...
                 chiW_new, omega_new', ...
                 METHOD, EXTRAPVAL);
```

E.2 Rotating Wave Spectrum Data

The following sample of MATLAB code can be used to rotate a wave spectrum in order to convert it from using wave directions, χ_W to using relative directions, β_S (see Annex A) for an arbitrary ship heading, χ_A. Note that the wave spectrum must first be in an ‘open’ form (see Annex D.3.1).

```
omega;      % Row vector containing wave frequencies
chiW;       % Row vector containing 'coming from' wave directions [degrees]
waveSpec;   % Wave spectrum matrix: waveSpec(omega,chiW)
chiA;       % Arbitrary ship heading angle [degrees]

METHOD     = 'linear'; % Use linear interpolation
EXTRAPVAL  = NaN;      % Assign Not-A-Number (NAN) to out-of-range values

% First check that the wave spectrum is open
if (chiW(end) - chiW(1)) == 360
  error('Wave spectrum must be in an open form');
end

rotateAngle = mod((180-chiA),360);
tempChi     = chiW + rotateAngle;
betaS       = chiW;
```

```
% Expand range of waveSpec to ensure angles can be covered
tempChi  = [(tempChi - 360), tempChi, (tempChi + 360)];
tempSpec = [waveSpec, waveSpec, waveSpec];

waveSpec = interp2(tempChi, omega, tempSpec, ...
                 betaS, omega', ...
                 METHOD, EXTRAPVAL);

% The now rotated spectrum consists of the following:
omega;     % Row vector containing wave frequencies
betaS;     % Row vector containing relative directions [degrees]
waveSpec;  % Wave spectrum matrix: waveSpec(omega,betaS)
```

E.3 Resizing RAO Data

The following is sample MATLAB code which can be used to interpolate a three dimensional complex-valued RAO using the three approaches used in this report. Note that the RAO data must be in a 'closed set' prior to interpolation (see Annex D.3.4).

```
Vs;    % Row vector containing RAO ship speeds
omega; % Row vector containing RAO wave frequencies
betaS; % Row vector containing RAO relative directions
RAO;   % Complex-value RAO array: RAO(Vs,omega,betaS)

Vs_new;    % Row vector containing new set of ship speeds
omega_new; % Row vector containing new set of wave frequencies
betaS_new; % Row vector containing new set of relative directions

METHOD     = 'linear'; % Use linear interpolation
EXTRAPVAL  = NaN;      % Assign Not-A-Number (NAN) to out-of-range values

% First check that the RAO is closed
if (betaS(end) - betaS(1)) ~= 360
   error('RAO must be in a closed form for interpolation');
end

% Options for approach:
%    'real & imaginary', 'magnitude & phase', or 'magnitude-squared'
interpApproach = 'real & imaginary';

switch interpApproach

  case 'real & imaginary'

    % The is the default method used by interp3

    RAO_new = interp3(omega, Vs, betaS, RAO, ...
      omega_new, Vs_new, betaS_new, ...
      METHOD, EXTRAPVAL);

  case 'magnitude & phase'
```

```
    r     = abs(RAO);    % get magnitudes of complex RAO values
    alpha = angle(RAO);  % get angles of complex RAO values [radians]

    % Unwrap alpha along each of its dimensions
    alpha = unwrap(alpha,[],1);
    alpha = unwrap(alpha,[],2);
    alpha = unwrap(alpha,[],3);

    % Interpolate magnitudes
    r_new  = interp3(omega, Vs, betaS, r, ...
      omega_new, Vs_new, betaS_new, ...
      METHOD, EXTRAPVAL);

    % Interpolate phase angles
    alpha_new  = interp3(omega, Vs, betaS, alpha, ...
      omega_new, Vs_new, betaS_new, ...
      METHOD, EXTRAPVAL);

    % Recombine into complex numbers
    RAO_new = r_new .* exp(alpha_new .* 1i);

    clear r alpha r_new alpha_new

  case 'magnitude-squared'

    mag2 = abs(RAO).^2;

    RAO_new = interp3(omega, Vs, betaS, mag2, ...
      omega_new, Vs_new, betaS_new, ...
      METHOD, EXTRAPVAL);

    clear mag2

  otherwise

    error(['unrecognized interpApproach: ',interpApproach]);

end
```

E.4 Rotating RAO Data

The following sample of MATLAB code can be used to rotate a RAO data set in order to convert it from using relative directions, β_S (see Annex A) to wave directions, χ_W for an arbitrary ship heading, χ_A. Note that the RAO must first be in an 'open' form (see Annex D.3.3). Note this example uses linear interpolation of the rectangular components, the default approach for the MATLAB function `interp3`. Other approaches could also be used as given in Annex E.3.

```
Vs;    % Row vector containing RAO ship speeds
omega; % Row vector containing RAO wave frequencies
betaS; % Row vector containing RAO relative directions
```

```
RAO;   % Complex-value RAO array: RAO(Vs,omega,betaS)
chiA;  % Arbitrary ship heading angle [degrees]

METHOD     = 'linear'; % Use linear interpolation
EXTRAPVAL  = NaN;      % Assign Not-A-Number (NAN) to out-of-range values

% First check that the RAO is open
if (betaS(end) - betaS(1)) == 360
   error('RAO must be in an open form');
end

rotateAngle = mod((180+chiA),360);
tempBeta     = betaS + rotateAngle;
chiW         = betaS;

% Expand range to ensure angles can be covered
tempBeta  = [(tempBeta - 360), tempBeta, (tempBeta + 360)];
tempRAO   = cat(3, RAO, RAO, RAO);

% The following for perform interpolation seperately on
% real & imaginary parts. See sample MATLAB code in
% Annex B.3 to use other approaches.
RAO = interp3(omega, Vs, tempBeta, tempRAO, ...
      omega, Vs', chiW', ...
      METHOD, EXTRAPVAL);

% The now rotated RAO consists of the following:
Vs;    % Row vector containing RAO ship speeds
omega; % Row vector containing RAO wave frequencies
chiW;  % Row vector containing 'coming from' wave directions [degrees]
RAO;   % Complex-value RAO array: RAO(Vs,omega,chiW)
```

Acronyms

DRDC	Defence Research and Development Canada
FFT	Fast Fourier Transform
RAO	response amplitude operator
RCN	Royal Canadian Navy
RMS	Root Mean Square
w.r.t.	with respect to

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Response amplitude operator (RAO) data sets are used in many applications, such as for calculating RMS ship motions in arbitrary sea conditions. In the process of performing these calculations, there can be cases where the complex-numbers of a RAO data set must be interpolated. This report presents the results of a study examining different approaches to interpolating complex RAO data, such as using its real & imaginary parts, or by using its magnitudes & phase angles. Performing interpolation at different steps within a calculation procedure for RMS motions is also examined to evaluate the effects on output.

Les ensembles de données d'amplitude des réactions de l'opérateur (response amplitude operator, ou RAO) servent à de nombreuses applications dont le calcul des mouvements des navires de la MRC en conditions en mer arbitraires. Cependant, pour ces calculs, il arrive de devoir interpoler les nombres complexes de l'ensemble de données RAO. Le rapport scientifique présente les résultats d'une étude de diverses stratégies d'interpolation des données RAO complexes, comme séparer les parties réelles et imaginaires des nombres ou utiliser plutôt leur magnitude et leur angle de phase. L'étude s'est aussi penchée sur l'interpolation à diverses étapes du calcul des mouvements des navires de la MRC afin d'en cerner les répercussions sur les résultats obtenus.