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Monetary Policy, Trends in Real Interest Rates and Depressed Demand

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Abstract

Over the last few decades, real interest rates have trended downward in many countries. The most common explanation is that this reflects depressed demand due to demographic, technological and other real factors such as income inequality. In this paper we explore the claim that these trends may have been amplified by certain features of monetary policy. We show that when long-run asset demands by households are C-shaped in relation to real interest rates, a feature we motivate through bequest motives, monetary policy has the potential to affect steady-state properties even if money is neutral in the long run. In particular, we show that if monetary policy reacts aggressively to inflation, this supports a steady state where inflation is close to the central bank's target. However, the same aggressive policy simultaneously favours the emergence of, and the convergence to, a second stable and determinate steady state where both the real interest rate and inflation are lower and monetary policy is constrained by the effective lower bound. We discuss how fiscal policy can be used to escape this low-real-rate, low-inflation trap with the potential for a discontinuous response of long-run inflation.

Topics: Monetary policy; Fiscal policy; Economic models; Inflation and prices; Interest rates; Debt management
JEL codes: E2, E43, E44, E5, E52, E62, E63, H3, H6, H63

Résumé

Au cours des dernières décennies, les taux d'intérêt réels se sont inscrits en baisse dans de nombreux pays. L'explication la plus courante est que ces tendances reflètent la faiblesse de la demande, qui s'explique par des facteurs démographiques et technologiques et d'autres facteurs réels, tels que les inégalités de revenu. Dans la présente étude, nous examinons l'hypothèse selon laquelle ces tendances ont pu être amplifiées par certaines caractéristiques de la politique monétaire. Nous montrons que, quand la demande à long terme d'actifs par les ménages présente une relation en forme de C par rapport aux taux d'intérêt réels, ce que nous expliquons par la volonté des ménages de constituer un héritage, la politique monétaire est susceptible d'influer sur les propriétés de l'état stationnaire même si la monnaie est neutre à long terme. Plus précisément, notre étude révèle que, si la politique monétaire réagit vigoureusement à l'inflation, cela soutient un état stationnaire où l'inflation avoisine la cible de la banque centrale. Toutefois, la même politique vigoureuse favorise simultanément l'apparition, et la coexistence, d'un deuxième état stationnaire stable et déterminé où le taux d'intérêt réel et l'inflation sont tous deux moins élevés et où la politique monétaire est limitée par la valeur plancher. Nous expliquons également comment la politique budgétaire peut être utilisée pour s'échapper de cette trappe de bas taux d'intérêt réel et de basse inflation qui pourrait provoquer une réaction discontinue de l'inflation à long terme.

Sujets : Politique monétaire; Politique budgétaire; Modèles économiques; Inflation et prix; Taux d'intérêt; Gestion de la dette

Codes JEL : E2, E43, E44, E5, E52, E62, E63, H3, H6, H63

1 Introduction

In many advanced economies, real interest rates have trended down over the last few decades and debt levels have increased. At the same time, economic activity has often run below full employment, as reflected by below-target inflation (see Figure 1). The most common explanation for these trends is that advanced economies have experienced a secular fall in demand and that policy decisions to decrease interest rates and to increase government debt have been important mitigating factors that have helped offset this fall in demand. The forces cited for inducing such a fall in demand include reduced productivity growth, the aging of the population, and increased inequality.[1] While all these factors may be relevant, this one-way narrative from exogenous reductions in demand to policy response has nonetheless been put into question by many. In particular, several market commentators argue that monetary policy over the period possibly contributed to the long-term downward trend in real interest rates by decreasing interest rates aggressively in every downturn and being hesitant to increase them in upturns. This has also been highlighted by policymakers such as Borio, Disyatat, Juselius, and Rungcharoenkitkul (2017), who provide evidence that over a long history "persistent shifts in real interest rates coincide with changes in monetary regimes. ... All this points to an underrated role of monetary policy in determining real interest over long horizons."[2]

The goal of this paper is to help advance the discussion around the potential factors that have weighed down real interest rates and depressed demand over the last few decades. In particular, we aim to highlight features that could (or could not) cause aggressive monetary policy to contribute toward persistently low real interest rates and favour the emergence of a low-real-rate, low-inflation trap. We will also discuss how expansionary fiscal policy may

[1] A vast literature examines the sources of the decreasing trend in real interest rates. Borio, Disyatat, Juselius, and Rungcharoenkitkul (2017) provide an excellent survey of the literature on these issues. Several hypotheses about these sources have been proposed: demographics (Summers (2014), Eggertsson and Mehrotra (2014), and Eichengreen (2015)); a productivity slowdown (Gordon (2017)); a global saving glut and/or lack of safe assets (Bernanke (2005), Caballero, Farhi, and Gourinchas (2008), Gourinchas and Rey (2016), Gourinchas, Rey, and Sauzet (2020), and Acharya and Dogra (2021)); a decline in desired investment (Rachel and Smith (2017)); a rise in inequality (Mian, Straub, and Sufi (2020b), Auclert and Rognlie (2020), Fagereng, Blomhoff Holm, Moll, and Natvik (2019), and Rachel and Smith (2017)).

[2] Gourinchas and Rey (2016)'s and Gourinchas, Rey, and Sauzet (2020)'s focus on financial cycles, especially the leveraging cycle that accompanied the boom and bust in the 1930s and 2000s, for explaining the short-term real interest rate movements is consistent with the role of monetary policy. Their explanation centres on the relative demand for safe assets in the aftermath of a deleveraging shock. However, the association between the consumption-to-wealth ratio and subsequent short-term real risk-free interest rates could also be seen as reflecting the central bank's reaction function in boom and bust periods. That is, the abnormally low consumption-to-wealth ratio following financial busts tends to coincide with periods of aggressive monetary policy easing to support the economy, which in turn gives rise to the association with low short-term risk-free rates in the subsequent period.

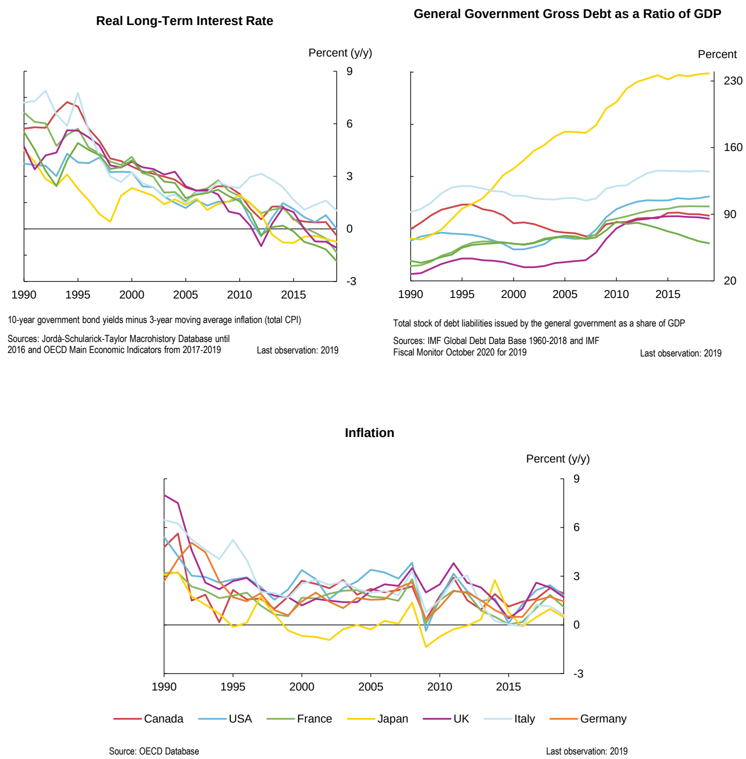

Figure 1: Long-term interest rates, government debt, and inflation for G7 countries from 1990 to 2019

help the economy exit the trap. Hopefully, clarifying such features will help direct future empirical work aimed at better understanding the causes of such a decline.

In the determination of real interest rates in the long run, the properties of asset demands are key. In a standard infinitely lived representative agent model, the long-run asset demand by households is stark. If the real interest rate is below the households' subjective discount rate (adjusted for growth), then households will not want to hold any assets in the long run. If it is slightly above the subjective discount rate, then households want to hold an infinite amount. Such a set-up exemplifies an asset demand function that is increasing in interest rates, albeit in an extreme form. While there are several modifications that can make such a long-run asset demand function less extreme, most tractable approaches maintain the property that the long-run asset demand, especially relative to consumption, is monotonically increasing in interest rates. However, there are many reasons to question our reliance on models where asset demand is monotonically increasing in real interest rates.

As is well known, some income effects associated with interest rates that could reverse this property. For example, when interest rates fall, households may want to hold more assets — not fewer — if at least part of their asset holdings are for precautionary motives, retirement motives or bequest motives. If asset demands are non-monotonic in interest rates, this opens the door to multiple equilibria and the possibility that policy — monetary or fiscal — could affect where the economy moves toward in the long run. The simplest example of such non-monotonicity is when the demand for assets has a C-shaped relationship with respect to real interest rates.

In this paper we will present an environment where asset demand functions are non-monotonic in real interest rates. We then use this environment to illustrate the potential macroeconomic implications of such a feature. In particular, we want to show how monetary policy can affect long-run outcomes in an environment where asset demands are non-monotonic even if money is essentially neutral in the long run.[3] The precise environment we build upon, which relies on bequest motives, is chosen for tractability so that our main results can be shown analytically and through the use of phase diagrams. There are certainly other environments that could generate C-shaped asset demand functions — but they are generally less tractable. Since we are not conducting a quantitative analysis and we aim instead to highlight a set of properties, we favour the choice of this highly tractable environment.

An important element of our analysis will be to show how a C-shaped asset demand can interact with a Taylor rule specification of monetary policy when the latter is subject to an effective lower bound (ELB) constraint. It is the interaction between these two forces that drives several of our results. From Benhabib, Schmitt-Grohé, and Uribe (2001), Benhabib, Schmitt-Grohé, and Uribe (2002), and related literature, we know that an ELB constraint can give rise to multiple equilibria.[4] However, most of this literature is not aimed at explaining changes in real interest rates, as the long-run real interest rate in the ELB regime is generally the same as the one in the non-ELB regime. Moreover, given that the equilibrium in the ELB regime in this literature is generally indeterminate, arguments related to learning put into question its relevance. In contrast, in our set-up, the long-run equilibrium that arises when monetary policy is constrained by the ELB will be shown to be

[3] A large literature supports the notion that money is neutral in the long run: see King and Watson (1997) for a survey. However, an emerging literature questioning such a view. For example, Jordà, Singh, and Taylor (2020) provide compelling evidence of the non-neutrality of money over long periods; that is, they show that monetary policy has real effects that last more than a decade.

[4] Expectations-driven liquidity traps have also been applied to fiscal policy, optimal monetary policy and open economy issues. See for example, Mertens and Ravn (2014), Bilbiie (2018), Nakata and Schmidt (2021), Aruoba, Cuba-Borda, and Schorfheide (2018), and Kollmann (2018).

both stable and determinate.[5] We will also show that the real interest rate that emerges when this constraint is binding is lower than when it is not binding. Therefore, a shift from a non-ELB-constrained equilibrium toward an ELB-constrained equilibrium is associated with a fall in real interest rates.[6]

Our focus in this paper will be mainly on the role of consumers, and especially on consumers' saving behaviour and desired wealth accumulation, in supporting or depressing demand in the long run in response to interest rate changes. Because of this focus on consumers, much of the analysis will abstract from capital accumulation. This is essentially equivalent to assuming that capital accumulation is not sensitive to interest rates. In general, as we will show, if capital accumulation were to increase substantially in response to a cut in interest rates, then investment spending could potentially offset some of the depressing effects of demand coming from household behaviour that we highlight in this paper.[7]

One of the challenges we face by wanting to focus on long-run implications of asset market equilibrium to understand the long-run outcomes is that, as mentioned above, asset demands are almost discontinuous in certain common set-ups. One way to capture richer saving behaviour is to follow Kumhof, Rancière, and Winant (2015), Mian, Straub, and Sufi (2020a), Michaillat and Saez (2018), De Nardi (2004), and Straub (2019), among others, and allow assets to directly affect utility. In this paper we complement this literature by re-examining when and to what extent bequest motives can rationalize introducing assets in the utility function, and when this can result in relative asset demands that are not monotonic in interest rates.[8]

[5] The set-up explored in Michaillat and Saez (2018) shares this feature. However, in contrast to their set-up, our approach generates a co-existence of a stable and determinate steady state at the ELB and one not at the ELB. This property is central to our results.

[6] Fernández-Villaverde, Marbet, Nuño, and Rachedi (2021) also consider how monetary policy can affect the long-run level of real interest rates. Specifically, they show, in a quantitative HANK model with an ELB constraint, that the interaction between the inflation target and wealth inequality is an important determinant for the level of real interest rates. However, their approach does not explore how monetary policy can affect the set of equilibria and their basin of attraction as we do here.

[7] Our choice of abstracting from capital accumulation can be seen as tilting our results — by design — to finding conditions where expansionary monetary policy (and/or counter-cyclical monetary policy) could be contractionary in the long run.

[8] When discussing how bequest motives may justify including assets in the utility function, it becomes clear that both the stock of assets and the flow of revenue from assets should enter utility. In the literature cited above, it is assumed that households get utility only from the stock of assets they bequeath. It does not matter whether the assets bequeathed have a high or low return. For example, the household gets the same utility from bequeathing a bond with a 1% real rate of interest or a 10% real rate of interest. In contrast, from a bequest perspective, these two situations may differ. In particular, we will show that if households have concave utility over the welfare of their offspring, this can rationalize having flow of return from assets entering utility, and it is this feature that creates C-shaped asset demands.

The main results of the paper can be divided into two categories. On the one hand, there is the issue of under what conditions asset demands could be non-monotonic in real interest rates, and, on the other hand, there are the implications of having long-run asset demands with such properties. We believe that the more interesting insights of the paper relate to the implications of C-shaped relative asset demands, while the results regarding how we generate this non-monotonic feature are secondary. In an environment where we show how bequest motives can generate a non-monotonic long-run asset demand, we find that when monetary policy cuts interest rates aggressively in response to slack in the economy, even a temporary fall in demand can generate dynamics by which interest rates converge to their ELB with the economy staying depressed after the initial shock is reversed. This type of hysteresis arises because the environment exhibits two stable and determinate equilibria. When inflation dynamics are governed by a Phillips curve, we show the existence of one high-real-interest-rate equilibrium where inflation is close to the central bank's target, and one low-real-interest-rate equilibrium where the nominal interest rate is at the ELB and inflation is well below target. The equilibrium at the lower bound is stable and determinate even if the Taylor principle is non-operative. In this environment, a strong anti-inflation stance by monetary policy is helpful to keep inflation close to target in the high-real-interest-rate regime; however, we will show that the same policy stance will make it more likely that, following bad temporary shocks, the economy converges to a stable low-real-rate, low-inflation trap where nominal interest rates are constrained by the ELB. We will also show how increases in public debt, even when they are eventually accompanied by increases in taxes to balance the budget, can be helpful to pull the economy out of a low-real-rate, low-inflation trap.[9] In our set-up, more debt does not depress output demand even if it can contribute to lower real interest rates. Increased debt does, nonetheless, have the potential to lead to a discontinuous response in long-run inflation when it manages to help the economy avoid a low-real-rate equilibrium.

Since we follow Mian, Straub, and Sufi (2020a) (MSS) in using bequests to motivate why households may have a strong demand for assets when interest rates are low, it is relevant to point out why we arrive at very different results in terms of the effects of both fiscal and monetary policy. There are two important points of departure. First, our analysis emphasizes how higher interest rates may incite households to hold fewer assets to satisfy their bequest motives since the flow of return from assets — not just the stock — is likely

[9]In a model with risky capital, Acharya and Dogra (2021) also show that an increase in government debt helps exit the zero lower bound by increasing the natural rate of interest and helps restore full employment. Similarly, Eggertsson and Mehrotra (2014), in an incomplete market environment with riskless capital, find that public debt issuance can restore full employment when monetary policy is constrained.

relevant for determining their desired bequests. It is this feature, along with the absence of human capital in the bequest motives and the marginal value of wealth decreasing as interest rates rise, that gives rise to a C-shaped relative demand for assets in our set-up and drives most of our results regarding monetary policy.[10] This feature finds support in the work by Greenwald, Leombroni, Lustig, and Van Nieuwerburgh (2021), which shows that "faced with lower returns on financial wealth, households with high levels of financial wealth must increase savings to afford the consumption that they planned before the decline in rates."[11] The second point relates to properties of steady state asset holdings. In our set-up, steady state asset demands are increasing in labour income; that is, in our formulation, when a household faces a given interest rate, its steady state holdings of financial wealth are increasing in the household's labour income (see our Proposition 1 and Lemma 1). In contrast, in the parameterization favoured by MSS, steady state holdings of financial assets are decreasing in a household's human capital — which is the discounted value of labour income — and therefore the steady state holdings of financial assets are decreasing in labour income when interest rates are fixed.[12] In our set-up, where steady state financial asset demands are increasing in labour income, debt does not have a depressing effect on demand at fixed interest rates even if it can decrease long-run real rates when interest rates are endogenously determined.[13] In fact, we show that increased debt can cause real interest rates to decrease when monetary policy is constrained by the ELB but can also lead to a

[10] An important dimension in which our set-up differs from that presented in MSS relates to properties of the steady state Euler conditions that express the demand for assets, holding consumption fixed when $\dot{c}_t = 0$. In the MSS set-up, the steady state demand for total assets, holding consumption fixed, is always increasing in interest rates, while in our set-up the demand for financial assets is C-shaped due to bequest motives responding to interest rates.

[11] Using proprietary data from the Office of the Superintendent of Financial Institutions — the regulatory agency of financial institutions in Canada — Betermier, Byrne, Fontaine, Ford, Ho, and Mitchell (2021) show that as interest rates decrease, big Canadian pension funds tend to increase their demand for bonds.

[12] The property that steady state financial asset holdings are decreasing in labour income in MSS may give the impression that savers in MSS would decrease their savings when their income rises. That is not the case. Savers in MSS increase savings when income rises and, in partial equilibrium, do not converge to a steady state holding. So the properties of the steady state asset holdings in MSS do not map easily to partial equilibrium behaviour. The properties of the steady state asset holdings only become relevant for understanding general equilibrium outcomes in MSS.

[13] This can be seen most simply by focusing on the steady state debt-market equilibrium condition of the form $A^{ss}(y,r) = D(r,\Omega)$, where $A^{ss}(y,r)$ is the steady state net demand function for debt from (saver) households, y is labour income of (saver) households, r is the real interest rate, $D(r)$ is an amount of debt offered on the market (either public debt or debt accumulated by non-saver households), and Ω is a debt shifter with $\frac{\partial D}{\partial \Omega} > 0$. We can use this condition to look at the effect of Ω on y at fixed prices and interest rates, that is, examine whether an exogenous increase in debt has a positive or negative effect on steady state demand at fixed prices and interest rates. In our set-up, this effect is positive because $\frac{\partial A^{LR}}{\partial y} > 0$. To get the result that increased debt can depress demand at fixed prices and interest rates, it has to be that $\frac{\partial A^{LR}}{\partial y} < 0$.

large increase in real interest rates and inflation when a sufficiently large increase in debt manages to push the economy out of a low-real-rate, low-inflation ELB trap.

The main results of the paper are in Sections 3 and 4. In these sections, we show how aggressive monetary policy could contribute to trend movements in real interest rates by facilitating a movement from a high-real-rate stable steady state to a low-real-rate, low-inflation stable steady state with depressed demand. As noted previously, the main feature driving the results is a relative asset demand function that is C-shaped in real interest rates. Section 2 sets the stage for these results by illustrating how such non-monotonic asset demands can arise. In particular, in Section 2.1, we discuss how bequest motives can justify including assets in households' utility payoff and what form this relationship should take. In Section 2.2, we examine households' consumption-savings decisions when households' bequest desires are modelled as suggested in Section 2.1. In Sections 3 and 4, we analyze the general equilibrium implications of having asset demands that are non-monotonic in real interest rates. To help clarify the forces at play, we begin by examining how this economy behaves when prices are fixed and economic activity is fully determined by demand. We then examine implications of allowing prices to adjust as governed by a Phillips curve. We compare equilibrium properties resulting from changing the extent to which monetary policy reacts aggressively to inflation. Throughout, we incorporate the possibility of a lower-bound constraint on interest rates. In Section 5, we discuss the implications of our model with respect to the natural rate of interest and offer some general takeaways regarding the main forces driving our results. Finally, Section 6 offers concluding comments.

2 Setting the Stage: Concave Bequest Motives and Assets in the Utility Function

In the standard representative agent set-up, saving behaviour is quite extreme. For example, if the interest rate is only slightly less than a household's subjective discount factor, it is optimal for the household to choose negative consumption growth forever with a complete depletion of their assets. As mentioned in the introduction, one way to make such features less extreme is to allow assets to directly enter the agent's utility function. Such a modification to preferences is often motivated by invoking bequest motives, but in most cases this link is not made very explicit. Accordingly, in the first part of this section, we will discuss how and when bequest motives may justify the reduced-form approach of introducing assets directly into utility. In particular, we will want to highlight that when bequest motives are used to justify including assets in the utility function, it implies that interest rates should also be included. This observation will be important in generating asset demands that are non-monotonic in interest rates, a feature that is key to the remaining analysis.

2.1 Asset in the utility function: a reduced form for bequest motives?

When thinking about bequests, a natural starting point is to assume that preferences are of the form:

$$\int_0^\infty e^{-(\delta+\rho)t} \left[\log(c_t) + \delta W(U_t)\right] dt,$$

where c_t is consumption, ρ is the discount rate, δ is the death rate, U_t is the discounted utility of offspring, and $W(\cdot)$ is a function that expresses how the utility of offspring enters parents' preferences. $W(\cdot)$ is an increasing function, likely concave. The utility level U_t satisfies the functional relationship $U_t = \int_0^\infty e^{-(\delta+\rho)t}[\log(c_t) + \delta W(U_t)]dt$. When $W(U_t) = U_t$, that is, when parents linearly internalize the utility of offspring, this problem is very tractable. For example, when the budget constraint is $\dot{a}_t = a_t r - c_t$, then $W(U_t) = U_t = \frac{\log(a_t)}{\rho+\delta} + \frac{r-\rho-\delta}{(\rho+\delta)^2} + \frac{\log(\rho+\delta)}{\rho+\delta}$, where a_t is the households' holding of an asset which pays the constant real interest rate r.

So in the case where $W(U) = U$, one could rewrite the preferences of households as follows:

$$\int_0^\infty e^{-(\delta+\rho)t} \left[\log(c_t) + \delta \left(\frac{\log(a_t)}{\rho+\delta} + \frac{r-\rho-\delta}{(\rho+\delta)^2} + \frac{\log(\rho+\delta)}{\rho+\delta}\right)\right] dt.$$

That is, the agent's preferences can be interpreted as if assets — and interest rates — enter their utility function. So this can be seen as offering a justification for including assets in the utility functions. Interestingly, when $W(\cdot)$ is linear, we see certain restrictions appear. Assets and interest rates enter in a separable fashion. Moreover, we observe that the curvature of the utility associated with assets (a_t) inherits a similar curvature to that associated with consumption. In fact, it is well known that when $W(\cdot)$ is linear, the household's problem could alternatively be written as if the household only cares about consumption but with an effective discount factor equal to ρ instead of $\rho + \delta$.

The more interesting case is when $W(\cdot)$ is strictly concave instead of linear. A concave $W(\cdot)$ function corresponds to a situation where the household cares about the utility of its offspring, but cares more about guaranteeing that its offspring does not have very low utility. The difficulty with this case is that it does not generally deliver a simple closed-form solution for U_t. Nonetheless, the above formulation does give us reason to conjecture that the household will still act as if both assets and interests rates enter their utility function.

For example, suppose that the first generation had a concave $W(\cdot)$ function but believed that all future generations would have a linear function. In this case, the household's problem for the first generation would take the form:

$$\int_0^\infty e^{-(\delta+\rho)t}\left[\log(c_t)+\delta W\left(\frac{\log(a_t)}{\rho+\delta}+\frac{r-\rho-\delta}{(\rho+\delta)^2}+\frac{\log(\rho+\delta)}{\rho+\delta}\right)\right]dt.$$

In this case, the first generation's problem would maintain the property that assets and interest rates enter the utility function, but they would no longer enter in a separable fashion. In fact, we would have higher interest rates decreasing the marginal value of assets. In addition, the curvature of the utility with respect to assets would no longer be governed simply by the curvature of the utility of consumption. It would instead also depend on the curvature of the $W(\cdot)$ function. To get a sense of what this may look like, it is helpful to consider a simple parameterization of $W(U)$. Since U may be a negative number, we need $W(\cdot)$ to be well defined, increasing, and concave over the range $(-\infty,\infty)$. Assuming that $W(\cdot)$ is of the CARA form, $W(U)=-\exp^{\gamma U}$ satisfies these conditions. Then the household's preferences in this special case can be written as:

$$\int_0^\infty e^{-(\delta+\rho)t}\left[\log(c_t)+\delta\varphi(-a_t^{-\gamma_1}\exp^{-\gamma r})\right]dt \qquad \gamma_1>0,\gamma>0,\varphi>0. \tag{1}$$

While the above argument does not give a full road map regarding how best to capture concave bequest motives, it gives a motivation for adopting a reduced-form approach that directly allows assets and interest rates to enter the household's payoff function.[14] Moreover, it gives us insight into the properties that the reduced form should take. Accordingly, in this paper, we will rely on this reasoning to capture bequest motives by directly assuming that agents act as if assets and interest rates enter their utility function in the form:

$$\int_0^\infty e^{-(\delta+\rho)t}\left[\log(c_t)+\delta V(a_t,r_t)\right]dt.$$

We assume that $V(a,r)$ satisfies the following properties,[15] which are all consistent with

[14]In a more general formulation, one may also want to allow labour income to enter the function $V(\cdot)$. This could involve the labour income of either the parent or the offspring, or both. We do not pursue this possibility here, but most of the results of the paper hold if we treat bequest as a superior good by having higher labour income of the parent increase the marginal value of assets bequeathed. This is relevant because it can help explain why the rich hold a disproportionate amount of assets.

[15]Note that for the last condition, if $\frac{-rV_{ar}}{V_a}$ is a constant, the condition is satisfied since $\frac{-V_{ar}}{V_a}$ would then be strongly decreasing in r. So this condition is much less demanding than having $\frac{-rV_{ar}}{V_a}$ be a constant.

equation 1.

Assumption 1.

$$(i) \quad V_a \equiv \frac{\partial V(a_t, r)}{\partial a_t} > 0, \quad (ii) \quad V_{aa} \equiv \frac{\partial^2 V(a_t, r)}{\partial^2 a_t} < 0,$$
$$(iii) \quad \lim_{a \to 0} V_a = -\infty, \quad (iv) \quad V_r \equiv \frac{\partial V(a_t, r)}{\partial r} \geq 0,$$
$$(v) \quad V_{ar} \equiv \frac{\partial^2 V(a_t, r)}{\partial a_t \partial r} \leq 0 \quad \textit{and} \quad (vi) \quad \frac{-V_{ar}}{V_a} \textit{ is non-increasing in } r.$$

It needs to be emphasized that in our motivation of the function $V(a_t, r)$, we considered a set-up where the interest rate paid on the asset a_t was constant. If we allowed interest rates to vary over time, bequest motives would likely give rise to a function $V(\cdot)$ that depends on the whole future path of interest rates. In what follows, we will allow interest rates to vary over time but we will maintain the assumption that agents' bequest motives can be captured by a function $V(a_t, r_t)$ that depends on asset holdings a_t and only on the contemporaneous interest rate r_t. This simplification, which does not affect the properties of the steady states, will allow us to illustrate many results using two-dimensional phase diagrams. However, this comes at the cost of a possibly less complete description of transitional dynamics. We believe this tradeoff is desirable.

In the introduction, we cited several papers that adopt a reduced-form approach to bequest motives. However, these papers all assume away the potential effect of interest rates in affecting how assets enter the household's bequest payoff. Accordingly, in this paper we will aim to make clear how properties of asset demands are modified when allowing for $V_{ar} < 0$ and how this effects general equilibrium properties. The case with $V_{ar} = 0$ is discussed in Appendix A.

The condition $V_{ar} < 0$ reflects that the marginal value of wealth decreases with interest rates since wealth and the return on wealth are substitutes when parents have concave preferences over their offspring's utility. To put it differently, the marginal value of leaving an asset a to an offspring is higher when that asset carries a lower interest rate r because the offspring will derive less consumption form an asset that has a low return. This implies that when interest rates are high (low), households have incentives to decrease (increase) their holdings of assets. We will refer to this bequest property as *bequest motives with income flow considerations*.

When possible, we will derive implications under Assumption 1. For some of the more detailed results, we will work with the specific functional form motivated by the above discussion where $V(a,r) = \varphi\left(-a^{-\gamma_1}\exp^{-\gamma r}\right)$, or alternatively written as $V(a,r) = \varphi\left(\frac{a^{1-\sigma}}{1-\sigma}\exp^{-\gamma r}\right)$ when $\sigma > 1$.[16]

2.2 Long-run asset demand when bequest motives include income flow considerations

This subsection presents the baseline decision problem for our representative household with concave bequest motives, which we will later incorporate into general equilibrium settings. The set-up is deterministic, and households derive utility from consumption and wealth. As in the previous subsection, a household dies at a rate δ and discounts future utility at rate ρ. Households maximize the inter-temporal utility function:

$$\int_0^\infty e^{-(\rho+\delta)t}\left[\log(c_t) + \delta V\left(a_t, r_t\right)\right] dt, \tag{2}$$

where c_t is households' per capita consumption, and a_t is households' per capita financial wealth. In addition to consumption, households derive utility from a warm-glow bequest motive that arrives at rate δ and is captured by the function $V\left(a_t, r_t\right)$, where Assumption 1 holds. Furthermore, we will denote the negative of the elasticity of $V_a(\cdot)$ with respect to wealth (a) as $\sigma(a,r) \equiv -\frac{aV_{aa}}{V_a}$.

The budget constraint of the household is given by:

$$\dot{a}_t = y_t^d + r_t a_t - c_t, \tag{3}$$

where y_t^d is the household's disposable labour income and r_t is the interest rate at date t.

Optimization problem. Households choose consumption c_t and financial assets a_t to maximize the inter-temporal utility function (2) subject to the budget constraint (3).

The household's Euler equation is therefore given by (4), with an associated transversality condition:

[16] Note that this functional form is less restrictive, for example, than assuming that $V(a,r) = \varphi\frac{(ar)^{1-\sigma}}{1-\sigma}$. When deriving implications using the functional form $V(a,r) = \varphi\left(-a^{-\gamma_1}\exp^{-\gamma r}\right)$, we would obtain similar results if we assumed that $V(a,r) = \varphi\frac{(ar)^{1-\sigma}}{1-\sigma}$, but this would not allow a simple treatment of negative real interest rates.

$$\frac{\dot{c}_t}{c_t} = r_t - \rho - \delta + \delta c_t V_a\left(a_t, r_t\right). \tag{4}$$

From the Euler equation, we can observe that, in addition to the standard substitution effects, there are income effects of interest rates when bequest motives include income flow considerations ($V_{ar} < 0$). The first component of the Euler equation, $r_t - \rho - \delta$, describes these standard substitution effects whereby lower interest rates favour more consumption today relative to future consumption. The second component, $\delta c_t V_a(a_t, r_t)$, represents the equivalent to an income effect working through bequest motives featuring income flow considerations. Specifically, when interest rates are low, households may want to postpone consumption today to guarantee a minimum level of utility to offspring. If bequest motives do not include income flow considerations ($V_{ar} = 0$) then, these income effects cease to exist, and as a result, only the standard substitution effects would be present. As we will show later, when bequest motives include income flow considerations, accounting for both the substitution and income channels of interest rates can lead to a non-monotonic relationship between interest rates and asset demands.

Assuming that disposable labour income and interest rates are constant over time, we can now examine both the conditions under which asset holdings for households converge to an interior solution and the properties of the resulting long-run asset holdings. We will denote the steady state of households' asset holdings by $A^{ss}(y^d, r)$. From equations (3) and (4), we can see that steady state asset holdings and consumption level are implicitly given by the following two equations:

$$c = y^d + ra, \tag{5}$$

$$r = \rho + \delta - \delta c V_a\left(a, r\right). \tag{6}$$

Lemma 1. *When $r < \rho + \delta$, households' asset holdings will converge to the steady state defined by equations (5) and (6) if and only if $\frac{ar}{c} \leq \frac{-aV_{aa}(a,r)}{V_a(a,r)} \equiv \sigma(a, r)$ when evaluated at this steady state.*

See Appendix A.1 for the proof.

Since we want to use the steady state defined by equations (5) and (6) to represent the long-run asset holdings of households, we need preferences to satisfy Lemma 1. A sufficient condition for $\frac{ar}{c} \leq \frac{-aV_{aa}(a,r)}{V_a(a,r)}$ is that $\frac{-aV_{aa}(a,r)}{V_a(a,r)} > 1$. Accordingly, we will generally assume either that the conditions of Lemma 1 are satisfied or that $\frac{-aV_{aa}(a,r)}{V_a(a,r)} > 1$.

A long-run demand for assets can be obtained by substituting the budget constraint into equation (6). The following expression implicitly defines the long-run properties of the resulting asset demand holding labour income constant, which we will denote by $A^{LR}(r, y)$:[17]

$$\frac{\rho + \delta - r}{\delta V_a(a, r)} - ra = y^d. \tag{7}$$

Equation (7) restricts how the long-run asset position is influenced by interest rate r and household labour income y^d. An implication of the stability condition in Lemma 1 is that the long-run asset holding function inferred from equation (7) is increasing in labour income y^d. If we were not imposing the stability condition, equation (7) would imply that steady state asset demand would be decreasing in labour income.We summarize this formally in Proposition 1 below. In addition, Proposition 1 indicates that if $V_{ar} = 0$, then households' long-run asset demands will be increasing monotonically in real interest rates, while if $V_{ar} < 0$, they will generally be non-monotonic.

Proposition 1. *If Lemma 1 is satisfied and $r < \rho + \delta$, then the long-run asset holdings of households $A^{LR}(y^d, r)$ are positive and increasing in income. Moreover, if $V_{ar} = 0$, they are increasing in interest rates, while if $V_{ar} < 0$, they are generally non-monotonic in interest rates.*

See Appendix A.2 for the proof.

The link between the condition $V_{ar} < 0$ and the possibility of non-monotonicity of asset demand with respect to interest rates, and especially the possibility of a C-shaped relationship, can be seen most clearly by focusing on equation (6). Equation (6) can be thought of as implicitly defining a demand for assets holding consumption constant. From this equation, it is clear that asset holdings will be monotonically increasing in interest rates holding consumption constant if bequest motives do not incorporate income flow considerations ($V_{ar} = 0$). In contrast, if bequest motives include income flow considerations ($V_{ar} < 0$), then the long-run demand for assets relative to consumption can be negatively related to interest rates. As shown in Corollary 1, under Assumption 1, this relative demand function is in fact C-shaped with respect to interest rates.

[17] Under the condition in Lemma 1, this asset demand is both the steady state of the system and the long run of the system.

Corollary 1. *When $V(a,r)$ satisfies Assumption 1 with $V_{ar} < 0$, the long-run asset demand relative to consumption, implicitly defined by (6), is C-shaped in the space (a,r).*

Since we want to highlight the implications of a non-monotonic relationship between asset demands and interest rates, we will continue with the assumption that bequest motives encompass income flow considerations ($V_{ar} < 0$), which we motivated by allowing households to have a concave payoff from offspring's utility. Recall that bequest motives entailing income flow considerations can lead to a non-monotonic relationship between interest rates and asset demands since, in addition to the standard substitution effects whereby higher interest rates favour higher savings, now there is the equivalent to an income effect working through bequest motives. When interest rates are low, households may want to save more to guarantee a minimum level of utility to offspring. This income effect channel of interest rates on savings can also appear in models emphasizing retirement savings or precautionary savings, but this generally involves a more complicated set-up.

3 General Equilibrium Properties when Bequest Motives include Income Flow Considerations: the Fixed Price Case

We now embed the household's problem of Section 2.2 in a general equilibrium setting where we include both monetary and fiscal elements. In our baseline, we populate this economy with only a representative consumer whose choice problem is the one presented in the previous section, with the assumption that the bequest motive captured by $V(\cdot)$ incorporates income flow considerations and satisfies Assumption 1. When useful, we will assume that this restricted bequest motive takes the form $V(a,r) = \varphi\left[\frac{a_t^{1-\sigma}\exp^{-\gamma r}}{1-\sigma}\right]$, where a higher value of φ represents a greater weight of bequest motives in utility. If φ is set to zero, this corresponds to a standard representative household set-up without a bequest motive. So when we focus on economies with low values of φ, which will be the preponderant case, this will correspond to economies that are not too far from more standard specifications.

The analysis in this section and the next will have fixed nominal wages (i.e., $w_t = w$), and accordingly employment l_t will be determined by demand. Output is produced using labour by a set of competitive firms. The production function is given by $y_t = Al_t$, where for simplicity productivity $A > 0$ is also assumed to be constant. Goods prices p_t are perfectly flexible and therefore competition between firms will ensure that the price of the output good is equal $\frac{w_t}{A}$, with real wages equal to A. The households budget constraint is given by:

$$c_t + \dot{a} = Al_t + r_t a_t - T_t,$$

where r_t is the real interest rate and T_t are lump sum taxes.

The government in this economy spends G_t, issues public debt B_t, and imposes lump sum income taxes T_t on households. The government must always satisfy the flow budget constraint:

$$T_t = G_t + r_t B_t - \dot{B}_t. \tag{8}$$

Household net labour market income is given by $y_t^d = Al_t - T_t = y_t - T_t$, where y_t is equal to aggregate labour income.

3.1 Effects of interest rates and debt accumulation on output in the long run

In this subsection we want to look at the effects of expansionary policy in this representative agent economy. Our aim is to explore how, under fixed nominal wages, (1) changes in interest rates affect output in the long run, and (2) an expansionary fiscal policy affects output in the long run when we include the need to balance the budget. To explore these two issues, we assume that monetary and fiscal policy are governed by simple targeting rules. Monetary policy targets an interest rate $\bar{r} < \rho + \delta$ according to:

$$\dot{r}_t = \lambda_1(\bar{r} - r_t) \qquad \lambda_1 > 0, \tag{9}$$

while fiscal policy targets a debt level $\bar{B} > 0$ according to:

$$\dot{B} = \lambda_2(\bar{B} - B_t) \qquad \lambda_2 > 0. \tag{10}$$

We now examine how this economy reacts to changes in either $\bar{r}$ or $\bar{B}$, starting from a steady state where $\bar{r}$ and $\bar{B}$ are equal to r_0 and B_0. For each of these interventions, either T_t or G_t will need to adjust over time to satisfy the government budget constraint. We will assume in this section that taxes T_t do the adjustment, but results are very similar (see appendix) if government spending G_t does the adjustment. Furthermore, in all the remaining sections of the paper, we will assume that $r_t < \rho + \delta$ as to focus on environments where the long-run asset demand by households is finite. Recall that in this set-up, economic activity $y_t = c_t + G_t$ is entirely demand determined, as wages are held fixed.

An equilibrium for this economy is defined as:

Definition 1. *Sequences $\{c_t, a_t, B_t, G_t, T_t, y_t, r_t\}$ such that households choose $\{c_t, a_t\}$ to maximize the intertemporal utility (2) subject to the budget constraint (3); $\{B_t, G_t, T_t\}$ are determined by equation (10) and the government budget constraint (8); interest rates are governed by (9); and asset markets clear at all times, that is, $B_t = a_t$. (The goods market clearing condition is satisfied by Walras's law.)*

Lemma 2 addresses the stability property of the competitive equilibrium.

Lemma 2. *The economy has a unique competitive steady state and this steady state is saddle path stable if Lemma 1 is satisfied.*

See Appendix A.3 for the proof.

Propositions 2 and 3 indicate how the steady state value of y changes in response to changes in either $\bar{r}$ or $\bar{B}$. The contents of these propositions are implied by the steady state condition for y, which is implicitly given by:

$$y - G = \frac{\rho + \delta - \bar{r}}{\delta V_a(\bar{B}, \bar{r})}.$$

Proposition 2. *When Lemma 1 is satisfied, there exists a cutoff real interest rate (smaller than $\rho + \delta$), such that if $\bar{r}$ is below this cutoff, a further decrease in $\bar{r}$ reduces steady state output. In contrast, if $\bar{r}$ is above the cutoff (but still smaller than $\bar{r} < \rho + \delta$), then a decrease in $\bar{r}$ increases steady state output.*

See Appendix A.4 for the proof.

We denote this cutoff interest rate by $\bar{r}^{cutoff}$. When real interest rates are above $\bar{r}^{cutoff}$ (e.g., close to $\rho+\delta$), Proposition 2 indicates that a decrease in interest rates is expansionary. However, the proposition indicates that the long-run effect of interest rates on output is non-monotonic. In particular, when interest rates are low (possibly negative), a decrease in $\bar{r}$ will depress demand in the long run. This change in sign reflects how the income flow considerations associated with bequest motives ($V_{ar} < 0$) can affect the long-run asset demand of households. When bequest motives feature income flow considerations, the effect of interest rates on the long-run asset demand of households becomes negative for sufficiently low interest rates. Hence, households would like to save more and accumulate assets in response to a fall in interest rates when interest rates are initially low. This reflects an income effect of interest rates on households' savings decisions, whereby to guarantee sufficient utility value of bequest to offspring, households want to save more when interest rates are low and are decreased.[18]

We now turn to examining the effect of fiscal policy in the form of increased public debt accumulation induced by a temporary decrease in taxes followed by a gradual increase in taxes to balance the budget.

Proposition 3. *When Lemma 1 is satisfied, an increase in $\bar{B}$ leads to an increase in the steady state output, even though this is associated with an increase in T in the long run when $\bar{r} > 0$.*

Figure 2 illustrates the dynamics of consumption following an increase in $\bar{B}$. Here we see that c increases both initially and in the long run. Since $y = c + G$, the effects on c shown in the figure are identical to that on y. If G were doing the adjustment, we would get a similar result even if it is not clear from the figure.

It may at first pass appear unusual that increased debt leads to increased output in the long run, even though in the long run the increased debt is associated with higher taxes in order to finance the new debt. The content of Proposition 3 can easily be seen by considering the long-run asset market equilibrium condition:

$$A^{LR}(y^d, r_0) = A^{LR}(y - G - r_0\bar{B}, r_0) = \bar{B}.$$

From Proposition 1, we know that $\frac{\partial A^{LR}}{\partial y^d} > 0$; then it follows directly that for the asset market to clear in the long run with higher debt, y must be higher despite higher taxes needed to balance the budget.[19] Note that we would get the same qualitative result if we

[18]This is easier to see when we assume that $V(a_t, r_t) = \varphi \frac{a_t^{1-\sigma}}{1-\sigma} \exp^{-\gamma r_t}$ with $\sigma > 1$ and $\gamma > 0$, in which case $\frac{\partial A^{LR}}{\partial \bar{r}} = \frac{\gamma \bar{B}^{\sigma} \exp^{\gamma \bar{r}}}{\delta \varphi}(\delta + \rho - \bar{r} - \frac{1}{\gamma}) - a$. If φ is not too large, then $\frac{\partial A^{LR}}{\partial \bar{r}} > 0$ for $\bar{r}$ close to $\rho + \delta$ but will be negative for somewhat lower rates.

[19]In the case of a change in $\bar{r}$, the long-run asset market equilibrium condition can be written as $A^{AL}(y - G - \bar{r}B_0, \bar{r}) = B_0$. From this equation it is difficult to see if a change in $\bar{r}$ is either expansionary or contractionary in the long run. Proposition 2 indicates that it depends on the initial level of $\bar{r}$. An easier case to understand from the asset market clearing condition is when taxes are held fixed and government spending adjusts to balance the budget. In this case, the asset market equilibrium condition is $A^{AL}(y - T, \bar{r}) = B_0$, which implies that $\frac{\partial y}{\partial \bar{r}} > 0$ if $\frac{\partial A^{LR}}{\partial y^d} > 0$ and $\frac{\partial A^{LR}}{\partial r} < 0$.

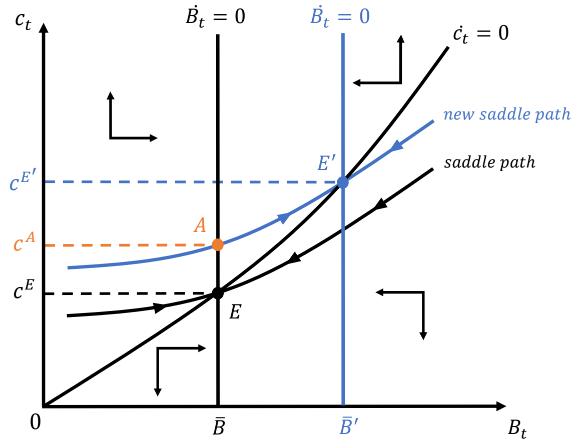

Figure 2: Phase diagram for an unanticipated permanent increase in government debt $\bar{B}$ to $\bar{B}' > \bar{B}$

assumed that taxes were fixed and government spending would do the adjustment. In this case, the long-run asset market equilibrium condition would be given by:

$$A^{LR}(y^d, r_0) = A^{LR}(y - T, r_0) = \bar{B}.$$

In this alternative case, government spending would be lower in the long-run equilibrium, but we would still need y to increase to clear the asset market with higher debt. Hence, in this environment (which does not depend on $V_{ar} < 0$), we have that an increase in government debt is an expansionary force even in the long run when we include budget balance considerations.

3.1.1 Allowing for interest elastic debt

In the above analysis we assumed that the debt supplied on the market does not respond to interest rates; that is, debt supply is interest rate inelastic. Here we want to briefly discuss potential implications of allowing for an elastic debt supply. The key relationship in our analysis is the steady state Euler equation condition $c = \frac{\rho+\delta-r}{\delta V_a(\bar{B},r)}$. When $V(\cdot)$ satisfies Assumption 1, and debt is interest rate inelastic, Proposition 2 indicates that this condition implies that c first increases in r for low values of r; and it reaches a maximum and then becomes decreasing in r. It is this non-monotonicity property that drives many of our results. However, if we were to assume that debt was interest rate elastic as given by a function $B(r)$, with $B'(r) < 0$, then this non-monotonicity may not arise. The new equilibrium condition would become $c = \frac{\rho+\delta-r}{\delta V_a(B(r),r)}$, and Assumption 1 would no longer be sufficient to imply a non-monotonic relationship between consumption and interest rates. In effect, if the elasticity of debt with respect to interest rates was sufficiently negative, then consumption could become monotonically decreasing in interest rates even when $V_{ar} < 0$. The intuition for this is straightforward. The possibility of a positive equilibrium relationship between consumption and interest rates is due to households wanting to hold more assets when interest rates are low. However, if debt supply has a greater response than asset demand to low interest rates, then even at low interest rates we would not have demand for assets outpacing supply in response to a fall in interest rates.[20] However, if the supply of debt were elastic, but not too elastic, then the non-monotonicity would still hold.

In the remaining sections of the paper we will maintain the assumption that debt is supplied inelastically. However, it is worth noting that all the results would continue to hold if we allowed debt to be supplied elastically as long as the elasticity was not strong enough to overturn the non-monotonicity between consumption and interest implied by the condition $c = \frac{\rho+\delta-r}{\delta V_a(B(r),r)}$.

We now turn to examining the behaviour of the economy when monetary policy reacts to the economic environment instead of following a targeting rule as captured by equation (10). In particular, in the next subsection, we maintain the fixed-price set-up and examine the implications of a monetary policy that aims to close the output gap. In the subsequent section, we introduce a Phillips curve and study the implications of monetary policy that follows a Taylor rule with a coefficient on inflation that is greater than 1 (i.e., satisfies the

[20]One extension of the model which is similar to allowing for interest elastic debt is to introduce a Lucas tree (see Appendix B). For example, suppose the Lucas tree produces a flow of fruit f. Then the effective aggregate supply of assets would become $\bar{B} + \frac{f}{r_t}$, which is equivalent to a form of interest elastic aggregate debt. While this can be easily introduced in our set-up, it implies that the effective asset supply goes to infinity as the real rate of interest approaches zero. Therefore such a framework would not allow for the possibility of negative real rates of interest. This feature can be overcome by introducing instead stochastic Lucas trees that die at rate ω, with the flow of new trees needed to keep the total stock constant being distributed in lump sum fashion to households. In this case, the equilibrium steady state Euler equation would take the form $c = \frac{\rho+\delta-r}{\delta V_a(\bar{B}+\frac{f}{r+\omega},r)}$. The shape of this $\dot{c}_t = 0$ curve would no longer have a simple hump shape. Instead it would be S-shaped under Assumption 1 as long as f is not too large; that is, the $\dot{c}_t = 0$ curve would first exhibit a negative slope when r is close to $-\delta$, then become positively sloped and finally return to being negatively sloped as r approached $\rho + \delta$. The main results of the paper can be shown to be robust to allowing for such Lucas trees, albeit with the need for some extra qualifications and the possibility of an extra steady state. If f was sufficiently large, then the $\dot{c}_t = 0$ curve could become monotonically decreasing, in which case the main results of the paper would no longer hold as they require non-monotonicity.

Taylor principle) when it is not constrained by an ELB constraint on the policy rate.

3.2 Counter-cyclical monetary policy and the emergence of secular stagnation

In the previous section we examined the effect of changes in $\bar{r}$ under the assumption that monetary policy followed the target rule given by equation (10). This allowed us to ask how a change in the policy interest rate could affect the economy in the long run when prices are fixed. In this section we want to explore a different question related to monetary policy. Instead of looking at whether exogenous interest rate changes could depress demand in the long run, we want to examine whether monetary policy may play a role in amplifying the effects of exogenous shocks that may depress demand. In order to keep the presentation tractable, in this section we will assume that $V(a_t, r_t)$ takes on the functional form motivated in Section 2, that is, $V(a_t, r_t) = \varphi \frac{a_t^{1-\sigma}}{1-\sigma} \exp^{-\gamma r_t}$ with $\sigma > 1$.

The type of monetary policy we want to consider is one where monetary authorities aim to keep y close to the full employment level, denoted by $\bar{y}$, by adjusting r according to:

$$\dot{r}_t = \theta(y_t - \bar{y}), \qquad \theta > 0.$$

We also want to account for the possibility of a potential lower bound on r. We will denote this lower bound by r^{ELB}, so that if $r_t = r^{ELB}$ and $y_t < \bar{y}$, then $\dot{r}_t$ is constrained to be zero.

The dynamics for this system can be represented by the following pair of dynamic equations:[21]

$$\frac{\dot{c}_t}{c_t} = r_t - \rho - \delta + \delta \varphi c_t \bar{B}^{-\sigma} \exp^{-\gamma r_t}$$

$$\dot{r}_t = \theta(c_t + G - \bar{y}) \quad \text{if} \quad r_t > r^{ELB} or \quad c_t + G > \bar{y}, \quad \text{with} \quad \dot{r}_t = 0 \quad \text{otherwise}, \tag{11}$$

where movements in c_t translate into one-to-one movements in y_t (since G is fixed).[22]

Proposition 4. *When $V(a_t, r_t) = \varphi \frac{a_t^{1-\sigma}}{1-\sigma} \exp^{-\gamma r_t}$ with $\sigma > 1$, monetary policy is given by equation (11), and r^{ELB} is not too constraining, there always exists a stable steady state*

[21] Asset market clearing implies that $a_t = \bar{B}$.

[22] In the background, taxes T_t are adjusting over time according to $T_t = G + r_t \bar{B}$.

with the real interest rate at the ELB and $y < \bar{y}$. Moreover, if φ is large, this is the unique stable steady state. If φ is sufficiently small, then there exists a second stable steady state with $r > r^{ELB}$ and $y = \bar{y}$.

See Appendix A.6 for the proof.

The properties of this system depend on φ, that is, they depend on the importance of bequests in utility. As summarized in Proposition 4, if r^{ELB} is not too constraining,[23] then this system can take on one of two configurations. The two possible configurations are represented in Figures 3 and 4. If φ is sufficiently small — so bequest motives are not too strong — then this economy will have three steady states: two (saddle path) stable steady states and one unstable steady state. This is illustrated in Figure 3, where we also include the equilibrium trajectory. The steady state to the right in the figure, denoted by E_1, corresponds to a case where the economy is at full capacity, $y_t = \bar{y}$, and the interest rate is above r^{ELB}. This equilibrium is stable, and we will refer to this as the full-employment steady state and we will denote the interest rate at this steady state as $\hat{r}$. There is another steady state with $y_t = \bar{y}$ and $r > r^{ELB}$; this is denoted as E_2, but this steady state is unstable. Finally, there is a third steady state, which is denoted E_3. This steady state corresponds to what we will call the depressed-economy steady state since $y_t < \bar{y}$ and $r = r^{ELB}$. It is a situation in which monetary authorities would like to reduce r_t further but are constrained by the ELB.

In contrast to the case where φ is small, when φ is sufficiently big, there is only one steady state as illustrated in Figure 4. In this case, the only steady state is a depressed-economy steady state with $y_t < \bar{y}$ and $r = r^{ELB}$. The intuition for why there is only a depressed-economy steady state when φ is sufficiently high is that the resulting demand for assets is so great relative to supply that activity needs to be depressed to clear the asset market.

3.2.1 Counter-cyclical monetary policy, hysteresis, and depressed demand under fixed prices

In order to illustrate how counter-cyclical monetary policy could contribute to depressed demand, let us start from a situation where φ is initially low and there are three steady states. The economy is initially at the stable full-employment steady state with $r = \hat{r}$. Now consider how the economy will adjust to a change in φ. We consider two cases: first a case

[23] If r^{ELB} is sufficiently high, then a third equilibrium configuration may emerge. This case is of limited interest, so we omit it here to simplify the analysis.

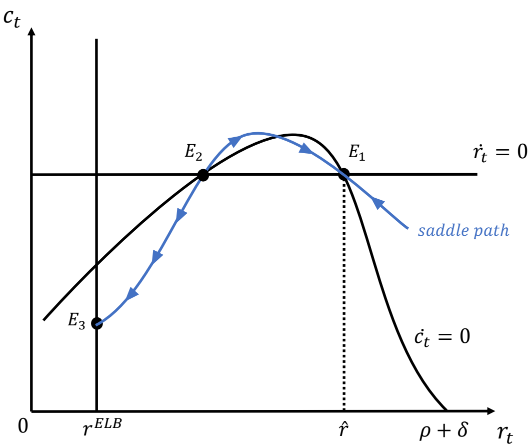

Figure 3: Equilibrium trajectories when φ is small and bequest motives include income flow considerations

where we have a permanent increase in φ, and then a case where the increase is temporary.

A large permanent rise in φ corresponds to switching from the situation depicted in Figure 3 to the one depicted in Figure 4. The induced dynamics are illustrated in Figure 5. In response to the shock, consumption will jump down from its full-employment compatible level (point a) onto the saddle path (point b) that converges to the new depressed-economy steady state (point d). Monetary policy responds to the fall in demand by reducing interest rates. Initially, this will lead to increased consumption as interest rates decline (between points b and c). However, over time this increase in consumption will reverse itself before it manages to reach back to the pre-shock level. The reduction in consumption arises despite the fall in interest rates because households would want to start saving more when interest rates become very low. Once consumption reverses from growth to contraction, it gradually declines (from point c to d). The process ends when r_t reaches the ELB, and consumption is below the level needed to maintain full employment. This is the new steady state since r cannot decrease further. Instead of reacting to this shock by decreasing interest rates according to the rule $\dot{r}_t = \theta(y_t - \bar{y})$, monetary authorities could have kept $r_t = \hat{r}$. In this case consumption would have jumped from point a to point e on the figure and then remained there. The initial contraction would be larger without the accommodating monetary policy (comparing point b to point e), but over the longer run, monetary accommodation could

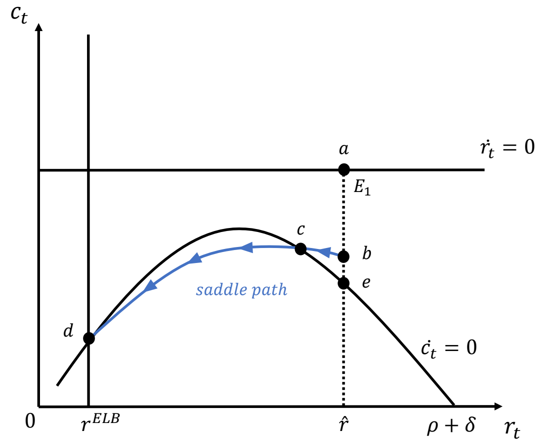

Figure 4: Equilibrium trajectory following a large permanent increase in φ

lead to a greater fall in output (comparing points d and e). In this sense, in reacting to the shock by decreasing interest rates, monetary policy in this economy could be interpreted as amplifying over the long run the negative demand effect associated with an increased desire to save.

The role of counter-cyclical monetary policy in contributing to depressed demand is more salient when considering a temporary increase in φ. In response to a temporary increase in φ, if monetary authorities did not change interest rates, consumption would initially fall and then gradually return to its full-employment compatible level. In contrast, if the shock is sufficiently large, and monetary policy acts aggressively (θ sufficiently large), the dynamics associated with a temporary shock will follow a path similar to that illustrated in Figure 5. There is an initial decline in consumption, followed by a rebound, and then a gradual decline to the depressed steady state. When monetary policy reacts strongly enough to excess capacity, the temporary shock causes the economy to switch from the full-employment steady state with $r = \hat{r}$ to a depressed-economy steady state with $r = r^{ELB}$. In this sense, monetary policy can be seen as contributing to a depressed steady state outcome, even though the driving force is a temporary change in φ. If monetary policy were to react less aggressively to the initial shock, the dynamics could instead look as in Figure 6, where the decline in consumption is only temporary and monetary policy actually manages to create

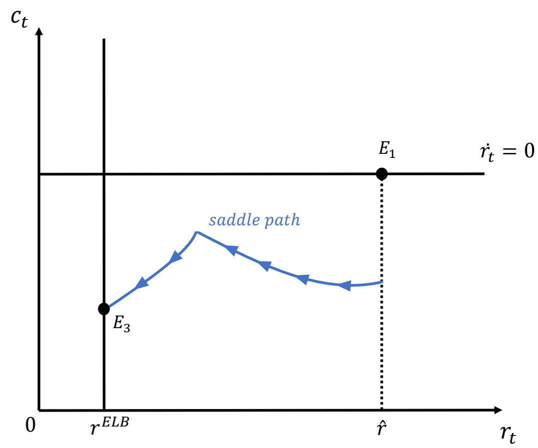

Figure 5: Equilibrium trajectory following a temporary increase in φ with aggressive monetary policy

a short-term boom.[24]

4 Allowing for Price Adjustments and Inflation

If the economy ends up in a depressed steady state with $r = r^{ELB}$, as was shown to potentially arise in the previous section under a fixed-wage assumption, it becomes unreasonable to maintain that wages, and therefore prices, would not eventually adjust. Accordingly, in this section we examine how the economy adjusts when agents have a bequest motive that features income flow considerations ($V_{ar} < 0$) and inflation dynamics are governed by a Phillips curve of the form:[25]

[24] In the case where the economy ends up at a depressed-economy steady state, because of either a temporary or a permanent increase in saving behaviour, it is worth noting that increased fiscal spending can bring the economy back toward full employment. In fact, if fiscal spending is sufficiently large, the economy can be brought back to full employment while r remains at r^{ELB}. In this situation, the economy is no longer depressed as fiscal deficits have managed to counter the effect of aggressive monetary policy.

[25] In most New Keynesian models, the Phillips curve is of the form $\dot{\pi}_t = \kappa(y_t - \bar{y}) + \kappa_2\pi_t$, where the presence of $\kappa_2 \neq 0$ gives rise to a non-vertical long-run Phillips curve. In the following, we choose to adopt a vertical Phillips curve formulation to ensure that our results do not depend on a non-neutrality of money coming from the Phillips curve. However, all our results can be easily extended to the case with a non-vertical long-run Phillips curve specification.

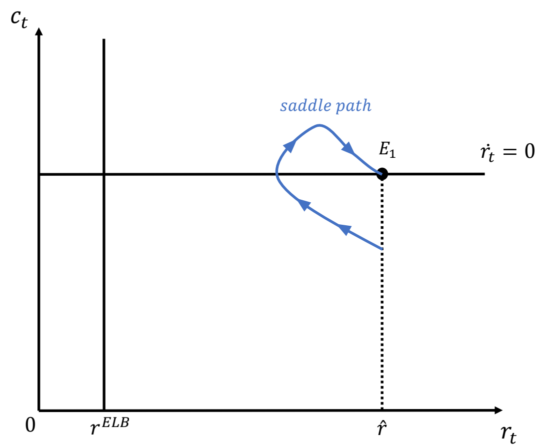

Figure 6: Equilibrium trajectory following a temporary increase in φ with less aggressive monetary policy

$$\dot{\pi}_t = \kappa(y_t - \bar{y}), \tag{12}$$

where π_t is the rate of inflation and $\bar{y} = A\bar{l}$ is the natural rate of output. This Phillips curve can be justified by assuming a downward nominal wage rigidity with nominal wages adjusting according to:[26]

$$\frac{\dot{w}_t}{w_t} = \pi_t^e + \kappa'(l_t - \bar{l}), \qquad \kappa' > 0,$$

where π_t^e is the expected inflation rate and $(l_t - \bar{l})$ represents the deviation of employment from full employment $\bar{l}$. If we further assume that expected inflation:

$$\pi_t^e = \pi_t + \Omega\dot{\pi},$$

then we get the Phillips curve specified by equation (12), where $\kappa = \frac{\kappa'}{-A\Omega}$. The parameter $\Omega > 0$ corresponds to extrapolative expectations, while $\Omega < 0$ corresponds to adaptive expectations.

[26]Note that in our model, wage inflation is equal to price inflation π_t.

With this formulation, when $\kappa > 0$, we have a backward-looking Phillips curve and π_t needs to be treated as a state variable. Alternatively, if $\kappa < 0$, we have a forward-looking Phillips curve and π_t needs to be treated as a jump variable. We will focus on the case with $\kappa > 0$ in the main text and leave to Appendix C the case with $\kappa < 0$. The characterization of stable steady states is similar in both cases. However, the narrative around inflation is potentially more compelling in the $\kappa > 0$ case as inflation is sluggish and without jumps.

Since we now allow for variable inflation, we now need to distinguish between real and nominal rates of interest. We will denote the nominal rate by i_t with the real rate given by $r_t = i_t - \pi_t$. We will begin by examining the properties of the system when i_t is set at the ELB so as to link up with the end of the previous section and clarify some important properties that can arise at the ELB. In the following subsection we adopt a Taylor rule specification for i_t with an ELB constraint in order to derive more general results. Throughout this section we will assume that $V(a_t, i_t - \pi_t)$ takes the functional form motivated in Section 2, that is, $\varphi \frac{a_t^{1-\sigma}}{1-\sigma} \exp^{-\gamma(i_t - \pi_t)}$, with $\sigma > 1$.

4.1 The dynamics of inflation and activity when i is at the ELB

In the presence of variable inflation, the household's Euler equation is given by:

$$\begin{aligned}\frac{\dot{c}_t}{c_t} &= (i^{ELB} - \pi_t - \rho - \delta) + \delta c_t V_a(\bar{B}, i^{ELB} - \pi) \\ &= (i^{ELB} - \pi_t - \rho - \delta) + \delta \varphi c_t \bar{B}^{-\sigma} e^{-\gamma(i^{ELB} - \pi_t)}. \end{aligned} \tag{13}$$

This Euler equation and the Phillips curve equation will govern the dynamics of c_t and π_t. For ease of presentation, we continue to assume that it is lump sum taxes T_t that adjust to ensure that the government budget constraint is satisfied when i_t and π_t change, while real government expenditures and debt stay constant.[27] The Phillips curve can therefore be rewritten as:

$$\dot{\pi}_t = \kappa(c_t + G - \bar{y}). \tag{14}$$

Equations (13) and (14) govern the dynamics of the economy when the interest rate is

[27] In terms of government finances, we are assuming that real government debt (B_t) is held constant so that the government's budget constraint becomes $G - T_t + (i_t - \pi_t)\bar{B} = 0$ (where G and T_t are real levels of government spending and taxes).

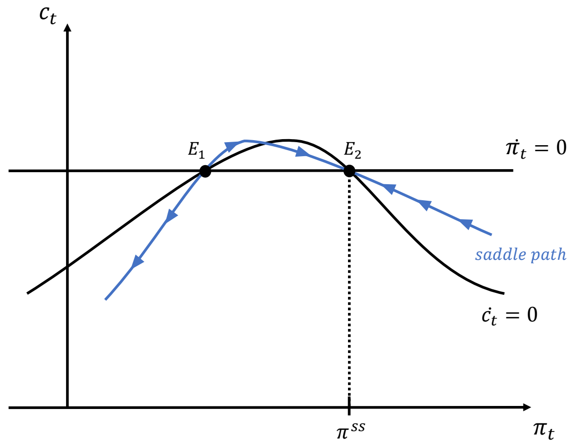

Figure 7: Equilibrium trajectories when i is unconditionally fixed at the ELB

set at the ELB. In this situation, there are two possible equilibrium configurations depending on the strength of bequest motives as captured by φ. If φ is sufficiently large relative to $\bar{B}$, there is too much asset demand (as opposed to goods demand) and the economy collapses with a deflation spiral. In this case, the economy would need a greater supply of assets to avoid the collapse. The other case, which will be our focus, is when φ is not too large. When φ is not too large, there exists a steady state with $\dot{\pi}_t = 0$ and $\dot{c}_t = 0$. The corresponding equilibrium trajectories are depicted in Figure 7, with features summarized in Proposition 5. With nominal interest rates at the ELB, two steady states arise, where only one is stable. The stable steady state is the one with the higher level of inflation (which could nonetheless be negative), and marked as E_2 in the figure. The interesting aspect to note is that this stable inflation steady state arises even if nominal interest rates are unresponsive to inflation, which implies that the Taylor principle does not hold. The reason that inflation can be stable with fixed nominal interest rates is related to Proposition 2. Around this steady state, activity decreases with lower real rates. So when the economy is operating below capacity and inflation is falling, a further decrease in inflation — which corresponds to a rise in real interest rates — is not contractionary. Instead, it favours consumption by decreasing the need for more bequests, and this makes the steady state stable. When deriving the Taylor principle, this possibility is generally not considered.

Proposition 5. *When $V(a_t, r_t) = \varphi \frac{a_t^{1-\sigma}}{1-\sigma} \exp^{-\gamma r_t}$ with $\sigma > 1$ and $i = i^{ELB}$, if φ is not too large, there is a unique stable steady state. At this steady state $\dot{\pi}_t = 0$ even if the Taylor principle is not satisfied. If φ is sufficiently large, the economy will collapse with c converging to zero and $\dot{\pi}_t < 0$.*

See Appendix A.7 for the proof.

There are two interesting comparisons to make between the analysis in this section and that of Section 3.2. In this section we see that, even when the interest rate is at the ELB, the process of price adjustment will favour a return to full employment. The economy will not remain in a depressed-demand state at the ELB as was possible in Section 3.2. However, a return to full employment may be long if prices adjust slowly. The second element to note is that the real interest rate ($i^{ELB} - \pi^{ss}$) that arises at the stable equilibrium is actually the same real interest rate that was shown to be unstable in Section 3.2 when monetary policy was conducted to close the output gap and prices were fixed. This is why we denoted this steady state by E_2 to link it with our previous notation. The unstable steady state, denoted by E_1 on Figure 7, involves the same high real interest rate as the stable steady state in Section 3.2. What this illustrates is that with $V_{ar} < 0$, there are likely to be two full-employment steady states, but which one is stable depends crucially on the nature of price adjustment and monetary policy. As we will show in the following section, it turns out that both a high-real-rate and a low-real-rate steady state can be stable when inflation dynamics are driven by a Phillips curve and we allow for a rather standard Taylor rule specification for monetary policy.

4.2 Aggressive Taylor rule and the emergence of a low-real-rate, low-inflation trap

Instead of setting the nominal interest rate unconditionally at the ELB, let us now consider the more general case where i_t is determined by a simple Taylor rule of the form:

$$i_t = \max\left\{0, i^T + \phi(\pi_t - \pi^T)\right\} \qquad \phi > 1, i^T = \rho + \delta + \pi^T. \tag{15}$$

The interest rate rule given by equation (15) implies that interest rates are set by feedback on inflation, which satisfies the Taylor principle when the ELB is not constraining (we set $i^{ELB} = 0$ for simplicity). In this rule, i^T represents a target nominal interest rate and π^T represents a target level for inflation. The parameter ϕ captures the aggressiveness

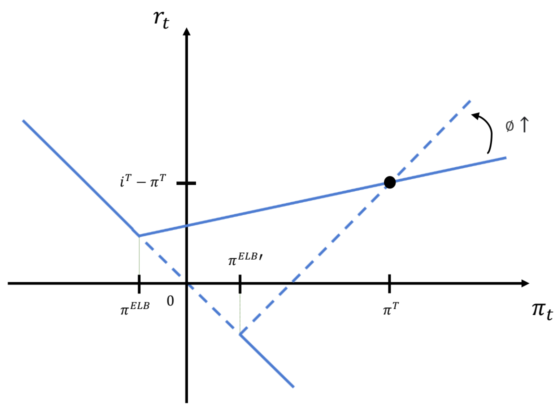

Figure 8: Link between real interest rates and inflation under the Taylor rule

of monetary policy toward trying to keep inflation close to π^T. To give more structure, we will assume that $i^T = \rho + \delta + \pi^T$; that is, the central bank targets a real rate equal to $\rho + \delta$. Note that a higher value of ϕ implies that the ELB constraint will become binding at higher levels of inflation. In particular, the ELB constraint will become binding at the inflation level $\pi^{ELB} \equiv \frac{(\phi-1)\pi^T-(\rho+\delta)}{\phi}$, which is increasing in ϕ. Moreover, when $\phi > 1$, the real interest rate ($r_t = i_t - \pi_t$) is increasing in inflation when $\pi > \pi^{ELB}$ (the ELB constraint is not binding). However, when the ELB is binding ($\pi < \pi^{ELB}$), the real rate decreases with inflation. This is illustrated in Figure 8. It can also be seen from the figure that as ϕ rises, the range of inflation rates for which the ELB constraint binds increases, but the positive link between inflation and the real rate when the ELB constraint is not binding becomes stronger.

The household's Euler equation is again given by:

$$\frac{\dot{c}_t}{c_t} = (i_t - \pi_t - \rho - \delta) + \delta c_t V_a(\bar{B}, i_t - \pi_t) \tag{16}$$

and the Phillips curve continues to be given by:

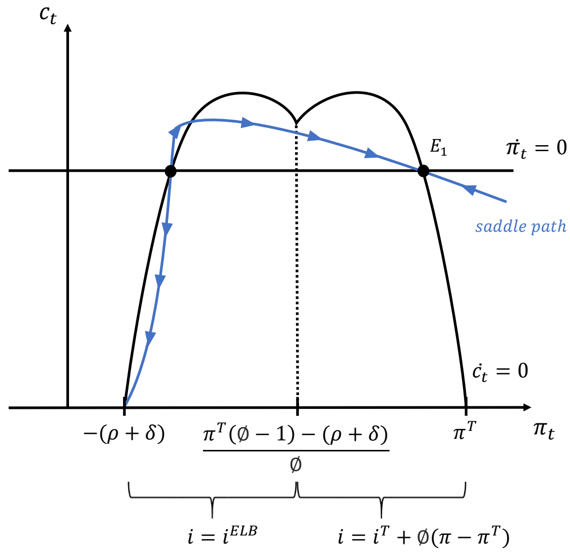

Figure 9: Equilibrium trajectories when monetary policy follows a not too aggressive Taylor rule: one stable steady state

$$\dot{\pi}_t = \kappa(c_t + G - \bar{y}) \qquad \kappa > 0. \tag{17}$$

Equations (15), (16) and (17) govern the dynamics of i_t, c_t, π_t under our Taylor rule specification for monetary policy. This dynamic system can have three configurations. As in the previous section, if φ is very high relative to $\bar{B}$, there is no steady state with stable inflation. There are more desired savings relative to assets in the system and the economy will collapse with spiralling deflation. Although this configuration could be of interest, we will not focus on it here. Instead we will focus on the case where φ is not too large relative to $\bar{B}$, which ensures that a stable inflation steady state always exists. In this case, the economy can have either one stable steady state or two stable steady states. As we will show, whether there are one or two stable steady states depends on the strength of monetary policy (as governed by ϕ). The case with one stable steady state is represented in Figure 9, while the case with two stable steady states is represented in Figure 10. The equilibrium dynamic trajectories are also illustrated in these figures.

In Figure 9, E_1 is the only stable steady state. E_1 is a high-real-interest-rate, high-inflation steady state, with inflation close to target. There is also a low-inflation steady

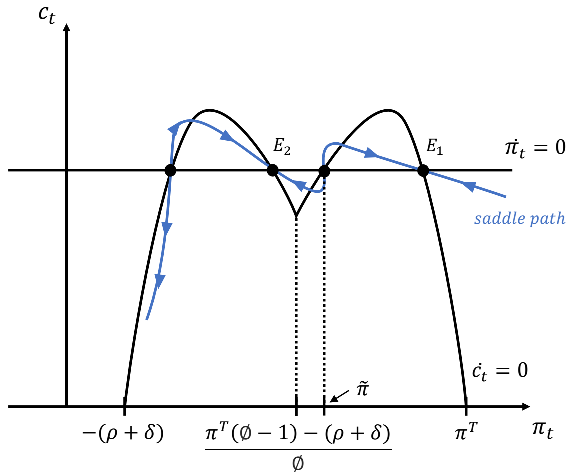

Figure 10: Equilibrium trajectories when monetary policy follows a sufficiently aggressive Taylor rule: two stable steady states

state in Figure 9, but it is not stable. The nominal interest rate at the unstable steady state in Figure 9 is in the ELB region since the level of inflation arising from that equilibrium point is less than $\pi^{ELB} \equiv \frac{\pi^T(\phi-1)-(\rho+\delta)}{\phi}$. This type of configuration, where there is an unstable steady state at the ELB and a stable steady state with $i > i^{ELB}$, echoes what arises in a standard infinitely lived representative agent environment without our bequest motives (see Benhabib, Schmitt-Grohé, and Uribe (2001)).[28] For comparison, note that the stable steady state that arose at the ELB in Figure 7 is not present in Figure 9, since at the level of inflation needed for this steady state to arise, the Taylor rule specifies an $i > i^{ELB}$. In contrast, in Figure 10 we now have two stable steady states. The high-real-interest, high-inflation stable steady state, denoted E_1, remains, but now we also have one low-real-rate, low-inflation stable steady state denoted E_2. The E_2 steady state is in the ELB region, while the E_1 steady state remains in the region where $i > i^{ELB}$ and where the Taylor principle is operative. Proposition 6 expresses this possibility. In this setting, given the two stable steady states, the system will exhibit *hysteresis*.[29] If inflation starts above the level $\tilde{\pi}$ denoted on Figure 10, the system will converge to E_1, while if it starts below, it

[28]Recall that we are assuming a backward-looking Phillips curve in the main body of the text. When assuming a forward-looking Phillips curve, this equilibrium would exhibit indeterminacy.

[29]In the case where the parameter κ in the Phillips curve is negative, the same two equilibria are determinate stable, and the system would jump to one of them instead of exhibiting hysteresis.

will tend to converge to E_2. In this set-up we can consider the effects of shocks, especially φ shocks, as we did in Section 3.2. For example, if the economy were to start at E_1, and there was a large temporary rise in φ, the steady state equilibrium E_1 could temporarily disappear — the reason being that there would then be too much demand for assets relative to supply, which depresses demand. As a result, there would be a contractionary period with deflation. Once the shock reverses itself, the level of inflation would be starting from a lower level. If this new inflation level was below $\tilde{\pi}$, the economy would converge to the long-run equilibrium at E_2 even if it was at equilibrium point E_1 before the temporary shock to φ.

Proposition 6. *When $V(a_t, r_t) = \varphi \frac{a_t^{1-\sigma}}{1-\sigma} \exp^{-\gamma r_t}$ with $\sigma > 1$ and monetary policy is given by equation (15), the equilibrium can exhibit two stable steady states with $\dot{\pi}_t = 0$: one high-inflation, high-real-rate steady state with $i > i^{ELB}$, and one low-inflation, low-real-rate steady state with $i = i^{ELB}$.*

See Appendix A.8 for the proof.

In this setting, we can highlight the potential role that aggressive monetary policy — as captured by high values of ϕ — has in making the low-inflation equilibrium outcome in Figure 10 more likely, that is, making it more likely that the economy converges to an ELB outcome with a low (stable) inflation rate and a low real rate of interest.[30] Recall that we are always assuming that $\phi > 1$, so the Taylor principle is active when not constrained by the ELB. If monetary policy is not too aggressive in the sense of ϕ not being much greater than 1, then the equilibrium configuration will take the form we represented in Figure 9.[31] So in this case with monetary policy not too aggressive (but still satisfying the Taylor principle when above the ELB), the economy can only converge to the E_1 equilibrium. This has the desired outcome of supporting inflation close to target. However, as ϕ is increased, this will increase the range of inflation that leads monetary authorities to set i at the ELB. An increase in ϕ can therefore be seen as changing the equilibrium configuration from that depicted in Figure 9 to that depicted in Figure 10. In fact, as ϕ gets very big, the equilibrium configuration will move toward that depicted in Figure 11. As can be seen in Figure 11,

[30] We also examined the effect on equilibrium outcomes of changing the inflation target π^T. Details are available upon request. Among other results, we find that increasing π^T favours the status quo; that is, we find that the basin of attraction of neither the stable ELB equilibrium nor the non-ELB equilibrium decreases when π^T increases. Accordingly, if an economy were caught in a low-inflation, low-real-rate trap, increasing π^T would not help the economy exit this trap.

[31] For this precise equilibrium configuration, we are assuming that $\phi > \frac{\gamma(\rho+\delta+\pi^T)}{\gamma(\rho+\delta+\pi^T)-1}$ and $\gamma > \frac{1}{\rho+\delta+\pi^T}$.

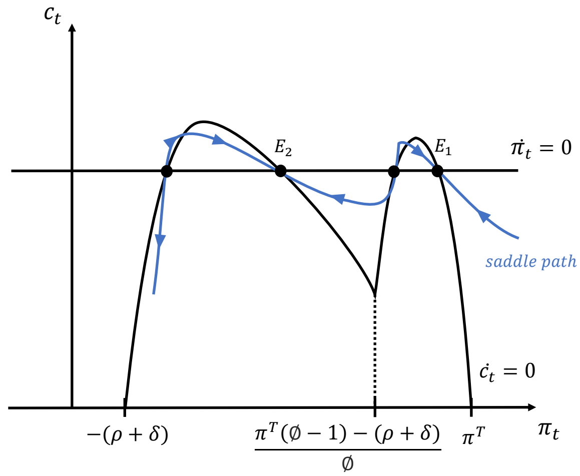

Figure 11: Equilibrium trajectories when monetary policy follows a very aggressive Taylor rule: two stable steady states

when monetary policy is very aggressive in reacting to deviations of inflation from target, the range of inflation rates that support the higher-inflation equilibrium E_1 becomes arbitrarily small. This implies that when such an economy is subjected to shocks, even if it starts at the high-real-rate, high-inflation equilibrium, it is very likely to end up at the low-inflation ELB equilibrium. In this sense, a high ϕ policy of reducing interest rates aggressively in response to deviation of inflation from target can contribute to the economy ending up at the ELB with low inflation and a low real rate of interest.[32] It is worth emphasizing that at this equilibrium, inflation is low (possibly negative), but it is nonetheless stable even if the Taylor principle does not hold (see Cochrane (2017)).[33] Proposition 7 confirms that the existence of the E_2 equilibrium depicted in Figure 10 actually depends on $\phi > 0$ being sufficiently large. If ϕ is not sufficiently large, the configuration depicted in Figure 10 cannot arise.

[32]In Appendix B, we consider an extension to examine the robustness of the results in the paper to incorporating effects of asset revaluation when real interest rates fall. We show that as long as the targeted inflation rate in the Taylor rule is relatively low, the main results of the paper extend without modification to this richer environment. However, when the targeted inflation rate becomes sufficiently high, the forces highlighted in the main text are still at play, but more complex equilibrium configurations can arise.

[33]A downward spiral in inflation is nonetheless possible in this set-up if inflation gets sufficiently close to $-(\rho + \delta)$.

Proposition 7. *When* $V(a_t, r_t) = \varphi \frac{a_t^{1-\sigma}}{1-\sigma} \exp^{-\gamma r_t}$ *with* $\sigma > 1$ *and monetary policy is given by equation (15), the existence of a stable low-real-rate, low-inflation steady state at the ELB is only possible if* $\phi > 1$ *is sufficiently large (holding other parameters fixed).*

See Appendix A.9 for the proof.

4.2.1 Real factors and the emergence of the low-real-rate, low-inflation trap

In the previous discussion, we emphasized how more aggressive monetary policy can simultaneously favour the emergence of a low-real-rate equilibrium at the ELB while also expanding its basin of attraction. We now want to briefly discuss the important role of real factors in allowing for such an equilibrium outcome. In particular, countries or regions that may have fallen into a low-inflation trap (such as Japan and Europe) do not appear to have a substantially more aggressive monetary policy than elsewhere. So monetary policy is unlikely to be the sole or main driver. Instead, these are countries that are generally viewed as having real factors that favour savings, and these are the factors that are commonly thought to contribute to demand being depressed and monetary policy being pushed to the ELB. In our set-up, real factors that favour savings play a very similar role to monetary policy in favouring the emergence of the low-real-rate, low-inflation trap. This is most easily seen by varying ρ.[34] As indicated in Proposition 8, for a given monetary policy stance parameterized by ϕ, ρ has to be sufficiently low for an equilibrium configuration such as in Figure 10 to arise. In Figure 12 we depict the effect of a change in ρ on the equilibrium configuration. As illustrated in the figure, a higher ρ will make the E_2 equilibrium that arises with a low ρ disappear. So if an economy finds itself in a low-real-rate equilibrium like E_2 in the figure, it is both because ρ is sufficiently low and ϕ is sufficiently high. In this sense, monetary policy can be viewed as contributing to the emergence of a low-real-rate, low-inflation trap, but it cannot be seen as the driving factor. Real factors affecting savings are also key.

Proposition 8. *When* $V(a_t, r_t) = \varphi \frac{a_t^{1-\sigma}}{1-\sigma} \exp^{-\gamma r_t}$ *with* $\sigma > 1$ *and monetary policy is given by equation (15), the existence of a stable low-real-rate, low-inflation trap at the ELB is only possible if* ρ *is sufficiently small (holding other parameters fixed).*

See Appendix A.10 for the proof.

[34]We could alternatively enrich the model to consider the effect of growth. The effect of higher growth would then have a very similar effect to a higher level of ρ.

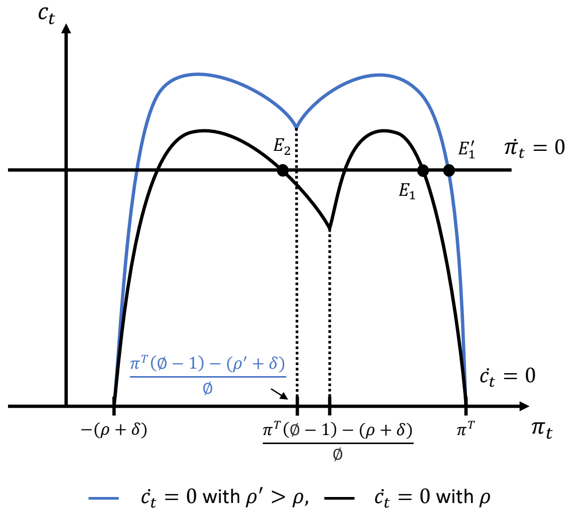

Figure 12: Equilibrium trajectories when the discount rate ρ increases to $\rho' > \rho$

4.2.2 Exiting the low-real-rate, low-inflation trap using expansionary fiscal policy

When the economy is at the ELB in a low-inflation trap, as represented by the equilibrium outcome E_2 shown in Figures 10 and 11, fiscal policy can be used to raise the rate of inflation and escape the trap. An increase in debt $\bar{B}$ corresponds to an upward shift in the $\dot{c} = 0$ in these figures. This implies that the long-run equilibrium point E_2 will move to the right when $\bar{B}$ is larger, implying higher inflation. This is expressed in Proposition 9. However, it must be noted that the effect of changes in $\bar{B}$ on long-run inflation is discontinuous. As debt rises, there will come a point where the E_2 equilibrium will cease to exist. At that point, the only stable equilibrium will be E_1. Hence, the long-run rate of inflation in such an economy can change discretely in response to a large fiscal expansion. A sufficiently large increase in $\bar{B}$ can create a switch from the long-run equilibrium E_2 to the long-run equilibrium E_1. Fiscal policy is therefore pushing the economy out of the low-real-rate, low-inflation steady state, but that is coming at the cost of a discrete jump in long-run inflation. Acharya and Dogra (2021), Eggertsson and Mehrotra (2014), and Mian, Straub, and Sufi (2020a) also find that rising public debt leads to an escape from the ELB, although the exit is not discrete. Moreover, when this jump in long-run inflation occurs, it also implies that the real interest rate jumps by an even greater amount since, as the economy emerges from the ELB constraint, the Taylor principle implies that nominal interest rates increase more than inflation.

Proposition 9. *When* $V(a_t, r_t) = \varphi \frac{a_t^{1-\sigma}}{1-\sigma} \exp^{-\gamma r_t}$ *with* $\sigma > 1$ *and monetary policy is given by equation (15), the inflation rate at the ELB stable steady state is increasing in government debt* B*. However, when* B *becomes too large, the equilibrium at the ELB will cease to exist and long run inflation will exhibit a discontinuous jump.*

See Appendix A.11 for the proof.

4.2.3 Equilibrium inflation and real interest rates when monetary policy is not aggressive

Up to now we have been assuming that monetary policy satisfies the Taylor principle when it is not constrained by the ELB (i.e., $\phi > 1$). However, there are historical examples where monetary policy has likely not satisfied the Taylor principle. It is therefore of interest to explore the equilibrium properties when the Taylor principle is not satisfied, that is, when the interest rate is given by:

$$i_t = \max\left\{0, \rho + \delta + \pi^T + \phi(\pi_t - \pi^T)\right\} \qquad \phi < 1, \quad \pi^T > 0. \tag{18}$$

In this case, as indicated in Proposition 10, generally there exists a unique stable steady state outcome. This contrasts with the case without bequest motives, where generally there does not exist a stable steady state. This unique equilibrium always happens with inflation running above target. This implies that the nominal interest in a steady state is never constrained by the ELB. Although the nominal interest rate is above the ELB, the configuration of equilibrium dynamics looks very similar to that described in Figure 7, when monetary policy was assumed to be unconditionally constrained by the ELB. The reason for the similarity is that in both cases the Taylor principle is not satisfied. Moreover, as in Figure 7, it can be verified that the stable steady state corresponds to what we have been calling our low-real-rate steady state. Finally, it is interesting to note that at this stable steady state, inflation is higher the closer ϕ gets to 1, with the additional property that as ϕ converges to 1, inflation goes off to infinity.

Proposition 10. *When* $V(a_t, r_t) = \varphi \frac{a_t^{1-\sigma}}{1-\sigma} \exp^{-\gamma r_t}$ *with* $\sigma > 1$ *and monetary policy is given by equation (18), then if* φ *is not too large, there is a unique stable steady state. At this steady state, inflation is above target,* $\pi > \pi^T$*, and* $y = \bar{y}$*. As* ϕ *converges to 1, inflation converges to infinity.*

See Appendix A.12 for the proof.

4.2.4 Welfare

Our analysis highlighted the possibility of two full-employment stable equilibria when monetary policy follows a truncated Taylor rule: one low-real-rate, low-inflation equilibrium, and one high-real-rate, high-inflation equilibrium. In both cases, the economy is at full employment with consumption being identical. From a welfare point of view, the high-real-rate, high-inflation equilibrium would nonetheless be preferred by consumers since $V(a, r)$ will be higher. Households would experience less utility in the lower-real-rate equilibrium because they feel poorer in terms of the value of bequest they can transmit. However, it is worth noting that this welfare calculation abstracts from any gains that may be associated with having the possibility of lower taxes in this equilibrium versus the higher taxes needed to finance the same spending and debt level in the higher-real-rate equilibrium. While examining the value of having more fiscal space in the low-inflation, low-real-rate equilibrium is beyond the scope of this paper, our analysis does open the possibility that the low-real-rate, low-inflation equilibrium may have desirable welfare properties.

5 Further Discussion

5.1 Natural rate of interest

While there are different concepts for the natural rate of interest, for our purpose we define the natural rate of interest as the real rate, denoted r^*, consistent with a long-run equilibrium where output is at full capacity. In a standard infinitely lived representative agent model, this rate can be inferred directly from the household's Euler equation for consumption. The main determinants of the natural rate of interest in such a case are the subjective discount rate of households and the long-run growth rate of the economy. This natural rate does not depend on fiscal policy. In contrast, in our environment with bequest motives, the natural rate of interest depends on the amount of debt in the economy, the level of government spending, and the level of taxes. In particular, the natural rate of interest is implicitly defined by the following condition if government spending ensures that the government budget constraint ($G + r^*\bar{B} - T = 0$) is satisfied:[35]

[35] Alternatively, it can be thought in this case as being implicitly defined by the asset market clearing condition $A^{LR}(y - T, r^*) = \bar{B}$.

$$\bar{y} - T + r^*\bar{B} = \frac{\rho + \delta - r^*}{\delta V_a(\bar{B}, r^*)}, \tag{19}$$

or alternatively by the following condition if taxes are adjusting to satisfy the government balance:

$$\bar{y} - G = \frac{\rho + \delta - r^*}{\delta V_a(\bar{B}, r^*)}. \tag{20}$$

Hence, in the first case, r^* becomes a function of the amount of debt in the economy and the level of taxes, while in the second case it is a function of debt and government spending. In both cases r^* is not a point as in more standard models but is instead a locus. When $V_{ar} < 0$, the locus is non-monotonic (in the natural interest rate - debt space) as the long-run asset demand is C-shaped. For each level of debt, there are two values for r^*.[36] Moreover, for the case where $V(a, r)$ takes the functional form $\varphi \frac{a^{1-\sigma}}{1-\sigma} \exp^{-\gamma r}$ with $\sigma > 1$, we can plot the locus of natural rates r^* as a function of debt levels and present different loci for different values of φ. This is illustrated in Figure 13. In the figure we see that as we decrease φ, the natural rates of interest tend to equal either something close to $\rho + \delta$ or something close to zero.

5.2 Proposing some general observations

In this last section we want to suggest why our analysis may offer a takeaway that goes well beyond the particular behavioural model we explored. In particular, the key element driving our results regarding potential forces favouring secular stagnation is a long-run demand for assets that is non-monotonic in the real interest rate. When the long-run demand for assets is C-shaped as represented in Figure 14 — regardless of the reason why it takes this form — then for a given supply of assets there are two equilibrium points, with only the higher-real-rate equilibrium likely stable. That is not especially surprising. The more intriguing observation arises when looking at the same asset demand as a function of inflation when interest rates are determined endogenously by a constrained Taylor rule of the form $i_t = \max\{i^{ELB}, i^T + \phi(\pi_t - \pi^T)\}$ with $\phi > 1$. Such a Taylor rule causes the effective

[36] We are not the first to show that the natural rate of interest can be a function of debt. Mian, Straub, and Sufi (2020a) and Acharya and Dogra (2021) find that the natural rate of interest depends on the amount of debt, but their analysis does not feature the C-shaped aspect of the long-run asset demand. In our model, r^* depends on whether interest rates were initially low or high.

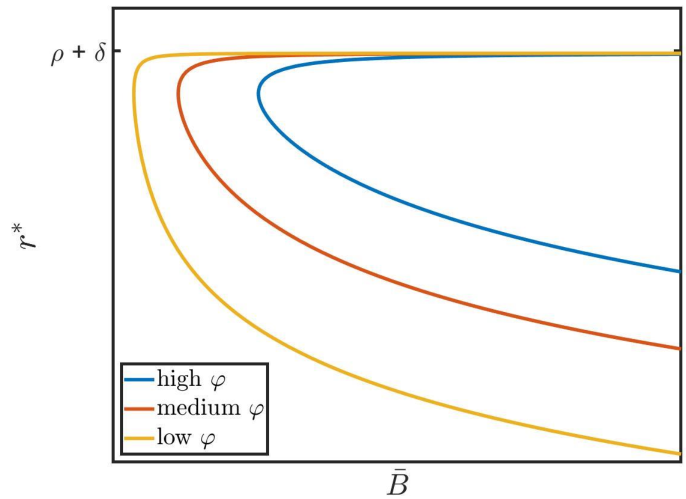

Figure 13: Natural rate of interest as a function of public debt, for different values of bequest motives φ when $V_{ar} < 0$

asset demand to mirror itself over a range of inflation levels, as presented in Figure 15.[37] The reason for the mirroring is that inflation and real rates of interest are positively related for a high level of inflation when i is above the ELB and the Taylor principle is satisfied, but are negatively related for lower levels of inflation when $i = i^{ELB}$. This potentially gives rise to four equilibrium points, instead of two, as represented in the figure. However, as we have shown in the paper with an explicit dynamic structure, only two of these four equilibrium points are likely stable: the high-real-rate, high-inflation outcome denoted E_1 and a low-real-rate, low-inflation outcome denoted E_2. In the absence of explicit microfoundations, the stability of these two equilibrium points could alternatively be argued less formally on *tâtonnement*-type arguments directly from the figure. We emphasize these two stable steady states as they can potentially open the door for aggressive inflation-targeting monetary policy to potentially have long-run non-neutral effects through favouring a move from E_2 to E_1 in response to shocks.

In the above discussion we continued to focus on a supply of assets that is exogenously

[37] In Figure 15 we are assuming as previously that $i^T = \rho + \delta + \pi^T$.

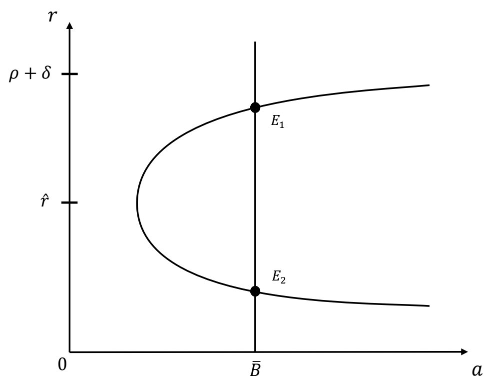

Figure 14: C-shaped long-run asset demand

set by the fiscal authority. We want to finish here by further indicating how the equilibrium configuration is likely to be modified when asset supply endogenously responds to interest rates.[38] The only assumption we want to make about this asset supply is that it is decreasing in the real interest rate. Assuming that monetary policy continues to follow our constrained Taylor rule, asset demand will continue to mirror itself over the inflation range when $A^{LR}(y^s, r)$. For the supply of assets, even if we assume it is (monotonically) decreasing in real interest rates, it will also become non-monotonic as a function of inflation. The supply of assets will decrease with higher inflation when i is not constrained by the ELB but will increase with higher inflation when i is set at the ELB. The resulting demand and supply of assets are shown in Figure 16. The points of sharp inflection for both the asset demand locus and the debt supply locus are associated with a level of inflation where i is exactly equal to the i^{ELB}. In this set-up, we can again have a situation with four long-run equilibrium points as depicted in the figure. However, only two of them are likely to be stable. The two equilibrium points that are likely stable are again those marked by E_1 and E_2 in the figure. The one notable new property that arises from this extension, when compared to the case with a fixed supply of assets, relates to the level of debt between the points E_1 and E_2. The point E_2 is now not only a lower-real-rate, lower-inflation equilibrium point relative to E_1, it is also a point with a higher debt level. Note, however, that the higher debt is not a force that would be causing the low-real-rate, low-inflation state of E_2, it would just be adjusting to it.

[38]See Appendix B for an extension of the model with Lucas trees that is similar to allowing to interest elastic debt.

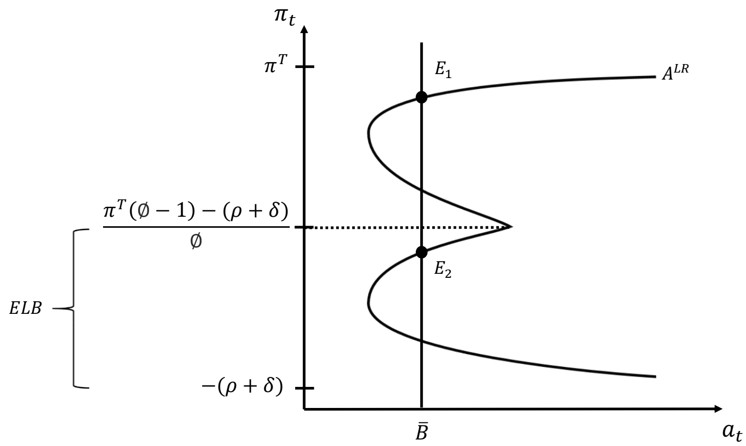

Figure 15: Asset demand as a function of π with fixed supply of assets

6 Conclusion

The idea that monetary policy may have contributed to the secular decline in interest rates and inflation by depressing demand is a popular theme among many financial market participants and economic commentators. However, evaluating this type of claim is difficult without first specifying the mechanisms that could in theory generate such an outcome. In this paper we explored the extent to which non-monotonic demand for assets, motivated by bequest motives, could support/rationalize such claims. If households have an asset demand that is C-shaped with respect to real interest rates, we showed how this can cause aggressive monetary policy to contribute to a long-term decline in both the real interest rate and inflation by favouring a switch in long-run equilibrium. In fact, we show how this environment can generate a low-real-interest-rate, low-inflation trap where the nominal interest rate is at the ELB. In this trap, inflation is stable and determinate, despite the fact that monetary policy does not satisfy the Taylor principle. We also discussed how fiscal policy can be used to exit the low-inflation equilibrium and how such an exit may induce a discontinuous response in inflation. We leave it to further work to evaluate the empirical relevance of non-monotonic asset demands and their implications.

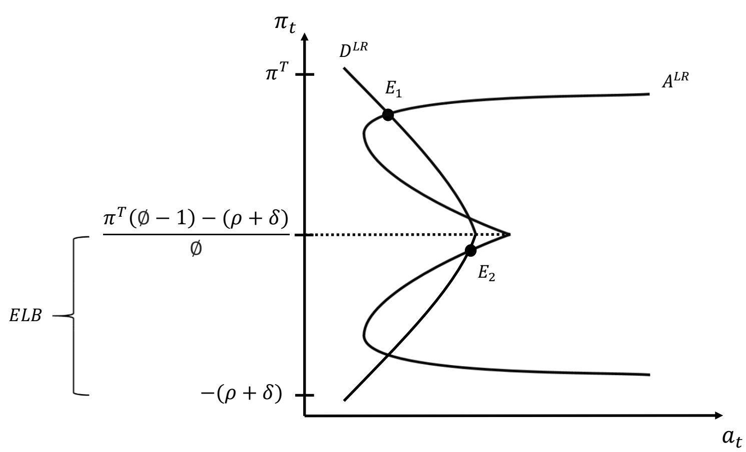

Figure 16: Asset demand as a function of π with elastic supply of assets

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Appendix

A Proofs of Propositions and Lemmas

A.1 Proof of Lemma 1

Let's start by presenting the household optimization problem.

$$\max_{\{c_t, a_t\}} \int_0^\infty e^{-(\rho+\delta)t} \{\log(c_t) + \delta V(a_t, r_t)\} \, dt \tag{A1}$$

s.t.

$$\dot{a}_t = y_t^d + r_t a_t - c_t, \tag{A2}$$

for $t \geq 0$ and a_0 given. Equation (A2) is the budget constraint.

The current-value Hamiltonian is:

$$\log(c_t) + \delta V(a_t, r_t) + \gamma_t \left(y_t^d + r_t a_t - c_t\right),$$

where γ_t is a co-state variable. Taking the first order conditions with respect to c_t and a_t, we obtain the following household's Euler equation:

$$\dot{c}_t = (r_t - \rho - \delta)\, c_t + \delta c_t^2 V_a(a_t, r_t). \tag{A3}$$

The system of ordinary differential equations (ODEs) is given by two equations, (A2) and (A3). The steady state of asset holdings $a > 0$ and consumption $c > 0$ are:

$$\begin{aligned} r &= \rho + \delta - \delta c V_a(a, r), \\ c &= y^d + ra. \end{aligned}$$

With the ODEs, we now seek to examine the stability of this steady state. Let's linearize the ODEs around the steady state (a, c), where $\hat{a}_t \equiv a_t - a$ and $\hat{c}_t \equiv c_t - c$ are deviations (in levels and not in logs) from the steady state.

$$\begin{aligned} \dot{\hat{a}}_t &= r\hat{a}_t - \hat{c}_t \\ \dot{\hat{c}}_t &= \left(\delta c_t^2 V_{aa}(a, r)\right) \hat{a}_t + \left(r - \rho - \delta + 2\delta c V_a(a, r)\right) \hat{c}_t. \end{aligned}$$

Let's put this ODE system in the form of a matrix:

$$\begin{pmatrix} \dot{\hat{a}}_t \\ \dot{\hat{c}}_t \end{pmatrix} = \underbrace{\begin{bmatrix} r & -1 \\ \frac{\varphi\delta}{\rho} c_t^2 V_{aa}(a,r) & \frac{\varphi\delta}{\rho} c V_a(a,r) \end{bmatrix}}_{J \text{ Jacobian evaluated at the steady state}} \begin{pmatrix} \hat{a}_t \\ \hat{c}_t \end{pmatrix}, \tag{A4}$$

where the Jocobian J of the ODEs is evaluated at the steady state. The determinant of J is:

$$det(J) = r\delta c V_a(a,r) + \delta c_t^2 V_{aa}(a,r).$$

After some simple algebra, the determinant $det(J)$ becomes:

$$det(J) = \delta r c V_a(a,r) \left(1 + \frac{c}{ra} \frac{a V_{aa}(a,r)}{V_a(a,r)} \right). \tag{A5}$$

For the steady state to be a saddle point, the determinant must be negative ($det(J) < 0$) (that is the eigenvalues $(\hat{\lambda}_1, \hat{\lambda}_2)$ must of opposite signs since $det(J) = \hat{\lambda}_1 \hat{\lambda}_2$). From equation (A5), since $V_a(a,r) > 0$ the steady state is a saddle point if and only if the following condition holds:

$$\frac{\sigma(a,r)c}{ra} > 1, \tag{A6}$$

where $\sigma(a,r) \equiv -\frac{a V_{aa}(a,r)}{V_{ar}(a,r)}$ is the negative of the elasticity of the marginal utility of wealth. The higher such elasticity the higher the likelihood for the stability condition (A6) to hold.

Q.E.D.

A.2 Proof of Proposition 1

Let's recall the implicit definition of the steady state household asset holdings $a \equiv A^{LR}(y^d, r)$:

$$\frac{(\rho + \delta - r)}{\delta V_a(a,r)} - ra = y^d.$$

Taking successively the derivative of both sides of this equation with respect to y^d and r, we have:

$$\frac{\partial a}{\partial y^d} = \frac{1}{r\left(\frac{\sigma(a,r)c}{ra} - 1\right)}, \tag{A7}$$

$$\frac{\partial a}{\partial r} = -\frac{\frac{1}{\delta V_a} + a + \frac{cV_{ar}}{V_a}}{r\left(-\frac{\sigma(a,r)c}{ra} + 1\right)}. \tag{A8}$$

The stability condition in Lemma 1 implies that $\frac{\partial a}{\partial y^d} > 0$, and if $V_{ar} = 0$, then $\frac{\partial a}{\partial r} > 0$. Therefore the long-run financial asset holdings function $A^{LR}(r, y^d)$ is increasing in income y^d and is rising in interest rate r if $V_{ar} = 0$. If $V_a r < 0$ (by Assumption 1), then $A^{LR}(r, y^d)$ is non-monotonic in r. *Q.E.D.*

A.3 Proof of Lemma 2

The steady state asset market clearing condition is given by $A^{LR}(y - T, \bar{r}) = \bar{B}$, where $\bar{r}$ and $\bar{B} > 0$ are constant policy targets and T is a fixed lump sum tax. From Proposition 1, the long-run asset holding $A^{LR}(y - T, \bar{r})$ is increasing in income y. Since $\bar{B}$ is constant and positive, there is a unique $y \equiv y^{SS}$ that clears the asset market. This implies that there is a unique steady state.

To show that the unique steady state equilibrium is saddle, we consider two cases. In the first case, we assume that the interest rate is fixed at the target rate $\bar{r}$ and the dynamics of government debt are still governed by equation (10). The dynamics of the equilibrium are characterized by equation (10) and the Euler equation (4), which in a form of a matrix are given by:

$$\begin{pmatrix} \dot{\hat{c}}_t \\ \dot{\hat{b}}_t \end{pmatrix} = \underbrace{\begin{bmatrix} \delta c V_a(a,r) & \delta c^2 V_{aa}(a,r) \\ 0 & -\lambda_2 \end{bmatrix}}_{J_2 \text{ Jacobian evaluated at the steady state}} \begin{pmatrix} \hat{c}_t \\ \hat{b}_t \end{pmatrix}, \tag{A9}$$

where the Jocobian J_2 of the ODEs is evaluated at the steady state. The determinant of J_2 is given by $det(J_2) = -\lambda_2 \delta c V_a(a,r) < 0$. The determinant is negative (i.e., the two eigenvalues are opposite signs), and as a result the unique steady state equilibrium is saddle.

Regarding the second case, we assume that public debt is constant. In this case, the dynamics of the equilibrium are characterized by equation (9) and the Euler equation (4). Similarly, the determinant is $det(J_1) = -\lambda_1 \delta c V_a(a,r) < 0$, implying a saddle stable steady state.

Q.E.D.

A.4 Proof of Proposition 2

With the tax rate T doing the adjustment to satisfy the government constraint, the steady state equilibrium condition for y is given by:

$$y - G = \frac{(\rho + \delta - \bar{r})}{\delta V_a}.$$

This implies:

$$\frac{\partial y}{\partial \bar{r}} = \frac{1}{\delta V_a}\left[\frac{-V_{ar}}{V_a}(\rho + \delta - \bar{r}) - 1\right].$$

Because $\frac{-V_{ar}}{V_a}$ is non-increasing in $\bar{r}$ (from Assumption 1), $\frac{-V_{ar}}{V_a}(\rho + \delta - \bar{r})$ goes from infinity to 0 as r goes from $-\infty$ to $\rho + \delta$. Hence there exists a unique cutoff of $\bar{r} < \rho + \delta$ for which $\left[\frac{-V_{ar}}{V_a}(\rho + \delta - \bar{r}) - 1\right] = 0$, with the properties below and above the cutoff as stated in the proposition. This cutoff interest rate is denoted by $\bar{r}^{cutoff}$. This is the end of the proof.

However, to ease the understanding of the content of Proposition 2, it is helpful to use a specific functional form such as $V = \varphi \frac{a^{1-\sigma}}{1-\sigma} \exp^{-\gamma r}$, with $\sigma > 1$. Then we have:

$$\frac{\partial y}{\partial \bar{r}} = \frac{\bar{B}^{\sigma} \exp^{\gamma \bar{r}}}{\delta \varphi}\left(\rho + \delta - \bar{r} - \frac{1}{\gamma}\right),$$

where $\bar{r}^{cutoff} \equiv \rho + \delta - \frac{1}{\gamma}$ is the cutoff. If $\bar{r}$ is below this cutoff, then $\frac{\partial y}{\partial \bar{r}}$ is positive, and above the cutoff it is negative.

Q.E.D.

A.5 Proof of Proposition 3

Similarly as in Section A.4, we obtain the derivative of output y with respect to debt $\bar{B}$:

$$\frac{\partial y}{\partial \bar{B}} = -\frac{(\rho + \delta - \bar{r}) V_{aa}(\bar{B}, \bar{r})}{\delta (V_a)^2}.$$

From Assumption 1 and $\bar{r} < \rho + \delta$, we have $\frac{\partial y}{\partial \bar{B}} > 0$, implying that an increase in $\bar{B}$ increases the steady state output. *Q.E.D.*

A.6 Proof of Proposition 4

It is helpful to recall the system of the two equations that govern the dynamics of c_t and r_t:

$$\frac{\dot{c}_t}{c_t} = r_t - \rho - \delta + \delta \varphi c_t \bar{B}^{-\sigma} \exp^{-\gamma r_t}$$

$$\dot{r}_t = \theta(c_t + G - \bar{y}) \quad \text{if} \quad r_t > r^{ELB} or \quad c_t + G > \bar{y}, \quad \text{with} \quad \dot{r} = 0 \quad \text{otherwise.}$$

We assume that r^{ELB} is not too constraining, meaning that there exists a sufficiently small $\tilde{r} < \rho + \delta - 1/\gamma$ such that $r^{ELB} < \tilde{r}$.

In the steady state, $\dot{c}_t = 0$, which implies that $\rho + \delta - r = \delta\varphi c \bar{B}^{-\sigma} e^{-\gamma r}$, where $r < \rho + \delta$. Rearranging this equation, the $\dot{c}_t = 0$ curve is given by

$$c = \frac{(\rho + \delta - r)\bar{B}^{\sigma} e^{\gamma r}}{\delta\varphi} \equiv H_c(r).$$

Similarly, the $\dot{r}_t = 0$ curve is

$$c = \bar{y} - G \equiv H_r(r),$$

which is a horizontal line in the space r and c with an intercept $\bar{y} - G > 0$.

A steady state equilibrium is found when the two curves intersect, that is, when:

$$\bar{y} - G = \frac{(\rho + \delta - r)\bar{B}^{\sigma} e^{\gamma r}}{\delta\varphi}. \tag{A10}$$

To show the existence of a steady state, we start by discussing the properties of the $\dot{c}_t = 0$ curve $H_c(r)$. The derivative $H_c'(r)$ is given by:

$$H_c'(r) = \frac{\bar{B}^{\sigma} e^{\gamma r}}{\delta\varphi} \left(\gamma(\rho + \delta - r) - 1\right).$$

This implies that at $r = \rho + \delta - \frac{1}{\gamma} \equiv r^{opt}$, the function $H_c(r)$ attains its optimum. If $r < \rho + \delta - 1/\gamma$ ($r > \rho + \delta - 1/\gamma$), $H_c'(r) > 0$ ($H_c'(r) < 0$). Similarly, basic algebra shows that $H_c(r)$ is concave (convex) when $r > \rho + \delta - 2/\gamma$ ($r < \rho + \delta - 2/\gamma$). Moreover, $H_c(\rho + \delta) = 0$, $\lim_{r \to -\infty} H_c(r) = 0$.

A necessary condition for the existence of a solution to equation (A10) when $r > r^{ELB}$ is that the maximum consumption $H_c(r^{opt})$ must be at least equal to $\bar{y} - G$, that is, $H_c(r^{opt}) \geq \bar{y} - G$. This condition is given by $\varphi \leq \frac{e^{\gamma(\rho+\delta)-1}\bar{B}^{\sigma}}{(\bar{y}-G)\delta\gamma}$.

If φ is sufficiently small and $r > r^{ELB}$, then there are two steady state equilibria since the function $H_c(r)$ is strictly increasing on the left of r^{opt} and strictly decreasing on the right of r^{opt}. These two equilibria are denoted E_1 and E_2 respectively as follows:

1. High-real-interest-rate equilibrium E_1: $r > r^{opt} = \rho + \delta - \frac{1}{\gamma}$ and $c = \bar{y} - G$

2. Low-real-interest-rate equilibrium E_2: $r < r^{opt} = \rho + \delta - \frac{1}{\gamma}$ and $c = \bar{y} - G$

If φ is sufficiently large, there are no steady states with $r > r^{ELB}$, and the only steady state is when $r = r^{ELB}$ and $c < \bar{y} - G$. This is the depressed demand at the ELB steady state and we denote it E_3. In addition, since we assume that r^{ELB} is not too constraining, the steady state E_3 always exists for any finite $\varphi > 0$.

Stability analysis. We now examine the stability of the three equilibria. We start with steady states given by E_1 and E_2. Following the steps in Section A.3, we have the following two-dimensional dynamic system:

$$\begin{pmatrix} \dot{\hat{c}}_t \\ \\ \dot{\hat{r}}_t \end{pmatrix} = \underbrace{\begin{bmatrix} \rho+\delta-r & c\left(1-\gamma(\rho+\delta-r)\right) \\ \\ \theta & 0 \end{bmatrix}}_{J \text{ Jacobian evaluated at the steady state}} \begin{pmatrix} \hat{c}_t \\ \\ \hat{r}_t \end{pmatrix},$$

where $\hat{x}_t \equiv x_t - x$ means the deviation of a variable x_t from its steady state x. The determinant $det(J) = -\theta c\left(1-\gamma(\rho+\delta-r)\right)$. If $r > \rho+\delta-\frac{1}{\gamma}$, then $det(J) < 0$, implying that the steady state E_1 is saddle stable. If $r < r^{opt}$, then $det(J) > 0$, meaning that the steady state E_2 is unstable.

We now turn to the stability analysis of the steady state E_3. Given that at E_3, $r = r^{ELB}$ and $c < \bar{y} - G$, we have that $\dot{r}_t = 0$. As a result, the system is one-dimensional in c_t. Therefore, we need only to evaluate the stability for E_3 when the $\dot{c} = 0$ curve ($H_c(r)$) is evaluated at $r = r^{ELB}$. Since c_t is a jump variable, we need the $\dot{c} = 0$ curve to be increasing in r, which is the case (i.e., $H_c'(r^{ELB}) > 0$). Hence, the steady state E_3 is stable.

Q.E.D.

A.7 Proof of Proposition 5

Equations (13) and (14) govern the dynamics of the economy when the interest rate is set at the ELB:

$$\frac{\dot{c}_t}{c_t} = (i^{ELB} - \pi_t - \rho - \delta) + \delta\varphi c_t \bar{B}^{-\sigma} e^{-\gamma(i^{ELB}-\pi_t)},$$

$$\dot{\pi}_t = \kappa(c_t + G - \bar{y}), \quad \text{with } \kappa > 0.$$

In the steady state $\dot{c}_t = 0$, implying that $\rho + \delta + \pi - i^{ELB} = \delta\varphi c\bar{B}^{-\sigma} e^{-\gamma(i^{ELB}-\pi)}$. Rearranging this equation, the $\dot{c}_t = 0$ curve is given by

$$c = \frac{(\rho+\delta+\pi-i^{ELB})\bar{B}^{\sigma} e^{\gamma(i^{ELB}-\pi)}}{\delta\varphi} \equiv H_c(\pi).$$

Taking the derivative of $H_c(\pi)$, we obtain the following properties. First, at $\pi = i^{ELB} - (\rho+\delta)+1/\gamma \equiv \pi^{opt}$, the function $H_c(\pi)$ attains its optimal. Second, if $\pi < \pi^{opt}$ ($\pi > \pi^{opt}$), then the function $H_c(\pi)$ is increasing (decreasing) in π. Third, if $\pi < i^{ELB} - (\rho+\delta) + 2/\gamma$

($\pi > i^{ELB} - (\rho + \delta) + 2/\gamma$), then the function $H_c(\pi)$ is concave (convex) in π. Finally, $H_c(i^{ELB} - \rho - \delta) = 0$ and $\lim_{\pi \to \infty} H_c = 0$.

Similarly, the $\dot{\pi}_t = 0$ curve is

$$c = \bar{y} - G \equiv H_r(\pi),$$

which is a horizontal line in the space π and c with an intercept $\bar{y} - G > 0$.

A steady state equilibrium exists when the two H_c and H_π intersect, that is, when

$$\bar{y} - G = \frac{(\rho + \delta + \pi - i^{ELB})\bar{B}^{\sigma} e^{\gamma(i^{ELB} - \pi)}}{\delta \varphi} \equiv H_c(\pi). \tag{A11}$$

A necessary condition for equation (A11) to hold is that $H_c(\pi^{opt}) > \bar{y} - G$, which implies that $\varphi < \frac{\bar{B}^{\sigma} e^{\gamma(\rho+\delta)-1}}{(\bar{y}-G)\delta\gamma}$.

If $\varphi < \frac{\bar{B}^{\sigma} e^{\gamma(\rho+\delta)-1}}{(\bar{y}-G)\delta\gamma}$ (i.e., small), there exist two steady state equilibria since H_c is strictly increasing at the left π^{opt} and strictly decreasing at the right of π^{opt}. The two steady states are characterized as follows:

1. Equilibrium E_1 with $\pi < i^{ELB} - (\rho + \delta) + 1/\gamma$ and $c = \bar{y} - G$. It is interesting to note that this steady state is identical to the equilibrium with a high real interest rate $i^{ELB} - \pi > \rho + \delta - 1/\gamma$ seen in Proposition 4 and in section A.6. It is for this reason that we also denote E_1 the present equilibrium.

2. Equilibrium E_2 with $\pi > i^{ELB} - (\rho+\delta) + 1/\gamma$ and $c = \bar{y} - G$. In the same fashion, this equilibrium corresponds to a low real interest rate steady state $i^{ELB} - \pi < \rho + \delta - 1/\gamma$ seen in Section A.6.

Stability analysis. We now examine the stability of the steady states given by E_1 and E_2. Following the steps as in section A.6 we have the following two-dimensional dynamic system:

$$\begin{pmatrix} \dot{\hat{c}}_t \\ \dot{\hat{\pi}}_t \end{pmatrix} = \underbrace{\begin{bmatrix} \pi + \rho + \delta - i^{ELB} & c\left(-1 + \gamma(\pi + \rho + \delta - i^{ELB})\right) \\ \kappa & 0 \end{bmatrix}}_{J \text{ Jacobian evaluated at the steady state}} \begin{pmatrix} \hat{c}_t \\ \hat{\pi}_t \end{pmatrix},$$

where $\hat{x}_t \equiv x_t - x$ means the deviation of a variable x_t from its steady state x. The determinant $det(J) = -\kappa c\left(-1 + \gamma(\pi + \rho + \delta - i^{ELB})\right)$. If $\pi > i^{ELB} - (\rho + \delta) + 1/\gamma$, then $det(J) < 0$, implying that the steady state E_2 is saddle stable. If $\pi < i^{ELB} - (\rho + \delta) + 1/\gamma$, then $det(J) > 0$, meaning that the steady state E_1 is unstable.

If $\varphi > \frac{\bar{B}^{\sigma} e^{\gamma(\rho+\delta)-1}}{(\bar{y}-G)\delta\gamma}$ (i.e., sufficiently large), then $\dot{\pi}_t < 0$. Consumption c converges to zero at $\pi = i^{ELB} - (\rho + \delta)$ since $H_c'(i^{ELB} - (\rho + \delta)) > 0$.

Q.E.D.

A.8 Proof of Proposition 6

Equations (15), (16) and (17) govern the dynamics of i_t, c_t, π_t under our Taylor rule specification for monetary policy:

$$i_t = \max\left\{0, i^T + \phi(\pi_t - \pi^T)\right\} \qquad \phi > 1, i^T = \rho + \delta + \pi^T.$$

Like in section A.7, we assume that φ is sufficiently small, which is a necessary condition for the existence of a steady state equilibrium.

Let's first consider the condition under which the ELB is binding, that is, when $i^T + \phi(\pi_t - \pi^T) < i^{ELB}$. Rearranging this inequality shows that when $\pi_t < \frac{i^{ELB} - i^T + \phi\pi^T}{\phi}$, the ELB constraint is binding. We denote this threshold by $\pi^{ELB} \equiv \frac{i^{ELB} - i^T + \phi\pi^T}{\phi}$. This constraint is defined for a given interest rate target i^T. If we use the definition $i^T = \rho + \delta + \pi^T$, then this condition becomes:

$$\pi_t < \frac{i^{ELB} - (\rho + \delta) + (\phi - 1)\pi^T}{\phi} \equiv \pi^{ELB}.$$

As can be seen, this inflation threshold π^{ELB} is increasing in ϕ, implying that the range of π for which the ELB constraint binds increases with ϕ.

For the proof the present proposition, it is helpful to consider two cases: the case where the ELB constraint is binding and the other when it is not binding.

Case 1: $\pi < \pi^{ELB}$ and $i = i^{ELB}$. This case is similar to the proof of Proposition 5 in section A.7. Recall that in this case at $\pi^{otp} = i^{ELB} - (\rho + \delta) + 1/\gamma$ the consumption at $\dot{c}_t = 0$ $(H_c(\pi))$ attains its maximum consumption. Hence, for an equilibrium to exist in this case, $\pi^{opt} < \pi^{ELB}$. After some simple algebra, two conditions emerge for an equilibrium to exist:

$$\phi > \frac{\gamma(\rho + \delta - i^{ELB})}{\gamma(\rho + \delta - i^{ELB}) - 1} \quad \text{and} \quad \gamma > \frac{1}{\rho + \delta - i^{ELB}}.$$

Since $\pi^{opt} < \pi^{ELB}$, $H_c(\pi^{ELB})$ decreases as π^{ELB} rises (e.g., as ϕ increases) given that $H_c'(\pi) < 0$ when $\pi > \pi^{opt}$.

For intermediate values of $\phi > \frac{\gamma(\rho+\delta-i^{ELB})}{\gamma(\rho+\delta-i^{ELB})-1}$, there are two steady states, but only equilibrium E_2, is characterized by a low real rate $(i^{ELB} - \pi < \rho + \delta - 1/\gamma)$), and low inflation is stable (see section A.7).

Case 2: $\pi > \pi^{ELB}$ and $i > i^{ELB}$. The steps of the proof are similar to those in Section A.7. The steady state consumption function is given by:

$$\tilde{H}_c(\pi) = \frac{(\rho + \delta - (\phi - 1)\pi + \phi\pi^T - i^T)\bar{B}^\sigma e^{\gamma(i^T + (\phi-1)\pi - \phi\pi^T)}}{\delta\varphi}.$$

Plugging the definition of i^T into the above equation implies that:

$$\tilde{H}_c(\pi) = \frac{(1 - \phi)(\pi - \pi^T)\bar{B}^\sigma e^{\gamma(\rho+\delta+(\phi-1)(\pi-\pi^T))}}{\delta\varphi},$$

where $\pi < \pi^T$ since $\phi > 1$ for consumption to be positive ($c > 0$).

If $\pi < \tilde{\pi}^{opt}$ ($\pi > \tilde{\pi}^{opt}$), then $\tilde{H}_c'(\pi) > 0$ ($\tilde{H}_c'(\pi) < 0$), where:

$$\tilde{\pi}^{opt} = \frac{\rho + \delta + \phi\pi^T - i^T - 1/\gamma}{\phi - 1} = \pi^T + \frac{1}{\gamma(1 - \phi)}.$$

Note that $\tilde{\pi}^{opt}$ is increasing in ϕ. We also need to ensure that $\tilde{\pi}^{opt} > \pi^{ELB}$, which is satisfied when:

$$\pi^T > \frac{\phi + \gamma(\phi - 1)(i^{ELB} - (\rho + \delta))}{\gamma(\phi - 1)}.$$

There exist two steady states: one with $\pi < \tilde{\pi}^{opt}$ and the other with $\pi > \tilde{\pi}^{opt}$. The latter has a high real interest rate (relative to the ELB case) where $i - \pi > \rho + \delta - 1/\gamma$, and we denoted such steady state by E_1. To see this, let us consider the following:

$$\begin{aligned}
&(i - \pi) - \left(\rho + \delta - \frac{1}{\gamma}\right) > 0 \\
&\rho + \delta + (\phi - 1)(\pi - \pi^T) - \left(\rho + \delta - \frac{1}{\gamma}\right) > 0 \\
&\pi > \pi^T + \frac{1}{(1 - \phi)\gamma}.
\end{aligned}$$

As a result, $i - \pi > \rho + \delta - 1/\gamma$ if and only if $\pi > \pi^T + \frac{1}{(1-\phi)\gamma} = \tilde{\pi}^{opt}$. This high real rate is also higher than the real interest rate in the ELB region.

The stability analysis of these two steady states is given by:

$$\begin{pmatrix} \dot{\hat{c}}_t \\ \dot{\hat{\pi}}_t \end{pmatrix} = \underbrace{\begin{bmatrix} -(\phi - 1)(\pi - \pi^T) & (\phi - 1)c\left[1 - \gamma(-(\phi - 1)(\pi - \pi^T))\right] \\ \kappa & 0 \end{bmatrix}}_{J \text{ Jacobian evaluated at the steady state}} \begin{pmatrix} \hat{c}_t \\ \hat{\pi}_t \end{pmatrix},$$

where $\hat{x}_t \equiv x_t - x$ means the deviation of a variable x_t from its steady state x. The determinant $det(J) = -\kappa(\phi - 1)c\left[1 - \gamma(-(\phi - 1)(\pi - \pi^T))\right]$. If $\pi > \tilde{\pi}^{opt}$, then $det(J) < 0$, implying that the steady state E_1 is saddle stable. If $\pi < \tilde{\pi}^{opt}$, then $det(J) > 0$, meaning that the first steady state is unstable.

Combining Case 1 and Case 2 shows that there are two stable steady states: the low-real-rate, low-inflation steady state (denoted by E_2) and the high-real-rate, high-inflation steady state (denoted by E_1).

Q.E.D.

A.9 Proof of Proposition 7

The proof is related to the proof of Proposition 6 in Section A.8. Like in Section A.8, for a solution at the ELB to exist we need the condition $\pi^{opt} < \pi^{ELB}$ to hold. This is met when:

$$\phi > \frac{\gamma(\rho+\delta-i^{ELB})}{\gamma(\rho+\delta-i^{ELB})-1} \quad \text{and} \quad \gamma > \frac{1}{\rho+\delta-i^{ELB}}.$$

Moreover, for a solution to exist in the decreasing portion of $H_c(\pi)$, we need to have:

$$H_c(\pi^{ELB}) \leq \bar{y} - G.$$

Recall that $\pi^{ELB} = \frac{i^{ELB}-(\rho+\delta)+(\phi-1)\pi^T}{\phi} \equiv F(\phi)$ and that π^{ELB} is increasing in ϕ (i.e., $F' > 0$). Note also that $H_c'(\pi) < 0$ for $\pi > \pi^{opt}$. As a result, taking the inverse of the function H_c, we obtain:

$$\pi^{ELB} \equiv F(\phi) \geq H_c^{-1}(\bar{y} - G),$$

and since F is increasing in ϕ, we get:

$$\phi \geq F^{-1}(H_c^{-1}(\bar{y} - G)).$$

This implies that, holding other parameters fixed, an equilibrium E_2 (low-real-rate, low-inflation steady state) exists if ϕ is above a cutoff. The equilibrium E_2 is also stable (see Section A.7). *Q.E.D.*

A.10 Proof of Proposition 8

This proof is also similar to the ones of Propositions 6 and 7:

$$c = \frac{(\rho+\delta+\pi-i^{ELB})\bar{B}^{\sigma}e^{\gamma(i^{ELB}-\pi)}}{\delta\varphi} \equiv H_c(\pi;\rho).$$

For given parameters, $H_c(\pi;\rho) \equiv \tilde{F}(\rho)$ is also increasing in ρ for any inflation rate π. At the low-real-rate, low-inflation equilibrium E_2, the function $H_c(\pi;\rho)$ is decreasing in π (that is, $\pi > \pi^{opt}$).

We start by assuming that π^{ELB} takes i^T as a given constant (that is, the definition of $i^T = \rho+\delta+\pi^T$ is not taken into account), implying that $\pi^{ELB} = \frac{i^{ELB}-i^T+\phi\pi^T}{\phi}$.

For an equilibrium E_2 to exist in the decreasing part of H_c, the following relationship must hold:

$$H(\pi^{ELB};\rho) \equiv \tilde{F}(\rho) \leq \bar{y} - G.$$

This implies that:

$$\rho \leq \tilde{F}^{-1}(\bar{y} - G).$$

As a result, for sufficiently low ρ, (i.e., ρ below a cutoff), the low-real-rate, low-inflation steady state (E_2) exists. The steady state equilibrium E_2 is also stable (see Section A.7).

When we take into account the definition of $i^T = \rho + \delta + \pi^T$, we obtain that $\pi^{ELB} = \frac{i^{ELB} - (\rho+\delta) + (\phi - 1)\pi^T}{\phi}$ is decreasing in ρ. That is, a decrease in ρ leads to an increase in π^{ELB}, which in turn leads $H_c(\pi^{ELB})$ to fall. This reinforces the result that a low-real-rate, low-inflation equilibrium exists only for sufficiently small ρ.

Q.E.D.

A.11 Proof of Proposition 9

Based on the proof in Section A.8, the stable ELB equilibrium is such that $\pi^{opt} < \pi < \pi^{ELB}$ and:

$$\frac{(\rho + \delta - i^{ELB})\bar{B}^{\sigma} e^{\gamma(i^E LB - \pi)}}{\delta \varphi} = \bar{y} - G.$$

This implies that:

$$\frac{\partial \pi}{\partial \bar{B}} = \frac{\sigma(\rho + \delta + \pi - i^{ELB})}{\bar{B}\left[\gamma(\rho + \delta + \pi - i^{ELB}) - 1\right]}.$$

Since $\pi > \pi^{opt} = 1/\gamma + i^{ELB} - (\rho + \delta)$ in this equilibrium, $\frac{\partial \pi}{\partial \bar{B}} > 0$. As a result, (low)inflation at the stable ELB equilibrium is increasing in $\bar{B}$.

Note also that $H_c(\pi)$ increases with $\bar{B}$ for any inflation rate π and that π^{ELB} is independent of $\bar{B}$.

Let's find a cutoff above which the ELB equilibrium ceases to exist, which arises when $H_c(\pi^{ELB}) > \bar{y} - G$. Rearranging this inequality leads to:

$$\bar{B} > \frac{\delta\varphi(\bar{y} - G)}{(\rho + \delta + \pi^{ELB} - i^{ELB})e^{\gamma(i^{ELB} - \pi^{ELB})}} \equiv \bar{B}^{cutoff}.$$

Therefore, when $\bar{B} > \bar{B}^{cutoff}$, the ELB equilibrium ceases to exist. If $\bar{B} < \bar{B}^{cutoff}$, $\lim_{\bar{B} \to \bar{B}^{cutoff}}(\pi) = \pi^{ELB}$. At $\bar{B}^{cutoff}$, there is a discontinuity and the stable ELB equilibrium disappears.

Q.E.D.

A.12 Proof of Proposition 10

We assume that φ is small. Equations (18), (16) and (17) govern the dynamics of i_t, c_t, π_t with the Taylor rule not satisfying the Taylor principle. Such a Taylor rule is given by:

$$i_t = \max\left\{0, \rho + \delta + \pi^T + \phi(\pi_t - \pi^T)\right\} \qquad \phi < 1, \quad i^T = \rho + \delta + \pi^T, \quad \pi^T > 0.$$

Consider the case where $i_t > i^{ELB} = 0$. The steady state consumption ($\dot{c}_t = 0$) is given by:

$$c = \tilde{H}_c(\pi) = \frac{(\rho + \delta - (\phi - 1)\pi + \phi\pi^T - i^T)\bar{B}^\sigma e^{\gamma(i^T + (\phi-1)\pi - \phi\pi^T)}}{\delta\varphi}.$$

Plugging the definition of i^T into the above equation implies that:

$$\tilde{H}_c(\pi) = \frac{(1-\phi)(\pi - \pi^T)\bar{B}^\sigma e^{\gamma(i^T + (\phi-1)\pi - \phi\pi^T)}}{\delta\varphi}.$$

For consumption to be positive, we must have $\pi > \pi^T$ since $\phi < 1$.[39] $H_c' \geq 0$ if and only if $(1-\phi)[1-\gamma(\rho+\delta+(1-\phi)\pi-i^T+\phi\pi^T)] \geq 0$, that is, if $(1-\phi)[1-\gamma(-(\phi-1)(\pi-\pi^T))] > 0$. Rearranging this equation with $\phi < 1$, we obtain that when:

$$\pi < \frac{1}{1-\phi}\left[1/\gamma - (\rho+\delta) + i^T - \phi\pi^T\right] = \pi^T + \frac{1}{\gamma(1-\phi)} \equiv \tilde{\pi}^{opt},$$

the steady state consumption increases in π ($\tilde{H}'_c > 0$). If $\pi > \tilde{\pi}^{opt}$, then $\tilde{H}'_c < 0$. The optimum consumption is achieved at $\tilde{\pi}^{opt}$.

The steady state equilibrium is found when $\tilde{H}_c(\pi) = \bar{y} - G$. There are two equilibria when $i > i^{ELB}$, $\tilde{\pi}^{opt} > \pi^{ELB}$ and $\tilde{H}_c(\pi^{ELB}) < \bar{y} - G$. These two equilibria are:

1. The equilibrium with $\pi < \tilde{\pi}^{opt}$ and $c = \bar{y} - G$.
2. The equilibrium with $\pi > \tilde{\pi}^{opt}$ and $c = \bar{y} - G$. We denote this E_2.

The second equilibrium with $\pi > \tilde{\pi}^{opt}$ corresponds to the low-real-rate equilibrium where $i - \pi < \rho + \delta - 1/\gamma$. To show this, we have (with $\phi < 1$):

$$(i - \pi) - \left(\rho + \delta - \frac{1}{\gamma}\right) < 0$$
$$\rho + \delta + (\phi - 1)(\pi - \pi^T) - \left(\rho + \delta - \frac{1}{\gamma}\right) < 0$$
$$\pi > \pi^T + \frac{1}{(1-\phi)\gamma} = \tilde{\pi}^{opt}.$$

[39] It can also be shown that if $\pi < \pi^T$, $r > \rho + \delta$, which is impossible.

Hence the low real interest rate corresponds to the case when $\pi > \pi^{opt}$ and $\phi < 1$ and it is denoted E_2.

The stability analysis of these two steady states is given by:

$$\begin{pmatrix} \dot{\hat{c}}_t \\ \dot{\hat{\pi}}_t \end{pmatrix} = \underbrace{\begin{bmatrix} -(\phi-1)(\pi-\pi^T) & (\phi-1)c\left[1-\gamma(-(\phi-1)(\pi-\pi^T))\right] \\ \kappa & 0 \end{bmatrix}}_{J \text{ Jacobian evaluated at the steady state}} \begin{pmatrix} \hat{c}_t \\ \hat{\pi}_t \end{pmatrix},$$

where $\hat{x}_t \equiv x_t - x$ means the deviation of a variable x_t from its steady state x. The determinant $det(J) = -\kappa(\phi-1)c\left[1-\gamma(-(\phi-1)(\pi-\pi^T))\right]$. If $\pi < \tilde{\pi}^{opt}$, then $det(J) > 0$, implying that the steady state with $\pi < \tilde{\pi}^{opt}$ is unstable. If $\pi > \tilde{\pi}^{opt}$, then $det(J) < 0$, meaning that the low-real-rate steady state denoted E_2 is stable.

Note that the unique stable steady state is such that $\pi > \pi^T + \frac{1}{\gamma(1-\phi)} > \pi^T$ since $\phi < 1$. As ϕ converges to 1, $\pi^T + \frac{1}{\gamma(1-\phi)}$ converges to infinity and consequently π goes to infinity.

Therefore, there is a unique *stable* steady state when $\phi < 1$, and such steady state is characterized by $\pi > \pi^T$ and $y = \bar{y}$, and as ϕ converges to 1, π converges to infinity.

Q.E.D.

B Extending the Model to Include Productive Assets: Lucas Trees

In this appendix we introduce an outside asset to our set-up, where the price of the asset adjusts to changes in interest rates, making the effective supply of assets endogenous. We want to use this extension to examine the robustness of the results in the paper to incorporating effects of asset revaluation when real interest rates fall. As we will discuss, as long as the targeted inflation rate in the Taylor rule is relatively low, the main results of the paper extend without modification to this richer environment. However, when the inflation rate targeted becomes sufficiently high, the forces highlighted in the main text are still at play, but more complex equilibrium configurations can arise.

To introduce a second asset to our set-up, suppose there is a mass one of Lucas trees that produce a flow f (fruit) of goods. If these trees lasted forever, and the real rate of interest were fixed at r, their value would be $\frac{f}{r}$. However, in order to allow for a risk premium and negative real rates, assume that trees die at flow rate ω and that dead trees are continuously replaced with new trees redistributed in lump sum fashion to households. This ensures that the total mass of trees remains constant. In equilibrium, the price of a Lucas tree will adjust so that households always want to hold one unit of Lucas trees. The consumption Euler equation will accordingly become:

$$\frac{\dot{c}_t}{c_t} = r_t - \rho - \delta + \delta c_t V_a(\bar{B} + z_t f, r_t),$$

where z_t is the price of a Lucas tree and it satisfies the standard asset pricing equation:

$$\frac{\dot{z}_t}{z_t} = \frac{f}{z_t} - (r_t + \omega).$$

The goods market equilibrium condition can now be expressed as $y_t + f = c_t + G$, where y_t are goods produced by labour and $y_t + f$ is the aggregate output. In this environment, y_t can again be viewed as being demand determined by the Euler equation. In steady state, condition $\dot{c}_t = 0$ therefore implies that:

$$c = \frac{\rho - \delta - r}{\delta V_a(\bar{B} + \frac{f}{r+\omega}, r)}.$$

Assumption 1 is no longer sufficient to guarantee that the $\dot{c}_t = 0$ curve is hump shaped in r. In fact, in the presence of Lucas trees, the $\dot{c}_t = 0$ curve will not be hump shaped in r as before, because it will have a negative slope for r close to $\rho + \delta$ and will have a negative slope for r close to $-\omega$. Nonetheless, the $\dot{c}_t = 0$ curve will still tend to be non-monotonic in r if V_{ar} is sufficiently negative. To illustrate when this will happen, let us return to assuming that $V(a, r)$ is of the form $\varphi \frac{a^{1-\sigma} \exp^{-\gamma r}}{1-\sigma}$. In this case, if $\gamma > \frac{1}{\rho+\delta+\omega}$ and f is not too large, then the $\dot{c}_t = 0$ curve will take the S-shaped as given in Figure B1. If f is very large or if γ is small ($\gamma < \frac{1}{\rho+\delta+\omega}$), then the $\dot{c}_t = 0$ curve will likely be negatively sloped throughout. Obviously, if $\dot{c}_t = 0$ becomes monotonic declining, the novel results highlighted in the text would no longer hold. Hence, we will assume in the remainder that the $\dot{c}_t = 0$ curve is as in Figure B1. In this figure we have superimposed the long-run goods market equilibrium condition $c = \bar{y} + f - G$. The important element to note in this figure is that there are now potentially three real interest rates compatible with full-capacity use. The two previous equilibria denoted E_1 and E_2 remain, but now a third equilibrium can arise. This third equilibrium, denoted E_4, has an associated real interest rate, denoted r_4.[40] This equilibrium arises with both very low real interest and high asset demands on the part of households. In the absence of Lucas trees, at r_4 there would be an excess demand for assets. However, with the Lucas trees, the high demand for assets at r_4 is satisfied by the large valuation of Lucas trees.

Assuming $\dot{c}_t = 0$ is as in Figure B1, we can then move on to combine the Euler equation with a Phillips curve and a Taylor rule to look at the joint determination of c and π as we did before. There are now two cases to consider. The easy case is when r_4 is small relative to the inflation target π^T in the Taylor rule, that is, when $r_4 < -\pi^T$. In such a case, all the main results from Section 4 carry over. In particular, if monetary policy is not very aggressive, then there can be only one stable steady state equilibrium, and that corresponds to the high-real-rate equilibrium E_1. As monetary policy gets more aggressive, the equivalent of equilibrium E_2 will appear as a stable steady state, and as monetary policy becomes gradually more aggressive, the basin of attraction of this E_2 equilibrium will expand while that of E_1 will become arbitrarily small. In this sense, the analysis in the main text is robust to including Lucas trees as long as r_4 and π^T are sufficiently small.

[40] Note that r_4 may well be negative.

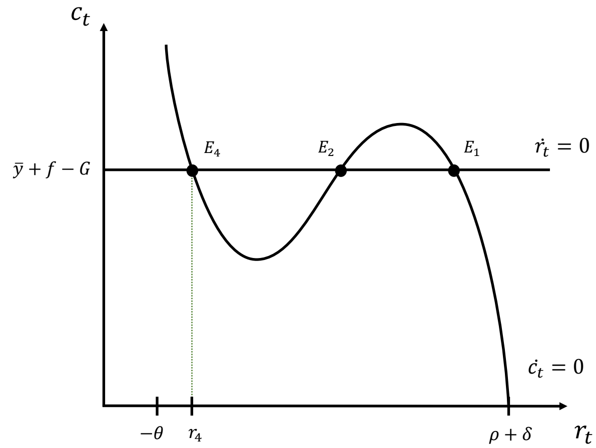

Figure B1: Equilibrium trajectories in the presence of Lucas trees when V_{ar} is sufficiently negative and f is not too large

Now if $r_4 > -\pi^T$, then the equilibrium configuration can get more complex than that presented in the main text. For example, it can take the form as given in Figure B2.[41] In this case, it is possible to have three stable steady states with different levels of inflation and different real rates. The high-real-rate equilibrium corresponding to E_1 remains. As before, an ELB equilibrium with a low real rate will also be present when monetary policy is sufficiently aggressive. But now we get the possibility of a third equilibrium; this one implements the real rate r_4 and is not in the ELB region. This equilibrium has a low real rate — even lower than that of the E_2 equilibrium — even though the nominal interest rate is positive. So the price of Lucas trees at the E_4 equilibrium, which is given by $z = \frac{f}{\omega+i-\pi}$ in steady state, will be higher in the E_4 equilibrium than in both the E_2 and E_1 equilibria. With such a configuration, if the economy were to start in the E_1 equilibrium and be subject to a set of demand shocks, it could go first from E_1 to E_4, with a drop in inflation and a rise in asset prices. This would be followed later by a move from E_4 to E_2 with a further drop in inflation, but now it would be associated with a fall in asset prices. The last switch could appear as if the fall in asset prices were depressing demand and leading to a fall in inflation; however, a better interpretation would be that the lower inflation at E_2 versus E_4 is causing higher real rates and thereby depressing asset prices.

[41]The more complex configuration presented in Figure B2 also requires — in addition to $r_4 > -\pi^T$ — that monetary policy be sufficiently aggressive. In particular, it requires that $\phi > \frac{\pi^T+\rho+\delta}{\pi^T-\omega}$.

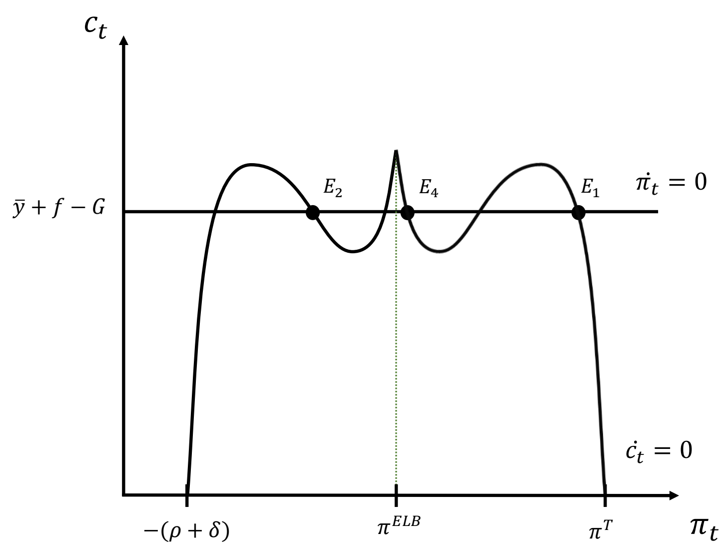

Figure B2: Equilibrium trajectories in the presence of Lucas trees when the inflation target is sufficiently high: three stable steady states

C Accounting For a Forward-Looking Phillips Curve

So far, our analysis has focused on a backward-looking Phillips curve. In this appendix, we consider a forward-looking Phillips curve in our set-up and discuss its implications. The Phillips curve is specified as:

$$\dot{\pi}_t = \kappa(y_t - \bar{y}).$$

With this specification, when $\kappa < 0$, we have a forward-looking Phillips curve and π_t needs to be treated as a jump variable and this is the focus of this appendix. In contrast, if $\kappa > 0$, we have a backward-looking Phillips curve and π_t is treated as a state variable as in the main text. The characterization of steady states is similar in both situations. However, the stability of these steady states differ. The low-real-rate, low-inflation steady state E_1 and the high-real-rate, high-inflation steady state E_2 that were saddle in the presence of a backward-looking Phillips cure are now a source when the Phillips curve is forward-looking. In other words, when $\kappa < 0$, the two equilibria are determinate stable and the system would jump to one of them instead of exhibiting hysteresis as in the backward-looking Phillips curve case.

To show this, first note that because π_t is a jump variable when the Phillips curve is forward-looking, stability requires that the steady states to be a source.[42] A system that is a source implies a determinate equilibrium. This means that the only non-explosive equilibrium trajectory is to jump to the steady state and remain there; if the economy jumps elsewhere, inflation or demand would become infinite. To examine the implications for $\kappa < 0$, it is enough to focus the discussion around Figures 9 and 10 (as well as the proofs of Proposition 6). We consider two cases: the binding ELB and non-binding ELB constraint.

Case 1: binding ELB; $\pi < \pi^{ELB}$ and $i = i^{ELB}$. The dynamics of c_t and π_t are governed by the Phillips curve and the Euler equation. Recall that there are two potential steady states: the first one is when $c = \bar{y} - G$ and $\pi < i^{ELB} - (\rho + \delta) + 1/\gamma$ while the second one is when $c = \bar{y} - G$ and $\pi > i^{ELB} - (\rho + \delta) + 1/\gamma$ which corresponds to the E_2 steady state. From the proof of Proposition 6 in Section A.8, the two-dimensional dynamical system in a matrix form is given by:

$$\begin{pmatrix} \dot{\hat{c}}_t \\ \\ \dot{\hat{\pi}}_t \end{pmatrix} = \underbrace{\begin{bmatrix} \pi + \rho + \delta - i^{ELB} & c\left(-1 + \gamma(\pi + \rho + \delta - i^{ELB})\right) \\ \\ \kappa & 0 \end{bmatrix}}_{J \text{ Jacobian evaluated at the steady state}} \begin{pmatrix} \hat{c}_t \\ \\ \hat{\pi}_t \end{pmatrix},$$

where $\hat{x}_t \equiv x_t - x$ means the deviation of a variable x_t from its steady state x. The trace $tr(J) = \pi + \rho + \delta - i^{ELB} > 0$ since $\pi > -\rho - \delta$. The determinant $det(J) = -\kappa c\left(-1 + \gamma(\pi + \rho + \delta - i^{ELB})\right)$. If $\pi > i^{ELB} - (\rho + \delta) + 1/\gamma$, then $det(J) > 0$, implying that the low-real-rate, low-inflation steady state E_2 is a source. If $\pi < i^{ELB} - (\rho + \delta) + 1/\gamma$, then $det(J) < 0$, meaning that the first steady state is saddle. Note that the existence of E_2 depends on the aggressiveness of monetary policy. When monetary policy is not too aggressive (but with $\phi > 1$), as depicted in Figure 9, and inflation is less than π^{ELB}, only the first steady state with $\pi < i^{ELB} - (\rho + \delta) + 1/\gamma$ exists. In such a situation, this single steady state is saddle.

Case 2: non-binding ELB; $\pi > \pi^{ELB}$ and $i > i^{ELB}$. Similarly, there are also two potential steady states in this case: the first is with $\pi < \pi^T + \frac{1}{\gamma(1-\phi)}$ and $c = \bar{y} - G$ and the second one is $\pi > \pi^T + \frac{1}{\gamma(1-\phi)}$ and $c = \bar{y} - G$ which equals to the steady state E_1. In the same fashion, the two-dimensional system governing the dynamics of c_t and π_t is:

[42] Mathematically, a steady state is a source when the determinant and the trace of the Jacobian J evaluated at the steady state are both positive.

$$\begin{pmatrix} \dot{\hat{c}}_t \\ \dot{\hat{\pi}}_t \end{pmatrix} = \underbrace{\begin{bmatrix} -(\phi-1)(\pi-\pi^T) & (\phi-1)c\left[1+\gamma(\phi-1)(\pi-\pi^T)\right] \\ \kappa & 0 \end{bmatrix}}_{J \text{ Jacobian evaluated at the steady state}} \begin{pmatrix} \hat{c}_t \\ \hat{\pi}_t \end{pmatrix}.$$

The trace $tr(J) = -(\phi-1)(\pi-\pi^T)$ and $tr(J) > 0$ since $\pi < \pi^T$ and $\phi > 1$. The determinant $det(J) = -\kappa(\phi-1)c\left[1+\gamma(\phi-1)(\pi-\pi^T)\right]$. If $\pi > \pi^T + \frac{1}{\gamma(1-\phi)}$, then $det(J) > 0$, implying that the steady state E_1 is a source since $tr(J) > 0$. If $\pi < \pi^T + \frac{1}{\gamma(1-\phi)}$, then $det(J) < 0$, meaning that the first steady state is saddle and this equilibrium would exhibit an indeterminacy. The existence of the first steady state with $\pi < \pi^T + \frac{1}{\gamma(1-\phi)}$ depends also on the aggressiveness of monetary policy (ϕ). When monetary policy is not too aggressive but with $\phi > 1$ as depicted in Figure 9, only the high-real-rate, high-inflation steady state E_1 exists and it is determinate stable.

The takeaways from these two cases can be summarized as follows: First, when monetary policy is not too aggressive (but with $\phi > 1$) as depicted in Figure 9, there are two steady states. Only the high-real-rate, high-inflation steady state E_1 is a source and is therefore determinate stable. There is also the low-inflation steady state in the ELB region which is saddle and exhibits indeterminacy. Second, when monetary policy is aggressive (with $\phi > 1$), as displayed in Figure 10, both the low-real-rate, low-inflation steady state E_2 and the high-real-rare, high-inflation steady state E_1 are a source and determinate stable. The system would jump to one of the steady states instead of exhibiting hysteresis as observed with a backward-looking Phillips curve.