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DIMENSIONAL ANALYSIS OF THERMAL PLUMES

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Résumé

Dans ce rapport, on a employé rigoureusement la méthode des dimensions pour établir l'étendue d'une "plume" en fonction de l'écoulement de l'eau chaude dans un lac. On a vérifié la nouvelle fonction et a déterminé un des constants par les données disponibles. Les resultats sont montrés par un graphique dont on peut utilizer pour prédire à l'avenir la surface d'une plume.

In the report, dimensional analysis has been rigourously employed to find a new functional relationship between the plume surface area and the discharge of warm water into a lake. The function has been verified and one of the proportionality constants determined from available data. The results are given in a graph which can be used to predict plume surfaces in the future.

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DIMENSIONAL ANALYSIS OF THERMAL PLUMES

BACKGROUND

Because of severe theoretical difficulties, thermal plume data are usually analyzed by statistical or dimensional methods. The result is an empirical equation which more or less describes the relationships between the chosen parameters. Empirical models have been developed by Ashbury and Frigo (1971), Elliott and Harkness (1972), Palmer (1972), and Kenny (1973). Each of these investigators have obtained different equations.

Ashbury and Frigo (1971) assumed the relationship

$$\frac{A}{Q_o^a} = f\left(\frac{\theta}{\theta_o}\right) \quad \text{--} \quad (1)$$

where A = plume area contained within the isotherm with excess temperatore θ

Q_o = plant volumetric discharge rate

θ = excess temperature in the plume

θ_o = excess temperature at the discharge, and

a = an exponent to be determined from the data.

Data from a number of different sites were used and they found that the most consistent grouping was obtained by using a = 1. A plot of (A/Q_o) versus (θ/θ_o) was presented as their model of plume surface area.

Elliott and Harkness (1972) used the same variables for their analysis but chose an equation of the form

$$\left(1 - \frac{\theta}{\theta_o}\right) = c_1 (A/Q_o\theta_o)^{c_2} \quad \text{------------------------------} \quad (2)$$

where c_1 and c_2 are coefficients to be determined. The above equation was fitted to data from the Lakeview generating station. The data were segregated into three seasonal groups and different coefficients were obtained for each group. Values of c_1 varied from 0.307 to 0.354, while the exponent c_2 varied from 0.364 to 0.712.

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Palmer (1972) also used data from the Lakeview site but proposed equations of the form

$$\ln\left(\frac{Q_o}{A}\right) = c_1 + c_2 \ln\theta \quad \text{------------------------------} \quad (3)$$

However, the scatter of the data about his prediction equations were very large.

Kenny (1973) proposed a model in which the area contained within the $1^{\circ}C$ excess temperature isotherm, A_1, was related to the source strength S of the power plant. The source strength is the amount of heat discharged into the lake per unit time expressed as BTU/sec. Correlation of A_1 with S for data from the Lakeview site alone was not successful. However, by including data from various other generating stations and using averaged values of the plume areas and source strength for each station obtained from a number of surveys, the following equation was obtained:

$$A_1 = 111.13 \quad S^{0.8117} \quad \text{------------------------------} \quad (4)$$

This equation was then used to obtain an estimate of the lake area which would be affected by thermal plumes by the year 2000.

The equations listed above all fitted the data because they were derived from regression analysis which enabled the exponents and constants to be varied until a fit is obtained.

However, it is obvious that all of the equations cannot be applicable, especially when extrapolating to a different range of variables. Furthermore, the equations are not dimensionally correct, and the proportionality constants and empirical exponents include the effects of parameters not considered in the analysis. As a result, there is no agreement concerning the relationship between variables. For example, in Kenney's equation, A_1 varies with S to the power 0.8117, but Elliott and Harkness' equation can be rearranged to show that A_1 varies with S to the power 1.

A review of the available models indicates that a formal dimensional analysis was not undertaken.

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A primary selection of the minimum set of independent variables affecting the observed result permits the rigorous development of the correct dimensional grouping and often illustrates the degree of dependence of the variables. The method of analysis is shown clearly by Rouse (1959), Huntley (1967) and in detail by Yalin (1971).

DIMENSIONAL ANALYSIS

The selection of a set of independent variables which must affect and describe the phenomenon is the critical first step in the analysis.

In this instance, certain a priori decisions were made. Firstly, the initial part of the plume, where it spouts into the lake, was not taken into account. The initial jet is a region where jet type mixing and momentum effects predominate and to consider this effect variables such as jet velocity, channel width, and channel depth would have to be included. But, since the initial jet portion has a small area compared to the plume, it can probably be safely neglected from the analysis. Secondly, the observed area of warmer water is also clearly affected by loss of heat to the atmosphere as well as by mixing. But measurements have shown that the portion of the temperature drop or rise attributable to atmospheric loss or gain is small compared to the mixing process. Thus variables such as air temperature, surface heat transfer coefficient, or wind velocity may also be neglected. Finally, therefore, the simplified model and the minimum set of variables can be considered.

The dependent parameter under investigation is the area A lying within a certain excess temperature isotherm. Clearly the area measured will be a function of the excess temperature θ, and the volumetic discharge Q_o and the initial excess temperature θ_o. Another parameter to take into account the characteristics of the receiving body of water is required. It was decided to use a heat diffusivity term ε which represents the turbulence and mixing conditions of the water body receiving the discharge. Densities are not included because they are a function of temperature. The effects of long-shore current caused by wind or by

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atmospheric pressure gradients on the spreading of the plume are also contained in the term ε which has the dimension length squared divided by time.

The independent set of parameters having been selected, then by dimensional analysis one may write,

$$f(A, \theta_o, Q, \theta, \varepsilon) = 0 \quad (5)$$

The parameters listed in equation (5) are believed to be the minimum independent set necessary to describe the phenomenon under the a priori decisions. Equation 5 contains all of the parameters employed in the previous phenomenological models with the addition of the heat diffusivity term "ε".

The five parameters in equation (5) contain three basic dimensions -- length, time, and temperature, and from the theory of dimensions one knows that this problem involves two dimensionless variables. A simple application of dimensional analysis results in the equation,

$$\frac{A\varepsilon^2}{Q_o^2} = c_1\left(\frac{\theta}{\theta_o}\right)^{c_2} \quad (6)$$

where c_1 and c_2 are dimensionless constants. This equation shows that the area within a certain excess temperature isotherm should increase as the square of the volumetric flow rate, decrease as the square of the diffusivity, and should depend on θ/θ_o to some power c_2. If there is data on all the parameters involved, equation (6) is very simple to verify. Plots of $A\varepsilon^2/Q^2{}_o$ versus θ/θ_o for different tests should all collapse on one line which would verify the dimensionless relationship and allow the constants c_1 and c_2 to be determined. However, there is no value for ε available for any of the thermal plume data, which makes it impossible to obtain both constants c_1 and c_2. Nevertheless, it is still possible to estimate the exponent c_2 by using the available data and writing equation (6) in the form

$$\frac{A}{Q_o^2} = \frac{c_1}{\varepsilon^2}\left(\frac{\theta}{\theta_o}\right)^{c_2} \quad (7)$$

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Since the value of Q_o and ε should be constant for a particular test, a logarithmic plot of (A/Q_o^2) versus (θ/θ_o) for any one test should give a straight line with slope equal to c_2. Plots for other test runs should produce a series of parallel lines with different intercepts as illustrated in Figure 1. This would indicate that the exponent c_2 is indeed the same for all the different plumes. The separation between the lines is the effect the variations in heat diffusivity ε from one test to the next.

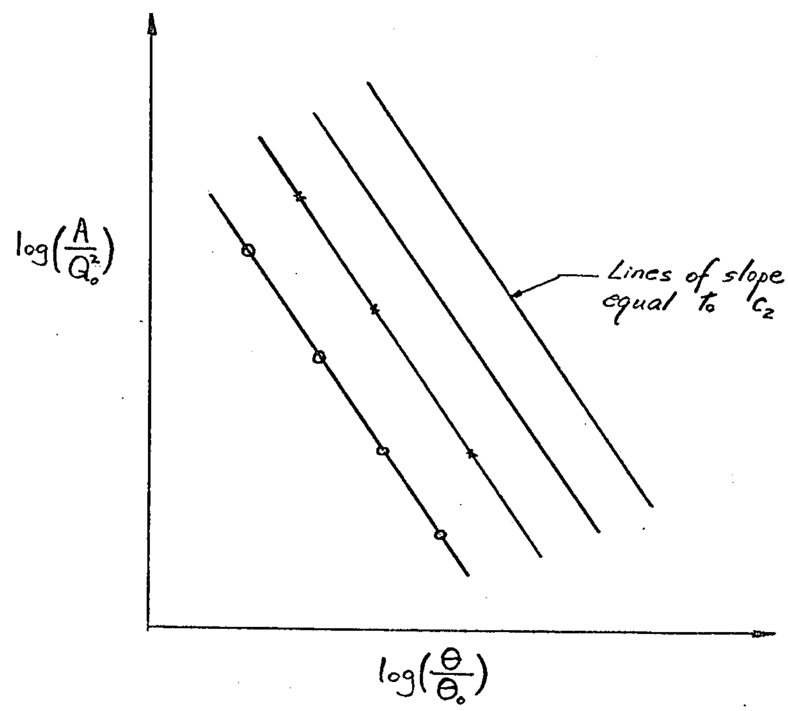

Figure 1. Anticipated plot based on the results of dimensional analysis

DATA ANALYSIS AND DISCUSSION

The data used in this analysis are listed in Table 1 and include those compiled by Ashbury and Frigo (1971) which came from thermal plume experiments from seven different generating stations and also the data from the Lakeview generating station published by Bryce et al. (1971).

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As discussed in the previous section, the results of this dimensional analysis do not apply to the region where the momentum of the discharge is governing the plume spread. In order to be reasonably sure that jet mixing effect is minimal, only those data points in which the excess temperature has decreased to less than half the initial excess temperature were used.

Plots of A/Q_o^2 versus θ/θ_o were made on log-log paper, using those data sets which had at least three or more points in the range $\theta/\theta_o < 0.5$. These plots are shown in Figure 2 from which it can be seen that data from the various plumes do seem to fall more or less on parallel straight lines. The slope which appeared to best fit the data was used to draw parallel lines through the various sets and agreement was quite good with one or two exceptions. The fitted lines are described by the relationship,

$$\frac{A}{Q_o^2} = \frac{c_1}{\varepsilon^2} \left(\frac{\theta}{\theta_o}\right)^{-1.167} \quad \text{-----------------------------} \quad (8)$$

Therefore, from the dimensional analysis, and using the experimental data, the relationship given by equation (9) has been deduced.

$$\frac{A\varepsilon^2}{Q_o^2} = c_1 \left(\frac{\theta}{\theta_o}\right)^{-1.167} \quad \text{-----------------------------} \quad (9)$$

It can be observed from Figure 2 that for the same fractional drop in excess temperature, say $\theta/\theta_o = 0.1$, the value of A/Q_o^2 varies from about 0.8 to 200 because A varies inversely as the square of the diffusivity ε, which can be significantly different from one location to the next and even from day to day in the same location. The more turbulent the water, the faster would heat be transferred to the ambient water and the smaller would the surface area of the plume be. If $\theta/\theta_o = 0.1$ is defined as the edge of a plume, then the area of a plume is proportional to the square of the discharge rate and the proportionality constant can vary as much as two hundred times in magnitude, depending on the state of the turbulence at the particular location at that particular time. Comparing equation (8) with the model of Elliott and Harkness (1972) which presumes without

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any justification that for any value of θ/θ_0 A is proportional to $(Q_0\theta_0)$, it can be seen that large errors may result when one tries to extrapolate to different discharges.

When considering practical problems of heat sources, the concept of a source strength S as employed by Kenny is a useful idea because of its physical meaning and relationship to power generation. It was decided, therefore, to manipulate equation (6) into a form including S. Since $S = Q_0\gamma Cp\theta_0$, then substitution in (6) gives

$$\frac{A\varepsilon^2}{S^2}(\gamma C_p\theta_o)^2 = c_1\left(\frac{\theta}{\theta_o}\right)^{c_2} \text{ ---------------------------- } \quad (10)$$

Equation (1) may also be obtained directly from dimensional analysis by using S, γ and C_p initially in the set of independent variables and it is of interest to note that the exponent of S is obtained from the analysis and not from plotting data. Thus one can be fairly certain that A does vary with the square of S and the relationship is valid. If θ, the isotherm excess temperature is now set to <u>one</u> degree, then equation (11) is obtained

$$A_1 = \frac{c_3}{\varepsilon^2}\,\frac{S^2}{\theta_o^{2+c_2}} \text{ ---------------------------------- } \quad (11)$$

where A_1 is the area contained by the 1^o isotherm and $c_3 = c_1/\gamma^2 c_p^2$.

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Using the value of c_2 obtained in Figure 2, values of $A_1\theta_o^{2+c_2}$ versus S may be plotted for different plumes. On a log-log plot, a set of lines of slope equal to 2 with different intercepts corresponding to different values of ε^2 is expected. Figure 3 shows the plot and it seems that lines of slope 2 agree fairly well with the data. From the largest and smallest intercepts it can be shown that the ratio of the largest to smallest diffusivity was about 10. This is a likely ratio and is also mentioned by Kenny in his recommendations. Predictions of plume areas contained by the one degree isotherm may be made from Figure 3, provided S and θ_o are known and a choice is made about the value of c_3/ε^2. Because of the usefulness of the source strength idea, it was decided to plot the area A versus the source strength S for the average line in Figure 3 which gives a value of c_3/ε^2 close to 10^{-4}. Using equation (11), and setting c_2 at -1.167, solutions for θ_o equal to 6^o, 12^o and 24^o were obtained. The results are plotted on Figure 4 along with available data and the averaged points given by Kenny.

It should be noted that plotting area A versus S will increase the data scatter because for a given value of S, the value A may vary because of differences in the source temperature θ_o in addition to differences of diffusivity in the lake. Thus it is not a good idea to plot the area versus the source strength S because it tends to aggravate the scatter making useful analysis difficult. If the source strength parameter S is used, it is preferable to plot observations as in Figure 3, which utilizes the parameter $A_1\theta^{2+c_2}$ where c_2 has been estimated to be -1.167. Future field programmes should, therefore, endeavour to obtain data from variations of θ_o

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and for as wide as possible variations in diffusivity conditions as estimated from wind and current observations. In this way, upper and lower limits of the plume areas may be calculated as desired.

CONCLUSIONS

1. The correct functional relationship between the plume area A, the isotherm θ, the source temperature θ_o, and the discharge rate Q_o is given by

$$A = \frac{c_1}{\varepsilon^2} Q_o^2 \left(\frac{\theta}{\theta_o}\right)^{c_2}$$

   Analysis of available data indicates that c_2 is about -1.167.

2. The above relationship can be manipulated to include the source strength S which is a convenient physical parameter and the relationship is given by

$$A = \frac{c_3}{\varepsilon^2} \frac{S}{\theta_o^2} \left(\frac{\theta}{\theta_o}\right)^{c_2}$$

   It follows that if θ is set at one degree, then

$$A_1 = \frac{c_3}{\varepsilon^2} \frac{S^2}{\theta_o^{0.833}} \qquad (12)$$

3. It should be remembered that S is also a function of θ_o and that for a fixed source strength, the area of the plume will decline as θ_o is increased. Thus, if a management objective is to reduce plume areas, it would be better to reduce the discharge and increase the source

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temperature. There are practical limits to this which probably lie outside the analyzed data. It is believed that large increases in temperature would produce density differences sufficiently great to reduce mixing and offset gains by increasing temperature. In addition, the industrial process involved has upper limits for the cooling water temperature for thermodynamic reasons. Further investigation to establish limits would be necessary.

4. It is recommended that predictions of upper and lower limits of area may be correctly estimated by using Figure 3. New data should also be analyzed by the same methods and plotted as for Figure 2 or Figure 3.

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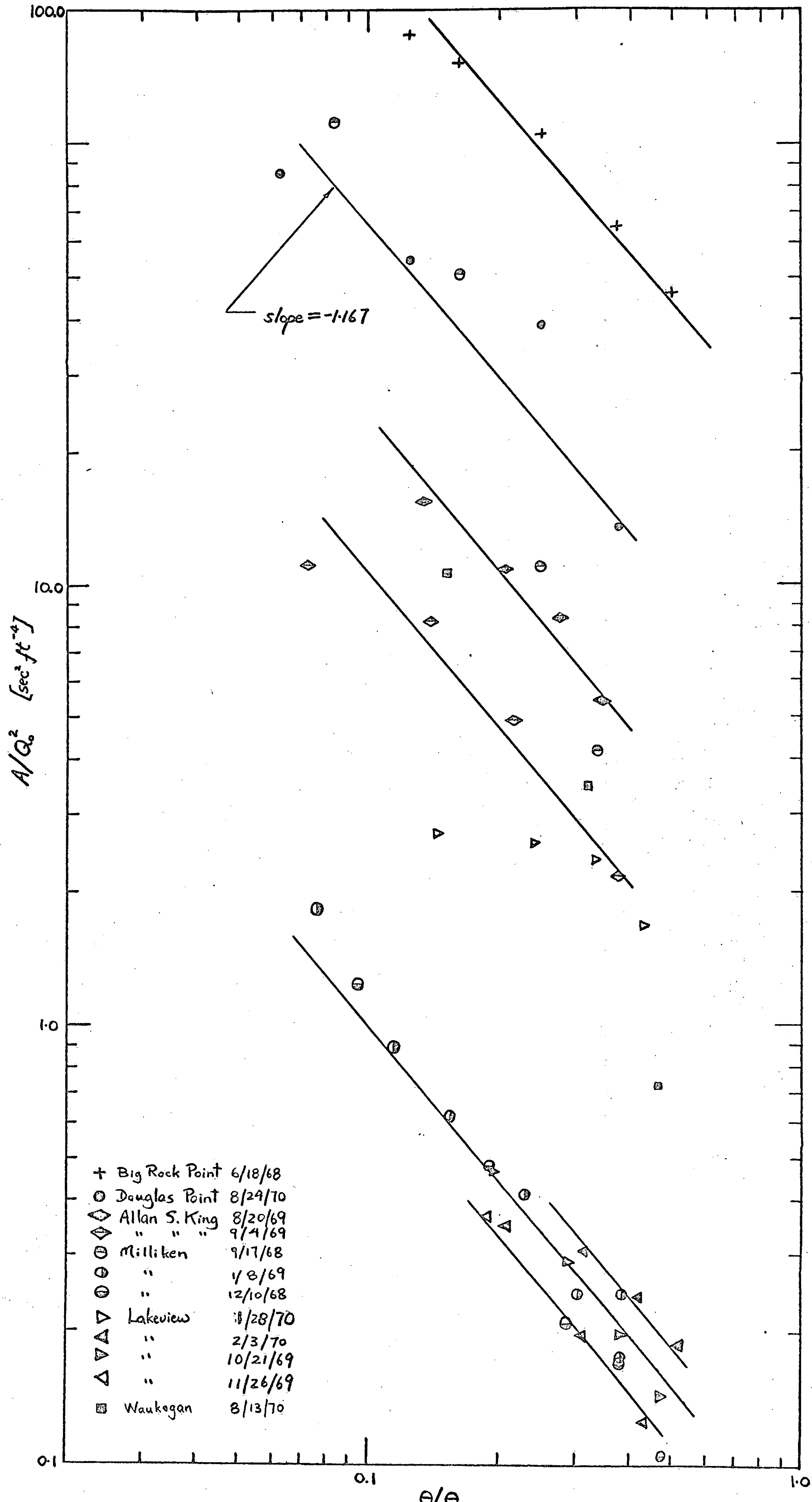

FIG. 2 Plot of Θ/Θ_0 versus A/Q_0^2 for Lake Thermal Plumes.

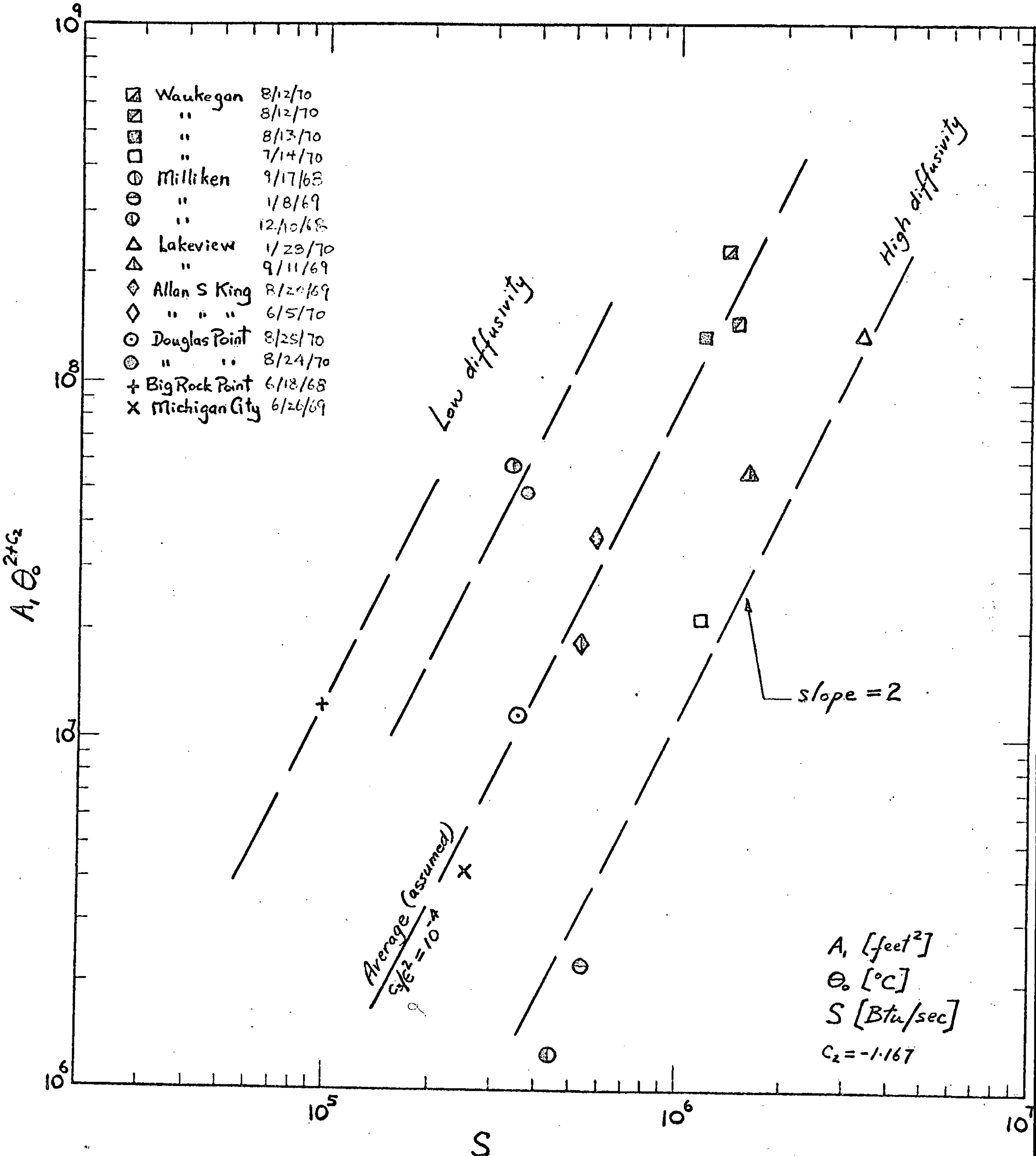

FIG. 3 Plot of S, the source strength, versus the parameter $A_1\theta_0^{2+c_2}$ obtained for the one degree isotherm.

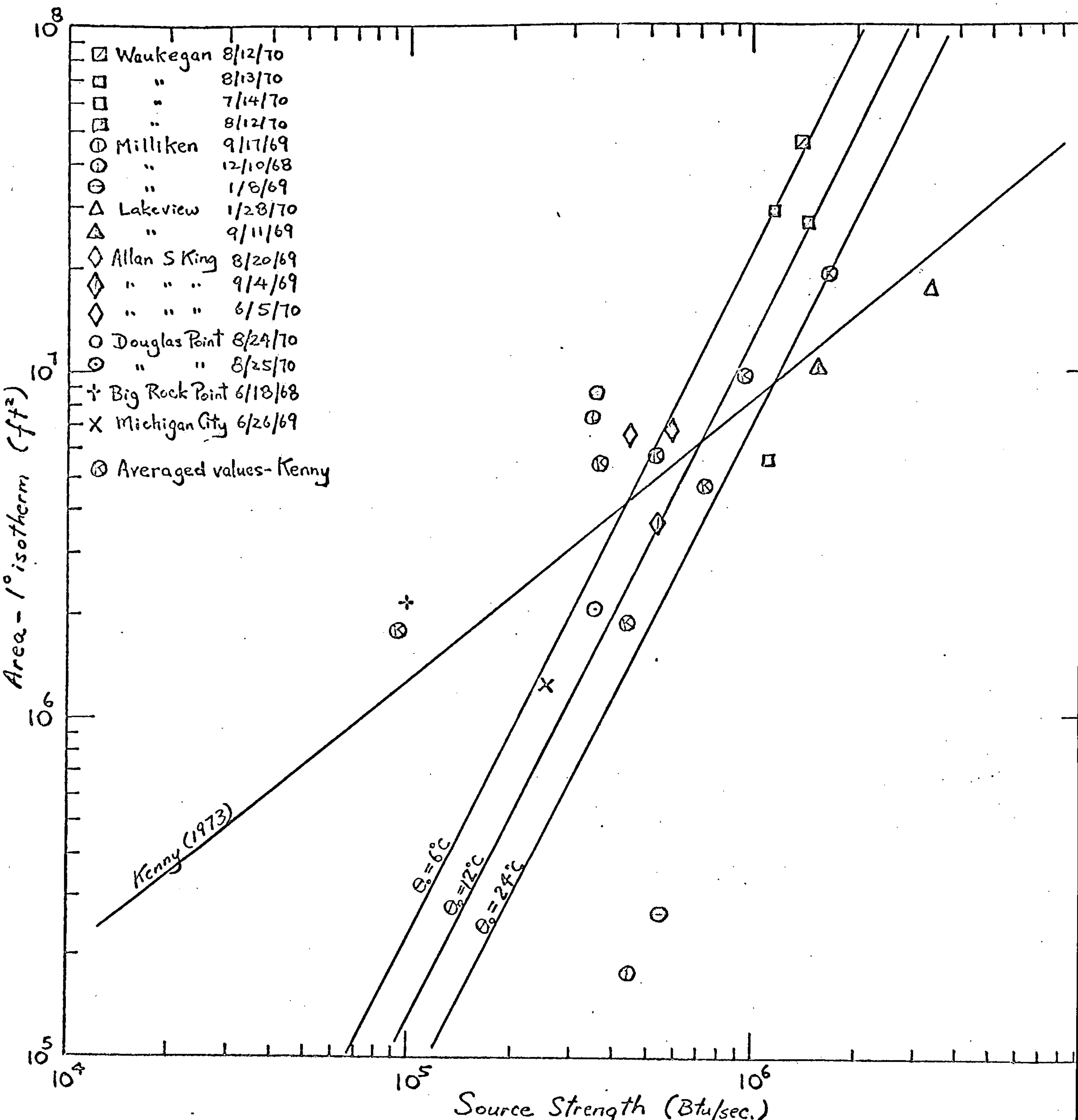

FIG. 4 Plot of Source Strength S Versus the Plume Area Contained by the 1° Isotherm.

TABLE 1. Data on Thermal Plumes gathered from various literature sources.

PLANT & DATE	Q_o C.F.S.	Θ_o °C	Θ °C	$\frac{\Theta}{\Theta_o}$	A ft^2	A/Q_o^2	S BTU/sec	A_1 ft^2	$A_1\Theta_o^{0.833}$
Big Rock Paint 6/18/68	111.4	8.0	3.0	0.375	8.153×10^5	65.69	9.98×10^4	2.18×10^6	1.23×10^7
			2.0	0.250	1.347×10^6	108.54			
			1.3	0.163	1.90×10^6	153.10			
			1.0	0.125	2.181×10^6	175.75			
Milliken 9/17/68	254.0	12.0	5.0	0.417	6.2×10^4	0.961	3.41×10^5	7.42×10^6	5.87×10^7
			4.0	0.333	2.72×10^5	4.216			
			3.0	0.250	7.22×10^5	11.19			
			2.0	0.167	3.29×10^6	50.99			
			1.0	0.083	7.42×10^6	115.0			
Milliken 12/10/68	377.0	10.5	5.0	0.476	1.477×10^4	0.104	4.43×10^5	1.77×10^5	1.25×10^6
			4.0	0.381	2.46×10^4	0.173			
			3.0	0.286	2.95×10^4	0.208			
			2.0	0.190	6.89×10^4	0.485			
			1.0	0.095	1.77×10^5	1.245			
Milliken 1/8/69	377.0	13.0	5.5	0.423	9.846×10^3	0.069	5.49×10^5	2.66×10^5	2.25×10^6
			5.0	0.385	2.462×10^4	0.173			
			4.0	0.308	3.466×10^4	0.242			
			3.0	0.231	5.908×10^4	0.416			
			2.0	0.154	8.862×10^4	0.625			
			1.5	0.115	1.28×10^5	0.90			
			1.0	0.077	2.658×10^5	1.87			
Allan S. King 8/20/69	660.0	7.86	3.83	0.488	7.041×10^5	1.616	5.81×10^5	6.71×10^6	3.74×10^7
			2.72	0.346	2.339×10^6	5.369			
			2.17	0.276	3.669×10^6	8.423			
			1.61	0.205	4.729×10^6	10.856			
			1.06	0.134	6.708×10^6	15.399			

TABLE 1. Data on Thermal Plumes gathered from various literature sources.

PLANT & DATE	Q_o C.F.S.	Θ_o ^oC	Θ ^oC	$\frac{\Theta}{\Theta_o}$	A ft^2	A/Q_o^2	S BTU/sec	A_1 ft^2	$A_1\Theta_o^{0.833}$
Allan S. King 9/4/69	660.0	7.18	3.22 2.67 1.56 0.999 0.555	0.449 0.371 0.217 0.139 0.073	6.641×10^5 9.437×10^5 2.108×10^6 3.574×10^6 4.9×10^6	1.525 2.166 4.839 8.205 11.25	5.31×10^5	3.57×10^6	1.85×10^7
Douglas Paint 8/24/70	397.0	8.0	3.0 2.0 1.0 0.5	0.375 0.250 0.125 0.063	2.123×10^6 6.13×10^6 8.637×10^6 1.35×10^7	13.47 38.89 54.80 85.66	3.56×10^5	8.64×10^6	4.88×10^7
Douglas Paint 8/25/70	397.0	8.0	3.0 2.0 1.0 0.5	0.375 0.25 0.125 0.063	1.2×10^5 4.2×10^5 2.06×10^6 1.824×10^7	0.76 2.66 13.07 115.7	3.56×10^5	2.06×10^6	1.16×10^7
Lakeview 1/28/69	2520.0	11.67	5.0 3.89 2.78 1.666 .555	0.428 0.333 0.238 0.143 0.043	1.04×10^7 1.499×10^7 1.632×10^7 1.734×10^7 1.818×10^7	1.68 2.36 2.57 2.73 2.86	3.29×10^6	1.77×10^7	1.37×10^8
Lakeview 10/21/69	1740.	11.67	5.55 4.44 3.33 2.22	0.476 0.381 0.286 0.19	4.38×10^6 6.02×10^6 8.79×10^6 1.43×10^7	1.45 1.99 2.90 4.72			

TABLE 1. Data on Thermal Plumes gathered from various literature sources.

PLANT & DATE	Q_o C.F.S.	Θ_o ^oC	Θ ^oC	$\frac{\Theta}{\Theta_o}$	A ft^2	A/Q_o^2	S BTU/sec	A_1 ft^2	$A_1 \Theta_o^{0.833}$
Lakeview 11/26/69	2030.	8.89	3.89 2.78 1.67	0.437 0.313 0.187	$5.245\text{x}10^6$ $8.055\text{x}10^6$ $1.52\ \text{x}10^7$	1.27 1.95 3.69			
Lakeview 2/3/70	2190.	10.5	5.5 4.39 3.28 2.17	0.524 0.418 0.312 0.206	$8.95\ \text{x}10^6$ $1.157\text{x}10^7$ $1.409\text{x}10^7$ $1.68\ \text{x}10^7$	1.87 2.41 3.06 3.50			
Waukegan 8/13/70	1624.	6.4	3.0 2.0 1.0	0.469 0.313 0.156	$1.917\text{x}10^6$ $9.18\ \text{x}10^6$ $2.905\text{x}10^7$	0.727 3.48 11.01	$1.16\text{x}10^6$	$2.9\text{x}10^7$	$1.36\text{x}10^8$
Lakeview 9/11/69	1750.	7.88	1.0	0.127	$1.01\ \text{x}10^7$		$1.55\text{x}10^6$	$1.01\text{x}10^7$	$5.64\text{x}10^7$
Waukegan 7/14/70	1871.	5.3					$1.11\text{x}10^6$	$5.49\text{x}10^6$	$2.2\text{x}10^7$
Waukegan 8/12/70	1730.	7.5					$1.45\text{x}10^6$	$2.69\text{x}10^7$	$1.44\text{x}10^8$
Waukegan 8/12/70	1730.	7.1					$1.37\text{x}10^6$	$4.60\text{x}10^7$	$2.35\text{x}10^8$

TABLE 1. Data on Thermal Plumes gathered from various literature sources.

PLANT & DATE	Q_o C.F.S.	Θ_o °C	Θ °C	$\frac{\Theta}{\Theta_o}$	A ft^2	A/Q_o^2	S BTU/sec	A_1 ft^2	$A_1\Theta_o^{0.833}$
Allan S. King 6/5/70	457.	8.47					4.34×10^5	$6.6.10^6$	3.91×10^7
Michigan City 6/26/69	537.	4.2					2.53×10^5	1.25×10^6	4.13×10^6