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COMMON FLUID DYNAMIC EQUATIONS IN CURVILINEAR CO-ORDINATES

by

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PREFACE

This manuscript contains most of the commonly used expressions and equations of fluid dynamics expressed in curvilinear co-ordinate systems. While listing the expressions for the metric coefficients and for the various vector operations such as gradient, divergence, curl and laplacian, it is also shown how they are derived from first principles. This manuscript provides a convenient source of fluid dynamic equations and expressions as represented in a curvilinear co-ordinate system. Both research scientists and engineers should find it useful when working with non-orthogonal systems. Practising engineers may also find it useful for understanding reports by scientists and mathematicians.

PREFACE

Le présent document renferme la plupart des formules et des équations de la dynamique des fluides généralement utilisées dans les systèmes de coordonnées curilignes. On y trouve la liste des formules des coefficients métriques et des diverses opérations vectorielles, comme le gradient, la divergence, le rotationnel et le laplacien; on y montre aussi comment les formules sont dérivées de principes premiers. Ce document expose de façon pratique des équations et des formules de la dynamique des fluides utilisées dans le système de coordonnées curilignes. C'est un instrument utile aux chercheurs (hommes de science et ingénieurs) pour l'étude de systèmes non orthogonaux: il aidera également les ingénieurs exerçants à comprendre les rapports des hommes de science et des mathématiciens.

1.0 CURVILINEAR CO-ORDINATE SYSTEMS

Let the co-ordinates of a point P in a curvilinear co-ordinate system be

$$\alpha, \beta, \gamma$$

and the same in a cartesian co-ordinate system be

$$x, y, z$$

and let us assume that there are unique relationships available connecting these two sets of co-ordinates as follows:

$$x = f_x(\alpha\,;\ \beta\,;\gamma)$$

$$y = f_y(\alpha\,;\ \beta\,;\gamma) \qquad (1)$$

$$z = f_z(\alpha\,;\ \beta\,;\gamma)$$

Let $\hat{i}$, $\hat{j}$, $\hat{k}$ be the unit vectors in the cartesian co-ordinate system and let $\hat{e}_\alpha$, $\hat{e}_\beta$, $\hat{e}_\gamma$ be the same in a curvilinear co-ordinate system.

Considering the point P (see Figure 1), the position vector of P in cartesian co-ordinates is

$$x\hat{i} + y\hat{j} + z\hat{k}$$

and the same in curvilinear co-ordinates can be expressed as:

$$\hat{r} = f_{\hat{r}}(\alpha\,;\ \beta;\gamma)$$

By definition:

$$\hat{e}_\alpha = \frac{\partial f_{\hat{r}}}{\partial \alpha} \Bigg/ \left| \frac{\partial f_{\hat{r}}}{\partial \alpha} \right|$$

$$\hat{e}_\beta = \frac{\partial f_{\hat{r}}}{\partial \beta} \Big/ \left| \frac{\partial f_{\hat{r}}}{\partial \beta} \right| \tag{2}$$

and

$$\hat{e}_\gamma = \frac{\partial f_{\hat{r}}}{\partial \gamma} \Big/ \left| \frac{\partial f_r}{\partial \gamma} \right|$$

The magnitudes of the tangent vectors, i.e. $\partial f_{\hat{r}}/\partial\alpha$, $\partial f_{\hat{r}}/\partial\beta$, $\partial f_{\hat{r}}/\partial\gamma$, are called the scale factors or the metric coefficients and are usually denoted by h_α, h_β and h_γ, i.e.

$$h_\alpha = \left| \frac{\partial f_{\hat{r}}}{\partial \alpha} \right|$$

$$h_\beta = \left| \frac{\partial f_{\hat{r}}}{\partial \beta} \right| \tag{3}$$

$$h_\gamma = \left| \frac{\partial f_{\hat{r}}}{\partial \gamma} \right|$$

The metric coefficients can be evaluated in terms of known functions f_x, f_y and f_z as follows:

$$\hat{r} = f_{\hat{r}}(\alpha\, ;\, \beta\, ;\gamma\,)$$

$$d\hat{r} = \frac{\partial f_{\hat{r}}}{\partial \alpha} .\, d\alpha + \frac{\partial f_{\hat{r}}}{\partial \beta} .\, d\beta + \frac{\partial f_{\hat{r}}}{\partial \gamma} .\, d\gamma$$

From (2) and (3)

$$d\hat{r} = e_\alpha h_\alpha d\alpha + e_\beta . h_\beta . d\beta + e_\gamma . h_\gamma . d\gamma \tag{4}$$

Again $\quad \hat{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$d\hat{r} = \hat{i}\, dx + \hat{j} dy + \hat{k} dz$$

$$= \hat{i} \left[\frac{\partial f_x}{\partial \alpha} d\alpha + \frac{\partial f_x}{\partial \beta} d\beta + \frac{\partial f_x}{\partial \gamma} d\gamma \right] +$$

$$\hat{j}\left[\frac{\partial f_y}{\partial \alpha}\, d\alpha + \frac{\partial f_y}{\partial \beta}\, d\beta + \frac{\partial f_y}{\partial \gamma}\, d\gamma\right] +$$

$$\hat{k}\left[\frac{\partial f_z}{\partial \alpha}\, d\alpha + \frac{\partial f_z}{\partial \beta}\, d\beta + \frac{\partial f_z}{\partial \gamma}\, d\gamma\right] +$$

$$= \left.\begin{aligned} &\hat{i}\left[\frac{\partial f_x}{\partial \alpha} + \hat{j}\,\frac{\partial f_y}{\partial \alpha} + \hat{k}\,\frac{\partial f_z}{\partial \alpha}\right] d\alpha\ + \\ &\hat{i}\left[\frac{\partial f_x}{\partial \beta} + \hat{j}\,\frac{\partial f_y}{\partial \beta} + \hat{k}\,\frac{\partial f_z}{\partial \beta}\right] d\beta\ + \\ &\hat{i}\left[\frac{\partial f_x}{\partial \gamma} + \hat{j}\,\frac{\partial f_y}{\partial \gamma} + \hat{k}\,\frac{\partial f_z}{\partial \gamma}\right] d\gamma \end{aligned}\right\} \qquad (5)$$

Comparing (4) and (5) we get:

$$\left.\begin{aligned} \hat{e}_\alpha h_\alpha &= \hat{i}\,\frac{\partial f_x}{\partial \alpha} + \hat{j}\,\frac{\partial f_y}{\partial \alpha} + \hat{k}\,\frac{\partial f_z}{\partial \alpha} \\ \hat{e}_\beta h_\beta &= \hat{i}\,\frac{\partial f_x}{\partial \beta} + \hat{j}\,\frac{\partial f_y}{\partial \beta} + \hat{k}\,\frac{\partial f_z}{\partial \beta} \\ \hat{e}_\gamma h_\gamma &= \hat{i}\,\frac{\partial f_x}{\partial \gamma} + \hat{j}\,\frac{\partial f_y}{\partial \gamma} + \hat{k}\,\frac{\partial f_z}{\partial \gamma} \end{aligned}\right\} \qquad (6)$$

If the unit vectors $\hat{e}_\alpha$, $\hat{e}_\beta$, $\hat{e}_\gamma$ are mutually perpendicular, i.e. if the curvilinear co-ordinate system is <u>orthogonal</u> then we will have

$$\hat{e}_\alpha \cdot \hat{e}_\alpha = \hat{e}_\beta \cdot \hat{e}_\beta = \hat{e}_\gamma \cdot \hat{e}_\gamma = 1 \qquad (7)$$

and the metric coefficients h_α, h_β and h_γ can be evaluated from (6) by taking the <u>dot</u> products as follows:

i.e. $(\hat{e}_\alpha h_\alpha)\cdot(\hat{e}_\alpha h_\alpha) = \left(\hat{i}\,\frac{\partial f_x}{\partial \alpha} + \hat{j}\,\frac{\partial f_y}{\partial \alpha} + \hat{k}\,\frac{\partial f_z}{\partial \alpha}\right)\cdot\left(\hat{i}\,\frac{\partial f_x}{\partial \alpha} + \hat{j}\,\frac{\partial f_y}{\partial \alpha} + \hat{k}\,\frac{\partial f_z}{\partial \alpha}\right)$

$$h_\alpha^2 = \left(\frac{\partial f_x}{\partial \alpha}\right)^2 + \left(\frac{\partial f_y}{\partial \alpha}\right)^2 + \left(\frac{\partial f_z}{\partial \alpha}\right)^2$$

similarly $$h_\beta^2 = \left(\frac{\partial f_x}{\partial \beta}\right)^2 + \left(\frac{\partial f_y}{\partial \beta}\right)^2 + \left(\frac{\partial f_z}{\partial \beta}\right)^2 \qquad (8)$$

and $$h_\gamma^2 = \left(\frac{\partial f_x}{\partial \gamma}\right)^2 + \left(\frac{\partial f_y}{\partial \gamma}\right)^2 + \left(\frac{\partial f_z}{\partial \gamma}\right)^2$$

Equation (8) fascilitates the evaluation of the metric coefficients from the known functions f_x, f_y and f_z for any orthogonal curvilinear co-ordinate system.

Examples:

(a) to evaluate the metric coefficients for a cylindrical co-ordinate system. The cylindrical co-ordinate system superimposed on the cartesian co-ordinate system is shown in Figure 2.

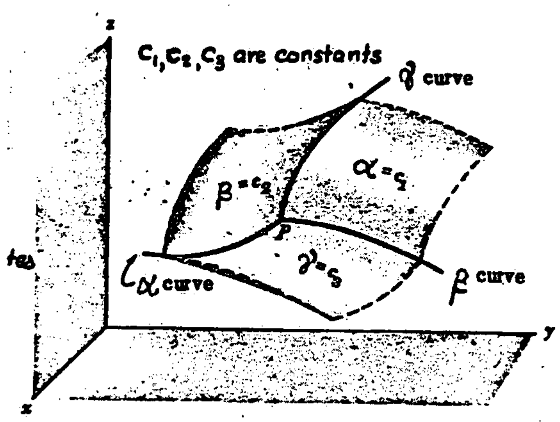

Figure 1

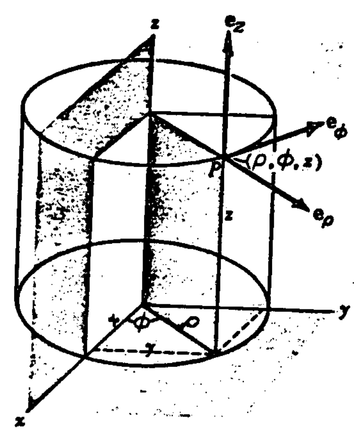

Figure 2

Referring to Figures 1 and 2.

$$\alpha = \rho$$
$$\beta = \phi \qquad (9)$$
and
$$\gamma = z$$

and again referring to Figure 2.

$$\left.\begin{aligned} x &= f_x(\rho;\phi;z) = \rho\cos\phi \\ y &= f_y(\rho;\phi;z) = \rho\sin\phi \\ z &= f_z(\rho;\phi;z) = z \end{aligned}\right\} \qquad (10)$$

and

$$h_\rho^2 = \left(\frac{\partial f_x}{\partial \rho}\right)^2 + \left(\frac{\partial f_y}{\partial \rho}\right)^2 + \left(\frac{\partial f_z}{\partial \rho}\right)^2$$
$$= (\cos\phi)^2 + (\sin\phi)^2 + 0 = \cos^2\phi + \sin^2\phi$$
$$= 1$$

or $\boxed{h_\rho = 1}$

Similarly:

$$h_\phi^2 = \left(\frac{\partial f_x}{\partial \phi}\right)^2 + \left(\frac{\partial f_y}{\partial \phi}\right)^2 + \left(\frac{\partial f_z}{\partial \phi}\right)^2$$

$$= (-\rho \sin \phi)^2 + (\rho \cos \phi)^2 + 0$$

$$= \rho^2 (\sin^2 \phi + \cos^2 \phi)$$

$$= \rho^2$$

or $\boxed{h_\phi = \rho}$

and

$$h_z^2 = \left(\frac{\partial f_x}{\partial z}\right)^2 + \left(\frac{\partial f_y}{\partial z}\right)^2 + \left(\frac{\partial f_z}{\partial z}\right)^2$$

$$= 0 + 0 + 1$$

or $\boxed{h_z = 1}$

Therefore, for a cylindrical co-ordinates system, the metric coefficients are:

$$\left.\begin{aligned} h_\rho &= 1 \\ \rho_\phi &= \rho \\ h_\rho &= 1 \end{aligned}\right\} \qquad (11)$$

(b) to evaluate the metric coefficients for a spherical co-ordinate system. The spherical co-ordinate system superimposed on the cartesian co-ordinate system is shown in Figure 3.

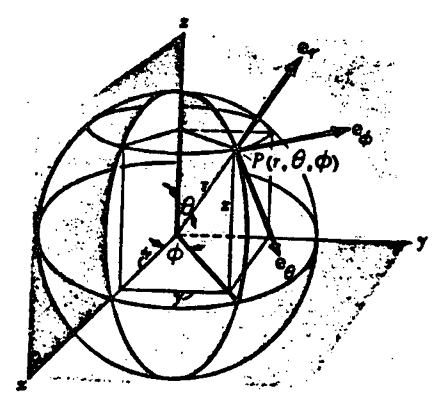

Figure 3

In this case:

$$\alpha = r \;;\; \beta = \theta \;;\; \gamma = \phi$$

and

$$x = f_x(r\,;\theta\,;\phi) = \sin\theta\cos\phi$$

$$y = f_y(r\,;\theta\,;\phi) = \sin\theta\sin\phi$$

$$z = f_z(r\,;\theta\,;\phi) = \cos\theta$$

$$h_r^2 = \left(\frac{\partial f_x}{\partial r}\right)^2 + \left(\frac{\partial f_y}{\partial r}\right)^2 + \left(\frac{\partial f_z}{\partial r}\right)^2$$

$$= (\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + (\cos\theta)^2$$

$$= \sin^2\theta\,(\cos^2\phi + \sin^2\phi) + \cos^2\theta$$

$$= \sin^2\theta + \cos^2\theta$$

$$= 1$$

or $\boxed{h_r = 1}$

$$h_\theta^2 = \left(\frac{\partial f_x}{\partial\theta}\right)^2 + \left(\frac{\partial f_y}{\partial\theta}\right)^2 + \left(\frac{\partial f_z}{\partial\theta}\right)^2$$

$$= (r\cos\phi\cos\theta)^2 + \{r\sin\phi\;(-\cos\theta)\;{}^2 + \;\; r\,(-\sin\theta)\}^2$$

$$= r^2\cos^2\theta\,(\cos^2\phi + \sin^2\phi) + r^2\sin^2\theta$$

$$= r^2(\cos^2\theta + \sin^2\theta)$$

$$= r^2$$

or $\boxed{h_\theta = r}$

$$h_\phi^2 = \left(\frac{\partial f_x}{\partial \phi}\right)^2 + \left(\frac{\partial f_y}{\partial \phi}\right)^2 + \left(\frac{\partial f_z}{\partial \phi}\right)^2$$

$$= \{r \sin\theta(-\sin\phi)\}^2 + (r \sin\theta \cos\phi)^2 + 0$$

$$= r^2 \sin^2\theta (\sin^2\phi + \cos^2\phi)$$

$$= r^2 \sin^2\theta$$

$$\therefore \boxed{h_\phi = r \sin\theta}$$

The metric coefficients for the spherical co-ordinates are:

$$\left.\begin{aligned} h &= 1 \\ h_\theta &= r \\ h_\phi &= r\sin\theta \end{aligned}\right\} \qquad (12)$$

(c) To evaluate the metric coefficients for a meandering co-ordinate system defined by Y. Chang*.

This co-ordinate system is devised to study flow processes in meandering channels. According to this system the channel consists of 90° circular bends connected by straight reaches with the circular bends alternating as shown in Figure 4. The x axis is taken along the centreline of the channel, y is vertical and z axis is perpendicular to both x and y.

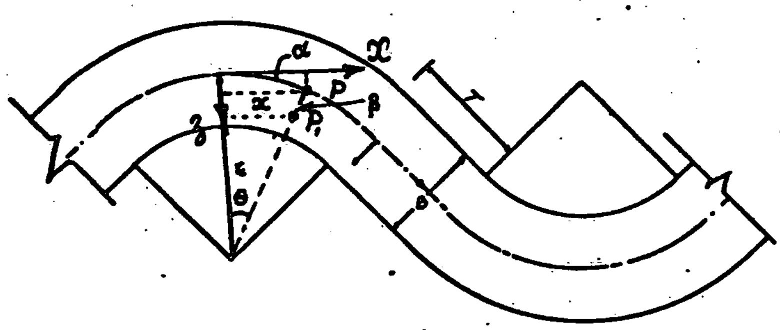

Figure 4

* Y. Chang, " Lateral mixing in meandering channels", Ph. D. Thesis, The University of Iowa, Engineering, Hydraulics, 1971.

Referring to Figure 4

$$x = \alpha \;;\; z = \beta \;;\; y = \gamma$$

and the functional relationships between these can be established as follows:

<u>Case 1</u>: When the bending is to the right

$$\alpha = r_c \theta$$

and

$$x = (r_c - \beta)\sin\theta = (r_c - \beta)\sin\left(\frac{\alpha}{r_c}\right) = fx(\alpha;\beta;\gamma)$$

$$z = r_c - (r_c - \beta)\cos\theta$$

$$= r_c - (r_c - \beta)\cos\left(\frac{\alpha}{r_c}\right) = f_z(\alpha;\beta;\gamma)$$

$$y = \gamma = f_y(\alpha;\beta;\gamma)$$

Therefore:

$$h_\alpha^2 = \left(\frac{\partial f_x}{\partial\alpha}\right)^2 + \left(\frac{\partial f_y}{\partial\alpha}\right)^2 + \left(\frac{\partial f_z}{\partial\alpha}\right)^2$$

$$= \left[\frac{r_c - \beta}{r_c}.\cos\left(\frac{d}{r_c}\right)\right]^2 + \left[\frac{r_c - \beta}{r_c}.\sin\left(\frac{\alpha}{r_c}\right)\right]^2 + 0$$

$$= \left[\frac{r_c - \beta}{r_c}\right]^2 \left(\cos^2\left(\frac{\alpha}{r_c}\right) + \sin^2\left(\frac{\alpha}{r_c}\right)\right)$$

$$= \left[\frac{r_c - \beta}{r_c}\right]^2$$

or

$$\boxed{h_\alpha = \left[\frac{r_c - \beta}{r_c}\right]}$$

$$h_\beta^2 = \left(\frac{\partial f_x}{\partial \beta}\right)^2 + \left(\frac{\partial f_y}{\partial \beta}\right)^2 + \left(\frac{\partial f_z}{\partial \beta}\right)^2$$

$$= \left(-\sin\frac{\alpha}{r_c}\right)^2 + \left(\cos\frac{\alpha}{r_c}\right)^2 + 0$$

$$= \sin^2\left(\frac{\alpha}{r_c}\right) + \cos^2\left(\frac{\alpha}{r_c}\right)$$

$$= 1$$

or $\boxed{h_\beta = 1}$

$$h_\gamma^2 = \left(\frac{\partial f_x}{\partial \gamma}\right)^2 + \left(\frac{\partial f_y}{\partial \gamma}\right)^2 + \left(\frac{\partial f_z}{\partial \gamma}\right)^2$$

$$= 0 + 0 + 1$$

$$= 1$$

or $\boxed{h_\gamma = 1}$

Therefore, the metric coefficients for a meandering co-ordinate system when the curve is towards the right are:

$$h_\alpha = \frac{r_c - \beta}{r_c}$$

$$h_\beta = 1$$

$$h_\gamma = 1$$

It can be easily shown that when the bend curves to the left the metric coefficients become:

$$h_\alpha = \frac{r_c + \beta}{r_c}$$

$$h_\beta = 1$$

$$h_\gamma = 1$$

Of course, for straight reach, the meandering co-ordinate system converges to cartesian co-ordinate system for which the metric coefficients are unity.

1.1 To derive expressions for gradient, divergence, curl and laplacian in the curvilinear co-ordinate system.

1.1.1 Gradient

Let $\nabla\phi = f_\alpha \hat{e}_\alpha + f_\beta \hat{e}_\beta + f_\gamma \hat{e}_\gamma$, where f_α, f_β and f_γ are to be determined. Using the expression for $d\hat{r}$ given by equation (4) i.e. by using $d\hat{r} = \hat{e}_\alpha h_\alpha d\alpha + \hat{e}_\beta h_\beta d\beta + \hat{e}_\gamma h_\gamma d\gamma$ and using the identity $\nabla\phi . d\hat{r} = d\phi$ we get:

$$d\phi = f_\alpha h_\alpha d\alpha + f_\beta h_\beta d\beta + f_\gamma h_\gamma d\gamma \tag{13}$$

But we know

$$d\phi = \frac{\partial \phi}{\partial \alpha} \alpha + \frac{\partial \phi}{\partial \beta} \beta + \frac{\partial \phi}{\partial \gamma} . d\gamma \tag{14}$$

Comparing (13) and (14) we get:

$$\left.\begin{aligned} f_\alpha &= \frac{1}{h_\alpha} \frac{\partial \phi}{\partial \alpha} \\ f_\beta &= \frac{1}{h_\beta} \frac{\partial \phi}{\partial \beta} \\ f_\gamma &= \frac{1}{h} \frac{\partial \phi}{\partial \gamma} \end{aligned}\right] \tag{15}$$

Substituting (15) in the expression for gradient

i.e.

$$\nabla \phi = \frac{\hat{e}_\alpha}{h_\alpha} \frac{\partial \phi}{\partial \alpha} + \frac{e_\beta}{h_\beta} \frac{\partial \phi}{\partial \beta} + \frac{\hat{e}_\gamma}{h_\gamma} \frac{\partial \phi}{\partial \gamma} \tag{16}$$

or

$$\nabla\phi = \left(\frac{\hat{e}_\alpha}{h_\alpha} \frac{\partial}{\partial \alpha} + \frac{\hat{e}_\beta}{h_\beta} \frac{\partial}{\partial \beta} + \frac{\hat{e}_\gamma}{h_\gamma} \frac{\partial}{\partial \gamma}\right) \phi \tag{17}$$

From equation (17)we get:

$$\nabla = \frac{\hat{e}_\alpha}{h_\alpha} \frac{\partial}{\partial \alpha} + \frac{\hat{e}_\beta}{h_\beta} \frac{\partial}{\partial \beta} + \frac{\hat{e}_\gamma}{h_\gamma} \frac{\partial}{\partial \gamma} \tag{18}$$

Equation (18) gives the expression for the del operator in the curvilinear co-ordinate system while equation (17) or (16) gives the gradient of ϕ in the same system of co-ordinates.

1.1.2 Divergence

Let $\hat{A} = A_\alpha \hat{e}_\alpha + A_\beta \hat{e}_\beta + A_\gamma \hat{e}_\gamma$ be a vector in the curvilinear co-ordinate system. The divergence of this vector in the curvilinear co-ordinates becomes:

$$\nabla . \hat{A} = \left(\frac{\hat{e}_\alpha}{h_\alpha}\frac{\partial}{\partial \alpha} + \frac{\hat{e}_\beta}{h_\beta}\frac{\partial}{\partial \beta} + \frac{\hat{e}_\gamma}{h_\gamma}\frac{\partial}{\partial \gamma}\right) . \left(A_\alpha \hat{e}_a + A_\beta \hat{e}_\beta + A_\gamma \hat{e}_\gamma\right) \tag{19}$$

$$\begin{aligned} = & \left(\frac{1}{h_\alpha}\frac{\partial A\alpha}{\partial \alpha} + \frac{1}{h_\beta}\frac{\partial A_\beta}{\partial \beta} + \frac{1}{h_\gamma}\frac{\partial A_\gamma}{\partial \gamma}\right) + \\ & A_\alpha\left[\frac{\hat{e}_\alpha}{h_\alpha}\frac{\partial \hat{e}_\alpha}{\partial \alpha} + \frac{\hat{e}_\beta}{h_\beta}\frac{\partial \hat{e}_\alpha}{\partial \beta} + \frac{\hat{e}_\gamma}{h_\gamma}\frac{\partial \hat{e}_\alpha}{\partial \gamma}\right] + \\ & A_\beta\left[\frac{\hat{e}_\alpha}{h_\alpha}\frac{\partial \hat{e}_\beta}{\partial \alpha} + \frac{\hat{e}_\beta}{h_\beta}\frac{\partial \hat{e}_\beta}{\partial \beta} + \frac{\hat{e}_\gamma}{h_\gamma}\frac{\partial \hat{e}_\beta}{\partial \gamma}\right] + \\ & A_\gamma\left[\frac{\hat{e}_\alpha}{h_\alpha}\frac{\partial \hat{e}_\gamma}{\partial \alpha} + \frac{\hat{e}_\beta}{h_\beta}\frac{\partial \hat{e}_\gamma}{\partial \beta} + \frac{\hat{e}_\gamma}{h_\gamma}\frac{\partial \hat{e}_\gamma}{\partial \gamma}\right] \end{aligned} \tag{20}$$

Note that in equation (20) the expressions within the square brackets do not vanish unlike the case of the cartesian co-ordinate system. The terms within the square brackets express the rate of change of the unit vectors along the co-ordinate axes. Even though the magnitude of the unit vectors does not change, the directions change with the co-ordinate axes for the culvilinear case. Hence, in order to evaluate the divergence of a vector in curvilinear co-ordinates, the derivatives of the unit vectors with respect to co-ordinate axes have to be evaluated first. This is done in Appendix A and the result expressed in matrix form is as follows:

$$\begin{bmatrix} \frac{\partial \hat{e}_\alpha}{\partial \alpha} & \frac{\partial \hat{e}_\alpha}{\partial \beta} & \frac{\partial \hat{e}_\alpha}{\partial \gamma} \\ \frac{\partial \hat{e}_\beta}{\partial \alpha} & \frac{\partial \hat{e}_\beta}{\partial \beta} & \frac{\partial \hat{e}_\beta}{\partial \gamma} \\ \frac{\partial \hat{e}_\gamma}{\partial \alpha} & \frac{\partial \hat{e}_\gamma}{\partial \beta} & \frac{\partial \hat{e}_\gamma}{\partial \gamma} \end{bmatrix} = \begin{bmatrix} \left(-\frac{\hat{e}_\beta}{h_\beta}\frac{\partial h_\alpha}{\partial \beta} - \frac{\hat{e}_\gamma}{h_\gamma}\frac{\partial h_\alpha}{\partial}\right) & \frac{\hat{e}_\beta}{h_\alpha}\frac{\partial h_\beta}{\partial \alpha} & \frac{\hat{e}_\gamma}{h_\alpha}\frac{\partial h_\gamma}{\partial \alpha} \\ \frac{\hat{e}_\alpha}{h_\beta}\frac{\partial h_\alpha}{\partial \beta} & \left(-\frac{\hat{e}_\alpha}{h_\alpha}\frac{\partial h_\beta}{\partial \alpha} - \frac{\hat{e}_\gamma}{h_\gamma}\frac{\partial h_\beta}{\partial \gamma}\right) & \frac{\hat{e}_\gamma}{h_\beta}\frac{\partial h_\gamma}{\partial \beta} \\ \frac{\hat{e}_\alpha}{h_\gamma}\frac{\partial h_\alpha}{\partial \gamma} & \frac{e_\beta}{h_\gamma}\frac{\partial h_\beta}{\partial \gamma} & \left(-\frac{e_\alpha}{h_\alpha}\frac{\partial h_\gamma}{\partial \alpha} - \frac{\hat{e}_\beta}{h_\beta}\frac{\partial h_\gamma}{\partial \beta}\right) \end{bmatrix}$$

(21)

Substituting the values of the derivatives of unit vectors appearing in equation (20) and rearranging terms (Appendix B) the expression for the divergence in curvilinear co-ordinate systems can be obtained as:

$$\nabla.\hat{A} = \frac{1}{h_a h_\beta h_\gamma} \left[\frac{\partial}{\partial\alpha} (h_\beta h_\gamma A_\alpha) + \frac{\partial}{\partial\beta} (h_\alpha h_\gamma A_\beta) + \frac{\partial}{\partial\gamma} (h_\alpha h_\beta A_\gamma) \right] \quad (22)$$

In a similar fashion, the expression for the curl of a vector $\hat{A}$ in the curvilinear co-ordinate system can be derived (Appendix C) which is

$$\nabla X \hat{A} = \frac{1}{h_\alpha h_\beta h_\gamma} \begin{bmatrix} h_\alpha \hat{e}_\alpha & h_\beta \hat{e}_\beta & h_\gamma \hat{e}_\gamma \\ \frac{\partial}{\partial\alpha} & \frac{\partial}{\partial\beta} & \frac{\partial}{\partial\gamma} \\ h_\alpha A_\alpha & h_\beta A_\beta & h_\gamma A_\gamma \end{bmatrix} \quad (23)$$

1.1.3 Laplacian

The laplacian of a scalar ϕ in a curvilinear co-ordinate system can be evaluated using equations (16) and (22) and expressing laplacian $\nabla^2\phi$ as

$$\nabla^2\phi = \nabla.(\nabla\phi)$$

The expression for $\nabla^2\phi$ becomes:

$$\nabla^2\phi = \frac{1}{h_\alpha h_\beta h_\gamma} \left[\frac{\partial}{\partial\alpha} \left(\frac{h_\beta h_\gamma}{h_\alpha} \frac{\partial\phi}{\partial\alpha} \right) + \frac{\partial}{\partial\beta} \left(\frac{h_\alpha h_\gamma}{h_\beta} \frac{\partial\phi}{\partial\beta} \right) + \frac{\partial}{\partial\gamma} \left(\frac{h_\alpha h_\beta}{h_\gamma} \frac{\partial\phi}{\partial\gamma} \right) \right] \quad (24)$$

1.2 Using equations (18), (16), (22), (23) and (24) some of the commonly used equations in fluid dynamics will be expressed in curvilinear co-ordinates as follows:

1.2.1 Material derivative D/Dt.

Definition of D/Dt in terms of vector notation is:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \hat{V}.\nabla \quad (25)$$

where $\hat{V}$ is the velocity vector given by

$$\hat{V} = \hat{e}_\alpha . u + \hat{e}_\beta v + \hat{e}_\gamma w$$

substituting for ∇ we get:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\hat{e}_\alpha u + \hat{e}_\beta v + \hat{e}_\gamma w) . (\frac{\hat{e}_a}{h_\alpha} \frac{\partial}{\partial \alpha} + \frac{\hat{e}_\beta}{h_\beta} \frac{\partial}{\partial \beta} + \frac{\hat{e}_\gamma}{h_\gamma} \frac{\partial}{\partial \gamma})$$

$$= \frac{\partial}{\partial t} + \frac{u}{h_\alpha} \frac{\partial}{\partial \alpha} + \frac{v}{h_\beta} \frac{\partial}{\partial \beta} + \frac{w}{h_\gamma} \frac{\partial}{\partial \gamma} \qquad (26)$$

1.2.2 Continuity equation

For a compressible flow the continuity equation in vector notation is:

$$\frac{D}{Dt} \rho + \rho(\nabla . V) = 0 \qquad (27)$$

where ρ is the density. Using equations (26) and (22), equation (27) takes the following form in curvilinear co-ordinates:

$$\frac{\partial \rho}{\partial t} + \frac{u}{h_\alpha} \frac{\partial \rho}{\partial \alpha} + \frac{v}{h_\beta} \frac{\partial \rho}{\partial \beta} + \frac{w}{h_\gamma} \frac{\partial \rho}{\partial \gamma} + \frac{\rho}{h_\alpha h_\beta h_\gamma} (\frac{\partial}{\partial \alpha} h_\beta h_\gamma u + \frac{\partial}{\partial \beta} h_\alpha h_\gamma u + \frac{\partial}{\partial \gamma} h_\alpha h_\beta w) = 0 \qquad (28)$$

For imcompressible flows, the above equation reduces to:

$$\frac{\partial}{\partial \alpha} (h_\beta h_\gamma u) + \frac{\partial}{\partial \beta} (h_\alpha h_\gamma v) + \frac{\partial}{\partial \gamma} (h_\alpha h_\beta w) = 0 \qquad (29)$$

1.2.3 Velocity potential ϕ

The velocity potential ϕ of a velocity vector V is defined as:

$$\hat{V} = -\nabla \phi$$

i.e

$$\hat{e}_\alpha u + \hat{e}_\beta v + \hat{e}_\gamma w = -(\frac{\hat{e}_\alpha}{h_\alpha} \frac{\partial \phi}{\partial \alpha} + \frac{\hat{e}_\beta}{h_\beta} \frac{\partial \phi}{\partial \beta} + \frac{\hat{e}_\gamma}{h_\gamma} \frac{\partial \phi}{\partial \gamma})$$

comparing terms we get:

$$
\left.
\begin{aligned}
u &= -\frac{1}{h_\alpha}\frac{\partial \phi}{\partial \alpha} \\
v &= -\frac{1}{h_\beta}\frac{\partial \phi}{\partial \beta} \\
w &= -\frac{1}{h_\gamma}\frac{\partial \phi}{\partial \gamma}
\end{aligned}
\right\} \qquad (30)
$$

1.2.4 Stream function ψ for a two-dimensional flow

By definition $\hat{V}.\nabla\psi = 0$

i.e.

$$
(\hat{e}_\alpha u + \hat{e}_\beta v) \,.\, \left(\frac{\hat{e}_\alpha}{h_\alpha}\frac{\partial \psi}{\partial \alpha} + \frac{\hat{e}_\beta}{h_\beta}\frac{\partial \psi}{\partial \beta}\right) = 0 = \frac{u}{h_\alpha}\frac{\partial \psi}{\partial \alpha} + \frac{v}{h_\beta}\frac{\partial \psi}{\partial \beta} = 0 \quad (31)
$$

Equation (31) will be satisfied by defining ψ as follows:

$$
\left.
\begin{aligned}
u &= -\frac{1}{h_\beta}\frac{\partial \psi}{\partial \beta} \\
v &= \frac{1}{h_\alpha}\frac{\partial \psi}{\partial \alpha}
\end{aligned}
\right\} \qquad (32)
$$

1.2.5 Circulation Γ

Circulation Γ in a flow field along a closed line c is given by

$$
\Gamma = \oint_c \hat{V}.\, d\hat{r} \qquad (33)
$$

using equation (4) Γ in curvilinear co-ordinates becomes:

$$
\begin{aligned}
\Gamma &= \oint_c (\hat{e}_\alpha u + \hat{e}_\beta v + \hat{e}_\gamma w).\,(e_\alpha h_\alpha d\alpha + \hat{e}_\beta h_\beta d\beta + \hat{e}_\gamma h_\gamma d\gamma) \\
&= \oint_c (u h_\alpha d\alpha + v h_\beta d\beta + w h_\gamma d\gamma) \qquad (34)
\end{aligned}
$$

1.2.6 Vorticity $\hat{\omega}$

The vorticity vector $\hat{\omega}$ is defined as

$$\hat{\omega} = \frac{1}{2} \nabla X \hat{V} \tag{35}$$

using equation (23) the equation (35) can be expanded as:

$$(\hat{e}_\alpha \xi + \hat{e}_\beta \eta + \hat{e}_\gamma \zeta) = \hat{e}_\alpha \frac{1}{2h_\beta h_\gamma} \{ \frac{\partial}{\partial \beta}(h_\gamma w) - \frac{\partial}{\partial \gamma}(h_\beta v) \} + \hat{e}_\beta \frac{1}{2h_\alpha h_\gamma} \{ \frac{\partial}{\partial \alpha}(h_\gamma w) - \frac{\partial}{\partial \gamma}(h_\alpha u) \} + \hat{e}_\gamma \frac{1}{2h_a h_\beta} \{ \frac{\partial}{\partial \alpha}(h_\beta v) - \frac{\partial}{\partial \beta}(h_\alpha u) \}$$

comparing terms, the components of the vorticity vector w become:

$$\xi = \frac{1}{2h_\beta h_\gamma} \{ \frac{\partial}{\partial \beta}(h_\gamma w) - \frac{\partial}{\partial \gamma}(h_\beta v) \} \tag{36a}$$

$$\eta = \frac{1}{2h_\alpha h_\gamma} \{ \frac{\partial}{\partial \alpha}(h_\gamma w) - \frac{\partial}{\partial \gamma}(h_\alpha u) \} \tag{36b}$$

$$\rho = \frac{1}{2h_\alpha h_\beta} \{ \frac{\partial}{\partial \alpha}(h_\beta v) - \frac{\partial}{\partial \beta}(h_\alpha u) \} \tag{36c}$$

1.2.7 Laplace equation for incompressible flows

The Laplace equation which gives the velocity potential ϕ for an incompressible flow is

$$\nabla^2 \phi = 0 \tag{37}$$

Using equation (24), the Laplace equation in curvilinear co-ordinates becomes

$$\frac{\partial}{\partial \alpha}\left(\frac{h_\beta h_\gamma}{h_\alpha} \frac{\partial \phi}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{h_\alpha h_\gamma}{h_\beta} \frac{\partial \phi}{\partial \beta}\right) + \frac{\partial}{\partial \gamma}\left(\frac{h_\beta h_\alpha}{h_\gamma} \frac{\partial \phi}{\partial \gamma}\right) = 0 \tag{38}$$

1.2.8 Euler equation

When tangential stresses are neglected the equation of motion of a fluid can be expressed in vector notation as follows:

$$\frac{D}{Dt} \hat{V} = \frac{G}{\rho} - \frac{1}{\rho} \nabla P \tag{39}$$

where G is the gravitational body force per unit volume and P is the pressure. This equation in component form in curvilinear co-ordinates becomes:

$$\frac{\partial}{\partial t} u + \frac{u}{h_\alpha}\frac{\partial u}{\partial \alpha} + \frac{v}{h_\beta}\frac{\partial u}{\partial \beta} + \frac{w}{h_\gamma}\frac{\partial u}{\partial \gamma} = \frac{G_\alpha}{\rho} - \frac{1}{\rho}\cdot\frac{1}{h_\alpha}\frac{\partial P}{\partial \alpha} \quad (40a)$$

$$\frac{\partial}{\partial t} v + \frac{u}{h_\alpha}\frac{\partial v}{\partial \alpha} + \frac{v}{h_\beta}\frac{\partial v}{\partial \beta} + \frac{w}{h_\gamma}\frac{\partial v}{\partial \gamma} = \frac{G_\beta}{\rho} - \frac{1}{\rho}\cdot\frac{1}{h_\beta}\frac{\partial P}{\partial \beta} \quad (40b)$$

$$\frac{\partial}{\partial t} w + \frac{u}{h_\alpha}\frac{\partial w}{\partial \alpha} + \frac{v}{h_\beta}\frac{\partial w}{\partial \beta} + \frac{w}{h_\gamma}\frac{\partial w}{\partial \gamma} = \frac{G_\gamma}{\rho} - \frac{1}{\rho}\cdot\frac{1}{h_\gamma}\frac{\partial P}{\partial \gamma} \quad (40c)$$

1.2.9 Navier-Stokes equation

The Navier-Stokes equation for an incompressible flow with constant viscosity will be expressed in curvilinear co-ordinates as follows. The equation in vector notation is

$$\frac{D\hat{V}}{Dt} = \frac{G}{\rho} - \frac{1}{\rho}\nabla P - \nu\,(\nabla \times \nabla \times \hat{V}) \quad (41)$$

where ν is the kinematic viscosity of the fluid.

When expressed in component form, equation (41) is equivalent to:

$$\frac{\partial}{\partial t} u + \frac{u}{h_\alpha}\frac{\partial u}{\partial \alpha} + \frac{v}{h_\beta}\frac{\partial u}{\partial \beta} + \frac{w}{h_\gamma}\frac{\partial u}{\partial \gamma} = \frac{G_\alpha}{\rho} - \frac{1}{\rho}\cdot\frac{1}{h_\alpha}\frac{\partial P}{\partial \alpha} -$$
$$\nu\Bigg[\frac{h_\gamma}{h_\beta}\left\{\frac{\partial^2}{\partial\alpha\partial\beta}(h_\beta v) - \frac{\partial^2}{\partial\beta^2}(h_\alpha u)\right\} +$$
$$\frac{h_\beta}{h_\gamma}\left\{\frac{\partial^2}{\partial\alpha\partial\gamma}(h_\gamma w) - \frac{\partial^2}{\partial\gamma^2}(h_\alpha u)\right\} +$$
$$h_\alpha \frac{\partial}{\partial\beta}\left(\frac{h_\gamma}{h_\alpha h_\beta}\right)\left\{\frac{\partial}{\partial\alpha}(h_\beta v) - \frac{\partial}{\partial\beta}(h_\alpha u)\right\} +$$
$$h_\alpha \frac{\partial}{\partial\gamma}\left(\frac{h_\beta}{h_\alpha h_\gamma}\right)\left\{\frac{\partial}{\partial\alpha}(h_\gamma w) - \frac{\partial}{\partial\alpha}(h_\alpha u)\right\}\Bigg] \quad (41a)$$

$$\frac{\partial}{\partial t} v + \frac{u}{h_\alpha}\frac{\partial v}{\partial \alpha} + \frac{v}{h_\beta}\frac{\partial v}{\partial \beta} + \frac{w}{h_\gamma}\frac{\partial v}{\partial \gamma} = \frac{G_\beta}{\rho} - \frac{1}{\rho}\cdot\frac{1}{h_\beta}\frac{\partial P}{\partial \beta} -$$
$$\nu\Bigg[\frac{h_\alpha}{h_\gamma}\left\{\frac{\partial^2}{\partial\beta\partial\gamma}(h_\gamma w) - \frac{\partial^2}{\partial\gamma^2}(h_\beta v)\right\} +$$

$$\frac{h_\gamma}{h_\alpha}\left\{\frac{\partial^2}{\partial\beta\partial\alpha}(h_\alpha u) - \frac{\partial^2}{\partial\alpha^2}(h_\beta v)\right\} +$$

$$h_\beta \frac{\partial}{\partial\gamma}\left(\frac{h_\alpha}{h_\beta h_\gamma}\right)\left\{\frac{\partial}{\partial\beta}(h_\gamma w) - \frac{\partial}{\partial\gamma}(h_\beta v)\right\} +$$

$$h_\beta \frac{\partial}{\partial\alpha}\left(\frac{h_\gamma}{h_\beta h_\alpha}\right)\left\{\frac{\partial}{\partial\beta}(h_\alpha u) - \frac{\partial}{\partial\alpha}(h_\beta v)\right\} \Bigg] \qquad (41b)$$

$$\frac{\partial}{\partial t} w + \frac{u}{h_\alpha}\frac{\partial w}{\partial\alpha} + \frac{v}{h_\beta}\frac{\partial w}{\partial\beta} + \frac{w}{h_\gamma}\frac{\partial w}{\partial\gamma} = \frac{G}{\rho} - \frac{1}{\rho}\cdot\frac{\partial P}{\partial\gamma} -$$

$$\nu\Bigg[\frac{h_\beta}{h_\alpha}\left\{\frac{\partial^2}{\partial\gamma\partial\alpha}(h_\alpha u) - \frac{\partial^2}{\partial\alpha^2}(h_\gamma w)\right\} +$$

$$\frac{h_\alpha}{h_\beta}\left\{\frac{\partial^2}{\partial\gamma\partial\beta}(h_\beta v) - \frac{\partial^2}{\partial\beta^2}(h_\gamma w)\right\} +$$

$$h_\gamma \frac{\partial}{\partial\alpha}\left(\frac{h_\beta}{h_\gamma h_\alpha}\right)\left\{\frac{\partial}{\partial\gamma}(h_\alpha u) - \frac{\partial}{\partial\alpha}(h_\gamma w)\right\} +$$

$$h_\gamma \frac{\partial}{\partial\alpha}\left(\frac{h_\alpha}{h_\gamma h_\alpha}\right)\left\{\frac{\partial}{\partial\gamma}(h_\beta v) - \frac{\partial}{\partial\beta}(h_\gamma w)\right\}\Bigg] \qquad (41c)$$

1.2.10 Boundary layer equations

Prandtl's boundary layer equations in culvilinear co-ordinates take the following forms:

$$\frac{\partial u}{\partial t} + \frac{u}{h_\alpha}\frac{\partial u}{\partial\alpha} + \frac{v}{h_\beta}\frac{\partial u}{\partial\beta} + \frac{w}{h_\gamma}\frac{\partial u}{\partial\gamma} + \frac{uv}{h_\alpha h_\beta}\frac{\partial h_\alpha}{\partial\beta} - \frac{v^2}{h_\alpha h_\beta}\frac{\partial h_\beta}{\partial\alpha} = -\frac{1}{\rho}\cdot\frac{1}{h_\alpha}\frac{\partial P}{\partial\alpha} + \nu\frac{\partial^2 u}{\partial\gamma^2} \qquad (42a)$$

$$\frac{\partial v}{\partial t} + \frac{u}{h_\alpha}\frac{\partial v}{\partial\alpha} + \frac{v}{h_\beta}\frac{\partial v}{\partial\beta} + \frac{w}{h_\gamma}\frac{\partial v}{\partial\gamma} + \frac{uv}{h_\alpha h_\beta}\frac{\partial h_\beta}{\partial\alpha} - \frac{u^2}{h_\alpha h_\beta}\frac{\partial h_\alpha}{\partial\beta} = -\frac{1}{\rho}\frac{1}{h_\beta}\frac{\partial P}{\partial\beta} + \nu\frac{\partial^2 v}{\partial\gamma^2} \qquad (42b)$$

$$\frac{\partial P}{\partial\gamma} = 0 \qquad (42c)$$

$$\frac{\partial\rho}{\partial t} + \frac{1}{h_\alpha}\frac{\partial}{\partial\alpha}(\rho u) + \frac{1}{h_\beta}\frac{\partial}{\partial\beta}(\rho v) + \frac{1}{h_\gamma}\frac{\partial}{\partial\gamma}(\rho w) + \frac{\rho u}{h_\alpha h_\beta}\frac{\partial h_\beta}{\partial\alpha} + \frac{\rho v}{h_\alpha h_\beta}\frac{\partial h_\alpha}{\partial\beta} = 0 \qquad (42d)$$

1.2.11 Diffusion equation

The concentration C of a diffusing substance is given by the diffusion equation which in vector notation can be expressed as:

$$\frac{D}{Dt} C = \nabla . (\varepsilon_i \nabla C) \qquad (43)$$

where ε_i is the diffusion coefficient in the ith direction.

In the curvilinear co-ordinate system, equation (43) becomes:

$$\frac{\partial C}{\partial t} + \frac{u}{h_\alpha}\frac{\partial C}{\partial \alpha} + \frac{v}{h_\beta}\frac{\partial C}{\partial \beta} + \frac{w}{h_\gamma}\frac{\partial C}{\partial \gamma} = \frac{1}{h_\alpha h_\beta h_\gamma}\left[\frac{\partial}{\partial \alpha}\left(\frac{h_\beta h_\gamma}{h_\alpha}\varepsilon_\alpha \frac{\partial C}{\partial \alpha}\right)\right.$$

$$\left. + \frac{\partial}{\partial \beta}\left(\frac{h_\alpha h_\gamma}{h_\beta}\varepsilon_\beta \frac{\partial C}{\partial \beta}\right) + \frac{\partial}{\partial \gamma}\left(\frac{h_\alpha h_\beta}{h_\gamma}\varepsilon_\gamma \frac{\partial C}{\partial \gamma}\right)\right] \qquad (44)$$

2.0 REFERENCES

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APPENDIX A

Let us consider the operation $\nabla(\hat{e}_\alpha . \hat{e}_\alpha) = \nabla(1) = 0$. Using the vector identity:

$$\nabla(\hat{a}.\hat{b}) = (\hat{a}.\nabla)\hat{b} + (\hat{b}.\nabla)\hat{a} + \hat{a}\times(\nabla\times\hat{b}) + \hat{b}\times(\nabla\times\hat{a})$$

and identifying $\hat{e}_\alpha = \hat{a} = \hat{b}$ we get

$$\nabla(\hat{e}_\alpha . \hat{e}_\alpha) = (\hat{e}_\alpha . \nabla)\hat{e}_\alpha + (\hat{e}_\alpha . \nabla)\hat{e}_\alpha + \hat{e}_\alpha \times (\nabla \times \hat{e}_\alpha) + \hat{e}_\alpha \times (\nabla \times \hat{e}_\alpha) = 0$$

i.e.

$$2(\hat{e}_\alpha . \nabla)\hat{e}_\alpha + 2\hat{e}_\alpha \times (\nabla \times \hat{e}_\alpha) = 0$$

or

$$(\hat{e}_\alpha . \nabla)\hat{e}_\alpha = -\hat{e}_\alpha \times (\nabla \times \hat{e}_\alpha) \tag{A1}$$

LHS of equation (A1) can be expanded as follows:

$$(\hat{e}_\alpha . \nabla)\hat{e}_\alpha = \hat{e}_\alpha . \left(\frac{\hat{e}_\alpha}{h_\alpha}\frac{\partial}{\partial \alpha} + \frac{\hat{e}_\beta}{h_\beta}\frac{\partial}{\partial \beta} + \frac{\hat{e}_\gamma}{h_\gamma}\frac{\partial}{\partial \gamma}\right)\hat{e}_\alpha = \frac{1}{h_\alpha}\frac{\partial \hat{e}_a}{\partial \alpha} \tag{A2}$$

In order to evaluate the RHS of equation (A1), we have to know $(\nabla \times \hat{e}_\alpha)$. This can be evaluated as follows. Consider the gradient of α

i.e.

$$\nabla\alpha = \left(\frac{\hat{e}_\alpha}{h_\alpha}\frac{\partial}{\partial \alpha} + \frac{\hat{e}_\beta}{h_\beta}\frac{\partial}{\partial \beta} + \frac{\hat{e}_\gamma}{h_\gamma}\frac{\partial}{\partial \gamma}\right)\alpha = \frac{\hat{e}_\alpha}{h_\alpha} \tag{A3}$$

If we use the vector identity $\nabla \times (\nabla\phi) = 0$, then we will have

$$\nabla \times \left(\frac{\hat{e}_\alpha}{h_\alpha}\right) = 0 \tag{A4}$$

Equation (A4) can be expanded using the vector identity:

$$\nabla \times \phi\,\hat{a} = \phi\,\nabla \times \hat{a} - \hat{a} \times \nabla\phi$$

Identifying $\hat{e}_\alpha$ with $\hat{a}$ and $1/h_\alpha$ with ϕ, we get:

$$\nabla \times \left(\frac{\hat{e}_\alpha}{h_\alpha}\right) = \frac{1}{h_\alpha} \nabla \times \hat{e}_\alpha - \hat{e}_\alpha \times \nabla\left(\frac{1}{h_\alpha}\right) = 0$$

$$= \frac{1}{h_\alpha} \nabla \times \hat{e}_\alpha - \hat{e}_\alpha \times \frac{h_\alpha \nabla(1) - 1 \nabla(h_\alpha)}{h_\alpha^2} = 0$$

$$= \frac{1}{h_\alpha} \nabla \times \hat{e}_\alpha + \frac{\hat{e}_\alpha \times \nabla(h_\alpha)}{h_\alpha^2} = 0$$

$$= \nabla \times \hat{e}_\alpha + \frac{\hat{e}_a \times \nabla(h_\alpha)}{h_\alpha} = 0$$

$$\text{or} \quad \nabla \times \hat{e}_\alpha = -\hat{e}_\alpha \times \frac{\nabla(h_\alpha)}{h_\alpha} \tag{A5}$$

The RHS of equation (A1) becomes:

$$-\hat{e}_\alpha \times \left[-\hat{e}_\alpha \times \frac{\nabla(h_\alpha)}{h_\alpha}\right] = \hat{e}_\alpha \times \frac{(\hat{e}_\alpha \times \nabla(h_\alpha))}{h_\alpha} \tag{A6}$$

This can be expanded as follows:

$$\hat{e}_\alpha \times \frac{\hat{e}_\alpha \times \left(\frac{\hat{e}_\alpha}{h_\alpha}\frac{\partial}{\partial \alpha} + \frac{\hat{e}_\beta}{h_\beta}\frac{\partial}{\partial \beta} + \frac{\hat{e}_\gamma}{h_\gamma}\frac{\partial}{\partial \gamma}\right) h_\alpha}{h_\alpha}$$

$$= \hat{e}_a \times \frac{\hat{e}_\alpha \times \left(\frac{\hat{e}_\alpha}{h_\alpha}\frac{\partial h_\alpha}{\partial \alpha} + \frac{\hat{e}_\beta}{h_\beta}\frac{\partial h_\alpha}{\partial \beta} + \frac{\hat{e}_\gamma}{h_\gamma}\frac{\partial h_\gamma}{\partial \gamma}\right)}{h_\alpha}$$

Using $\hat{e}_\alpha \times \hat{e}_\alpha = \hat{e}_\beta \times \hat{e}_\beta = \hat{e}_\gamma \times \hat{e}_\gamma = 0$ and $\hat{e}_\alpha \times \hat{e}_\beta = \hat{e}_\gamma$ $\hat{e}_\alpha \times \hat{e}_\gamma = \rho\, \hat{e}_\beta$ and so on we get

$$= e_a \times \frac{1}{h_\alpha}\left[\frac{\hat{e}_\beta}{h_\beta}\frac{\partial h_\alpha}{\partial \beta} - \frac{\hat{e}_\beta}{h_\gamma}\frac{\partial h_\alpha}{\partial \gamma}\right]$$

$$= \left[-\frac{\hat{e}_\beta}{h_\alpha h_\beta}\frac{\partial h_a}{\partial \beta} - \frac{\hat{e}_\gamma}{h_\alpha h_\nu}\frac{\partial h_\alpha}{\partial \gamma}\right]$$

Therefore equation (A1) can be written as:

$$\frac{1}{h_\alpha}\frac{\partial \hat{e}_\alpha}{\partial \alpha} = -\frac{\hat{e}_\beta}{h_\alpha h_\beta}\frac{\partial h_\alpha}{\partial \beta} - \frac{\hat{e}_\gamma}{h_\alpha h_\gamma}\frac{\partial h_\alpha}{\partial \gamma} \qquad \text{(A7)}$$

Similarly, considering $\nabla(\hat{e}_\beta.\hat{e}_\beta)=(1)=0$ and $\nabla(\hat{e}_\gamma.\hat{e}_\gamma)=(1)=0$ and following the same procedure as above we can get two more expressions as follows:

$$\frac{1}{h_\beta}\frac{\partial \hat{e}_\beta}{\partial \beta} = -\frac{\hat{e}_\gamma}{h_\beta h_\gamma}\frac{\partial h_\beta}{\partial \gamma} - \frac{\hat{e}_\alpha}{h_\beta h_\alpha}\frac{\partial h_\beta}{\partial \alpha} \qquad \text{(A8)}$$

$$\frac{1}{h_\gamma}\frac{\partial \hat{e}_\gamma}{\partial \gamma} = -\frac{\hat{e}_\alpha}{h_\gamma h_\alpha}\frac{\partial h_\gamma}{\partial \alpha} - \frac{\hat{e}_\beta}{h_\gamma h_\beta}\frac{\partial h_\gamma}{\partial \beta} \qquad \text{(A9)}$$

There are still six more derivatives to be evaluated. For this we consider the operation $\nabla(\hat{e}_\alpha.\hat{e}_\beta)=(0)=0$ and again using the vector identity:

$$\nabla(\hat{a}.\hat{b}) = (\hat{a}.\nabla)\hat{b} + (\hat{b}.\nabla)\hat{a} + \hat{a}\times(\nabla\times\hat{b}) + \hat{b}\times(\nabla\times\hat{a})$$

we get

$$(\hat{e}_\alpha.\hat{e}_\beta) = (\hat{e}_\alpha.\nabla)\hat{e}_\beta + (\hat{e}_\beta.\nabla)\hat{e}_\alpha + \hat{e}_\alpha\times(\nabla\times\hat{e}_\beta) + \hat{e}_\beta\times(\nabla\times\hat{e}_\alpha) = 0$$

or

$$(\hat{e}_\alpha.\nabla)\hat{e}_\beta + (\hat{e}_\beta.\nabla)\hat{e}_\alpha = -\left[\hat{e}_\alpha\times(\nabla\times\hat{e}_\beta) + \hat{e}_\beta\times(\nabla\times\hat{e}_\alpha)\right] \qquad \text{(A10)}$$

Expanding the RHS and LHS of equation (A10) in the same manner as has been done for equation (A1), the following equation can be obtained:

$$\frac{1}{h_\alpha}\frac{\partial \hat{e}_\beta}{\partial \alpha} + \frac{1}{h_\beta}\frac{\partial \hat{e}_\alpha}{\partial \beta} = \frac{\hat{e}_a}{h_\alpha h_\beta}\frac{\partial h_\alpha}{\partial \beta} + \frac{\hat{e}_\beta}{h_\alpha h_\beta}\frac{\partial h_\beta}{\partial \alpha} \qquad \text{(A11)}$$

Taking dot product with the unit vector $\hat{e}_\alpha$, equation (31) becomes:

$$\hat{e}_a \cdot \left(\frac{1}{h_\alpha}\frac{\partial \hat{e}_\beta}{\partial \alpha} + \frac{1}{h_\beta}\frac{\partial \hat{e}_\alpha}{\partial \beta}\right) = \hat{e}_\alpha \cdot \left(\frac{\hat{e}_\alpha}{h_\alpha h_\beta}\frac{\partial h_\alpha}{\partial \beta} + \frac{\hat{e}_\beta}{h_\alpha h_\beta}\frac{\partial h_\beta}{\partial \alpha}\right)$$

i.e.

$$\frac{\hat{e}_\alpha}{h_\alpha} \cdot \frac{\partial \hat{e}_\beta}{\partial \mu} + \frac{\hat{e}_\alpha}{h_\beta} \frac{\partial \hat{e}_\alpha}{\partial \beta} = \frac{1}{h_\alpha h_\beta} \frac{\partial h_\alpha}{\partial \beta}$$

$$= \frac{\hat{e}_\alpha}{h_\alpha} \frac{\partial \hat{e}_\beta}{\partial \alpha} + \frac{1}{2} \frac{1}{h_\beta} \frac{\partial}{\partial \beta} (\hat{e}_\alpha . \hat{e}_\alpha) = \frac{1}{h_\alpha h_\beta} \frac{\partial h_\alpha}{\partial \beta}$$

$$= \frac{\hat{e}_\alpha}{h_\alpha} \frac{\partial \hat{e}_\beta}{\partial \alpha} = \frac{1}{h_\alpha h_\beta} \frac{\partial h_\alpha}{\partial \beta} \qquad \text{(A12)}$$

Taking dot product with $\hat{e}_\alpha$ again equation (A12) becomes:

$$\frac{1}{h_\alpha} \frac{\partial \hat{e}_\beta}{\partial \alpha} = \frac{\hat{e}_\alpha}{h_\alpha h_\beta} \frac{\partial h_\alpha}{\partial \beta} \qquad \text{(A13)}$$

Similarly multiplying (dot product) equation (A11) with $\hat{e}_\beta$ and following similar operations as above the following expression can be derived:

$$\frac{1}{h_\beta} \frac{\partial \hat{e}_\alpha}{\partial \beta} = \frac{\hat{e}_\beta}{h_\alpha h_\beta} \frac{\partial h_\beta}{\partial \alpha} \qquad \text{(A14)}$$

By considering the operations $\nabla(\hat{e}_\beta . \hat{e}_\gamma) = \nabla(0) = 0$ and $\nabla(\hat{e}_\gamma . \hat{e}_\alpha) = (0) = 0$ and following similar steps starting from equation (A10) to equation (A14) above, four more equations containing the derivatives of the unit vectors can be obtained which are shown below:

$$\left.\begin{aligned} \frac{1}{h_\gamma} \frac{\partial e_\alpha}{\partial \gamma} &= \frac{\hat{e}_\gamma}{h_\gamma h_\alpha} \frac{\partial h_\gamma}{\partial \alpha} \\ \frac{1}{h_\gamma} \frac{\partial \hat{e}_\beta}{\partial \gamma} &= \frac{\hat{e}_\gamma}{h_\gamma h_\beta} \frac{\partial h_\gamma}{\partial \beta} \\ \frac{1}{h_\alpha} \frac{\partial \hat{e}_\gamma}{\partial \alpha} &= \frac{\hat{e}_\alpha}{h_\alpha h} \frac{\partial h_\alpha}{\partial \gamma} \\ \frac{1}{h_\beta} \frac{\partial \hat{e}}{\partial \beta} &= \frac{\hat{e}_\beta}{h_\beta h} \frac{\partial h_\beta}{\partial} \end{aligned}\right\} \qquad \text{(A15)}$$

The equations (A7), (A8), (A13), (A14) and (A15) which give the values of the derivatives of the unit vectors can be expressed compactly in a matrix form as follows:

$$
\begin{bmatrix}
\frac{\partial \hat{e}_\alpha}{\partial \alpha} & \frac{\partial \hat{e}_\alpha}{\partial \beta} & \frac{\partial \hat{e}_\alpha}{\partial \gamma} \\
\frac{\partial \hat{e}_\beta}{\partial \alpha} & \frac{\partial \hat{e}_\beta}{\partial \beta} & \frac{\partial \hat{e}_\beta}{\partial \gamma} \\
\frac{\partial \hat{e}_\gamma}{\partial \alpha} & \frac{\partial \hat{e}_\gamma}{\partial \beta} & \frac{\partial \hat{e}_\gamma}{\partial \gamma}
\end{bmatrix}
=
\begin{bmatrix}
\left(-\frac{\hat{e}_\beta}{h_\beta}\frac{\partial h_\alpha}{\partial \beta} - \frac{\hat{e}_\gamma}{h_\gamma}\frac{\partial h_\alpha}{\partial \gamma}\right) & \frac{\hat{e}_\beta}{h_\alpha}\frac{\partial h_\beta}{\partial \alpha} & \frac{\hat{e}_\gamma}{h_\alpha}\frac{\partial h_\gamma}{\partial \alpha} \\
\frac{\hat{e}_\alpha}{h_\beta}\frac{\partial h_\alpha}{\partial \beta} & \left(-\frac{\hat{e}_\alpha}{h_\alpha}\cdot\frac{\partial h_\beta}{\partial \alpha} - \frac{\hat{e}_\gamma}{h_\gamma}\frac{\partial h_\beta}{\partial \gamma}\right) & \frac{\hat{e}_\gamma}{h_\beta}\cdot\frac{\partial h_\gamma}{\partial \beta} \\
\frac{\hat{e}_\alpha}{h_\gamma}\frac{\partial h_\alpha}{\partial \gamma} & \frac{\hat{e}_\beta}{h_\gamma}\frac{\partial h_\beta}{\partial \gamma} & \left(-\frac{\hat{e}_\alpha}{h_\alpha}\frac{\partial h_\gamma}{\partial \alpha} - \frac{\hat{e}_\beta}{h_\beta}\frac{\partial h_\gamma}{\partial \beta}\right)
\end{bmatrix}
\tag{A16}
$$

The above equation is the required result which would be used to evaluate the divergence of a vector in a curvilinear co-ordinate system. In fact, as indicated earlier, this result is needed for any operation which involves vector differentiation.

APPENDIX B

Using equation (A16), the divergence of a vector in a curvilinear co-ordinate system become: (see equation (20))

$$\nabla . \hat{A} = \left[\frac{1}{h_\alpha}\frac{\partial A_\alpha}{\partial \alpha} + \frac{1}{h_\beta}\frac{\partial A_\beta}{\partial \beta} + \frac{1}{h_\gamma}\frac{\partial A_\gamma}{\partial \gamma} \right] +$$

$$A_\alpha \left[\frac{\hat{e}_\alpha}{h_\alpha} . \left(- \frac{\hat{e}_\beta}{h_\beta}\frac{\partial h_\alpha}{\partial \beta} - \frac{\hat{e}_\gamma}{h_\gamma}\frac{\partial h_\alpha}{\partial \gamma} \right) + \frac{\hat{e}_\beta}{h_\beta} . \left(\frac{\hat{e}_\beta}{h_\alpha}\frac{\partial h_\beta}{\partial \alpha} \right) + \frac{\hat{e}_\gamma}{h_\gamma} . \left(\frac{\hat{e}_\gamma}{h_\alpha}\frac{\partial h_\gamma}{\partial \alpha} \right) \right] +$$

$$A_\beta \left[\frac{\hat{e}_\alpha}{h_\alpha} . \left(\frac{\hat{e}_\alpha}{h_\beta}\frac{\partial h_\alpha}{\partial \beta} \right) + \frac{\hat{e}_\beta}{h_\beta} . \left(- \frac{\hat{e}_\alpha}{h_\alpha}\frac{\partial h_\beta}{\partial \alpha} - \frac{\hat{e}_\gamma}{h_\gamma}\frac{\partial h_\beta}{\partial \gamma} \right) + \frac{\hat{e}_\gamma}{h_\gamma} . \left(\frac{\hat{e}_\gamma}{h_\beta}\frac{\partial h_\gamma}{\partial \beta} \right) \right] +$$

$$A_\gamma \left[\frac{\hat{e}_\alpha}{h_\alpha} . \left(\frac{\hat{e}_\alpha}{h_\gamma}\frac{\partial h_\alpha}{\partial \gamma} \right) + \frac{\hat{e}_\beta}{h_\beta} . \left(\frac{\hat{e}_\beta}{h_\beta} . \frac{\hat{e}_\beta}{h_\gamma}\frac{\partial h_\beta}{\partial \gamma} \right) + \frac{\hat{e}_\gamma}{h_\gamma} . \left(- \frac{\hat{e}_\alpha}{h_\alpha}\frac{\partial h_\gamma}{\partial \alpha} - \frac{\hat{e}_\beta}{h_\beta}\frac{\partial h_\gamma}{\partial \beta} \right) \right]$$

$$= \left[\frac{1}{h_\alpha}\frac{\partial A_\alpha}{\partial \alpha} + \frac{1}{h_\beta}\frac{\partial A_\beta}{\partial \beta} + \frac{1}{h_\gamma}\frac{\partial A_\gamma}{\partial \gamma} \right. + A_\alpha \left[\frac{1}{h_\beta h_\alpha}\frac{\partial h_\beta}{\partial \alpha} + \frac{1}{h_\gamma h_\alpha}\frac{\partial h_\gamma}{\partial \alpha} \right] +$$

$$A_\beta \left[\frac{1}{h_\alpha h_\beta}\frac{\partial h_\alpha}{\partial \beta} + \frac{1}{h_\gamma h_\beta}\frac{\partial h_\gamma}{\partial \beta} \right] + A_\gamma \left[\frac{1}{h_\alpha h_\gamma}\frac{\partial h_\alpha}{\partial \gamma} + \frac{1}{h_\beta h_\gamma}\frac{\partial h_\beta}{\partial \gamma} \right]$$

$$= \frac{1}{h_\alpha h_\beta h_\gamma} \left[\left(h_\beta h_\gamma \frac{\partial A_\alpha}{\partial \alpha} + h_\alpha h_\gamma \frac{\partial A_\beta}{\partial \beta} + h_\alpha h_\beta \frac{\partial A_\gamma}{\partial \gamma} \right) + A_\alpha \left(h_\gamma \frac{\partial h_\beta}{\partial \gamma} + \right. \right.$$

$$\left. \left. h_\beta \frac{\partial h_\gamma}{\partial \alpha} \right) + A_\beta \left(h_\gamma \frac{\partial h_\alpha}{\partial \beta} + h_\alpha \frac{\partial h_\gamma}{\partial \beta} \right) + A_\gamma \left(h_\beta \frac{\partial h_\alpha}{\partial \gamma} + h_\alpha \frac{\partial h_\beta}{\partial \gamma} \right] \right.$$

$$= \frac{1}{h_\alpha h_\beta h_\gamma} \left[h_\beta h_\gamma \frac{\partial A_\alpha}{\partial \alpha} + A_\alpha \frac{\partial}{\partial \alpha}(h_\beta h_\gamma) + h_\alpha h_\gamma \frac{\partial A_\beta}{\partial \beta} + A_\beta \frac{\partial}{\partial \beta}(h_\alpha h_\gamma) + \right.$$

$$\left. h_\alpha h_\beta \frac{\partial A_\gamma}{\partial \gamma} + A_\gamma \frac{\partial}{\partial \gamma}(h_\alpha h_\beta) \right]$$

$$= \frac{1}{h_\alpha h_\beta h_\gamma} \left[\frac{\partial}{\partial \alpha}(h_\beta h_\gamma A_\alpha) + \frac{\partial}{\partial \beta}(h_\alpha h_\gamma A_\beta) + \frac{\partial}{\partial}(h_\alpha h_\beta A_\gamma) \right] \qquad \text{(B1)}$$

APPENDIX C

curl:

$$
\begin{aligned}
\nabla \times \hat{A} &= \left(\frac{\hat{e}_\alpha}{h_\alpha}\frac{\partial}{\partial\alpha} + \frac{\hat{e}_\beta}{h_\beta}\frac{\partial}{\partial\beta} + \frac{\hat{e}_\gamma}{h_\gamma}\frac{\partial}{\partial\gamma}\right) \times \left(A_\alpha \hat{e}_\alpha + A_\beta \hat{e}_\beta + A_\gamma \hat{e}_\gamma\right) \\
&= \left(\frac{\hat{e}_\alpha}{h_\alpha} \times \hat{e}_\alpha \frac{\partial}{\partial\alpha} A_\alpha + A_\alpha \frac{\hat{e}_\alpha}{h_\alpha} \times \frac{\partial \hat{e}_\alpha}{\partial\alpha}\right) + \\
&\quad \left(\frac{\hat{e}_\alpha}{h_\alpha} \times \hat{e}_\beta \frac{\partial A_\beta}{\partial\alpha} + A_\beta \frac{\hat{e}_\alpha}{h_\alpha} \times \frac{\partial \hat{e}_\beta}{\partial\alpha}\right) + \\
&\quad \left(\frac{\hat{e}_\alpha}{h_\alpha} \times \hat{e}_\gamma \frac{\partial A}{\partial\alpha} + A_\gamma \frac{\hat{e}_\alpha}{h_\alpha} \times \frac{\partial \hat{e}_\gamma}{\partial\alpha}\right) + \\
&\quad \left(\frac{\hat{e}_\beta}{h_\beta} \times \hat{e}_\alpha \frac{\partial A_\alpha}{\partial\beta} + A_\alpha \frac{\hat{e}_\beta}{h_\beta} \times \frac{\partial \hat{e}_\alpha}{\partial\beta}\right) + \\
&\quad \left(\frac{\hat{e}_\beta}{h_\beta} \times \hat{e}_\beta \frac{\partial A_\beta}{\partial\beta} + A_\beta \frac{\hat{e}_\beta}{h_\beta} \times \frac{\partial \hat{e}_\beta}{\partial\beta}\right) + \\
&\quad \left(\frac{\hat{e}_\beta}{h_\beta} \times \hat{e}_\gamma \frac{\partial A_\gamma}{\partial\beta} + A_\gamma \frac{\hat{e}_\beta}{h_\beta} \times \frac{\partial \hat{e}_\gamma}{\partial\beta}\right) + \\
&\quad \left(\frac{\hat{e}_\gamma}{h_\gamma} \times \hat{e}_\alpha \frac{\partial A_\alpha}{\partial\gamma} + A_\alpha \frac{\hat{e}_\gamma}{h_\gamma} \times \frac{\partial \hat{e}_\alpha}{\partial\gamma}\right) + \\
&\quad \left(\frac{\hat{e}_\gamma}{h_\gamma} \times \hat{e}_\beta \frac{\partial A_\beta}{\partial\gamma} + A_\beta \frac{\hat{e}_\gamma}{h_\gamma} \times \frac{\partial \hat{e}_\beta}{\partial\gamma}\right) + \\
&\quad \left(\frac{\hat{e}_\gamma}{h_\gamma} \times \hat{e}_\gamma \frac{\partial A}{\partial\gamma} + A_\gamma \frac{\hat{e}_\gamma}{h_\gamma} \times \frac{\partial \hat{e}_\gamma}{\partial\gamma}\right) \\
&= \left(\frac{\hat{e}_\gamma}{h_\alpha}\frac{\partial A_\beta}{\partial\alpha} - \frac{\hat{e}_\beta}{h_\alpha}\frac{\partial A_\gamma}{\partial\alpha}\right) - \left(\frac{\hat{e}_\gamma}{h_\beta}\frac{\partial A_\alpha}{\partial\beta} - \frac{\hat{e}_\alpha}{h_\beta}\frac{\partial A_\gamma}{\partial\beta}\right) + \\
&\quad \left(\frac{\hat{e}_\beta}{h_\gamma}\frac{\partial A_\alpha}{\partial\gamma} - \frac{\hat{e}_\alpha}{h_\gamma}\frac{\partial A_\beta}{\partial\gamma}\right) + \\
&\quad A_\alpha \left[\frac{\hat{e}_\alpha}{h_\alpha} \times \frac{\partial \hat{e}_\alpha}{\partial\alpha} + \frac{\hat{e}_\beta}{h_\beta} \times \frac{\partial \hat{e}_\alpha}{\partial\beta} + \frac{\hat{e}_\gamma}{h_\gamma} \times \frac{\partial \hat{e}_\alpha}{\partial\gamma}\right] +
\end{aligned}
$$

$$A_\beta \left[\frac{\hat{e}_\alpha}{h_\alpha} \times \frac{\partial \hat{e}_\beta}{\partial \alpha} + \frac{\hat{e}_\beta}{h_\beta} \times \frac{\partial \hat{e}_\beta}{\partial \beta} + \frac{\hat{e}_\gamma}{h_\gamma} \times \frac{\partial \hat{e}}{\partial \gamma} \right] +$$

$$A_\gamma \left[\frac{\hat{e}_\alpha}{h_\alpha} \times \frac{\partial \hat{e}_\gamma}{\partial \alpha} + \frac{\hat{e}_\beta}{h_\beta} \times \frac{\partial \hat{e}_\gamma}{\partial \beta} + \frac{\hat{e}_\gamma}{h_\gamma} \times \frac{\partial \hat{e}_\gamma}{\partial \gamma} \right]$$

(C1)

Using equation (A16) for the values of the derivatives of the unit vectors, equation (C1) can be expressed as:

$$\nabla \times \hat{A} = \left(\frac{\hat{e}_\gamma}{h_\alpha} \frac{\partial A_\beta}{\partial \alpha} - \frac{\hat{e}_\beta}{h_\alpha} \frac{\partial A_\gamma}{\partial \alpha}\right) - \left(\frac{\hat{e}_\gamma}{h_\beta} \frac{\partial A_\alpha}{\partial \beta} - \frac{e_\alpha}{h_\beta} \frac{\partial A_\gamma}{\partial \beta}\right) + \left(\frac{\hat{e}_\beta}{h_\gamma} \frac{\partial A_\alpha}{\partial \gamma} - \frac{\hat{e}_\alpha}{h_\gamma} \frac{\partial A_\beta}{\partial \gamma}\right) +$$

$$A_\alpha \left[-\frac{\hat{e}_\gamma}{h_\alpha h_\beta} \frac{\partial h_\alpha}{\partial \beta} + \frac{\hat{e}_\beta}{h_\alpha h_\gamma} \frac{\partial h_\alpha}{\partial \gamma} \right] +$$

$$A_\beta \left[\frac{\hat{e}_\gamma}{h_\beta h_\alpha} \frac{\partial h_\beta}{\partial \alpha} - \frac{\hat{e}_\alpha}{h_\beta h_\gamma} \frac{\partial h_\beta}{\partial \gamma} \right] +$$

$$A_\gamma \left[-\frac{\hat{e}_\beta}{h_\gamma h_\alpha} \frac{\partial h_\gamma}{\partial \alpha} + \frac{\hat{e}_\alpha}{h_\gamma h_\beta} \frac{\partial h_\gamma}{\partial \beta} \right]$$

$$= \frac{1}{h_\alpha h_\beta h_\gamma} \left[\left(h_\beta h_\gamma \hat{e} \frac{\partial A_\beta}{\partial \alpha} + h_\gamma \hat{e}_\gamma A_\beta \frac{\partial h_\beta}{\partial \alpha}\right) - \left(h_\beta h_\gamma \hat{e}_\beta \frac{\partial A_\gamma}{\partial \alpha} + \right.\right.$$

$$\left. h_\beta \hat{e}_\beta A_\gamma \frac{\partial h_\gamma}{\partial \alpha}\right) - \left(h_\alpha h_\gamma \hat{e}_\gamma \frac{\partial A_\alpha}{\partial \beta} + h_\gamma \hat{e}_\gamma A_\alpha \frac{\partial h_\alpha}{\partial \beta}\right) +$$

$$\left(h_\alpha h_\gamma \hat{e}_\alpha \frac{\partial A_\gamma}{\partial \beta} + h_\alpha \hat{e}_\alpha A_\gamma \frac{\partial h_\gamma}{\partial \beta}\right) + \left(h_\alpha h_\beta \hat{e}_\beta \frac{\partial A_\alpha}{\partial \gamma} + \right.$$

$$\left.\left. h_\beta \hat{e}_\beta A_\alpha \frac{\partial h_\alpha}{\partial \gamma}\right) - \left(h_\alpha h_\beta \hat{e}_\alpha \frac{\partial A_\beta}{\partial \gamma} + h_\alpha \hat{e}_\alpha A_\beta \frac{\partial h_\beta}{\partial \gamma}\right) \right]$$

$$= \frac{1}{h_\alpha h_\beta h_\gamma} \Big[\{ h_\gamma \hat{e}_\gamma \frac{\partial}{\partial \alpha} (h_\beta A_\beta) - h_\gamma \hat{e}_\gamma \frac{\partial}{\partial \beta} (h_\alpha A_\alpha) \} -$$

$$\{ h_\beta \hat{e}_\beta \frac{\partial}{\partial \alpha} (h_\gamma A_\gamma) - h_\beta \hat{e}_\beta \frac{\partial}{\partial \gamma} (h_\alpha A_\alpha) \} +$$

$$\{ h_\alpha e_\alpha \frac{\partial}{\partial \beta} (h_\gamma A_\gamma) - h_\alpha \hat{e}_\alpha \frac{\partial}{\partial \gamma} (h_\beta A_\beta) \} \Big]$$

$$= \frac{1}{h_\alpha h_\beta h_\gamma} \Big[h_\alpha \hat{e}_\alpha \{ \frac{\partial}{\partial \beta} (h_\gamma A_\gamma) - \frac{\partial}{\partial \gamma} (h_\beta A_\beta) \} -$$

$$h_\beta \hat{e}_\beta \{ \frac{\partial}{\partial \alpha} (h_\gamma A_\gamma) - \frac{\partial}{\partial \gamma} (h_\alpha A_\alpha) \}$$

$$h_\gamma \hat{e}_\gamma \{ \frac{\partial}{\partial \alpha} (h_\beta A_\beta) - \frac{\partial}{\partial \beta} (h_\alpha A_\alpha) \} \Big] \tag{C2}$$

Expressing (C2) in matrix form, the curl of a vector in curvilinear co-ordinates becomes:

$$\nabla x \hat{A} = \frac{1}{h_\alpha h_\beta h_\gamma} \begin{bmatrix} h_\alpha \hat{e}_\alpha & h_\beta \hat{e}_\beta & h_\gamma \hat{e}_\gamma \\ \frac{\partial}{\partial \alpha} & \frac{\partial}{\partial \beta} & \frac{\partial}{\partial \gamma} \\ h_\alpha A_\alpha & h_\beta A_\beta & h_\gamma A_\gamma \end{bmatrix} \tag{C3}$$