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by

M. Romanowski and S. M. Kasner

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Ch. I General theory of equal arms balances

1. Introduction

The equal arms balance is a balance in which the distances of the outer knife-edges to the central knife are made equal with the utmost accuracy. The three knife-edges are, as exactly as possible, in the same plane.

The center of gravity of the beam must be somewhat below the central knife as, otherwise, the beam would have no defined position of equilibrium. The positions of the knives and of the center of gravity of the beam are generally adjustable. The effects of the adjustment cannot be evaluated directly but only through the modifications they produce in the functioning of the balance during weighing operations. This is the reason why a beam (initially adjusted by the constructor) should never be touched by anyone except by a person of appropriate training and skill, capable of *interpreting* correctly the modifications of the balance's behaviour. As all factors influencing the quality of a balance are strongly correlated with each other, to bring a high sensitivity balance to a state of utmost perfection, represents a serious mechanical and metrological achievement.

The pans are practically of equal weights, so that when the beam is "free", i.e., rests only on the central knife, the position of the beam is very close to horizontality; such is also the case when the balance is loaded with two nominally equal masses.

In the category of metrological instruments designated by the general term of "comparators" (i.e., instruments which can evaluate differences only between nominally equal quantities), the equal arms balance is still today, in spite of all progresses accomplished in various domains of science and technology, one of the most reliable and the most sensitive.

The reliability and the sensitivity of a balance is, however, not founded on the static properties of the beam when the latter is in its equilibrium position but on the dynamic properties of small oscillations (about the equilibrium position).

Historically, the balance was the first high precision instrument devised by man; its supremacy remained completely unchallenged for centuries, until the advent of some modern methods, e.g., those used in the measurement of lengths by interferometric techniques. Although balances built two hundred years ago are relatively crude in comparison with the modern instruments, some of those manufactured by reputed constructors (e.g., Rueprecht) in the middle of the nineteenth century are almost equivalent to the modern balances. Modern techniques for constructing knives and planes and for checking them optically permit, however, an easier production (at a lower cost) of very high quality equal arms balances.

The fact that a balance is used in dynamic conditions (i.e., when its beam is swinging) necessitates a continuous supervision of its functioning during the comparisons. This point is treated in Chapter IV. Another memorandum treats in detail the question how statistical methods, based on the theory of least squares, can be applied to the results of weighings and how they lead to a reliable estimation of the accuracy of the weighings.

2. Equation of equilibrium

For the theoretical study of equilibrium conditions, the beam (Fig. 1) can be reduced to those points which are actually necessary for establishing the equations of the balance (Fig. 2). These points are:

P' and P'': points representing the edges of the outer knives (edges upward), i.e., points on which act the forces produced by the compared masses.

C: central knife (edge downward), point which represents the line about which the beam can pivot. C should be (as close as it is technically possible to realize) located midway between P' and P''. Nominally therefore, $l' = CP' = l'' = CP''$. The edges of P' and P'' and C should be coplanar.

G: center of gravity of the beam. It is located on the line passing through C and perpendicular to P'CP''. The distance GC will be designated by r, G being below C. The mass of the beam, assumed concentrated in G, will be denoted by m.

Let us assume that the beam takes the position represented in Fig. 2. The forces which act on the beam and which determine its actual position in space are:

(a) downward forces exerted by the gravitational field on the masses M_1, M_2, and the corresponding upward buoyancy forces f_1, f_2 due to the fact that the masses are surrounded by air.

(b) downward force acting on the beam, which can be reduced to the gravitional force applied to G. No buoyancy force has to be introduced concerning the beam as this force is constant and independent of the position of the beam in space.

In the theory that follows, the forces acting on the beam will be:

$$F_1 = M_1 g - f_1,$$

$$F_2 = M_2 g - f_2,$$

$$F_3 = mg.$$

The combined effect of the moments of F_2 and F_3 counterbalances the moment of F_1. The angle of deflection of r from verticality will be designated by θ. In the sequel, conventions will be introduced concerning this angle; for the time being θ will be considered as positive.

The conditions represented in Fig. 2 lead to the following equation:

$$(1) \qquad F_1\ l' \cos\theta = F_2\ l'' \cos\theta + (mgr)\sin\theta .$$

The angle θ being small, it is permissible to introduce the approximations

$$\cos\theta \doteq 1 \text{ and } \sin\theta \doteq \theta \text{ radians.}$$

Hence

$$(2) \qquad F_1\ l' = F_2\ l'' + (mgr)\ \theta$$

From this equation *alone*, it is not possible to determine precisely the relationship between F_1 and F_2. Other equations are necessary to eliminate all those parameters which cannot be determined accurately. Two methods can be used for this purpose and will be analyzed successively.

Ch. II Interchange method

1. Fundamental law

If, after the equation (2) had been obtained, the masses M_1 and M_2 are interchanged, the beam takes the position indicated in Fig. 3, the angle of deflection being now η, directed in the opposite sense and similar but *not* identical to θ. The difference between θ and η is due to the small but not completely negligible difference between l' and l'': in a rigorously symmetrical balance, the interchange of masses would lead to $\eta = \theta$. In high quality instruments, the difference between l' and l'' may be extremely small but, nevertheless, these lengths can never be considered as rigorously equal. The equation that corresponds to Fig. 3 is:

$$(3) \qquad F_2\ l' \cos\eta + (mgr)\sin\eta = F_1\ l'' \cos\eta$$

and, in a simplified form,

$$(4) \qquad F_2\ 1' + (mgr)\ \eta = F_1\ 1''.$$

Thus, the equations (2) and (4) form the basic system:

$$F_1\ 1' = F_2\ 1'' + (mgr)\ \theta,$$
$$F_2\ 1' + (mgr)\ \eta = F_1\ 1'',$$

which can be solved for $(F_1 - F_2)$:

$$(F_1 - F_2)\ 1' - (mgr)\ \eta = -(F_1 - F_2)\ 1'' + (mgr)\ \theta,$$
$$(F_1 - F_2)(1' + 1'') = (mgr)(\theta + \eta),$$
$$F_1 - F_2 = \frac{mgr}{1' + 1''}(\theta + \eta).$$

The factor $\frac{mgr}{1' + 1''}$ is a positive quantity; it will be designated by the symbol k:

$$(5) \qquad F_1 - F_2 = k\ (\theta + \eta).$$

This important relation is called the *fundamental law* of the interchange method. The analysis of its second term will be done in two steps; first we shall deal with the angles of deflection and, secondly, with the proprotionality factor k.

2. Measurement of the angles of deflection

Instead of measuring the angles θ and η from the vertical axis CC′ as origin, it is more convenient to adopt an arbitrary origin, for instance the axis CX located as in Fig. 4. The total angle $(\theta + \eta)$ is then equivalent to:

$$(6) \qquad (\theta + \eta) = a' - a''$$

The sign of the difference $F_1 - F_2$ will be the same as that of $(a' - a'')$ if, with respect to CX, the angle a is a monotonic increasing function of F_1. The reader must always bear in mind that a' is the angle that corresponds to the *first* weighing i.e., that made *before* the exchange of masses.

The angles θ and η being small, various devices are used for evaluating them with sufficient accuracy and without too much strain on the observer. One of the most commonly used devices consists of a long thin pointer fixed normally to the beam. Its fine lower extremity oscillates in front of a short, finely engraved scale, or graticule. To each angle of deflection, say a, corresponds a well defined number which is proportional to the angle's value and which the observer can read on the scale, with the possibility of interpolating between two consecutive marks to one or two tenths of a division.

To avoid misunderstandings and to maintain uniform patterns in the records of the Laboratory the following rules have been adopted.

a. The left-hand pan is designated by the symbol P′ and the right-hand pan by the symbol P″.

b. In the "first" or "direct" weighing, the mass placed in P′ is termed "first" mass and the mass placed in P″, the "second" mass. The terms "first" and "second" always refer to the first (direct) comparison, even after the masses had been interchanged.

c. The difference between two compared forces or two compared masses are always defined as follows:

Difference between forces = First force − Second force,
Difference between masses = First mass − Second mass.

Thus, in the theory given above,

First mass $= M_1$, i.e. first force $= F_1$;
Second mass $= M_2$, i.e. second force $= F_2$.

The differences are equal to:

$$(M_1 - M_2) \text{ or } (F_1 - F_2).$$

d. That extremity of the scale which is the closest to the left-hand pan (P′) is considered as the *"zero-end"*. The reading on the scale that corresponds to a' is designated by R′ and is termed "first" (or "direct") reading; similarly, R″ corresponds to a'' and is termed "second" (or "reversed") reading. The differences $(F_1 - F_2)$ and $(R' - R'')$ are therefore always of the same sign. It is implicitely assumed that the pointer never goes farther to the left than the zero end.

The distance between R′ and R″ is designated by the symbol D. It is equal to:

$$D = R' - R'' \qquad (7)$$

and the *fundamental law* takes the form:

$$F_1 - F_2 = KD. \qquad (8)$$

The total proportionality factor K represents now a combination of the factor k (5) and the proportionality factor between angles and scale readings.

3. Determination of the proportionality factor (K)

The determination of K is made by means of a small known mass μ generally called "sensitivity mass". This mass must be just large enough to reverse the relative positions of the pans: if it is added to P″ in Fig. 2 it should reverse the relative positions of the pans, i.e. P′ lower than P″. In other words, added to the apparently "lighter" of the two sides it should make it just a little "heavier". The force produced by μ will be:

$$\phi = \mu g - \phi' \qquad (9)$$

ϕ' being the buoyancy force. The latter is generally so small that it can be neglected. To denote the readings obtained *before* the introduction of the sensitivity mass μ the subscript o will be used: R'_o, R''_o, D_o; the readings obtained *after* the introduction of μ will be denoted by the symbols R'_μ, R''_μ, D_μ.

In the sequence of operations leading to the elimination of K and to the expression of the difference $F_1 - F_2$ in terms of observed quantities, the two following cases must be considered:

$F_1 > F_2$ and $F_1 < F_2$.

First Case: $F_1 > F_2$,

(1) Place M_1 (first mass) in P′ and M_2 (second mass) in P″. Determine the rest point R'_o. This is the "direct" comparison.

(2) Add, without arresting the balance, a small mass μ to M_2 (i.e., in P″) so as to make $(F_2 + \phi)$ a little larger than F_1:

$$F_2 + \phi > F_1. \text{ Rest point: } R'_\mu.$$

(3) Arrest the beam and interchange the masses: now the mass on P′ is $(M_2 + \mu)$ and is acted upon by the force $(F_2 + \phi)$. M_1 is on P″, force F_1. This is the "reversed" comparison, with respect to (2); rest point R''_μ.

(4) Without arresting the beam, remove the mass μ. This constitutes the reversed comparison, with respect to (1); rest point R''_0.

Second case: $F_1 < F_2$,

(1) Place M_1 (first mass) in P′ and M_2 (second mass) in P″. Determine the rest point R'_0. This is the "direct" comparison.

(2) As now $F_1 < F_2$, add, without arresting the balance, a small mass μ to M_1 so as to make $F_1 + \mu$ a little larger than F_2. This gives

$$F_2 - \phi < F_1.$$

(3) Arrest the beam and interchange the masses; now the mass on P″ is $(M_1 + \mu)$ and is acted upon by the force $F_1 + \phi$; M_2 is on P′, force F_2. This is the reversed comparison with respect to (2). Rest point R''_μ.

(4) Without arresting the beam, remove the mass μ. This constitutes the "reversed" comparison with respect to (1); rest point R''_0.

The system of equations to which these operations lead is:

from 1 and 4: $$F_1 - F_2 = KD_0; \; D_0 = R'_0 - R''_0.$$

from 2 and 3: $$F_1 - (F_2 \pm \phi) = KD_\mu; \; D_\mu = R'_\mu - R''_\mu.$$

The elimination of K gives:

$$F_1 - F_2 = \pm \phi \frac{D_0}{D_0 - D_\mu} \qquad (10)$$

The sign + should be taken when $F_1 > F_2$ and the sign −, when $F_1 < F_2$.

Because of the relation (9) the sign before μ in the equations (14), (15) and (16) is the same as the sign of ϕ in equation (10).

The difference $R'_0 - R'_\mu$ indicates how sensitive the balance is. The sensitivity S is defined by:

$$S = \frac{\phi}{|R'_0 - R'_\mu|} = \frac{\mu g - \phi'}{|R'_0 - R'_\mu|},$$

but, as ϕ is generally negligible,

$$S = \frac{\mu g}{|R'_0 - R'_\mu|} \qquad (11)$$

In practice, the sensitivity is referred to the mass, i.e., it is equal to

$$\sigma = \frac{S}{g} = \frac{\mu}{|R'_0 - R'_\mu|} \qquad (12)$$

It indicates therefore what mass should be added to one of the pans to provoque a variation of one division on the scale.

In the course of a complete weighing, the sensitivity is determined twice: from $(R'_0 - R'_\mu)$ and from $(R''_0 - R''_\mu)$. The obtained values should be practically identical and any significant discrepancy should be considered as a valid reason for rejecting the result and for investigating the underlying cause of the malfunctioning of the balance.

The calculation of the difference $M_1 - M_2$ in terms of the observed difference $F_1 - F_2$ requires the following data:

a) the *actual* density a of the air, i.e., at the time of the comparisons and at the location of the balance;

b) the volumes V_1 and V_2 at the *actual* temperature of the objects. These volumes may be computed from the corresponding masses by the following formulae in which the density δ is also a function of temperature:

$$M = V\delta$$

$$V = \frac{M}{\delta}$$

Hence,

$$V_1 = \frac{M_1}{\delta_1}, V_2 = \frac{M_2}{\delta_2}$$

and

$$(13) \qquad f_1 = V_1 ag = \frac{M_1}{\delta_1} ag$$

$$f_2 = V_2 ag = \frac{M_2}{\delta_2} ag$$

Introducing this into the expressions of F_1 and F_2 we obtain:

$$F_1 = M_1 g - V_1 ag = g(M_1 - V_1 a),$$

$$F_2 = M_2 g - V_2 ag = g(M_2 - V_2 a),$$

and, by substituting into (10) the latter can be given the form [1]

$$(14) \qquad M_1 - M_2 = \mu \frac{D_o}{D_o - D_\mu} + a (V_1 - V_2),$$

or

$$(15) \qquad M_1 - M_2 = \mu \frac{D_o}{D_o - D_\mu} + a\left(\frac{M_1}{\delta_1} - \frac{M_2}{\delta_2}\right)$$

This equation, solved for M_1 takes the form:

$$(16) \qquad M_1 = \mu \frac{D_o}{D_o - D_\mu} \cdot \frac{\delta_1}{\delta_1 - a} + M_2 \frac{\delta_1 (\delta_2 - a)}{\delta_2 (\delta_1 - a)}.$$

The term

$$(17a) \qquad \beta = a\left(\frac{M_1}{\delta_1} - \frac{M_2}{\delta_2}\right)$$

generally referred to as "buoyancy term" is always small so that any reasonable approximation for M_1 and M_2 will be sufficient for numerical calculations. Actually, except in some specific cases, M_1 and M_2 can be adequately approximated by the same nominal mass M^*; in practice, therefore, the buoyancy term takes the form

[1] For the sign before μ, refer to equation (10)

$$\beta = aM^*\left(\frac{1}{\delta_1} - \frac{1}{\delta_2}\right) \tag{17b}$$

and, the equation, (15) can be written

$$M_1 - M_2 = \mu \frac{D_o}{D_o - D_\mu} + a M^*\left(\frac{1}{\delta_1} - \frac{1}{\delta_2}\right) \tag{15'}$$

The use of M^* is always justified in the comparisons of standards of mass which, even in the lowest class, are well adjusted to their nominal values. If, however, M_1 is unknown then the observer is compelled to assume that its nominal mass is the same as that of the standards of mass M_2. If, however, the density δ_1 is very different from δ_2, this may require that he checks the accuracy with which M^*, based on M_2, represents also M_1. This is illustrated below by a hypothetical case in which, to avoid unnecessary complications, it will be assumed that in the equations of the substitution method all balances are realized exactly, i.e., without our having to introduce the sensitivity mass μ (in other words, $\mu = 0$).

A brass standard, very closely adjusted to 100 g, is placed in P_1 and "exactly" counterbalanced. Then it is removed and replaced by a block of aluminum which is adjusted until it also produces an "exact" equilibium. Calling

M_s = mass of brass = 100 g,

δ_s = density of brass = 8.4 g/cm^3,

M_x = mass of Al,

δ_x = density of Al = 2.7 g/cm^3,

a = density of air = 0.0012 g/cm^3

and assuming that it is permissible (as a first approximation) to use $M^* = 100$ as the common nominal mass, we have (by 15′)

$$M_x = 100 + 0.0012 \times 100\left(\frac{1}{2.7} - \frac{1}{8.4}\right)$$

$$M = 100 + 0.12\,(0.37 - 0.12)$$

$$M = 100.03 \text{ g.}$$

If this value instead of 100, is now introduced into (15′), for the nominal mass of M , we obtain

$$M_x = 100 + 0.0012\left(\frac{100.03}{2.7} - \frac{100}{8.7}\right),$$

$$M = 100.03 + 0.000012 = 100.030012.$$

The error amounts to about 0.1 ppm. An error of such magnitude is actually negligible, except in very high accuracy weighings (accuracy which is required only on true first quality standards of mass).

The above numerical calculations can be generalized as follows. We have

$$M_x = M^* + \triangle M_x,$$

$$M_s = M^* + \triangle M_s;$$

so that

$$\beta = a\left(\frac{M^* + \triangle M_x}{\delta_x} - \frac{M^* + \triangle M_s}{\delta_s}\right)$$

$$\beta = a\,M^*\left(\frac{1}{\delta_x} - \frac{1}{\delta_s}\right) + a\left(\frac{\Delta M_x}{\delta_x} - \frac{\Delta M_s}{\delta_s}\right) = \beta^* + \Delta\beta,$$

with

$$\Delta\beta = a\left(\frac{\Delta M_x}{\delta_x} - \frac{\Delta M_s}{\delta_s}\right)$$

Here

$$\Delta\beta = 0.0012\left(\frac{0.03}{2.7} - \frac{0}{8.4}\right) = 0.000012$$

The calculation given above represents a typical case of what is generally called "second approximation". The observer should always be ready (in all approximation procedures) to perform a second approximation in order to prove that the first approximation has been sufficient.

Note: in metrology the term "nominal" has a precise meaning; e.g., if the nominal mass is equal to 100 g this assumes that the mass is not different from 100 g by more than the tolerance permited for the class to which the standard belongs.

Example I

A weight made of stainless steel, the mass of which will be designated by M_1, is compared by the interchange method with brass standards of mass M_2 equal to 100.01 g. The nominal value M^* is therefore 100 g. The densities are $\delta_1 = 7.8$ and $\delta_2 = 8.4$ g/cm^3, respectively; the sensitivity mass μ is equal to 0.002 g (2 milligrams).

The sequence of four operations, given in Section 3, Ch. II is here the following.

First (direct) comparison: M_1 in P′ and M_2 in P″

Observed rest point: $R'_o = 11.0$

Second (direct) comparison (with sensitivity mass μ): M_1 in P′ and $M_2 + \mu$ in P″.

Observed rest point: $R'_\mu = 7.0$

Third (reversed) comparison (with sensitivity mass μ): $M_2 + \mu$ in P′ and M_1 in P″.

Observed rest point: $R''_\mu = 13.0$

Fourth (reversed) comparison (mass μ removed): M_2 in P′ and M_1 in P″.

Observed rest point $R''_o = 9.0$

Hence

$$D_o = R'_o - R''_o = 11.0 - 9.0 = +2.0$$
$$D_\mu = R'_\mu - R''_\mu = 7.0 - 13.0 = -6.0$$

and

$$\mu \cdot \frac{D_0}{D_0 - D_\mu} = 0.002 \left(\frac{2.0}{+2.0 + 6.0} \right) = +0.0005$$

Calculation of β term:

The density a of the air at the time of comparisons has been determined from the temperature, the barometric pressure and the relative humidity. According to the tables published by the National Physical Laboratory* it was found equal to a = 0.0012 g/cm^3.

Hence

$$\beta = 0.0012 \times 100.01 \left(\frac{1}{7.8} - \frac{1}{8.4} \right) = +\ 0.00108 \text{ g.}$$

and

$$\overline{M_1} = 100.01 + 0.0005 + 0.00108 = 100.010508$$

Ch. III Substitution method

1. Equation of the substitution method

The interchange method is generally preferred by metrologists for all calibrations of standards of mass because it uses the balance in a completely symmetrical manner. In more complex operations (which, from a metrological standpoint are somewhat less precise than the comparisons of mass standards) the substitution method, as compared with the interchange method, may require less operations. Such is for instance the case in pycnometry (determination of density of liquids).

All general conventions and nomenclature being the same as in Ch. II, the mass to be determined is denoted by M_x and placed on the pan P′; it is counterbalanced as exactly as possible by a counterweight placed on the pan P″. It will be assumed that this operation will leave the beam in the position indicated in Fig. 5, i.e., that the rest point R_x will be in the lower part of the scale. Applying directly the approximations of Ch. I, Section 2, we can put the equilibrium equation under the form

$$(18) \qquad M_x g l' - V_x a g l' + (mrg)\,\theta = F_0 l'', \text{ (rest point: } R_x\text{)},$$

where θ is considered as positive, V_x is the volume of M_x and F_0 the sum of all forces which act on the pan P″. Now, we add a small "sensitivity mass" μ, the effect of which should be to produce a deflection θ' of the same order of magnitude as θ, but in opposite sense with respect to CC′. The resulting equation, in which θ' is also considered as positive and to which corresponds a new rest point R_μ, is

$$(19) \qquad (M_x + \mu)\, g l' - V_x a g l' = (mrg)\,\theta' + F_0 l''; \text{ (rest point: } R_\mu\text{)}.$$

Hence, on substracting (18) from (19), we find

$$(20) \qquad mr = \frac{l'\mu}{\theta + \theta'}$$

If we remove from the pan P′ the masses M_x and μ (for this the balance must be arrested) and replace them by appropriate standards of mass M_s, the resulting angle of deflection η will become similar to the angle θ of the equation (18). This will lead to

*Notes on Applied Science No. 7. N.P.L. (1962).

$$(21)\quad M_s\, gl' - V_s\, agl' + (mgr)\,\eta = F_o l''; \text{(rest point } R_S).$$

The system of equations (18) and (21) can be put under the form

$$(22)\qquad M_x - V_x\, a + \frac{(mr)\,\theta}{l'} = \frac{F_o l''}{l'},$$

$$M_s - V_s\, a + (mr)\,\eta = \frac{F_o l''}{l'},$$

which gives

$$M_x - M_s - a\,(V_x - V_s) + \frac{(mr)}{l'}(\theta - \eta) = 0,$$

and, by (20),

$$M_X - M_S - a\,(V_X - V_S) + \mu\,\frac{\theta - \eta}{\theta - \theta'} = 0.$$

The ratio of the angles may be expressed as above, in terms of the ratio of rest-point readings. From Fig. 5 we deduce (all angles θ, θ', η being positive) that

$\theta - \eta$ is proportional to $R_x - R_s$,

$\theta + \theta'$ is proportional to $R_\mu - R_X$, $(R_\mu > R_X)$.

Hence,

$$(23)\qquad M_X = M_S + \mu\,\frac{R_X - R_S}{R_\mu - R_X} + a\,(V_X - V_S);$$

$$(24)\qquad M_X = M_S + \mu\,\frac{R_X - R_S}{R_\mu - R_X} + a\left(\frac{M_X}{\delta_X} - \frac{M_S}{\delta_S}\right)$$

The form of this equation is the same as that of (15): it can therefore be used as it is, if the value of the mass M_X can be adequately approximated. Otherwise, the equation is given a form similar to (16), i.e.

$$(25)\qquad M_X = \mu\,\frac{R_X - R_S}{R_\mu - R_X}\cdot\frac{\delta_X}{\delta_X - a} + M_S\,\frac{\delta_X(\delta_S - a)}{\delta_S\,(\delta_X - a)}.$$

For the numerical calculation of the term

$$a\left(\frac{M_X}{\delta_X} - \frac{M_S}{\delta_S}\right)$$

the reader is referred to the discussion and the example that follow.

2. Application of the method to the determination of volumes and densities of liquids

The determination of volumes is generally made with water. Water is one of the best studied liquids and its density δ_W is given in various published tables as a function of temperature. The sequence of operations is the following.

a) The pycnometer *filled with water* is placed on the pan P′ and counterweighed until the rest point R_w finds itself on the lower part of the scale.

b) A sensitivity mass μ is added to P′. It should produce a rest point R_μ, on the upper part of the scale.

c) Water and μ are removed and replaced by standards of mass. The rest point R_s should be again in the lower part of the scale i.e., in the vicinity of R_w.

The resulting equation is identical to (25) i.e.,

$$M_W = \mu \frac{R_W - R_S}{R_\mu - R_W} \cdot \frac{\delta_W}{\delta_W - a} + M_S \frac{\delta_W(\delta_S - a)}{\delta_S(\delta_W - a)}. \tag{26}$$

The reader should notice that:

1° the parameters attached to the pycnometer (its mass and density) are completely eliminated from the equations as the container is always on the same pan of the balance.

2° the interchange method can also be used here, but it could lead to a greater number of operations as it would require that the mass of the pycnometer itself (i.e., the container alone) and the mass of the pycnometer filled with water be separately determined.

Example II

In the calibration of a pycnometer the observed values were the following:

M_s = 48.536 g; brass, δ_s = 8.4 g/cm³

R_w = 8.0

R_μ = 10.7

R_s = 6.4

t = 23.0°C

At 23°C the density of water (according to water density tables*) is equal to 0.9975382 g/cm³. The density of the air, determined as in example I., was found equal to $a = 0.0012$ g/cm³. The sensitivity mass μ being equal to 0.002 g. the equation (26) gives the volume V:

$$V = \frac{M_W}{\delta_W} = 0.002 \frac{8.0 - 6.4}{10.7 - 8.0} \cdot \frac{1}{0.997538 - 0.0012} + \frac{48.536}{8.4} \cdot \frac{(8.40 - 0.0012)}{(0.997538 - 0.0012)}$$

$$= \frac{0.0011852}{0.9963382} + 5.7780952 \cdot \frac{8.3988}{0.9963382}$$

$$= 0.0011895 + 48.7074225 = 48.7086120 \text{ cm}^3.$$

*Brit. J. Appl. Phys., Vol. 18

Ch. IV Dynamic and Physical Properties of Balances.

1. Oscillations of the beam.

An equal arms balance of the type called "comparator" is always used dynamically so that a rest point R is *not* the point at which the pointer really "stops" but the point which is the center of the pointer's oscillations. The observer records a certain number of extreme positions of the pointer, to the left and to the right, averages the readings on each side and deduces their mid-point. This point represents the value R. As the oscillations are always slightly damped, the observations of extreme points of oscillations must always start and finish on the same side, for instance:

	Left	Right
	4.0	
	4.1	8.1
	4.2	8.0
Means:	4.10	8.05

and therefore:

$$\text{mid point} = R = \frac{8.05 + 4.10}{2} = 6.08$$

If the beam were left oscillating indefinitely, it would sooner or later come to rest but perhaps not *exactly* at 6.08 because the inherent frictions would probably stop the beam at a point close to but not coinciding exactly with R. It may be said that the dynamic method is *more accurate* than the static method for determining rest-points.

The fact that a beam oscillates about the true rest-point is easily conceivable from an intuitive standpoint. Actually it is justified by the law of mechanics called "law of small oscillations". This law may be presented as follows. Let us consider a mechanical system in its static equilibrium position and let a be the corresponding value of a certain parameter. If a_o is disturbed by a small amount Δ_a, the system according to the law, will oscillate sinusoidally about a_o, like a pendulum oscillates about the vertical axis. The always present frictions produce damping but if the latter is weak the oscillations remain isochronal so that the movement of the beam becomes slower and slower when the amplitude decreases. Special care must then be taken to estimate correctly the readings of the extreme positions of the pointer.

The general rule which should always be born in mind is that the observer must constantly keep his eyes on the pointer so as to be able to detect all transient irregularities of the beam's oscillations. These irregularities may be produced by a speck of dust, a weak air draft etc.; only those observations are recorded which are absolutely irreproachable. The damping, in particular, must be regular and weak. If an irregularity is noticed or a disturbance occurs during oscillations (e.g., because of a slam of a door), the observations should be interrupted and then should start from the beginning.

In extremely high quality balances (as e.g., the main balance of the International Bureau of Weights and Measures, which compares only kilogrammes) the oscillations can be made very small. In principle the graticule with which a balance is equipped by the manufacturer, indicates the order of magnitude of the oscillations which, in general, should be restricted to the central portion of the scale.

A balance is always built for a certain well defined range of masses and the higher the precision the narrower the range. A laboratory must therefore possess a certain number of balances to cover the total range of masses it may have to calibrate.

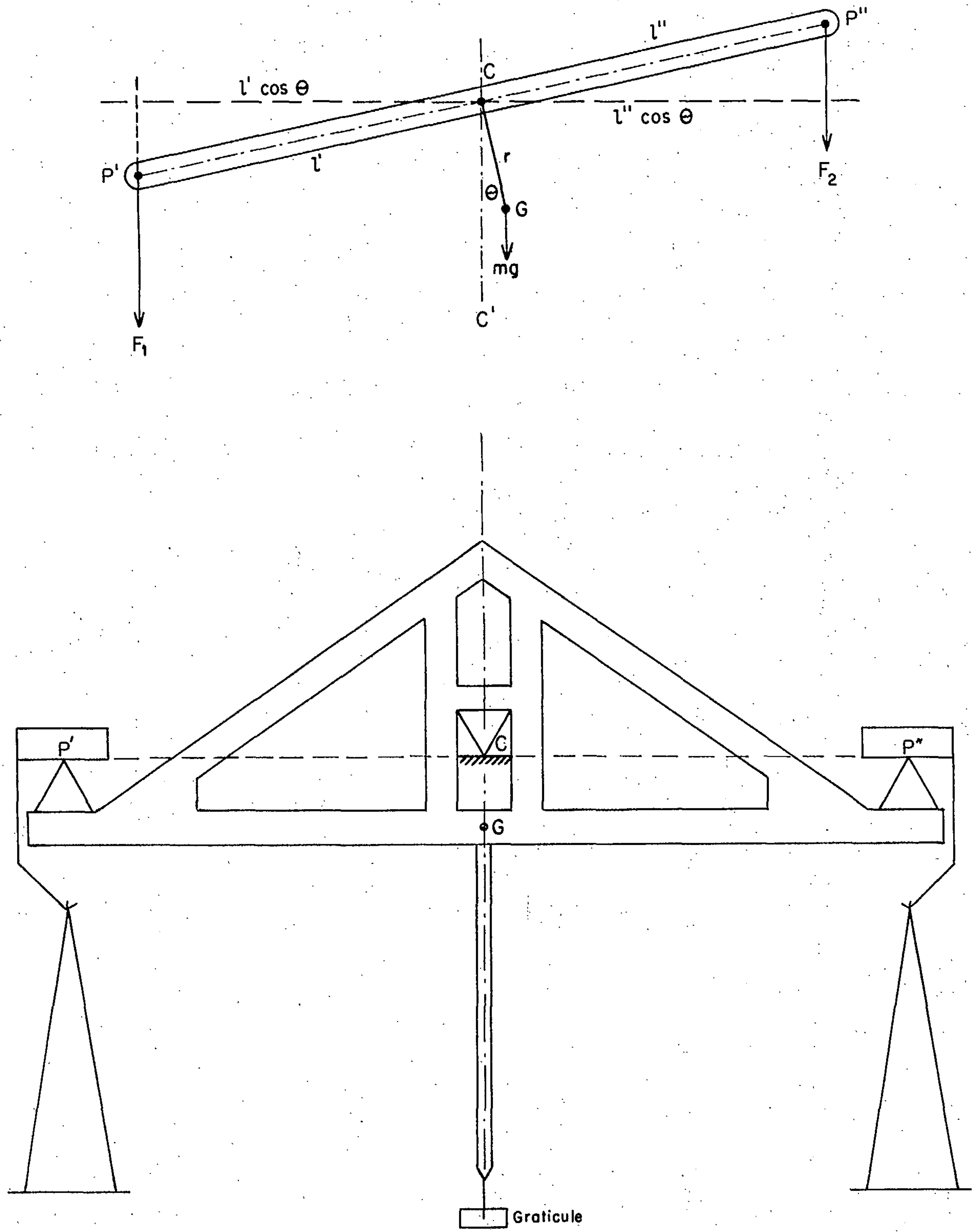
Note: an equal arms classical balance should never be treated as an "indicating" instrument: although there is a point on the scale which, in theory, corresponds to rigorously equal forces, this point has no absolute fixed position and cannot be used as reference point.

2. Physical Properties of the Balance.

One single comparison between two mass standards will not yield very much information about the overall precision of which the instrument is actually capable. To obtain a very high precision (in a metrological and a statistical sense), for instance as it is indicated by the calculation of standard deviations in certain complex intercomparisons patterns, the operator must be extremely conscious of the fact that a balance is a real *physical instrument,* placed in a real *physical environment.* Thus for instance, it had been assumed in the calculations leading to the equations that the lengths of the arms are constant: this is true only if the temperatures of various parts of the beam remain constant; it had also been assumed that the only forces which act on the beam are produced by the compared masses; when quantities as small as a fraction of a milligram are considered, it is obvious that very minute actions such as movements of the air, specks of dust, condensation of water etc., may play an appreciable role. The ultimate precision, which becomes apparent only at the end of statistical calculations, will be influenced by certain precautions the reason of which is not clearly visible beforehand.

The most important part of a balance is the line of contact between the central knife and the horizontal plane on which the knife rests. On a microscopic scale this contact is very far from any "geometrical idealization", as the pressures are enormous and produce elastic deformations, both in the knife and the plane. Every time a balance is arrested (which lifts the knife from the plane) the physical nature of the contact is more or less modified. This is why the sensitivity may not be identically the same before and after an interchange of masses. In the theory given in this Memorandum, it had been assumed that comparisons are accomplished with a minimum of arrests, and that the adding and the removal of the sensitivity mass μ may be made without arresting the beam. Some metrologists prefer not to follow the principle of the minimum of arrests but, on the contrary to arrest the beam several times in the course of a comparison. Their equations are thus based on the notion of "average sensitivity."

In some of the highest quality balances the contact between the knife and the plane is not broken when the beam is arrested; the latter is simply immobilized by means of appropriate felt pads.



THREE EDGE, EQUAL ARM, TWO PAN BALANCE

FIG. 1

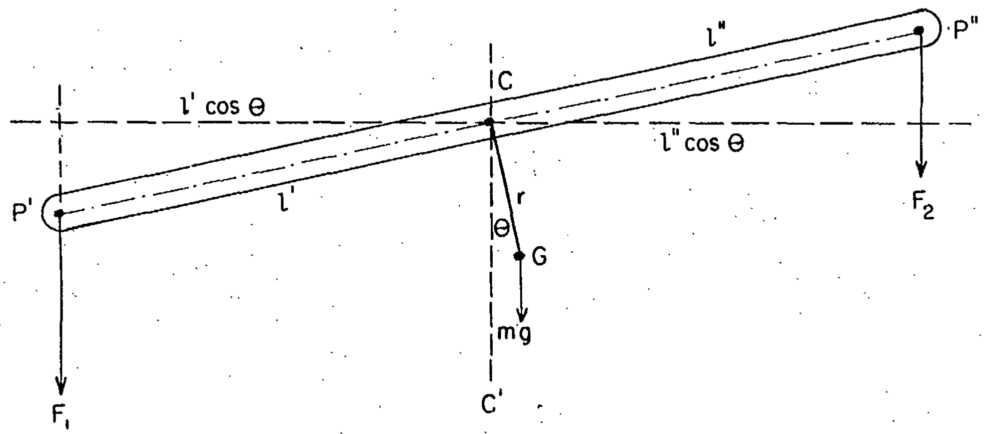


FIG. 2

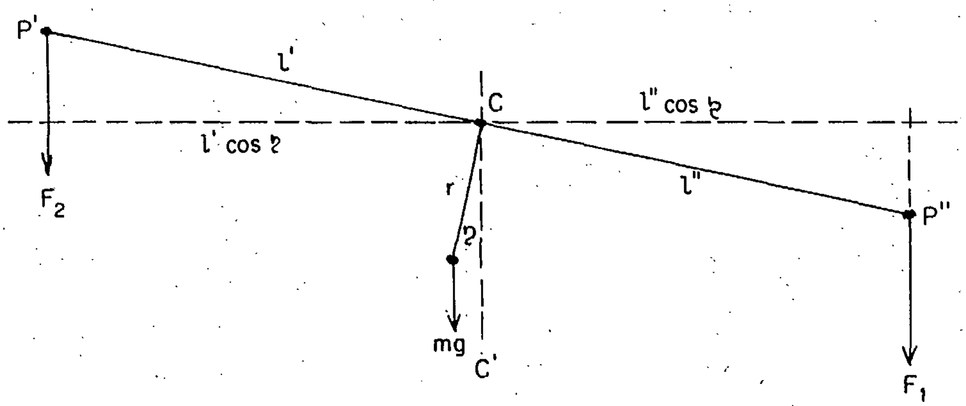


FIG. 3

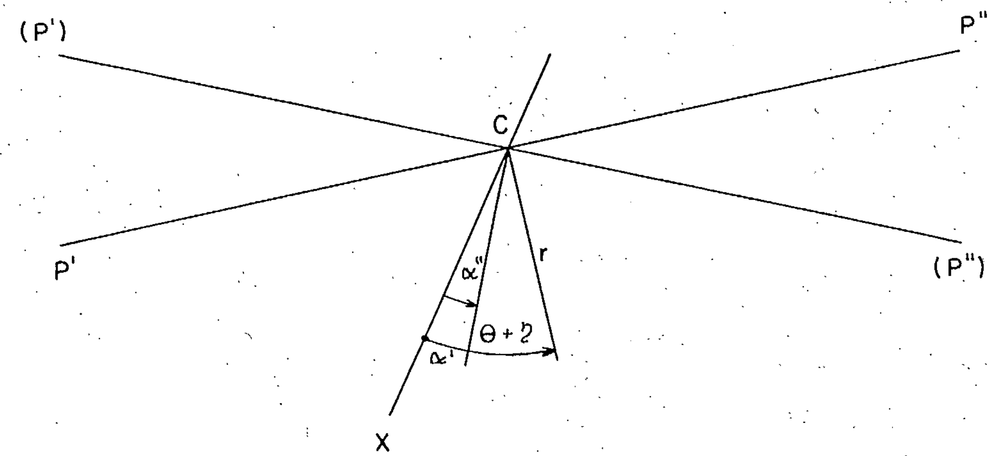


FIG. 4

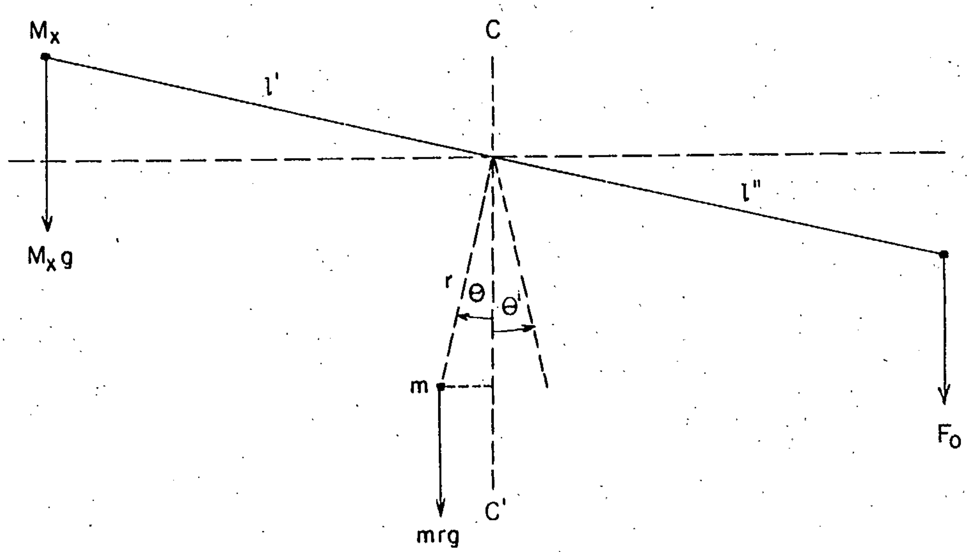


FIG. 5