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Theory of Likelihood Ratio Detection of Random Signals Dependent on Causally Filtered Wiener-plus-Poisson Noise

by A. Climescu-Haulica and A.F. Gualtierotti

CRC Report No. CRC-RP-01-01
Ottawa, August 2001

Canada

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ACKNOWLEDGEMENTS

The authors are grateful to Prof. C.R. Baker from Department of Statistics, University of North Carolina at Chapel Hill for useful discussions during the work reported herein. Also they wish to thank Dr. T.J. Willink and Dr. M. Sablatash from the Communications Research Centre for their helpful comments.

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This Research was supported by Swiss National Fund for Scientific Research (Contracts 21-43201-95 and 20-50534-97).

ABSTRACT

A general method of likelihood ratio computation is obtained for a filtered type of noise, with Gaussian and Poisson components. The idea is to call upon the Cramér-Hida representation of second order processes and to interpret it as a path transformation. This approach applies to underwater acoustics signal detection and potentially it is a tool to be used in mobile communication techniques.

RÉSUMÉ

Une méthode générale pour le calcul du rapport de vraisemblance est obtenue pour des types de bruit filtrés, avec des composantes Gaussiennes et Poissoniennes. L'idée est de faire appel à la représentation de Cramér-Hida et de l'interpréter comme une transformation des trajectoires. Cet approche s'applique à la détection des signaux acoustiques sous-marine et pourrait trouver usage dans les techniques de la communication mobiles.

EXECUTIVE SUMMARY

Detection of random signals based on likelihood ratio is optimal in the sense of the Neyman-Pearson criterion. As in general the signals can be given, *a priori,* only a broadly qualitative description and have a dynamical behavior, they are modeled as stochastic processes.

With the aim of fitting a general class of signals and noises, this report addresses the case of causally filtered Gaussian and Poisson noise components. It is assumed that the noise has paths of finite energy, i.e. they are continuous in quadratic mean. The signal is smoother than the noise, so it is assumed that it belongs to the reproducing kernel Hilbert space of the noise. That ensures the absolute continuity of the probability laws induced by the received signal and the noise and hence the existence of the likelihood ratio.

Explicitly, the likelihood ratio is obtained as a functional on the space of the received signal. Its computation is decoupled into two operations. The first one is the computation of the likelihood ratio for the unfiltered received signal and noise. This is basically a stochastic calculus problem and involves the use of a version of the Girsanov theorem as well as particular factorization results. The second one is the computation of the conditional probability law of the unfiltered noise with respect to the filtered noise. This conditional law depends mostly on the trace-class properties of the covariance operator of the filtered noise.

The likelihood ratio method described in this document has been adapted to sonar applications. In particular the active sonar in a reverberation limited environment benefits from this approach, as the reverberation has the characteristics of a causally filtered phenomenon.

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$\mathbb{R}_+$	positive real numbers set
$\overline{\mathbb{R}}_+$	positive real numbers set union with ∞
A^c	complement of the set A
$\overline{A}$	closure of the set A
	with respect to a considered topology
α	parameter in the interval $[0,1]$
Filtered Model	
X, X_α	received signal
S, S_α	part containing the transmitted signal
N, N_α	noise
Unfiltered Model	
Y, Y_α	received signal
B, B_α	noise
$\langle M \rangle$	conditional quadratic variation
	of a square integrable martingale M
$[M]$	quadratic variation
	of a square integrable martingale M
$V[M]$	variance function of the random variable M
$M(\cdot)$	values of the function M
	when applied to its entire definition set
$s \wedge t$	minimum between s and t
$\ll$	relationship of absolute continuity for measures
$\langle\ ,\ \rangle_{\mathcal{X}}$	inner product in the space $\mathcal{X}$
$\oplus$	set addition
$\oplus^{-1}$	inverse map associated with set addition
$[\![S, T]\!]$	random interval generated by the stopping times S and T
J^*	adjoint of the operator J
$\mathcal{R}(J)$	range of the operator J

$\otimes$	tensorial product
$\star$	convolution of probability measures

1.0 INTRODUCTION

A well known fact from the theory and practice of communication systems is that the simultaneous presence of environmental random fluctuations called "noise" makes the status of the signal uncertain at the receiver. *A priori* it may be unknown if the received signal contains information or is just noise. In techniques like sonar and radar, the answer to the question "does the received signal contain any information?" is the core of the application.

Signal detection theory appeared in the 1940's and seems to have been a consequence of the war efforts [1]. Its foundations are strongly connected with the "l'air du temps" brought by Norbert Wiener's work for a MIT project trying to predict the track of an airplane, as well as the publication of the first book on radar detection [2]. The idea, new at the time, was that the communication of information is a statistical problem and that the performance limits could be calculated from optimization criteria and a systematic approximation designed.

The first technique used in detection problems was the "matched filter"[1], derived independently by Wiener, Hansen, North, Van Vleck and Middleton. It was acknowledged that the "signal-to-noise ratio" was not the natural criterion for signal detection. Mark Kaç provided the connection with statistical hypothesis testing, noting that the Neyman-Pearson criterion is adequate for radar detection. The theory shows that the key quantity to compute is the likelihood ratio, useful also for applying a number of other criteria. Woodward [3] came to the likelihood ratio via a different route, inspired by the information-theoretic result that the relevant information is all preserved in the conditional probabilities of the hypotheses given the observations. Later, the so-called statistical decision theory introduced by Wald [4] was applied to signal detection problems. In all cases, the basic operation is to compare a likelihood ratio with a threshold, whose value is determined by the chosen criterion. A general

[1] Based on the maximization of the signal-to-noise ratio.

Figure 1.1: The three steps of a signal detection algorithm.

approach of the detection problem can be depicted as in fig. 1.1.

Informally, if $X(t)$, $S(t)$, and $N(t)$ are stochastic processes describing the received signal, the transmitted signal and the noise, respectively, then the detection problem consists, in terms of statistical hypotheses tests, of choosing between

$$\begin{cases} \mathbf{H}_0 : & X(t) & = & N(t) & 0 \leq t \leq T \\ \mathbf{H}_1 : & X(t) & = & S(t) + N(t) & 0 \leq t \leq T. \end{cases} \tag{1}$$

The strategy provided by Neyman-Pearson criterion assigns the detector to the likelihood ratio $\frac{dP_{S+N}}{dP_N}$. This is an optimal detector, in the sense that it minimizes the probability of non-detection, i.e. $P\left(\mathbf{H}_1 \text{ rejected } \mid \mathbf{H}_1 \text{ true}\right)$, for a given probability of false alarm $P\left(\mathbf{H}_0 \text{ rejected } \mid \mathbf{H}_0 \text{ true}\right)$. In particular this fits the case of radar or sonar detection, where it is hard to judge the implications of not detecting a target but the acceptable probability of false alarm can be determined.

The performance of this detection method is usually measured by means of the receiver operating characteristic (ROC), obtained by plotting the probability of detection versus the probability of false alarm.

Along with the expansion of application from radar to sonar, remote sensing and pattern recognition, the noise models evolved from white Gaussian noise [5][6][3] to coloured Gaussian noise and randomly modulated jump processes. Following these ideas, the present report contains the derivation, under minimal assumptions, of a likelihood detection formula for a random signal of unknown law, disturbed by a noise with filtered Wiener and Poisson components. Such models, as discussed at length in [7] and [8], are applicable when the noise is very nonstationary and the signal cannot be represented as a set of narrowband components. Typical examples come from the radar and sonar areas [9].

2.0 GENERAL DESCRIPTION OF THE MODEL

In all that follows, signals and noise are monitored over the time interval $[0, T]$. N denotes a zero-mean, mean-square continuous noise process with paths in $\mathcal{L}_2[0, T]$, the set of functions over $[0, T]$ whose square is integrable with respect to Lebesgue measure. S is a random signal, dependent on N, such that, for almost every $\omega \in \Omega$, with respect to a probability measure P, defined on a σ-field of subsets of Ω,

$$S(\omega, \cdot) \in H(N)$$

where $S(\omega, \cdot)$ denotes the signal path for event ω, and $H(N)$ is the reproducing kernel Hilbert space (RKHS) associated with N. The condition that signal paths belong to the RKHS of the noise is an operational form of the requirement that the signal be smoother than the noise. It also has the consequence, as the noise is mean-square continuous, that S has continuous paths, and thus that $S + N$ has paths in $\mathcal{L}[0, T]$.

Let P_N be the probability measure induced on the Borel sets of $L_2[0, T]$ by N, and P_{S+N} that induced by $S + N$. When N is Gaussian, or more generally spherically invariant, that is Gaussian with a random variance, no further mathematical restriction is needed to obtain that P_{S+N} is absolutely continuous with respect to P_N.

However, to have an "explicit" expression[2] for the likelihood, information on some derivative of the signal is required. Indeed, the likelihood is a functional

$$\Lambda : \; \mathcal{L}_2 [0, T] \longrightarrow I\!R$$

to be computed for every function $f \in \mathcal{L}_2 [0, T]$ (the received waveform), irrespective of the regime (P_N or P_{S+N}) that produces f. Λ is related to P_N and P_{S+N} through the expression:

$$P_{S+N} (df) = \Lambda (f) \, P_N (df) \, , \; f \in \mathcal{L}_2 [0, T] \, .$$

It turns out that the RKHS condition is enough to enable the derivation of the explicit expression for Λ only with respect to P_{S+N}, but not with respect to P_N. The latter requires the information about the derivative already mentioned, which amounts to demanding mutual absolute continuity of P_{S+N} and P_N. In its absence, an approximation to the likelihood, which is moreover signal dependent, is explicitly obtained (Proposition 12). Explicit expressions of the likelihood are useful in actual practice when computing its value using discretely collected data [7]. An explicit expression for the likelihood can be had by restricting the family of signals that are admitted. Such a sufficient condition may be stated as follows:

$$E \left[\exp \left\{ \frac{1}{2} \| S (\cdot, \cdot) \|^2_{H(N)} \right\} \right] < \infty .$$

Establishing the form of Λ may be achieved through a decoupling operation which involves the Cramér-Hida decomposition [10, 11] and a theory of stochastic calculus that is tailor-made for that decomposition. The relevant papers are [12], [8], [13] and [14].

As noises are frequently not purely Gaussian, nor for that matter spherically invariant, it is imperative to obtain Λ for noises that accomodate at least an explicitly impulsive component, such as a Poisson process. For example, the underwater acoustics noise [15] is such a noise which prompted the search for the method producing

[2]The quotation marks on the word "explicit" are meant as a reminder that though the analytical form of Λ must be explicit, its actual use requires extensive calibration by actual data, and detailed knowledge of the specifics of the detection environment that applies. Some indication of what that entails can be found in [7].

Λ. It is shown in this report how to obtain the analytic form of Λ when the noise N has the form:

$$N(\omega,t)=\int_0^t F(t,x)\,W(\omega,dx)$$

where F is a non-anticipative, non-random filter, and W is a white noise of the form:

$$W(\omega,t)=\frac{B_1(\omega,t)+\tilde{B}_2(\omega,t)}{\sqrt{2}}$$

where B_1 is a generalized Brownian motion, $\tilde{B}_2$ is a Poisson martingale, and B_1 and $\tilde{B}_2$ are independent and have the same variance function β[3]. The signal is still assumed to have paths in the RKHS of the noise.

When mutual absolute continuity holds, the analytic form of the likelihood Λ is then as follows. Let R_N be the covariance operator built from the covariance C_N of the noise N, and K be the closure in $L_2[0,T]$ of the range of the square root of R_N[4]. Let us define an operator $U:\ L_2[\beta]\longrightarrow L_2[0,T]$ with the property that

$$U=R_N^{\frac{1}{2}}J^{\star}$$

where

$$J:\ L_2[0,T]\longrightarrow L_2[\beta]$$

is a partial isometry with initial space K and final space $L_2[\beta]$ and $J^{\star}$ denotes its adjoint. The families $\{\lambda_n,\ n\in\mathbb{N}\}$ and $\{e_n,\ n\in\mathbb{N}\}$ are, respectively, the eigenvalues and orthonormal eigenvectors of R_N. Then the process M is defined as

$$M(f,t)=\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\langle UI_{[0,t]},e_k\rangle_{L_2[0,T]}\langle f,e_k\rangle_{L_2[0,T]}$$

[3]The Cramér-Hida decomposition stipulates that W is a process with orthogonal increments and variance σ_W^2. The specific form of W and σ_W^2 that is chosen finds its justification *a posteriori* in Girsanov's theorem - see Proposition 6. As shown later, the two sources of noise can be weighted by considering a W of the form: $W_\alpha=\sqrt{\alpha}\,B_1+\sqrt{1-\alpha}\,\tilde{B}_2,\ 0<\alpha<1$.

[4]Thus $K=\overline{\mathcal{R}\left(R_N^{\frac{1}{2}}\right)}\subseteq L_2[0,T]$, where $\mathcal{R}$ denotes the range of an operator and the overbar denotes the closure in $L_2[0,T]$ of a set. In what follows, K is always used for the set just defined.

where $f \in D[0,T]$ defines a function continuous to the right and with limits to the left. While M is obtained with the help of the Cramér-Hida representation, from stochastic calculus it follows that

$$\begin{aligned}\ln\left[\tilde{\Lambda}(f)\right] &= \int_0^T s(f,x)\, ev(f,dx) \\ &\quad -\frac{1}{4}\int_0^T s^2(f,x)\,\beta_1(dx) \\ &\quad -\frac{1}{\sqrt{2}}\int_0^T s(f,x)\,\hat{B}_2(f,dx)\end{aligned}$$

where $ev(f,t) = f(t)$, $\hat{B}_2$ is a centred counting process and s is a predictable process resulting from the condition $S(\omega,\cdot) \in H(N)$. M has paths in $D[0,T]$, and $\underline{M}$ is the path map of M ($\underline{M}(\omega) = \{M(\omega,t),\ t \in [0,T]\}$). Then, finally,

$$\Lambda(f) = \left\{\tilde{\Lambda} \circ \underline{M}\right\}(f).$$

It is an interesting fact that with such a model it is no need to worry about robust versions of the likelihood as defined by J.M.C. Clark [16].

Two remarks about the derivation should be made. As the stochastic calculus is used on $D[0,T]$, for processes which are adapted to the filtration generated by the evaluation maps and defined simultaneously for couples of probability measures not known *a priori* to be mutually absolutely continuous, the usual assumption of the "usual conditions"[5] of stochastic calculus being met is not warranted. Because no such assumptions are made in [17], that is the reference used for stochastic calculus. Secondly, most of the derivation is made under assumptions that are somewhat more general than those stated so far; the main reason for so doing is that the limits are better seen. In particular, it can be explicitly seen to what extent the Cramér-Hida framework and the RKHS requirement are essential for a realistic modeling of signal detection problems (Section 4.6).

Here is the rationale for the method of calculation which in turn structures the report. It can be written, as a path relation [12, Thm 1, p.163],

$$N = \Phi \circ W.$$

[5] The exact meaning is given in section 3.1.1.

As the requirement $S(\omega, \cdot) \in H(N)$ translates into

$$P\left(\int s^2 d\sigma_W^2 < \infty\right) = 1,$$

denoting absolute continuity as $\ll$, then

$$P_{\int s \, d\sigma_W^2 + W} \ll P_W$$

and consequently [12, Theorem 1, p.163],

$$S + N = \Phi\left(\int s \, d\sigma_W^2 + W\right).$$

From there it follows for some explicit Ψ involving N and W only

$$\frac{dP_{S+N}}{dP_N} = \frac{dP_{\int s \, d\sigma_W^2 + W}}{dP_W} \circ \Psi.$$

To have an explicit form of

$$\frac{dP_{\int s \, d\sigma_W^2 + W}}{dP_W}$$

$\int s \, d\sigma_W^2 + W$ has to be written in "innovations" form, that is, as a functional of its past. Furthermore, the computation of

$$\frac{dP_{\int s \, d\sigma_W^2 + W}}{dP_W}$$

must pass through a form of Girsanov's theorem which is established first as a mutual absolute continuity result following the introduction of an exponential martingale. Only thereafter can path conditions and absolute continuity be introduced.

In Section 3, the model used, and some of its properties, are presented and explained. Section 4 deals with the unfiltered problem, that is, the computation of

$$\tilde{\Lambda} = \frac{dP_{\int s \, d\sigma_W^2 + W}}{dP_W}$$

and has several parts. The first deals with the required version of Girsanov's change of measure. Then, assuming that $\int s\, d\sigma_W^2 + W$ can be written as a stochastic integral equation, the likelihood $\tilde{\Lambda}$ is obtained. In the last part of Section 4, it is shown how the requirement $P\left(\int s^2 d\sigma_W^2 < \infty\right) = 1$ suffices; firstly it is proved that the condition on the exponential martingale can be stated as signal paths properties, and weakened to absolute continuity, and then the required innovations representation is provided. Section 5 yields the function Ψ.

The systematic use of the methods that follow has its origins in [18], but some important ideas in the latter can be found already in [19]. "This is distinguished by its modelling of the problem on the space of sample paths, rather than on an underlying abstract probability space, and was found by us to be very useful" [12, p.160].

3.0 THE DETECTION MODEL

This section contains an extended description of the noise and signal models. The signal is assumed to be smoother as randomness than the noise which is nonstationary and non-Gaussian. Also, the complete list of the assumptions made along the presentation is included here.

3.1 THE NOISE

The noise, denoted N_α, is defined as the integral of a non-anticipative deterministic kernel with respect to a process with orthogonal increments, and may be looked at as a filtered white noise with independent Gaussian and Poisson components.

3.1.1 The integrator

As usual, $(\Omega, \mathcal{A}, P)$ is the reference probability space, all processes considered are defined on that space, and adapted to a filtration $\underline{\mathcal{A}} = \{\mathcal{A}_t, t \in [0,T]\}$ of $\mathcal{A}$, which satisfies the *usual conditions*: the filtration is right continuous, every null set belongs to all σ-fields $\mathcal{A}_t$ and every subset of a null set is $\mathcal{A}_t$ measurable. A generalized

Brownian motion is a Brownian motion for which the variance function is a non-negative, monotone non-decreasing and continuous function. Its paths are almost surely continuous, and those that are not may be taken as being continuous to the right [17, 4.3.5, p. 71]. It is denoted B_1 in the sequel, and β_1 represents its variance function:

$$V[B_1(\cdot,t)] = E\left[B_1^2(\cdot,t)\right] = \beta_1(t), \; 0 \le t \le T.$$

B_1 is a square integrable martingale and its compensator[6] [17, p.148] for fixed $t \in [0,T]$, is given by

$$\langle B_1 \rangle (\omega,t) = \beta_1(t),$$

almost surely with respect to P.

Let B_2 denote a Poisson process. Then $\beta_2(t)$, which stands for $E[B_2(\cdot,t)]$, is finite, and continuous for $t \ge 0$ [20, 2.4.1, p.41]. Let

$$\tilde{B}_2(\omega,t) = B_2(\omega,t) - \beta_2(t).$$

$\tilde{B}_2$ is a square integrable martingale. Its compensator [17, p.148] for fixed $t \in [0,T]$, is given by

$$\langle B_2 \rangle (\omega,t) = \beta_2(t)$$

almost surely with respect to P.

Furthermore, for fixed $t \in [0,T]$, almost surely with respect to P, the quadratic variation[7] of B_2 is given by

$$[B_2](\omega,t) = B_2(\omega,t).$$

It is assumed that B_1 and B_2 are independent. Then let $0 \le \alpha \le 1$ and set

$$\beta_\alpha(t) \doteq \alpha\beta_1(t) + (1-\alpha)\beta_2(t)$$

[6]The compensator (also called predictable increasing process or conditional quadratic variation) of a square integrable martingale M is denoted usually by $\langle M \rangle$ and is defined as the unique, up to the almost sure equality, predictable increasing process, such that $M^2 - \langle M \rangle$ is a local martingale.

[7]The quadratic variation (also called increasing process) of a square integrable martingale M is defined by $[M](\omega,t) = M^2(\omega,t) - M^2(\omega,0) - 2\int_0^t M(\omega,x)M(\omega,dx)$. For almost surely continuous square integrable martingales, $[M] = \langle M \rangle$.

and

$$B_\alpha(\omega,t) \doteq \sqrt{\alpha}\,B_1(\omega,t) + \sqrt{1-\alpha}\,\tilde{B}_2(\omega,t).$$

B_α is then a square integrable martingale and, for fixed $t \in [0,T]$, almost surely with respect to P,

$$\langle B_\alpha \rangle(\omega,t) = \beta_\alpha(t)$$

and

$$[B_\alpha](\omega,t) = \alpha\beta_1(t) + (1-\alpha)\,\beta_2(\omega,t).$$

3.1.2 The integrand

Let F denote a Borel measurable function over the rectangle $[0,T] \times [0,T]$ that has the following properties:

a. for t and x in $[0,T]$ fixed but arbitrary, such that $x > t$, $F(t,x) = 0$,

b. for $t \in [0,T]$ fixed but arbitrary, $\int_0^t F^2(t,x)\,\beta_\alpha(dx) < \infty$,

c. the map $t \mapsto [F(t,\cdot)]_\alpha \in L_2[\beta_\alpha]$ is continuous (where $[F(t,\cdot)]_\alpha$ is the equivalence class of $F(t,\cdot)$ in $L_2[\beta_\alpha]$),

d. $\{[F(t,\cdot)]_\alpha,\ t \in [0,T]\}$ generates $L_2[\beta_\alpha]$.

Remarks:

a. As
$$\int_0^t f^2(x)\,\beta_\alpha(dx) = \alpha\int_0^t f^2(x)\,\beta_1(dx) + (1-\alpha)\int_0^t f^2(x)\,\beta_2(dx)$$
whenever $0 < \alpha < 1$,
$$\mathcal{L}_2[\beta_\alpha] = \mathcal{L}_2[\beta_1] \cap \mathcal{L}_2[\beta_2].$$

b. The conditions that F must satisfy are those that ensure that N has a canonical representation of multiplicity one, in the sense of Cramér-Hida [10, 11]. A discussion of the nature of the restriction on the noise process that is thus introduced may be found in [21].

3.1.3 Noise model and properties

B_α may be considered as a prototype of a process with orthogonal increments. The stochastic process

$$N_\alpha(\omega, t) = \int_0^t F(t, x)\, B_\alpha(\omega, dx)\,,\ t \in [0, T]$$

can be defined by following the general construction of the integral with respect to a process with orthogonal increments [20, 7.4, p.160].

Then, for $t \in [0, T]$ fixed but arbitrary, $E[N_\alpha(\cdot, t)] = 0$, and the covariance C_{N_α} of N_α is then given by the following expression:

$$C_{N_\alpha}(s, t) = \int_0^{s\wedge t} F(t, x)\, F(s, x)\, \beta_\alpha(dx)\,,\ (s, t) \in [0, T] \times [0, T]\,.$$

As a consequence of the assumptions on F, the function $t \mapsto C_{N_\alpha}(t, t)$ is continuous for $t \in\ ]0, T[$. N_α is thus continuous in quadratic mean [20, 6.21, p.133] and its covariance is continuous [20, 6.2.2, p.133]. Furthermore, the paths of N_α are, almost surely with respect to P, in $\mathcal{L}_2[0, T]$.

Let $H(N_\alpha)$ denote the reproducing kernel Hilbert space of N_α. Then [22, p.97]

$$H(N_\alpha) = \left\{ \tilde{f}(t) = \int_0^t F(t, x)\, f(x)\, \beta_\alpha(dx)\,,\ f \in \mathcal{L}_2[\beta_\alpha] \right\}.$$

For the inner product $\langle \cdot\,, \cdot \rangle_{H(N_\alpha)}$ of $H(N_\alpha)$, it follows that whenever f and $g \in \mathcal{L}_2[\beta_\alpha]$ and

$$\tilde{f}(t) = \int_0^t F(t, x)\, f(x)\, \beta_\alpha(dx)\,,\ \tilde{g}(t) = \int_0^t F(t, x)\, g(x)\, \beta_\alpha(dx)$$

then

$$\langle \tilde{f}, \tilde{g} \rangle_{H(N_\alpha)} = \langle f, g \rangle_{L_2[\beta_\alpha]}.$$

The covariance operator $R_{N_\alpha} : L_2[0, T] \longrightarrow L_2[0, T]$ is computed using the formula

$$R_{N_\alpha}\left([f]_{L_2[0,T]}\right) = \left[\int_0^T C_{N_\alpha}(\cdot, x)\, f(x)\, dx\right]_{L_2[0,T]}\,,\ f \in \mathcal{L}_2[0, T]$$

where $[\cdot]$ denotes equivalence classes.

This operator is non-negative, self-adjoint and continuous, with finite trace [23, p.125]. It can thus be written as

$$R_{N_\alpha} = \sum_{i=1}^{\infty} \lambda_i \left[e_i \otimes e_i \right]$$

where, for an orthonormal family $\{e_n,\ n \in I\!N\}$,

$$R_{N_\alpha} e_n = \lambda_i e_n, \quad [e_n \otimes e_n] f = \langle f, e_n \rangle_{L_2[0,T]} e_n, \quad \lambda_i \geq 0, \quad \sum_{n=1}^{\infty} \lambda_n < \infty.$$

In an obvious way, in $L_2[P]$ it follows that

$$[N_\alpha(\cdot, t)]_{L_2[P]} = \left[N_\alpha^{(1)}(\cdot, t)\right]_{L_2[P]} + \left[N_\alpha^{(2)}(\cdot, t)\right]_{L_2[P]}$$

with

$$\begin{aligned} N_\alpha^{(1)}(\omega, t) &= \sqrt{\alpha} \int_0^t F(t, x) \, B_1(\omega, dx) \\ N_\alpha^{(2)}(\omega, t) &= \sqrt{1-\alpha} \int_0^t F(t, x) \, \tilde{B}_2(\omega, dx). \end{aligned}$$

$N_\alpha^{(1)}$ and $N_\alpha^{(2)}$ are independent, and therefore, on $L_2[0,T]$,

$$P_{N_\alpha} = P_{N_\alpha^{(1)}} \star P_{N_\alpha^{(2)}}$$

where $\star$ denotes convolution. P_{N_α} is the measure induced on $L_2[0,T]$ by P and the maps

$$\omega \mapsto \langle N_\alpha(\omega, \cdot), f \rangle_{L_2[0,T]}, \quad f \in \mathcal{L}_2[0,T].$$

Proposition 1

Let $\mathcal{S}_\alpha^{(1)}, \mathcal{S}_\alpha^{(2)}, \mathcal{S}_\alpha$ denote the supports in $L_2[0,T]$ of $P_{N_\alpha^{(1)}}$, $P_{N_\alpha^{(2)}}$ and P_{N_α}, respectively. Then

$$\mathcal{S}_\alpha = \overline{\mathcal{S}_\alpha^{(1)} + \mathcal{S}_\alpha^{(2)}}.^{8}$$

[8] The overbar denote here the closure in $L_2[0,T]$ of a set.

Proof: In a separable metric space, the support of a probability measure μ is the unique closed set of measure one that is contained in every closed set of measure one and that has the property that, for each of its points x and for every open set O containing it, $\mu(O) > 0$ [24, Thm 2.1, p.27].

Let $\oplus$ denote addition in $L_2[0,T]$[9]. Then

$$\begin{aligned} P_{N_\alpha}\left(\overline{\mathcal{S}_\alpha^{(1)} + \mathcal{S}_\alpha^{(2)}}\right) &\geq P_{N_\alpha}\left(\mathcal{S}_\alpha^{(1)} + \mathcal{S}_\alpha^{(2)}\right) \\ &= P_{N_\alpha^{(1)}} \star P_{N_\alpha^{(2)}}\left(\oplus^{-1}\left[\mathcal{S}_\alpha^{(1)} + \mathcal{S}_\alpha^{(2)}\right]\right) \\ &= P_{N_\alpha^{(1)}}\left(\mathcal{S}_\alpha^{(1)}\right) P_{N_\alpha^{(2)}}\left(\mathcal{S}_\alpha^{(2)}\right) \\ &= 1. \end{aligned}$$

Consequently, by definition $\mathcal{S}_\alpha \subseteq \overline{\mathcal{S}_\alpha^{(1)} + \mathcal{S}_\alpha^{(2)}}$. Let

$$\begin{aligned} F_\epsilon &= \left\{\|N_\alpha(\omega,\cdot) - x\|_{L_2[0,T]} \geq \epsilon\right\}, \\ G_\epsilon &= \left\{\left\|N_\alpha^{(1)}(\omega,\cdot) - u\right\|_{L_2[0,T]} \geq \frac{\epsilon}{2}\right\}, \\ H_\epsilon &= \left\{\left\|N_\alpha^{(2)}(\omega,\cdot) - v\right\|_{L_2[0,T]} \geq \frac{\epsilon}{2}\right\}. \end{aligned}$$

Suppose now $x = u + v \in \mathcal{S}_\alpha^{(1)} + \mathcal{S}_\alpha^{(2)}$, but $x \notin \mathcal{S}_\alpha$. Then, since $N_\alpha^{(1)}$ and $N_\alpha^{(2)}$ are independent,

$$P(F_\epsilon) \leq P(G_\epsilon \cup H_\epsilon) = P(G_\epsilon) + P(H_\epsilon) - P(G_\epsilon)\,P(H_\epsilon).$$

But there is an ϵ_0 such that, for $\epsilon \leq \epsilon_0$, $P(F_\epsilon) = 1$. However, for any $\epsilon > 0$,

$$P(G_\epsilon) < 1 - \lambda(\epsilon),\ \lambda(\epsilon) > 0,$$

and

$$P(H_\epsilon) < 1 - \mu(\epsilon),\ \mu(\epsilon) > 0.$$

Thus

$$1 = P(F_\epsilon) \leq (1 - \lambda(\epsilon)) + (1 - \mu(\epsilon)) - (1 - \lambda(\epsilon))(1 - \mu(\epsilon)),$$

[9] $\oplus^{-1}$ will denote the inverse map associated with $\oplus : L_2[0,T] \times L_2[0,T] \to L_2[0,T]$.

which is clearly impossible. Consequently $\mathcal{S}_\alpha^{(1)} + \mathcal{S}_\alpha^{(2)} \subseteq \mathcal{S}_\alpha$. *q.e.d.*

Remark: Proposition 1 and [25] yield that whenever $K = L_2[0,T]$ then $\mathcal{S}_\alpha = L_2[0,T]$.

Proposition 2

Let $U : L_2[\beta_\alpha] \longrightarrow L_2[0,T]$ denote the operator for which Uf is the equivalence class of $\langle F(t,\cdot), f\rangle_{L_2[\beta_\alpha]}$ in $L_2[0,T]$. Then

$$U = R_{N_\alpha}^{\frac{1}{2}} J^\star, \ \textit{where } J : L_2[0,T] \longrightarrow L_2[\beta_\alpha]$$

is a partial isometry onto $L_2[\beta_\alpha]$, with initial space K.

Proof: The right hand side of the equality that defines U is the equivalence class of a continuous function, and, as such, the latter is square integrable over $[0,T]$. U is thus well defined. It is continuous as

$$\|Uf\|_{L_2[0,T]}^2 \leq \left\{ \max_{t\in[0,T]} \|F(t,\cdot)\|_{L_2[\beta_\alpha]}^2 \right\} \|f\|_{L_2[\beta_\alpha]}^2.$$

Let $U^\star$ denote the adjoint of U. Then

$$U^\star : L_2[0,T] \longrightarrow L_2[\beta_\alpha], \ [U^\star f](t) = \int_t^1 F(x,t) f(x)\, dx.$$

A computation shows that $UU^\star = R_{N_\alpha}$. The polar decomposition yields then that

$$U^\star = J R_{N_\alpha}^{\frac{1}{2}},$$

where J is a partial isometry with initial space K and final space L such that

$$K = \overline{\mathcal{R}\left(R_{N_\alpha}^{\frac{1}{2}}\right)}, \text{ and } L = \overline{\mathcal{R}(U^\star)}.$$

Furthermore, if $Uf = 0$ in $L_2[0,T]$, by continuity,

$$\langle F(t,\cdot), f\rangle_{L_2[\beta_\alpha]} = 0, \;\; t \in [0,T],$$

and thus $f = 0$ in $L_2[\beta_\alpha]$. Hence, the null space of U is $\mathcal{N}(U) = \{0\}$ and consequently, since

$$\mathcal{N}(U) = \overline{\mathcal{R}(U^*)}^{\perp},$$

it follows that

$$\overline{\mathcal{R}(U^*)} = L_2[\beta_\alpha].$$

q.e.d.

Remark: The operator J is unitary as soon as $K = L_2[0,T]$. A sufficient condition is that the closure of the range of $R^{\frac{1}{2}}_{N^{(1)}_\alpha}$ be $L_2[0,T]$, i.e., that $P_{N^{(1)}_\alpha}$ has full support.

Corollary 1

Let K° be the range of $R^{\frac{1}{2}}_{N_\alpha}$, and define, on K°, the inner product

$$\langle R^{\frac{1}{2}}_{N_\alpha} f, R^{\frac{1}{2}}_{N_\alpha} g\rangle_{K^\circ} = \langle f, g\rangle_{L_2[0,T]}.$$

Then $L_2[\beta_\alpha]$ and K° are unitarily equivalent, and thus so are $H(N_\alpha)$ and K°.

Proof: K° is obviously a Hilbert space. Define $\tilde{U}: \; L_2[\beta_\alpha] \longrightarrow H(N_\alpha)$ by

$$\left\{\tilde{U}(f)\right\}(t) = \langle F(t,\cdot), f\rangle_{L_2[\beta_\alpha]}.$$

$\tilde{U}$ is a unitary operator. It thus suffices to show that $L_2[\beta_\alpha]$ and K° are unitarily equivalent. But from Proposition 2 it follows that, "setwise,"

$$\mathcal{R}(U) \subseteq K^\circ.$$

Hence, for $f \in L_2[\beta_\alpha]$,

$$\|Uf\|^2_{K^\circ} = \|J^\star f\|^2_{L_2[0,T]} = \|f\|^2_{L_2[\beta_\alpha]},$$

as $JJ^\star$ is the identity of $L_2[\beta_\alpha]$. $U: \ L_2[\beta_\alpha] \longrightarrow K^\circ$ is thus an isometry. It is onto, as

$$K = \overline{\mathcal{R}\left(R^{\frac{1}{2}}_{N_\alpha}\right)}$$

and, if Π_K denotes the projection onto K, then for $f \in L_2[0,T]$,

$$\begin{aligned} R^{\frac{1}{2}}_{N_\alpha}(f) &= R^{\frac{1}{2}}_{N_\alpha}\left[\Pi_K(f) + \Pi_{K^\perp}(f)\right] \\ &= R^{\frac{1}{2}}_{N_\alpha}\Pi_K(f) \\ &= R^{\frac{1}{2}}_{N_\alpha}J^\star J\Pi_K(f) \\ &= UJ\Pi_K(f), \end{aligned}$$

as the kernel of $R^{\frac{1}{2}}_{N_\alpha}$ is the orthogonal complement of the closure of its range. U is also injective as $Uf = Ug$ means $\tilde{U}f = \tilde{U}g$, almost surely with respect to Lebesgue measure, and that, $\tilde{U}f$ and $\tilde{U}g$ being continuous, they must then be equal. Consequently K° and $L_2[\beta_\alpha]$ are indeed unitarily equivalent. *q.e.d.*

Proposition 3

$$\mathcal{R}\left(R^{\frac{1}{2}}_{N_\alpha}\right) = \mathcal{R}\left(R^{\frac{1}{2}}_{N^{(1)}_\alpha}\right) + \mathcal{R}\left(R^{\frac{1}{2}}_{N^{(2)}_\alpha}\right).$$

Proof: $N^{(1)}_\alpha$ is a second order process, with covariance

$$C_{N^{(1)}_\alpha}(s,t) = \alpha\int_0^{s\wedge t} F(s,x)\,F(t,x)\,\beta_1(dx).$$

Analogously $N^{(2)}_\alpha$ is a second order process, with covariance

$$C_{N^{(2)}_\alpha}(s,t) = (1-\alpha)\int_0^{s\wedge t} F(s,x)\,F(t,x)\,\beta_2(dx).$$

Furthermore,

$$C_{N_\alpha}(s,t) = C_{N_\alpha^{(1)}}(s,t) + C_{N_\alpha^{(2)}}(s,t).$$

Thus [26, Thm 3.1, p.9]

$$H(N_\alpha) = H\left(N_\alpha^{(1)}\right) + H\left(N_\alpha^{(2)}\right),$$

with

$$\|f\|^2_{H(N_\alpha)} = \min\left\{ \begin{array}{l} \|f_1\|^2_{H\left(N_\alpha^{(1)}\right)} + \|f_2\|^2_{H\left(N_\alpha^{(2)}\right)}, \\ (f_1, f_2) \in H\left(N_\alpha^{(1)}\right) \times H\left(N_\alpha^{(2)}\right), \\ f_1 + f_2 = f \end{array} \right\}.$$

As $C_{N_\alpha}(s,t) \gg C_{N_\alpha^{(1)}}(s,t)$, in terms of reproducing kernels, $H(N_\alpha) \supseteq H\left(N_\alpha^{(1)}\right)$. But, by the Corollary to Proposition 2, on one side, $H(N_\alpha)$ and $\mathcal{R}\left(R^{\frac{1}{2}}_{N_\alpha}\right)$, and on the other, $H\left(N_\alpha^{(1)}\right)$ and $\mathcal{R}\left(R^{\frac{1}{2}}_{N_\alpha^{(1)}}\right)$ are related by bijections. Thus, $\mathcal{R}\left(R^{\frac{1}{2}}_{N_\alpha}\right) \supseteq \mathcal{R}\left(R^{\frac{1}{2}}_{N_\alpha^{(1)}}\right)$. These ranges being vector spaces, finally

$$\mathcal{R}\left(R^{\frac{1}{2}}_{N_\alpha}\right) \supseteq \mathcal{R}\left(R^{\frac{1}{2}}_{N_\alpha^{(1)}}\right) + \mathcal{R}\left(R^{\frac{1}{2}}_{N_\alpha^{(2)}}\right).$$

Now, writing $F_t(x) = F(t,x)$, it follows that

$$\langle F_t, f\rangle_{L_2[\beta_\alpha]} = \alpha\langle F_t, f\rangle_{L_2[\beta_1]} + (1-\alpha)\langle F_t, f\rangle_{L_2[\beta_2]},$$

and, *mutatis mutandis,* going to equivalence classes in $L_2[0,T]$,

$$U(f) = \alpha U_1(f) + (1-\alpha) U_2(f)$$

that is,

$$R^{\frac{1}{2}}_{N_\alpha} J^\star(f) = R^{\frac{1}{2}}_{N_\alpha^{(1)}} J_1^\star(f) + R^{\frac{1}{2}}_{N_\alpha^{(2)}} J_2^\star(f).$$

To end the proof, $f \in L_2[0,T]$ is written as $f_0 + f_0^{\perp}$, with $f_0 \in \mathcal{N}\left(R_{N_\alpha}^{\frac{1}{2}}\right)$. Then

$$f_0^{\perp} \in \mathcal{N}\left(R_{N_\alpha}^{\frac{1}{2}}\right)^{\perp} = K,$$

so that

$$R_{N_\alpha}^{\frac{1}{2}}(f) = R_{N_\alpha}^{\frac{1}{2}}\left(f_0^{\perp}\right) = R_{N_\alpha}^{\frac{1}{2}} J^{\star}\left(\tilde{f}_0^{\perp}\right) = R_{N_\alpha^{(1)}}^{\frac{1}{2}} J_1^{\star}\left(\tilde{f}_0^{\perp}\right) + R_{N_\alpha^{(2)}}^{\frac{1}{2}} J_2^{\star}\left(\tilde{f}_0^{\perp}\right)$$

where

$$\tilde{f}_0^{\perp} \in L_2[\beta_\alpha], \; J^{\star}\left(\tilde{f}_0^{\perp}\right) = f_0^{\perp}.$$

Consequently,

$$\mathcal{R}\left(R_{N_\alpha}^{\frac{1}{2}}\right) \subseteq \mathcal{R}\left(R_{N_\alpha^{(1)}}^{\frac{1}{2}}\right) + \mathcal{R}\left(R_{N_\alpha^{(2)}}^{\frac{1}{2}}\right).$$

q.e.d.

3.2 DEFINITION OF THE SIGNAL S

Let S denote a random signal, adapted to $\underline{\mathcal{A}}$. It is assumed that, almost surely with respect to P,

$$S(\omega,\cdot) \in H\left(N_\alpha^{(1)}\right).$$

As it can be seen further in the presentation (Propositions 5 and 6), the method used works for $S(\omega,\cdot) \in H(N_\alpha)$ only when $\beta_1 = \beta_2$. Nevertheless the assumptions which are made, though less natural and elegant for the problem at hand than those stated in the Introduction, cover that case also. They have the advantage of unmasking the role of each assumption.

It can be shown [12, Thm 3, Step 3, p.170] that the following representation is obtained:

$$S(\omega,t) = \alpha \int_0^t F(t,x)\, s(\omega,x)\, \beta_1(dx)$$

for some predictable s, with paths in $\mathcal{L}_2[\beta_1]$. Thus

$$P\left(\omega \in \Omega: \ \|s(\omega, \cdot)\|^2_{L_2[\beta_1]} < \infty\right) = 1$$

always holds.

Also, in what follows, s will usually be progressively measurable, except when a predictability assumption is required, and the assumption will be explicit. It is seen here that this is not a restriction.

In what follows, X_α represents the process $S_\alpha + N_\alpha$ and Y_α a process such that, for $t \in [0, T]$ fixed, almost surely with respect to P,

$$Y_\alpha(\omega, t) = \alpha \int_0^t s(\omega, x) \beta_1(dx) + B_\alpha(\omega, t).$$

3.3 SUMMARY LIST OF ASSUMPTIONS

Here is a list of recurrent assumptions which will be called upon in order to shorten the statement of many propositions. $\mathcal{D}$ denotes the σ-field of $D[0, T]$ generated by the evaluation maps

$$\{ev(f, t) = f(t), \ t \in [0, T], \ f \in D[0, T]\},$$

$\mathcal{D}_t$ that which is generated by the evaluation maps "up to time t," and $\underline{\mathcal{D}} = \{\mathcal{D}_t, \ t \in [0, T]\}$.

a. **A0**

The basic probability space is $(\Omega, \mathcal{A}, P)$, and the basic filtration is $\underline{\mathcal{A}}$. For $\underline{\mathcal{A}}$ the *usual assumptions* hold.

b. **A1**

$B_\alpha^{(\bullet)}$ is a process, defined on an appropriate probability space, with respect to an appropriate filtration, represented by the symbol $\bullet$ (which can be absent!). It has the following defining characteristics:

(a) $0 < \alpha < 1$

(b) $B_\alpha^{(\bullet)} = \sqrt{\alpha}\, B_1^{(\circ)} + \sqrt{1-\alpha}\, \tilde{B}_2^{(\bullet)}$

(c) $B_1^{(\bullet)}$ is generalized Brownian motion with variance function β_1 : it has continuous paths, almost surely and the non-continuous ones are continuous to the right; β_1 is continuous non-decreasing.

(d) $B_2^{(\bullet)}$ is a Poisson process with expectation β_2 and $\tilde{B}_2^{(\bullet)} = B_2^{(\bullet)} - \beta_2$.

(e) $B_1^{(\bullet)}$ and $B_2^{(\bullet)}$ are independent.

c. **A2**

s is a process, progressively measurable for $\underline{A}$, with the property[10]

$$P\left(\omega \in \Omega : \int_0^T s^2(\omega, x)\, \beta_1(dx) < \infty\right) = 1.$$

d. **A3**

Y_α is a process with paths in $D[0,T]$. It has the property that, for $t \in [0,T]$ fixed but arbitrary, almost surely with respect to P,

$$Y_\alpha(\omega, t) = \alpha \int_0^T s(\omega, x)\, \beta_1(dx) + B_\alpha(\omega, t).$$

e. **A4**

s is a process, progressively measurable for $\underline{D}$, with the property that

$$P\left(\omega \in \Omega : \int_0^T s^2(Y_\alpha(\omega, t), x)\, \beta_1(dx) < \infty\right) = 1.$$

f. **A5**

Y_α is a process with paths in $D[0,T]$. It has the property that, for $t \in [0,T]$ fixed but arbitrary, almost surely with respect to P,

$$Y_\alpha(\omega, t) = \alpha \int_0^T s(Y_\alpha(\omega, \cdot), x)\, \beta_1(dx) + B_\alpha(\omega, t).$$

[10]This assumption is the consequence of the requirement that $S(\omega, \cdot) \in H(N_\alpha)$.

g. **A6**

s is a process, progressively measurable for $\underline{\mathcal{D}}$, with the property that

$$P\left(\omega \in \Omega : \int_0^T s^2\left(B_\alpha(\omega, t), x\right) \beta_1(dx) < \infty\right) = 1.$$

h. **A7**

For ϕ, a deterministic, strictly positive, measurable function such that, simultaneously,

$$\int_0^T \phi(x)\, \beta_2(dx) < \infty \quad \text{and} \quad \int_0^T |\ln \phi(x)|\, \beta_2(dx) < \infty$$

$$\begin{aligned} \ln\{L_{\alpha,s,\phi}(\omega, t)\} &= -\sqrt{\alpha}\int_0^t s(\omega, x)\, B_1(dx) - \frac{\alpha}{2}\int_0^t s^2(\omega, x)\, \beta_1(dx) \\ &\quad + \int_0^t \ln[\phi(x)]\, B_2(\omega, dx) + \int_0^t [1 - \phi(x)]\, \beta_2(dx). \end{aligned}$$

Remark: The terms of $L_{\alpha,s,\phi}$ involving ϕ, B_2, and β_2 are basically those that yield the likelihood in the pure Poisson case (with deterministic intensity) [27, T2, p.165]. A likelihood L of the form

$$\ln[L] = -\int s\; dB_\alpha - \gamma \int s^2\; d[B_\alpha]$$

or

$$\ln[L] = -\int s\; dB_\alpha - \delta \int s^2\; d\langle B_\alpha\rangle$$

would require, to progress along Girsanov's route, an s with uniformly bounded jumps and, in the first case, jumps strictly smaller than one [28, Lemma 23.19, p.449]. On one hand, it is unlikely that such evidence would be readily available, and on the other, the simpler form that has been chosen for the initial likelihood provides sufficient evidence (Proposition 6) to confirm the fact that the part of the likelihood effective in the change of measure is its Gaussian component.

i. **A8**

$$E_P\left[L_{\alpha,s,\phi}(\cdot,T)\right] = 1.$$

Lemma 1

When **A0**, **A2**, **A4** *and* **A6** *hold, it can always be furthermore assumed, without the* usual assumptions, *that the maps*

$$\begin{aligned} t &\mapsto \int_0^t |\, s \,|\,(\omega, x)\,\beta_1(dx) \\ t &\mapsto \int_0^t s^2(\omega, x)\,\beta_1(dx) \\ t &\mapsto \int_0^t |\, s \,|\,(Y_\alpha(\omega,\cdot), x)\,\beta_1(dx) \\ t &\mapsto \int_0^t s^2(Y_\alpha(\omega,\cdot), x)\,\beta_1(dx) \end{aligned}$$

are all continuous in the extended real line.

Proof: All these statements are, *mutatis mutandis,* identical. It thus suffices, for example, to prove the fourth result. Now, as s is adapted, the process

$$\nu: \ (f,t) \mapsto \int_0^t s^2(f,x)\,\beta_1(dx)$$

is adapted. For $t \in [0,T]$ fixed but arbitrary, let

$$F_t = \{f \in D[0,T]: \ \nu(f,t) < \infty\}$$

and define

$$\tilde{s}(f,t) = I_{F_t}(f)\, s(f,t),$$

where the notation I_A holds for the indicator function of the set A [11].

Now, for a Borel subset G of $I\!R$,

$$\begin{aligned} \{(g,u) \in D[0,T] \times [0,t] &: \quad \tilde{s}(g,u) \in G\} \\ &= \quad [\{F_u \times [0,t]\} \cap \{(g,u) \in D[0,T] \times [0,t] : \ s(g,u) \in G\}] \\ &\cup \quad [\{F_u^c \times [0,t]\} \cap \{(g,u) \in D[0,T] \times [0,t] : \ 0 \in G\}] . \end{aligned}$$

Thus, since $F_u \in \mathcal{D}_u \subseteq \mathcal{D}_t$, and s is progressively measurable, $\tilde{s}$ is progressively measurable. But, with respect to P_{B_α} and P_{Y_α}, $\tilde{s}$ is indistinguishable from s as

$$\{f \in D[0,T] : \ \tilde{s}(f,\cdot) \neq s(f,\cdot)\} \subseteq \{f \in D[0,T] : \ \nu(f,T) = \infty\},$$

and, for instance, by assumption **A4**,

$$P_{Y_\alpha}(\{f \in D[0,T] : \ \nu(f,T) = \infty\}) = 0.$$

Let now

$$\tilde{\nu}(f,t) = \int_0^t \tilde{s}^2(f,x)\,\beta_1(dx).$$

The process $\tilde{\nu}$ is continuous to the left because of monotone convergence. For fixed $f \in D[0,T]$, it is not continuous to the right at $t < T$ if, for every sufficiently large positive integer n,

$$\tilde{\nu}(f,t) < \infty, \text{ but } \tilde{\nu}\left(f, t + \frac{1}{n}\right) = \infty.$$

Then, as a consequence of the definition of $\tilde{s}$, $\tilde{s}(f,u) = 0$, for $u \in \left[t + \frac{1}{n}, T\right]$, with the result that

$$\int_{t+\frac{1}{n}}^T \tilde{s}^2(f,x)\,\beta_1(dx) = 0,$$

and thus that

$$\int_{]t,T]} \tilde{s}^2(f,x)\,\beta_1(dx) = 0.$$

[11] $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 0$ if $x \notin A$.

So, for $u > t$, and $\tilde{\nu}(f,t) = \int_0^t \tilde{s}^2(f,x)\,\beta_1(dx)$, $\tilde{\nu}(f,u) = \tilde{\nu}(f,t)$, and $\tilde{\nu}$ is continuous to the right, and thus continuous. *q.e.d.*

4.0 ABSOLUTE CONTINUITY AND LIKELIHOOD RATIO FOR P_{B_α} AND P_{Y_α}.

In this section, it is proved that under some weak assumptions the probability law of the processes generating the noise is absolutely continuous with respect to the probability law of the process generating the signal. Further, the likelihood ratio between these two probability laws is calculated when it exists.

4.1 THE PROCESS $L_{\alpha,S,\phi}$

The process $L_{\alpha,s,\phi}$ serves as Radon-Nikodým derivative in a Girsanov-type change of measure operation. It is shown below that it is a semimartingale, a needed technical result.

For the proposition to be stated and proved, the following notation and definitions are needed. For any process U such that $U(\omega,t-)$ makes sense,

$$\{\Delta U\}(\omega,t) = U(\omega,t) - U(\omega,t-).$$

The process

$$U^s(\omega,t) = \sum_{x\le t} \{\Delta U\}(\omega,x)$$

is then called the process of the jumps of U. The process U^c is subsequently defined as

$$U^c(\omega,t) = U(\omega,t) - U^s(\omega,t).$$

The proof will furthermore require the following form of Itô's formula for semimartingales [17, p.194]:

$$F(U(\omega,t)) \;=\; F(U(\omega,0))$$

$$+\int_0^t F'(U(\omega,x-))\,U^c(\omega,dx)$$
$$+\frac{1}{2}\int_0^t F''(U(\omega,x-))\,\langle U^c\rangle(\omega,dx)$$
$$+\sum_{0\le x\le t}\{\Delta[F\circ U]\}(\omega,x).$$

Proposition 4

It is assumed that **A0**, **A1**, **A2** *and* **A7** *hold. Define the process M by the relation*

$$M(\omega,t)=\int_0^t s(\omega,x)\,B_1(\omega,dx).$$

Then it follows that

$$\begin{aligned} L_{\alpha,s,\phi}(\omega,t) &= 1 \\ &\quad -\sqrt{\alpha}\int_0^t L_{\alpha,s,\phi}(\omega,x-)\,M(\omega,dx) \\ &\quad -\int_0^t L_{\alpha,s,\phi}(\omega,x-)\,[1-\phi(x)]\,\tilde{B}_2(\omega,dx). \end{aligned}$$

Proof: For any semimartingale U, and admissible integrand f, [17, 7.3.18, p.169]

$$\left\{\Delta\int_0^\cdot f\,dU\right\}(\omega,t)=f(\omega,t)\,\{\Delta U\}(\omega,dt).$$

So, letting $Z(\omega,t)=\ln[L_{\alpha,s,\phi}(\omega,t)]$ and using the explicit form of Z (in **A7**) it follows that

$$\{\Delta Z\}(\omega,t)=\ln[\phi(t)]\,\{\Delta B_2\}(\omega,t)$$

and consequently, that

$$\begin{aligned} Z^s(\omega,t) &= \sum_{x\le t}\ln[\phi(x)]\,\{\Delta B_2\}(\omega,x) \\ &= \int_0^t \ln[\phi(x)]\,B_2(\omega,dx). \end{aligned}$$

Furthermore,

$$Z^c(\omega,t) = -\sqrt{\alpha} M(\omega,t) - \frac{\alpha}{2}\langle M\rangle(\omega,t) + \int_0^t [1-\phi(x)]\,\beta_2(dx)$$

so that $\langle Z^c\rangle(\omega,t) = \alpha\,\langle M\rangle(\omega,t)$. Itô's formula, in the format repeated above, applied to the function $F(x) = \exp[x]$ and the process Z, yields

$$F(Z(\omega,t)) = \exp[Z(\omega,t)] = L_{\alpha,s,\phi}(\omega,t)$$

and

$$\begin{aligned} L_{\alpha,s,\phi}(\omega,t) &= L_{\alpha,s,\phi}(\omega,0) \\ &\quad -\sqrt{\alpha}\int_0^t L_{\alpha,s,\phi}(\omega,x-)\,M(\omega,dx) \\ &\quad -\frac{\alpha}{2}\int_0^t L_{\alpha,s,\phi}(\omega,x-)\,\langle M\rangle(\omega,dx) \\ &\quad +\int_0^t L_{\alpha,s,\phi}(\omega,x-)\,[1-\phi(x)]\,\beta_2(dx) \\ &\quad +\frac{\alpha}{2}\int_0^t L_{\alpha,s,\phi}(\omega,x-)\,\langle M\rangle(\omega,dx) \\ &\quad +\sum_{0\le x\le t}\{\Delta L_{\alpha,s,\phi}\}(\omega,x). \end{aligned}$$

But

$$\begin{aligned} \{\Delta L_{\alpha,s,\phi}\}(\omega,t) &= \exp[Z(\omega,t)] - \exp[Z(\omega,t-)] \\ &= \exp[Z(\omega,t-)]\,\{\exp[\{\Delta Z\}(\omega,t)]-1\}, \end{aligned}$$

and, since

$$\begin{aligned} \exp[Z(\omega,t-)] &= L_{\alpha,s,\phi}(\omega,t-) \\ \{\Delta Z\}(\omega,t) &= \ln[\phi(t)]\,\{\Delta B_2\}(\omega,t), \end{aligned}$$

it follows, successively,

$$\begin{aligned} \exp[\{\Delta Z\}(\omega,t)] &= [\phi(t)]^{\{\Delta B_2\}(\omega,t)} \\ \exp[\{\Delta Z\}(\omega,t)] - 1 &= [\phi(t)-1]\,\{\Delta B_2\}(\omega,t) \end{aligned}$$

and thus, finally,

$$\sum_{0 \le x \le t} \{\Delta L_{\alpha,s,\phi}\}(\omega, x) = \sum_{0 \le x \le t} L_{\alpha,s,\phi}(\omega, x-)[\phi(x) - 1]\{\Delta B_2\}(\omega, x).$$

Now, using again the property stated at the beginning of this proof,

$$\begin{aligned} L_{\alpha,s,\phi}(\omega, t) &= 1 - \sqrt{\alpha}\int_0^t L_{\alpha,s,\phi}(\omega, x-)\, M(\omega, dx) \\ &\quad + \int_0^t L_{\alpha,s,\phi}(\omega, x-)[1 - \phi(x)]\, \beta_2(dx) \\ &\quad + \int_0^t L_{\alpha,s,\phi}(\omega, x-)[\phi(x) - 1]\, B_2(\omega, dx) \\ &= 1 - \sqrt{\alpha}\int_0^t L_{\alpha,s,\phi}(\omega, x-)\, M(\omega, dx) \\ &\quad - \int_0^t L_{\alpha,s,\phi}(\omega, x-)[1 - \phi(x)]\, \tilde{B}_2(\omega, dx). \end{aligned}$$

q.e.d.

Corollary 2

$L_{\alpha,s,\phi}$ is a positive local martingale, and thus a supermartingale. Consequently, $E[L_{\alpha,s,\phi}(\cdot, t)] \le 1,\ 0 \le t \le T$.

4.2 A VERSION OF GIRSANOV'S THEOREM

To use the change of measure method, it must be proved that the original process (signal-plus-noise) has the same law as the original noise, with respect to the constructed absolutely continuous probability measure. It follows from what is proved below that in the case considered the only possibility is $\phi \equiv 1$.

In what follows, it is assumed that **A8** holds. Consequently

$$E[L_{\alpha,s,\phi}(\cdot, t)] = 1,\ 0 \le t \le T.$$

This assumption allows the definition of a probability measure $Q_{\alpha,s,\phi}$ by setting

$$Q_{\alpha,s,\phi}(A) = \int_A L_{\alpha,s,\phi}(\omega, T)\, P(d\omega), \; A \in \mathcal{A}.$$

As an immediate consequence, the following obvious proposition is obtained:

Proposition 5

It is assumed that **A0**, **A1**, **A2**, **A7** *and* **A8** *hold. Then P and $Q_{\alpha,s,\phi}$, as defined above, are mutually absolutely continuous. Furthermore*

$$\frac{dQ_{\alpha,s,\phi}}{dP} = L_{\alpha,s,\phi}(\cdot, T) \quad \text{and} \quad \frac{dP}{dQ_{\alpha,s,\phi}} = \frac{1}{L_{\alpha,s,\phi}(\cdot, T)}.$$

Let the process $Z_{\alpha,s,\phi}$ be defined as follows:

$$Z_{\alpha,s,\phi}(\omega, t) = \alpha \int_0^t s(\omega, x)\, \beta_1(dx) + \sqrt{1-\alpha} \int_0^t [1 - \phi(x)]\, \beta_2(dx) + B_\alpha(\omega, t).$$

Set

$$U_{\alpha,s,\phi}(\omega, t) = \sqrt{\alpha} \int_0^t s(\omega, x)\, \beta_1(dx) + B_1(\omega, t)$$

and

$$V_{\alpha,s,\phi}(\omega, t) = B_2(\omega, t) - \int_0^t \phi(x)\, \beta_2(dx).$$

Obviously,

$$Z_{\alpha,s,\phi} = \sqrt{\alpha}\, U_{\alpha,s,\phi} + \sqrt{1-\alpha}\, V_{\alpha,s,\phi}.$$

Lemma 2

It is assumed that **A0**, **A1**, **A2**, **A7** *and* **A8** *hold. The process*

$$U_{\alpha,s,\phi}(\omega,t)=\sqrt{\alpha}\int_0^t s(\omega,x)\,\beta_1(dx)+B_1(\omega,t)$$

is then, with respect to $Q_{\alpha,s,\phi}$, *a generalized Brownian motion such that*

$$\langle U_{\alpha,s,\phi}\rangle^{Q_{\alpha,s,\phi}}=\beta_1,$$

where the notation $\langle U_{\alpha,s,\phi}\rangle^{Q_{\alpha,s,\phi}}$ *is chosen as a reminder of the measure that prevails.*

Proof: The reference measure being P, integration by parts yields again

$$\begin{aligned}U_{\alpha,s,\phi}(\omega,t)\,L_{\alpha,s,\phi}(\omega,t) &= U_{\alpha,s,\phi}(\omega,0)\,L_{\alpha,s,\phi}(\omega,0)\\ &\quad+\int_0^t U_{\alpha,s,\phi}(\omega,x-)\,L_{\alpha,s,\phi}(\omega,dx)\\ &\quad+\int_0^t L_{\alpha,s,\phi}(\omega,x-)\,U_{\alpha,s,\phi}(\omega,dx)\\ &\quad+[U_{\alpha,s,\phi},L_{\alpha,s,\phi}](\omega,t)\,.\end{aligned}$$

But

$$\begin{aligned}\int_0^t L_{\alpha,s,\phi}(\omega,x-)\,U_{\alpha,s,\phi}(\omega,dx) &= \sqrt{\alpha}\int_0^t L_{\alpha,s,\phi}(\omega,x-)\,s(\omega,x)\,\beta_1(dx)\\ &\quad+\int_0^t L_{\alpha,s,\phi}(\omega,x-)\,B_1(\omega,dx)\,,\end{aligned}$$

and using successively, the fact that processes of bounded variation do not contribute to quadratic variation [17, 7.3.13, p.167] properties of the stochastic integral ([17, 7.4.2, p.171] and [17, 7.4.3, p.174]) and Proposition 4,

$$\begin{aligned}-[U_{\alpha,s,\phi},L_{\alpha,s,\phi}] &= -[B_1,L_{\alpha,s,\phi}]\\ &= \sqrt{\alpha}\left[B_1,\int_0^{\cdot} L^{-}_{\alpha,s,\phi}\,dM\right]+\left[B_1,\int_0^{\cdot} L^{-}_{\alpha,s,\phi}\,(1-\phi)\,d\tilde{B}_2\right]\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\alpha}\int_0^{\cdot} L^{-}_{\alpha,s,\phi}\, d\left[B_1, M\right] + \int_0^{\cdot} L^{-}_{\alpha,s,\phi}\, \left(1-\phi\right) d\left[B_1, \tilde{B}_2\right] \\
&= \sqrt{\alpha}\int_0^{\cdot} L^{-}_{\alpha,s,\phi}\, s\, d\beta_1
\end{aligned}
$$

where $L^{-}_{\alpha,s,\phi}(\omega,t) = L_{\alpha,s,\phi}(\omega,t-)$. Finally,

$$
\begin{aligned}
U_{\alpha,s,\phi}(\omega,t)\, L_{\alpha,s,\phi}(\omega,t) &= \int_0^t U_{\alpha,s,\phi}(\omega,x-)\, L_{\alpha,s,\phi}(\omega,dx) \\
&\quad + \int_0^t L_{\alpha,s,\phi}(\omega,x-)\, B_1(\omega,dx).
\end{aligned}
$$

Thus, as P and $Q_{\alpha,s,\phi}$ are mutually absolutely continuous, $U_{\alpha,s,\phi}$ is (with respect to $Q_{\alpha,s,\phi}$) a continuous local martingale [17, 10.1.4, p.247] and [17, p.245]

$$
\langle U_{\alpha,s,\phi}\rangle^{Q_{\alpha,s,\phi}} = \langle U_{\alpha,s,\phi}\rangle^{P} = \langle B_1\rangle^{P} = \beta_1.
$$

Lévy's characterization [17, 9.1.1, p.204] then suffices to end the proof. *q.e.d.*

Lemma 3

It is assumed that **A0, A1, A2, A7** *and* **A8** *hold. The process B_2 is then a Poisson process, with respect to $Q_{\alpha,s,\phi}$, such that*

$$
E\left[B_2(\cdot,t)\right] = \int_0^t \phi(x)\, \beta_2(dx).
$$

Proof: Define the process $V_{\alpha,s,\phi}$ by the equality:

$$
V_{\alpha,s,\phi}(\omega,t) = B_2(\omega,t) - \int_0^t \phi(x)\, \beta_2(dx).
$$

As above, use the integration by parts formula to get:

$$
V_{\alpha,s,\phi}(\omega,t)\, L_{\alpha,s,\phi}(\omega,t) = V_{\alpha,s,\phi}(\omega,0)\, L_{\alpha,s,\phi}(\omega,0)
$$

$$
\begin{aligned}
&+\int_0^t V_{\alpha,s,\phi}(\omega, x-)\, L_{\alpha,s,\phi}(\omega, dx) \\
&+\int_0^t L_{\alpha,s,\phi}(\omega, x-)\, V_{\alpha,s,\phi}(\omega, dx) \\
&+\left[V_{\alpha,s,\phi}, L_{\alpha,s,\phi}\right](\omega, t)\,.
\end{aligned}
$$

The explicit expressions for $V_{\alpha,s,\phi}$ and $L_{\alpha,s,\phi}$ yield successively

$$
\begin{aligned}
\int_0^t L_{\alpha,s,\phi}(\omega, x-)\, V_{\alpha,s,\phi}(\omega, dx) &= \int_0^t L_{\alpha,s,\phi}(\omega, x-)\, B_2(\omega, dx) \\
&\quad - \int_0^t L_{\alpha,s,\phi}(\omega, x-)\, \phi(x)\, \beta_2(dx)
\end{aligned}
$$

and

$$
\begin{aligned}
-\left[V_{\alpha,s,\phi}, L_{\alpha,s,\phi}\right] &= -\left[B_2, L_{\alpha,s,\phi}\right] \\
&= \sqrt{\alpha}\left[B_2, \int_0^{\cdot} L^{-}_{\alpha,s,\phi}\, dM\right] + \left[B_2, \int_0^{\cdot} L^{-}_{\alpha,s,\phi}\, (1-\phi)\, d\tilde{B}_2\right] \\
&= \sqrt{\alpha}\int_0^{\cdot} L^{-}_{\alpha,s,\phi}\, d\left[B_2, M\right] + \int_0^{\cdot} L^{-}_{\alpha,s,\phi}\, (1-\phi)\, d\left[B_2, \tilde{B}_2\right] \\
&= \int_0^{\cdot} L^{-}_{\alpha,s,\phi}\, (1-\phi)\, dB_2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
V_{\alpha,s,\phi}(\omega, t)\, L_{\alpha,s,\phi}(\omega, t) &= \int_0^t V_{\alpha,s,\phi}(\omega, x-)\, L_{\alpha,s,\phi}(\omega, dx) \\
&\quad + \int_0^t L_{\alpha,s,\phi}(\omega, x-)\, \phi(x)\, \tilde{B}_2(\omega, dx)
\end{aligned}
$$

and $V_{\alpha,s,\phi}$ is a local martingale, with respect to $Q_{\alpha,s,\phi}$. Now, B_2 is also a counting process, with respect to $Q_{\alpha,s,\phi}$. As just shown

$$
B_2(\omega, t) - \int_0^t \phi(x)\, \beta_2(dx)
$$

is a local martingale, with respect to $Q_{\alpha,s,\phi}$. So $\int_0^t \phi(x)\, \beta_2(dx)$ is the compensator of B_2, with respect to $Q_{\alpha,s,\phi}$ [29, Thm 2.3.1, p.61]. As it has been assumed that

$\int_0^t \phi(x)\,\beta_2(dx) < \infty$,

$$B_2(\omega,t) - \int_0^t \phi(x)\,\beta_2(dx)$$

is a martingale, with respect to $Q_{\alpha,s,\phi}$, [29, Lemma 2.3.2, p.62]. But then B_2 is a Poisson process, with respect to $Q_{\alpha,s,\phi}$. [27, T5 p.25], such that

$$E_{Q_{\alpha,s,\phi}}\left[B_2(\cdot,t)\right] = \int_0^t \phi(x)\,\beta_2(dx).$$

q.e.d.

Corollary 3

It is assumed that **A0**, **A1**, **A2**, **A7** *and* **A8** *hold. The process* $Z_{\alpha,s,\phi}$ *is then a martingale, with respect to* $Q_{\alpha,s,\phi}$.

Lemma 4

It is assumed that **A0**, **A1**, **A2**, **A7** *and* **A8** *hold.* $U_{\alpha,s,\phi}$ *and* B_2 *are then independent processes with respect to* $Q_{\alpha,s,\phi}$.

Proof: The time points

$$0 \le t_1 < \cdots < t_m \le T, \quad 0 \le u_1 < \cdots < u_n \le T,$$

and the arbitrary real constants

$$\lambda_1, \ldots, \lambda_m, \ \mu_1, \ldots, \mu_n,$$

are fixed. The expressions

$$i\sum_{j=1}^{m} \lambda_j U_{\alpha,s,\phi}(\omega,t_j) \ \text{ and } \ i\sum_{k=1}^{n} \mu_k B_2(\omega,u_k)$$

can be written respectively as

$$H(\omega,T)=\int_0^T h(x)\,U_{\alpha,s,\phi}(\omega,dx) \text{ and } K(\omega,T)=\int_0^T k(x)\,B_2(\omega,dx),$$

where

$$h(x)=i\sum_{j=1}^{m}\lambda_j I_{[0,t_j]}(x), \text{ and } k(x)=i\sum_{l=1}^{n}\mu_l I_{[0,u_l]}(x).$$

It is thus sufficient to check that

$$E_{Q_{\alpha,s,\phi}}\left[\exp\left\{H(\cdot,T)+K(\cdot,T)\right\}\right]=E_{Q_{\alpha,s,\phi}}\left[\exp\left\{H(\cdot,T)\right\}\right]\ E_{Q_{\alpha,s,\phi}}\left[\exp\left\{K(\cdot,T)\right\}\right]$$

As the functions h and k are bounded, are continuous to the left, have limits to the right and are adapted, they are predictable and properly integrable, so that the processes H and K are semimartingales. Then Itô's formula for multiple processes with the expression

$$L(\omega,t)=\exp\left\{H(\omega,t)+K(\omega,t)\right\}$$

is used to get

$$\begin{aligned}
L(\omega,t)-L(\omega,0) \;=\;& \int_0^t L(\omega,x-)\,h(x)\,U_{\alpha,s,\phi}(\omega,dx)\\
&+\int_0^t L(\omega,x-)\,k(x)\,B_2(\omega,dx)\\
&+\frac{1}{2}\int_0^t L(\omega,x-)\,h^2(x)\left[U_{\alpha,s,\phi},U_{\alpha,s,\phi}\right]^c(\omega,dx)\\
&+\int_0^t L(\omega,x-)\,h(x)\,k(x)\left[U_{\alpha,s,\phi},B_2\right]^c(\omega,dx)\\
&+\frac{1}{2}\int_0^t L(\omega,x-)\,k^2(x)\left[B_2,B_2\right]^c(\omega,dx)\\
&+\sum_{0\le u\le t}\left\{L(\omega,u)-L(\omega,u-)\right.\\
&\left.-\,L(\omega,u-)\left[h(u)\left\{\Delta U_{\alpha,s,\phi}\right\}(\omega,u)+k(u)\left\{\Delta B_2\right\}(\omega,u)\right]\right\}.
\end{aligned}$$

However it follows that ($Q_{\alpha,s,\phi}$ being the prevailing probability):

$$\begin{aligned}
[U_{\alpha,s,\phi}, U_{\alpha,s,\phi}]^c(\omega,t) &= \beta_1(dx) \\
[U_{\alpha,s,\phi}, B_2]^c(\omega,t) &= 0 \\
[B_2, B_2]^c(\omega,t) &= 0 \\
\{\Delta U_{\alpha,s,\phi}\}(\omega,t) &= 0 \\
\{\Delta B_2\}(\omega,u) &= 0 \text{ or } 1.
\end{aligned}$$

Furthermore

$$\begin{aligned}
L(\omega,t) - L(\omega,t-) &= L(\omega,t-)\left\{e^{\{\Delta H\}(\omega,t)+\{\Delta K\}(\omega,t)} - 1\right\} \\
&= L(\omega,t-)\left\{e^{k(t)\{\Delta B_2\}(\omega,t)} - 1\right\} \\
&= L(\omega,t-)\left\{e^{k(t)} - 1\right\}\{\Delta B_2\}(\omega,t).
\end{aligned}$$

Combining the above, it follows

$$\begin{aligned}
L(\omega,t) - 1 &= \int_0^t L(\omega,x-)\,h(x)\,U_{\alpha,s,\phi}(\omega,dx) \\
&\quad + \int_0^t L(\omega,x-)\,k(x)\,B_2(\omega,dx) \\
&\quad + \frac{1}{2}\int_0^t L(\omega,x-)\,h^2(x)\,\beta_1(dx) \\
&\quad + \int_0^t L(\omega,x-)\left\{e^{k(x)} - k(x) - 1\right\}B_2(\omega,dx) \\
&= \int_0^t L(\omega,x-)\,h(x)\,U_{\alpha,s,\phi}(\omega,dx) \\
&\quad + \frac{1}{2}\int_0^t L(\omega,x-)\,h^2(x)\,\beta_1(dx) \\
&\quad + \int_0^t L(\omega,x-)\left\{e^{k(x)} - 1\right\}B_2(\omega,dx).
\end{aligned}$$

Let $\mathcal{L}(t) = E_{Q_{\alpha,s,\phi}}[L(\cdot,t)]$. On the last representation of L taking the expectation with respect to $Q_{\alpha,s,\phi}$, Lemma 2 and Lemma 3 yield the following equation

$$\mathcal{L}(t) = 1 + \frac{1}{2}\int_0^t \mathcal{L}(x-)\,h^2(x)\,\beta_1(dx) + \int_0^t \mathcal{L}(x-)\left\{e^{k(x)} - 1\right\}\phi(x)\,\beta_2(dx).$$

This can be rewritten as

$$\mathcal{L}(t) = \mathcal{L}(0) + \int_0^t \mathcal{L}(x-)\,\mu(dx)$$

with

$$\mu(dt) = \frac{1}{2}\,h^2(t)\,\beta_1(dt) + \left\{e^{k(t)} - 1\right\}\phi(t)\,\beta_2(dt)$$

an equation which has the unique solution [30, Thm A4.12, p.428]

$$\mathcal{L}(t) = \exp\left\{\frac{1}{2}\int_0^t h^2(x)\,\beta_1(dx) + \int_0^t \left\{e^{k(x)} - 1\right\}\phi(x)\,\beta_2(dx)\right\}.$$

Thus $\mathcal{L}(T) = E_{Q_{\alpha,s,\phi}}\left[e^{H(\cdot,T)}\right]\ E_{Q_{\alpha,s,\phi}}\left[e^{K(\cdot,T)}\right].$ *q.e.d.*

Remark: It is only at the end of the proof of Lemma 4, when solving the integral equation for $\mathcal{L}$, that $\phi(x)$ cannot be replaced with $\phi(\omega, x)$.

As a consequence of the above, the following proposition is obtained, which is a version of Girsanov's theorem, and applies later only when $\phi(t) = 1,\ t \in [0, T]$.

Proposition 6

It is assumed that **A0**, **A1**, **A2**, **A7** *and* **A8** *hold. Then, with respect to* $Q_{\alpha,s,1}$, $Z_{\alpha,f,1}$ *defined by*

$$Z_{\alpha,f,1} = \sqrt{\alpha}\ U_{\alpha,f,1} + \sqrt{1-\alpha}\ V_{\alpha,f,1}$$

satisfies

$$Q_{\alpha,s,1} \circ \underline{Z}_{\alpha,s,1}^{-1} = P \circ \underline{B}_{\alpha}^{-1}.$$

Remark: In what follows, Y_α will be used for $Z_{\alpha,s,1}$.

4.3 ABSOLUTE CONTINUITY AND RADON-NIKODÝM DERIVATIVES FOR P_{B_α} AND P_{Y_α}

The *implicit* form of the Radon-Nikodým derivatives for $\mathbf{P}_{B_\alpha}$ and $\mathbf{P}_{Y_\alpha}$ are derived as a direct consequence of the Girsanov theorem.

In what follows, $D[0,T]$ is the space of functions that are continuous to the right, and have limits to the left. The topology is Skorohod's topology whose Borel sets $\mathcal{D}$ are generated by the evaluation maps $ev(f,t) = f(t)$. If X is a process with paths in $D[0,T]$, the measure it induces on $D[0,T]$ is denoted P_X. Finally,

$$\mathcal{D}_t = \sigma\left(ev(\cdot,s),\ s \le t,\ t \in [0,T]\right), \text{ and } \underline{\mathcal{D}} = \{\mathcal{D}_t,\ t \in [0,T]\}.$$

Proposition 7

It is assumed that **A0, A1, A2, A7** *and* **A8** *hold. Then,* P_{Y_α} *and* P_{B_α} *are mutually absolutely continuous and, for* $f \in D[0,T]$,

a. almost surely with respect to P_{Y_α},

$$\frac{dP_{B_\alpha}}{dP_{Y_\alpha}}[f] = E_{P_{Y_\alpha}}\left[\ L_{\alpha,s,1}(\cdot,T) \mid \underline{Y}_\alpha = f\ \right],$$

b. almost surely with respect to P_{B_α},

$$\frac{dP_{Y_\alpha}}{dP_{B_\alpha}}[f] = E_{P_{B_\alpha}}\left[\ \tfrac{1}{L_{\alpha,s,1}(\cdot,T)} \mid \underline{Y}_\alpha = f\ \right].$$

Proof: Define $Q_{\alpha,s,1}$ as in Section 4.2. As $Q_{\alpha,s,1}$ and P are mutually absolutely continuous, $Q_{\alpha,s,1} \circ \underline{Y}_\alpha^{-1}$ and $P \circ \underline{Y}_\alpha^{-1}$ are mutually absolutely continuous. But Girsanov's theorem (Proposition 6) yields that $Q_{\alpha,s,1} \circ \underline{Y}_\alpha^{-1} = P_{B_\alpha}$, so that P_{Y_α} and P_{B_α} are mutually absolutely continuous. Let now A belong to $\mathcal{D}$. Then

$$\begin{aligned} P_{B_\alpha}(A) &= Q_{\alpha,s,1}(\underline{Y}_\alpha \in A) \\ &= \int_{\underline{Y}_\alpha^{-1}(A)} L_{\alpha,s,1}(\omega,T)\, P(d\omega) \\ &= \int_A E_{P_{Y_\alpha}}\left[L_{\alpha,s,1}(\cdot,T) \mid \underline{Y}_\alpha = f\right] P_{Y_\alpha}(df). \end{aligned}$$

The conditional expectation being adapted to $\mathcal{D}$, the conditional expectation in the last expression is the Radon-Nikodým derivative. Similarly,

$$\begin{aligned}
P_{Y_\alpha}(A) &= \int_{\underline{Y}_\alpha^{-1}(A)} \frac{1}{L_{\alpha,s,1}(\omega,T)} Q_{\alpha,s,1}(d\omega) \\
&= \int_A E_{Q_{\alpha,s,1}\circ\underline{Y}_\alpha^{-1}}\left[\tfrac{1}{L_{\alpha,s,1}(\cdot,T)} \,\middle|\, \underline{Y}_\alpha = f \right] Q_{\alpha,s,1}\circ\underline{Y}_\alpha^{-1}(df) \\
&= \int_A E_{P_{B_\alpha}}\left[\tfrac{1}{L_{\alpha,s,1}(\cdot,T)} \,\middle|\, \underline{Y}_\alpha = f \right] P_{B_\alpha}(df) .
\end{aligned}$$

q.e.d.

It can be shown, as in the case for which $B_\alpha = B_1$, that the following corollary holds.

Corollary 4

It is assumed that **A0**, **A1**, **A2** *and* **A7** *hold. Then* $E_P[L(\cdot,T)] < 1$, *and* P_{Y_α} *is absolutely continuous with respect to* P_{B_α}.

4.4 FACTORIZATIONS

Explicit expressions for the likelihood require that the Radon-Nikodým derivatives be "lifted" onto $D[0,T]$. This is achieved through factorization by Y_α of the different components of each Radon-Nikodým derivative of Proposition 7. When the evaluation maps are taken as processes with respect to lifted probabilities of the form P_U, the notation ev^{P_U} will be used for ev. $\sigma_t(Y_\alpha)$ is the σ-field generated by $\{Y_\alpha(\cdot,s),\ s \le t\}$, completed with the sets of measure zero, with respect to P, belonging to $\mathcal{A}_t$. $\underline{\sigma}(Y_\alpha)$ denotes the resulting filtration.

Proposition 8

It is assumed **A0** *and* **A1** *hold. Let* Y_α *denote a process with paths in* $D[0,T]$. *If* B_α *is adapted to* $\underline{\sigma}(Y_\alpha)$ *there exist processes* $B_1^{Y_\alpha}$, $B_2^{Y_\alpha}$, *and* $B_\alpha^{Y_\alpha}$ *defined on* $(D[0,T], \mathcal{D}, P_{Y_\alpha})$

and adapted to $\underline{\mathcal{D}}$, with paths in $D[0,T]$ such that, for[12]

$$B_\alpha^{Y_\alpha} = \sqrt{\alpha}\; B_1^{Y_\alpha} + \sqrt{1-\alpha}\; \tilde{B}_2^{Y_\alpha},$$

$$\begin{aligned} P_{Y_\alpha} \circ \left[\underline{B}_1^{Y_\alpha}\right]^{-1} &= P \circ \underline{B}_1^{-1}, \\ P_{Y_\alpha} \circ \left[\underline{B}_2^{Y_\alpha}\right]^{-1} &= P \circ \underline{B}_2^{-1} \\ P_{Y_\alpha} \circ \left[\underline{B}_\alpha^{Y_\alpha}\right]^{-1} &= P \circ \underline{B}_\alpha^{-1} \end{aligned}$$

and, for $t \in [0,T]$ fixed but arbitrary, almost surely with respect to P,

$$\begin{aligned} B_1(\omega,t) &= B_1^{Y_\alpha}(Y_\alpha(\omega,\cdot),t) \\ B_2(\omega,t) &= B_2^{Y_\alpha}(Y_\alpha(\omega,\cdot),t) \\ B_\alpha(\omega,t) &= B_\alpha^{Y_\alpha}(Y_\alpha(\omega,\cdot),t). \end{aligned}$$

Proof: The notation used is that of Section 3.1. Then

$$\begin{aligned} \{\Delta B_\alpha\}(\omega,t) &= \sqrt{1-\alpha}\,\{\Delta B_2\}(\omega,t), \\ B_\alpha^s(\omega,t) &= \sqrt{1-\alpha}\; B_2(\omega,t), \\ B_\alpha^c(\omega,t) &= \sqrt{\alpha}\; B_1(\omega,t) - \sqrt{1-\alpha}\; \beta_2(t). \end{aligned}$$

Consequently, B_α is adapted to $\underline{\sigma}(Y_\alpha)$, so then are B_α^s and B_α^c, and hence B_1 and B_2. Since $D[0,T]$ is a metric space, it can be checked that, as in the purely Gaussian case [18], there is a process $B_1^{Y_\alpha}$ defined on $(D[0,T], \mathcal{D}, P_{Y_\alpha})$, adapted to $\underline{\mathcal{D}}$, with paths in $C[0,T]$, such that, for $t \in [0,T]$ fixed, almost surely with respect to P,

$$B_1(\omega,t) = B_1^{Y_\alpha}(Y_\alpha(\omega,\cdot),t).$$

It thus suffices to obtain the analogous result for B_2.

As in the Gaussian case, there exists, for $t \in [0,T]$ fixed but arbitrary, a modification $\overline{B}_2(\cdot,t)$ of $B_2(\cdot,t)$, which is adapted to $\sigma_t^\circ(Y_\alpha)$, and for which it follows that

$$\overline{B}_2(\omega,t) = B_2^{Y_\alpha}(Y_\alpha(\omega,\cdot),t)$$

[12] $\tilde{B}_2^{Y_\alpha} = B_2^{Y_\alpha} - \beta_2$.

for some $B_2^{Y_\alpha}(\cdot, t)$ adapted to $\mathcal{D}_t$.

Let now $t_i^{(n)}$ denote the fraction $\frac{i}{2^n}T,\ 1 \leq i \leq 2^n$, and $\mathcal{T}_D^{(n)}$ the set

$$\left\{t_i^{(n)},\ 1 \leq i \leq 2^n\right\}.$$

It follows that

$$\mathcal{T}_D^{(n)} \subset \mathcal{T}_D^{(n+1)}$$

and that $\mathcal{T}_D$, defined by

$$\mathcal{T}_D = \bigcup_{n=1}^{\infty} \mathcal{T}_D^{(n)},$$

is a dense subset of $[0, T]$. By construction, the paths of $B_2^{Y_\alpha}$, restricted to $\mathcal{T}_D$, are, almost surely with respect to P_{Y_α}, restrictions of paths of B_2, a counting process associated with a Poisson process. So, given $n \in I\!N$, and $f \in D[0, T]$, the set $\mathcal{T}_n^{Y_\alpha}[f]$ is defined as follows:

$$\mathcal{T}_n^{Y_\alpha}[f] = \left\{t \in \mathcal{T}_D : B_2^{Y_\alpha}(f, t) \geq n\right\}.$$

The next step requires the following definitions:

$$T_n^{Y_\alpha}[f] = \begin{cases} T & \text{if } \ \mathcal{T}_n^{Y_\alpha}[f] = \emptyset \\ \inf \mathcal{T}_n^{Y_\alpha}[f] & \text{if } \ \mathcal{T}_n^{Y_\alpha}[f] \neq \emptyset \end{cases}$$

and, for $t \in [0, T]$,

$$\hat{B}_2^{Y_\alpha}(f, t) = \sum_{n=1}^{\infty} I_{[\![T_n^{Y_\alpha}, T]\!]}(f, t).$$

Because

$$\left\{f \in D[0, T] : \hat{B}_2^{Y_\alpha}(f, t) = n\right\} = \left\{f \in D[0, T] : T_n^{Y_\alpha}[f] \leq t < T_{n+1}^{Y_\alpha}[f]\right\},$$

$$\left\{\omega \in \Omega : \hat{B}_2^{Y_\alpha}(Y_\alpha(\omega, \cdot), t) = n\right\} = \left\{\omega \in \Omega : T_n^{Y_\alpha}[Y_\alpha(\omega, \cdot)] \leq t < T_{n+1}^{Y_\alpha}[Y_\alpha(\omega, \cdot)]\right\}.$$

Let

$$A = \left\{\omega \in \Omega : B_2^{Y_\alpha}(Y_\alpha(\omega, \cdot), s) < n,\ s \in \mathcal{T}_D\right\}.$$

As $\left\{\omega \in \Omega : B_2^{Y_\alpha}\left(Y_\alpha(\omega,\cdot), s\right) < n,\ s \in \mathcal{T}_D\right\}$

$$\begin{aligned} &= \left\{\omega \in \Omega : \overline{B}_2(\omega, s) < n,\ s \in \mathcal{T}_D\right\} \\ &= \left\{\omega \in \Omega :\ B_2^{Y_\alpha}\left(Y_\alpha(\omega,\cdot), s\right) < n,\ s \in \mathcal{T}_D\right\} \end{aligned}$$

it follows that

$$\begin{aligned} T_n^{Y_\alpha}\left[Y_\alpha(\omega,\cdot)\right] &= I_A(\omega)\, T + I_{A^c}(\omega) \inf\left\{s \in \mathcal{T}_D : B_2^{Y_\alpha}\left(Y_\alpha(\omega,\cdot), s\right) \geq n\right\} \\ &= I_A(\omega)\, T + I_{A^c}(\omega) \inf\left\{s \in \mathcal{T}_D : \overline{B}_2(\omega, s) \geq n\right\}. \end{aligned}$$

Let N denote a measurable set of measure zero, with respect to P, such that, for $\omega \in N^c$,

$$\overline{B}_2(\omega, s) = B_2(\omega, s),\ s \in \mathcal{T}_D.$$

Then, for $\omega \in N^c$,

$$T_n^{Y_\alpha}\left[Y_\alpha(\omega,\cdot)\right] = I_{A \cap N^c}(\omega)\, T + I_{A^c \cap N^c}(\omega) \inf\left\{s \in \mathcal{T}_D : B_2(\omega, s) \geq n\right\}.$$

As the process B_2 is separable and continuous in probability, every dense subset is a separator, so that

$$\inf\left\{s \in \mathcal{T}_D : B_2(\omega, s) \geq n\right\} = \inf\left\{t \in [0, T] : B_2(\omega, t) \geq n\right\}.$$

Define thus

$$\tilde{T}_n[\omega] = \begin{cases} T & \text{if } B_2(\omega, T) < n \\ \inf\left\{t \in [0, T] : B_2(\omega, t) \geq n\right\} & \text{if } B_2(\omega, T) \geq n. \end{cases}$$

Then, almost surely with respect to P

$$T_n^{Y_\alpha}\left[Y_\alpha(\omega, t)\right] = \tilde{T}_n[\omega]$$

and, consequently, for $t \in [0, T]$ fixed but arbitrary

$$\hat{B}_2^{Y_\alpha}\left(Y_\alpha(\omega,\cdot), t\right) = B_2(\omega, t).$$

Thus, with respect to P_{Y_α}, $\hat{B}_2^{Y_\alpha}$ is a Poisson process restricted to $[0,T]$, and $T_n^{Y_\alpha}$, being one of the times of discontinuity of $\hat{B}_2^{Y_\alpha}$, is a stopping time for $\underline{\mathcal{D}}$. In the sequel, $\hat{B}_2^{Y_\alpha}$ will be dentoted $B_2^{Y_\alpha}$, and $\tilde{B}_2^{Y_\alpha}$ will be the Poisson martingale

$$\left\{B_2^{Y_\alpha}(\omega,t) - \beta_2(t)\,,\ (\omega,t) \in \Omega \times [0,T]\right\}.$$

q.e.d.

Corollary 5

Let $\sigma_t^{Y_\alpha}(B_\alpha)$ be the σ-field generated by $\sigma_t^\circ(B_\alpha)$ and the sets of $\sigma_t^\circ(Y_\alpha)$ which have measure zero for P. Similarly, let $\sigma_t^{Y_\alpha}\left(B_\alpha^{Y_\alpha}\right)$ be the σ-field generated by $\sigma_t^\circ\left(B_\alpha^{Y_\alpha}\right)$ and the sets of $\mathcal{D}_t$ which have measure zero for P_{Y_α}. Then

$$\sigma_t^{Y_\alpha}(B_\alpha) = \underline{Y}_\alpha^{-1}\left\{\sigma_t^{Y_\alpha}\left(B_\alpha^{Y_\alpha}\right)\right\}.$$

Proposition 9

It is assumed that **A0**, **A1**, **A4** *and* **A5** *hold. There is then a process $B_\alpha^{Y_\alpha}$, defined on the base $(D[0,T], \mathcal{D}, P_{Y_\alpha})$, adapted to $\underline{\mathcal{D}}$, such that, for $t \in [0,T]$ fixed but arbitrary, almost surely with respect to P_{Y_α},*

$$ev^{P_{Y_\alpha}}(f,t) = \alpha \int_0^t s(f,x)\,\beta_1(dx) + B_\alpha^{Y_\alpha}(f,t)$$

with **A1** *true for $B_\alpha^{Y_\alpha}$.*

Proof: Define $B_\alpha^{Y_\alpha}$ as

$$B_\alpha^{Y_\alpha}(f,t) = ev^{P_{Y_\alpha}}(f,t) - \alpha \int_0^t s(f,x)\,\beta_1(dx)\,.$$

By definition, the map

$$t \mapsto B_\alpha^{Y_\alpha}(f,t)$$

is, almost surely with respect to P_{Y_α}, in $D[0,T]$. But the paths of $B_\alpha^{Y_\alpha}$ that are not in $D[0,T]$ can be taken as continuous to the right, thanks to Lemma 1. It is furthermore adapted to $\underline{\mathcal{D}}$. Finally, for $t \in [0,T]$ fixed but arbitrary, almost surely with respect to P,

$$B_\alpha^{Y_\alpha}(Y_\alpha(\omega,\cdot),t) = Y_\alpha(\omega,t) - \alpha\int_0^t s(Y_\alpha(\omega,\cdot),x)\,\beta_1(dx) = B_\alpha(\omega,t).$$

Thus, with respect to P_{Y_α}, $B_\alpha^{Y_\alpha}$ is a Lévy process. But

$$\left\{\Delta B_\alpha^{Y_\alpha}\right\}(Y_\alpha(\omega,\cdot),t) = \{\Delta B_\alpha\}(\omega,t) = \sqrt{1-\alpha}\,\{\Delta B_2\}(\omega,t),$$

so that the jump process of $B_\alpha^{Y_\alpha}$ is, with respect to P_{Y_α}, a Poisson process. Consequently, its continuous part is a generalized Brownian motion. *q.e.d.*

Proposition 10

It is assumed that **A0**, **A1**, **A4** *and* **A5** *hold. Let then the process M be defined on the base $(\Omega,\mathcal{A},P)$, and for the filtration $\underline{\sigma}^\circ(Y_\alpha)$, as*

$$M(\omega,t) = \int_0^t s(Y_\alpha(\omega,\cdot),x)\,B_1(\omega,dx).$$

A process M^{Y_α} defined on the base $(D[0,T],\mathcal{D},P_{Y_\alpha})$ and adapted to the filtration $\underline{\mathcal{D}}$ can be find with the following properties: its paths are continuous to the right and belong, almost surely with respect to P_{Y_α}, to $C[0,T]$. Furthermore, for a generalized Brownian motion $B_1^{Y_\alpha}$ with variance β_1, defined on $(D[0,T],\mathcal{D},P_{Y_\alpha})$ and adapted to $\underline{\mathcal{D}}$, for $t \in [0,T]$ fixed, almost surely with respect to P_{Y_α}

$$M^{Y_\alpha}(f,t) = \int_0^t s(f,x)\,B_1^{Y_\alpha}(f,dx)$$

and

$$M^{Y_\alpha}(Y_\alpha(\omega,\cdot),t) = M(\omega,t).$$

Proof: Firstly simple processes s of the form

$$s(f,t) = I_A(f)\,I_{]u,v]}(t),\ \ u<v,\ \ A \in \mathcal{D}_u$$

are considered. Setting $B = \underline{Y}_\alpha^{-1}[A]$, B then belongs to $\sigma_u^\circ(Y_\alpha)$ and by definition,

$$\int_0^t s(Y_\alpha(\omega,\cdot), x)\, B_1(\omega, dx) = I_A(Y_\alpha(\omega,\cdot))\{B_1(\omega, t\wedge v) - B_1(\omega, t\wedge u)\}.$$

Now, from Proposition 8 it follows that $B_1(\omega,t) = B_1^{Y_\alpha}(Y_\alpha(\omega,\cdot), t)$, so that setting

$$M^{Y_\alpha}(f,t) = \int_0^t s(f,x)\, B_1^{Y_\alpha}(f,dx)$$

the result for simple processes which are products of the appropriate indicators I_A and $I_{]u,v]}$ is obtained.

Let now $\mathcal{S}$ denote the class of processes s defined on $D[0,T] \times [0,T]$, which are progressively measurable for $\underline{\mathcal{D}}$, bounded and such that[13]

$$\{s \circ Y_\alpha\} \cdot B_1 = \left\{s \cdot B_1^{Y_\alpha}\right\} \circ Y_\alpha$$

as stated. $\mathcal{S}$ is a vector space containing all constants. It is closed for uniform and monotone convergence. If $\mathcal{S}_f$ denotes the subspace of $\mathcal{S}$ made of finite linear combinations of simple processes of the form

$$s(f,t) = I_A(f)\, I_{]u,v]}(t), \; u < v, \; A \in \mathcal{D}_u$$

then $\mathcal{S}_f$ is a subspace which is stable for multiplication. Hence, the monotone class theorem yields that $\mathcal{S}$ contains all bounded predictable processes, and thus all elementary processes in the sense of [17, p.72]. The properties of the stochastic integral suffice then to claim that the proposition's assertion is true. *q.e.d.*

Remark: The same proof yields, *mutatis mutandis,* the same result with B_1 replaced with B_α, and $B_1^{Y_\alpha}$ replaced with $B_\alpha^{Y_\alpha}$.

4.5 LIKELIHOODS FOR P_{B_α} AND P_{Y_α}

This section contains the likelihood formulae for the detection of Y_α when the noise is B_α. They only depend on the signal sent, the statistics of the noise and the received waveform.

[13] $\circ$ denotes composition and $\cdot$ stochastic integration.

Theorem 1

It is assumed that **A0**, **A1**, **A4**, **A5**, **A7** *with* $\phi = 1$, *and* **A8** *hold. Then:*

a. P_{Y_α} *and* P_{B_α} *are mutually absolutely continuous;*

b. *for almost every* $f \in D[0,T]$, *with respect to* P_{Y_α},

$$\ln\left[\frac{dP_{B_\alpha}}{dP_{Y_\alpha}}\right](f) = -\sqrt{\alpha}\int_0^T s(f,x)\, B_1^{Y_\alpha}(f,dx) - \frac{\alpha}{2}\int_0^T s^2(f,x)\,\beta_1(dx);$$

c. *for almost every* $f \in D[0,T]$, *with respect to* P_{Y_α},

$$\begin{aligned} -\ln\left[\frac{dP_{B_\alpha}}{dP_{Y_\alpha}}\right](f) &= \int_0^T s(f,x)\, ev^{P_{Y_\alpha}}(f,dx) \\ &\quad -\frac{\alpha}{2}\int_0^T s^2(f,x)\,\beta_1(dx) \\ &\quad -\sqrt{1-\alpha}\int_0^T s(f,x)\,\tilde{B}_2^{Y_\alpha}(f,dx); \end{aligned}$$

d. *for almost every* $f \in D[0,T]$, *with respect to* P_{B_α},

$$\begin{aligned} -\ln\left[\frac{dP_{B_\alpha}}{dP_{Y_\alpha}}\right](f) &= \int_0^T s(f,x)\, ev^{P_{B_\alpha}}(f,dx) \\ &\quad -\frac{\alpha}{2}\int_0^T s^2(f,x)\,\beta_1(dx) \\ &\quad -\sqrt{1-\alpha}\int_0^T s(f,x)\,\tilde{B}_2^{Y_\alpha,B_\alpha}(f,dx), \end{aligned}$$

where $\tilde{B}_2^{Y_\alpha,B_\alpha}$ *is the representation of* $\tilde{B}_2^{Y_\alpha}$ *with respect to* ${P_{B_\alpha}}$[14]*;*

[14] It is in fact the same process, as seen in the proof, but it is useful to keep in mind that it is a proven fact, not *a priori* obvious. It also helps to stress the fact that it is indeed a likelihood.

e. *for almost every* $f \in D[0,T]$, *with respect to* P_{B_α},

$$\begin{aligned}\ln\left[\frac{dP_{Y_\alpha}}{dP_{B_\alpha}}\right](f) &= \int_0^T s(f,x)\, ev^{P_{B_\alpha}}(f,dx) \\ &\quad -\frac{\alpha}{2}\int_0^T s^2(f,x)\,\beta_1(dx) \\ &\quad -\sqrt{1-\alpha}\int_0^T s(f,x)\,\tilde{B}_2^{Y_\alpha,B_\alpha}(f,dx);\end{aligned}$$

f. *for almost every* $f \in D[0,T]$, *with respect to* P_{Y_α},

$$\begin{aligned}\ln\left[\frac{dP_{Y_\alpha}}{dP_{B_\alpha}}\right](f) &= \int_0^T s(f,x)\, ev^{P_{Y_\alpha}}(f,dx) \\ &\quad -\frac{\alpha}{2}\int_0^T s^2(f,x)\,\beta_1(dx) \\ &\quad -\sqrt{1-\alpha}\int_0^T s(f,x)\,\tilde{B}_2^{Y_\alpha}(f,dx).\end{aligned}$$

Proof: As (**A4**)

$$P_{Y_\alpha}\left(f \in D[0,T] : \int_0^T s^2(f,x)\,\beta_1(dx) < \infty\right) = 1,$$

then the expression

$$\ln\left[\tilde{\Lambda}\right](f) = -\sqrt{\alpha}\int_0^T s(f,x)\,B_1^{Y_\alpha}(f,dx) - \frac{\alpha}{2}\int_0^T s^2(f,x)\,\beta_1(dx)$$

is well defined on $(D[0,T], \mathcal{D}, P_{Y_\alpha})$. Furthermore, from Proposition 10, $\tilde{\Lambda}(Y_\alpha(\omega,\cdot)) = L_{\alpha,s(Y_\alpha(\omega,\cdot),\cdot),1}(\omega,T)$. Q_{Y_α} is the probability obtained by setting

$$Q_{Y_\alpha} = Q_{\alpha,s(Y_\alpha(\omega,\cdot),\cdot),1} \circ \underline{Y}_\alpha^{-1}.$$

Note that **A2** is satisfied with respect to P_{Y_α}. Then Proposition 6 ensures $Q_{Y_\alpha} = P_{B_\alpha}$. Hence, the mutual absolute continuity of point (a) follows from Proposition 7 as well as the formula for the Radon-Nikodým derivative of P_{B_α} with respect to P_{Y_α} in point (b):

$$\frac{dP_{B_\alpha}}{dP_{Y_\alpha}}(f) = \tilde{\Lambda}(f)$$

for almost every $f \in D[0,T]$ with respect to P_{Y_α}.

Now (Proposition 9),

$$ev^{P_{Y_\alpha}}(f,t) = \alpha \int_0^t s(f,x)\,\beta_1(dx) + B_\alpha^{Y_\alpha}(f,dx),$$

and s is, by definition of the stochastic integral with respect to semimartingales, integrable with respect to $ev^{P_{Y_\alpha}}$ with

$$\begin{aligned}
\int_0^t s(f,x)\,ev^{P_{Y_\alpha}}(f,dx) &= \alpha \int_0^t s^2(f,x)\,\beta_1(dx) \\
&\quad + \sqrt{\alpha}\int_0^t s(f,x)\,B_1^{Y_\alpha}(f,dx) \\
&\quad + \sqrt{1-\alpha}\int_0^t s(f,x)\,\tilde{B}_2^{Y_\alpha}(f,dx).
\end{aligned}$$

Thus

$$\begin{aligned}
-\ln\left[\tilde{\Lambda}\right](f) &= \sqrt{\alpha}\int_0^T s(f,x)\,B_1^{Y_\alpha}(f,dx) \\
&\quad + \frac{\alpha}{2}\int_0^T s^2(f,x)\,\beta_1(dx) \\
&= \int_0^T s(f,x)\,ev^{P_{Y_\alpha}}(f,dx) \\
&\quad - \alpha\int_0^T s^2(f,x)\,\beta_1(dx) \\
&\quad - \sqrt{1-\alpha}\int_0^T s(f,x)\,\tilde{B}_2^{Y_\alpha}(f,dx) \\
&\quad + \frac{\alpha}{2}\int_0^T s^2(f,x)\,\beta_1(dx),
\end{aligned}$$

which is the required expression (c). This has to be re-expressed with respect to P_{B_α}. But stochastic integrals are invariant with respect to equivalent measures [17, p. 245] so that, since the evaluation is a semimartingale for both P_{B_α} and P_{Y_α},

$$\int_0^t s(f,x)\, ev^{P_{Y_\alpha}}(f,dx) = \int_0^t s(f,x)\, ev^{P_{B_\alpha}}(f,dx).$$

Furthermore, since the process defined by

$$\ln\left[\tilde{\Lambda}(f,t)\right] = -\sqrt{\alpha}\int_0^t s(f,x)\, B_1^{Y_\alpha}(f,dx) - \frac{\alpha}{2}\int_0^t s^2(f,x)\,\beta_1(dx)$$

is by definition almost surely continuous, the process $\tilde{B}_2^{Y_\alpha}$ which is, with respect to P_{Y_α}, an L_2-martingale, has with respect to P_{B_α} the representation [17, 10.1.6, p. 248]

$$\tilde{B}_2^{Y_\alpha}(f,t) = \tilde{B}_2^{Y_\alpha,B_\alpha}(f,t) + \int_0^t \frac{1}{\tilde{\Lambda}(f,x)}\left[\tilde{B}_2^{Y_\alpha},\tilde{\Lambda}\right](f,dx).$$

Now [17, 8.2.1. p. 183] $\tilde{\Lambda}$ is the solution of the equation

$$\tilde{\Lambda}(f,t) = 1 - \sqrt{\alpha}\int_0^t \tilde{\Lambda}(f,x)\, s(f,x)\, B_1^{Y_\alpha}(f,dx)$$

so that $\left[\tilde{B}_2^{Y_\alpha},\tilde{\Lambda}\right] = 0$, and consequently that $\tilde{B}_2^{Y_\alpha} = \tilde{B}_2^{Y_\alpha,B_\alpha}$. Thus (d) is also true.

For (e), it has to be noted that

$$\frac{dP_{Y_\alpha}}{dP_{B_\alpha}} = \frac{1}{\tilde{\Lambda}},$$

so, from (d),

$$\begin{aligned}\ln\left[\frac{dP_{Y_\alpha}}{dP_{B_\alpha}}\right](f) &= \int_0^T s(f,x)\, ev^{P_{B_\alpha}}(f,dx) \\ &\quad -\frac{\alpha}{2}\int_0^T s^2(f,x)\,\beta_1(dx) \\ &\quad -\sqrt{1-\alpha}\int_0^T s(f,x)\,\tilde{B}_2^{Y_\alpha,B_\alpha}(f,dx).\end{aligned}$$

To obtain (f) it suffices, as seen, to switch back from $ev^{P_{B_\alpha}}$ to $ev^{P_{Y_\alpha}}$, and from $\tilde{B}_2^{Y_\alpha,B_\alpha}$ to $\tilde{B}_2^{Y_\alpha}$. *q.e.d.*

4.6 PATH REQUIREMENTS FOR ABSOLUTE AND MUTUAL ABSOLUTE CONTINUITY

In the previous section the existence of the likelihood ratios and hence the mutual absolute continuity has been obtained under two conditions, namely that the random variable $L_{\alpha,s,1}(\cdot, T)$ has expectation one, and that the signal-plus-noise process be the solution of a stochastic differential equation. The first condition is hard to check in practice and, given the context, is not a natural assumption. It makes more sense in practical situations to verify the finite energy of the signal derivative. In the model, this is expressed in conditions in terms of the finiteness of the RKHS norm of the signal, or of some function of it. And that is then a path condition, instead of an expectation condition.

This section is thus devoted first to the investigation of mutual absolute continuity in terms of such path conditions. In the second part of the section, innovation representations of "signal-plus-noise" models are studied; this is the usual approach to transform the received signal into the solution of a stochastic differential equation.

4.6.1 Signal path conditions for absolute and mutual absolute continuity

In what follows the same assumptions that have been made to this point are kept. The first result is the next proposition (Proposition 11) which will be proved as a sequence of lemmas; it determines conditions for mutual absolute continuity in terms of square integrability of the derivative of the signal paths. As only assumption **A4**, and not assumption **A6** follows from the RKHS requirement, Proposition 11 must be weakened, and that leads to Proposition 12 which still calls on Proposition 11. Proposition 11 requires assumptions that are unlikely to be verifiable in practice, but its Corollary says that the Cramér-Hida framework is sufficient to ensure that these assumptions hold.

Proposition 11

It is assumed that **A0, A1, A4, A5** *and* **A6** *hold, and furthermore that* s *is* *predictable and that both*[15]

$$P_{B_\alpha}\left(f \in D[0,T]: \int_0^T |s|(f,x)\,\beta_2(dx) < \infty\right) = 1,$$

and

$$P_{Y_\alpha}\left(f \in D[0,T]: \int_0^T |s|(f,x)\,\beta_2(dx) < \infty\right) = 1$$

hold. Then Theorem 1 is valid.

Proof: The proof will be presented in a sequence of lemmas (Lemma 5 to Lemma 9), followed by a short conclusion (Epilogue to Proposition 11).

Remarks:

1. From Proposition 9, given the assumptions **A0**, **A1**, **A4** and **A5** of the present proposition, there exist on $(D[0,T], \mathcal{D}, P_{Y_\alpha})$

   (a) a generalized Brownian motion $B_1^{Y_\alpha}$, adapted to $\underline{\mathcal{D}}$, with

   $$V\left[B_1^{Y_\alpha}(\cdot,t)\right] = \beta_1(t)$$

   (b) a Poisson process $B_2^{Y_\alpha}$, adapted to $\underline{\mathcal{D}}$, independent of $B_1^{Y_\alpha}$, for which

   $$E\left[B_2^{Y_\alpha}(\cdot,t)\right] = \beta_2(t)$$

   such that, for $t \in [0,T]$ fixed but arbitrary, for almost every $f \in D[0,T]$, with respect to P_{Y_α},

   $$ev^{P_{Y_\alpha}}(f,t) = \alpha \int_0^t s(f,x)\,\beta_1(dx) + B_\alpha^{Y_\alpha}(f,t)$$

   with $B_\alpha^{Y_\alpha}(f,t) = \sqrt{\alpha}\; B_1^{Y_\alpha}(f,t) + \sqrt{1-\alpha}\; \tilde{B}_2^{Y_\alpha}(f,t)$ and $\tilde{B}_2^{Y_\alpha} = B_2^{Y_\alpha} - \beta_2$.

[15]These assumptions on s are needed in the proof of Lemma 8.

Furthermore, from Lemma 4, it follows that s can be replaced by $\tilde{s}$, for which the following properties holds:

(a) the map $\tilde{\nu}(f,t) = \int_0^t \tilde{s}^2(f,x)\,\beta_1(dx)$ is continuous in $\overline{\mathbb{R}}_+$;

(b) the probabilities
$$P_{B_\alpha}\left(f \in D[0,T] : \|\tilde{s}(f,\cdot)\|^2_{L_2[\beta_1]} < \infty\right)$$
and
$$P_{Y_\alpha}\left(f \in D[0,T] : \|\tilde{s}(f,\cdot)\|^2_{L_2[\beta_1]} < \infty\right)$$
are equal to 1;

(c) and, for $t \in [0,T]$ fixed but arbitrary, for almost every $f \in D[0,T]$, with respect to P_{Y_α},
$$ev^{P_{Y_\alpha}}(f,t) = \alpha \int_0^t \tilde{s}(f,x)\,\beta_1(dx) + B_\alpha^{Y_\alpha}(f,t).$$

2. Here is a brief sketch of the proof to help the understanding of subsequent technicalities.

The Girsanov's theorem requires that the exponential of some stochastic integral expression be one. Truncation of the signal, followed by a limiting argument, is the standard way to achieve such a result. But, to define a stopping time in the absence of the *usual conditions*, the continuity is needed, and the limiting argument that the stopping time converges to the observations' duration time. So the attention is restricted to a subset $\tilde{D}[0,T]$ of $D[0,T]$, which has measure one, with respect to P_{B_α} and P_{Y_α}, a restriction which is shown eventually not to matter. In the process, the evaluation map must also be truncated, hence the $\widetilde{ev}_n^{P_{Y_\alpha}}$ process. The latter allows a likelihood-type functional, $\tilde{\Psi}_n$ to be introduced on $\tilde{D}[0,T]$, with probability
$$\tilde{P}_{Y_\alpha}(A) = P_{Y_\alpha}\left(A \cap \tilde{D}[0,T]\right),\ A \in \mathcal{D}.$$

Actually, the likelihood functional of interest is Ψ, which, restricted to $\tilde{D}[0,T]$, is denoted $\tilde{\Psi}$. Then, it will be shown successively, that
$$\begin{aligned} E_{\tilde{P}_{Y_\alpha}}\left[\tilde{\Psi}_n\right] &= 1 \\ \tilde{P}_{Y_\alpha} - \lim_n \tilde{\Psi}_n &= \tilde{\Psi} \end{aligned}$$

and that, with respect to $\tilde{P}_{Y_\alpha}$, $\left\{\tilde{\Psi}_n,\ n \in \mathbb{N}\right\}$ is uniformly integrable.

Consequently it will be proven that $E_{\tilde{P}_{Y_\alpha}}\left[\tilde{\Psi}\right] = 1$, and then $E_{P_{Y_\alpha}}[\Psi] = 1$. This is the needed result, because it yields:

$$P_{Y_\alpha} \circ \left[\underline{ev}^{P_{Y_\alpha}}\right]^{-1} = P \circ \underline{B}_\alpha^{-1}.$$

To check uniform convergence, the probability $\tilde{Q}_n^{Y_\alpha}$, defined on $D[0,T]$, by the following relation:

$$\tilde{Q}_n^{Y_\alpha} = \tilde{P}_{Y_\alpha} \circ \left[\widetilde{ev}_n^{P_{Y_\alpha}}\right]^{-1}$$

is used.

But, since $\widetilde{ev}_n^{P_{Y_\alpha}}$ is the solution of a stochastic differential equation, there is, with respect to P_{B_α}, on $D[0,T]$, a Radon-Nikodým derivative,

$$\Phi_n \text{ such that } \Phi_n \circ \left[\widetilde{ev}_n^{P_{Y_\alpha}}\right]^{-1} = \tilde{\Psi}_n.$$

As Φ_n can be rewritten in terms of evaluation maps, the properties of martingale integrals with respect to P_{B_α} and P_{Y_α} can be used to obtain the required convergence.

The following steps restrict the problem to paths $f \in D[0,T]$ for which $\tilde{\nu}(f,T) < \infty$. The (strict) stopping time $T_n : D[0,T] \longrightarrow [0,T]$ is defined by the equality

$$T_n(f) = \begin{cases} T & \text{if } \{t \in [0,T] :\ \tilde{\nu}(f,t) \geq n\} = \emptyset \\ \inf\{t \in [0,T] :\ \tilde{\nu}(f,t) \geq n\} & \text{if } \{t \in [0,T] :\ \tilde{\nu}(f,t) \geq n\} \neq \emptyset. \end{cases}$$

It should be noted that $\lim_{n\to\infty} T_n(f) = T$ if and only if $t < T$ implies $\tilde{\nu}(f,t) < \infty$.

Further definitions are needed, as follows:

$$\begin{aligned} \tilde{D}[0,T] &= \{f \in D[0,T] :\ \tilde{\nu}(f,T) < \infty\}, \\ \tilde{D} &= D \cap \tilde{D}[0,T],\ D \in \mathcal{D}, \end{aligned}$$

$$
\begin{aligned}
\tilde{P}_{Y_\alpha}\left(\tilde{D}\right) &= P_{Y_\alpha}\left(D \cap \tilde{D}[0,T]\right), \\
\tilde{\mathcal{D}} &= \mathcal{D} \cap \tilde{D}[0,T], \\
\underline{\tilde{\mathcal{D}}} &= \underline{\mathcal{D}} \cap \tilde{D}[0,T].
\end{aligned}
$$

The process $\widetilde{ev}_n^{P_{Y_\alpha}}$ is subsequently defined on the base $\left(\tilde{D}[0,T], \tilde{\mathcal{D}}, \tilde{P}_{Y_\alpha}\right)$, with respect to the filtration $\underline{\tilde{\mathcal{D}}}$, as

$$
\widetilde{ev}_n^{P_{Y_\alpha}}(f,t) = \begin{cases} ev^{P_{Y_\alpha}}(f,t) & \text{if} \quad (f,t) \in [\![0, T_n[\![\\ ev^{P_{Y_\alpha}}(f,t) - \alpha \int_{T_n}^t \tilde{s}(f,x)\,\beta_1(dx) & \text{if} \quad (f,t) \in [\![T_n, T]\!]. \end{cases}
$$

This process can be rewritten as

$$
\widetilde{ev}_n^{P_{Y_\alpha}}(f,t) = f(t) - I_{[\![T_n,T]\!]}(f,t)\left\{\alpha \int_0^t I_{[\![T_n,T]\!]}(f,x)\,\tilde{s}(f,x)\,\beta_1(dx)\right\},
$$

and this shows first that $\left\{\widetilde{ev}_n^{P_{Y_\alpha}}(f,t),\ t \in [0,T]\right\} \in D[0,T]$, as $f \in \tilde{D}[0,T]$, and then that, on $[\![0, T_n]\!]$,

$$
\widetilde{ev}_n^{P_{Y_\alpha}}(f,t) = f(t) = ev(f,t).
$$

One last definition yields the progressively measurable, bounded process $\tilde{s}_n$, given by the relation

$$
\tilde{s}_n(f,t) = I_{[\![0,T_n]\!]}(f,t)\,\tilde{s}(f,t).
$$

Let $J : \tilde{D}[0,T] \longrightarrow D[0,T]$ be the (injection) map defined by the relation $J(f) = f$. If E is a Borel set of $\mathbb{R}$,

$$
\begin{aligned}
\left[ev(\cdot,t) \circ J\right]^{-1}(E) &= \left\{f \in \tilde{D}[0,T] : \ ev(J(f),t) \in E\right\} \\
&= \tilde{D}[0,T] \cap \{f \in D[0,T] : \ ev(f,t) \in E\} \\
&\in \tilde{\mathcal{D}}_t.
\end{aligned}
$$

Thus the restriction of $\tilde{s}_n$ to $\tilde{D}[0,T]$ has the measurability properties of $\tilde{s}_n$ as defined on $D[0,T]$, and it is therefore not necessary to introduce one more notation to distinguish one situation from the other. In particular, an integral of the form

$$
\int_0^t \tilde{s}_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot), x\right)\beta_1(dx)
$$

will be well defined for $f \in \tilde{D}[0,T]$.

Define now $\tilde{B}_\alpha^{Y_\alpha}$ as the restriction of $B_\alpha^{Y_\alpha}$ to $\tilde{D}[0,T]$. For

$$0 \leq t_1 < t_2 < t_3 < \cdots t_n \leq T,$$

and Borel sets of $I\!R$,

$$E_1, E_2, E_3, \ldots, E_n,$$

it follows that

$$\tilde{P}_{Y_\alpha}\left(f \in \tilde{D}[0,T]:\ \tilde{B}_\alpha^{Y_\alpha}(f,t_1) \in E_1, \ldots, \tilde{B}_\alpha^{Y_\alpha}(f,t_n) \in E_n\right) =$$

$$P_{Y_\alpha}\left(\tilde{D}[0,T] \cap \left\{f \in D[0,T]:\ B_\alpha^{Y_\alpha}(f,t_1) \in E_1, \ldots, B_\alpha^{Y_\alpha}(f,t_n) \in E_n\right\}\right) =$$

$$P_{Y_\alpha}\left(\left\{f \in D[0,T]:\ B_\alpha^{Y_\alpha}(f,t_1) \in E_1, \ldots, B_\alpha^{Y_\alpha}(f,t_n) \in E_n\right\}\right)$$

so that

$$\tilde{P}_{Y_\alpha} \circ \left[\underline{\tilde{B}}_\alpha^{Y_\alpha}\right]^{-1} = P_{Y_\alpha} \circ \left[\underline{B}_\alpha^{Y_\alpha}\right]^{-1}.$$

Then the following result can be stated:

Lemma 5

For every $f \in \tilde{D}[0,T]$,

$$T_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot)\right) = T_n(f),$$

and, for $t \in [0,T]$ fixed but arbitrary, almost surely with respect to $\tilde{P}_{Y_\alpha}$,

$$\widetilde{ev}_n^{P_{Y_\alpha}}(f,t) = \alpha \int_0^t \tilde{s}_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot), x\right) \beta_1(dx) + \tilde{B}_\alpha^{Y_\alpha}(f,t).$$

Proof: Let $D_t^{(n)} = \{f \in D[0,T] : T_n(f) = t\} \in \mathcal{D}_t$. The function $I_{D_t^{(n)}}$ then has the representation

$$I_{D_t^{(n)}}(f) = F\left(ev(f,t_i),\ 0 \le t_i \le t,\ i \in I\!N\right)$$

where the map $F : I\!R^\infty \longrightarrow I\!R$ is measurable. But, as $T_n(f) = t$, as seen above, for $i \in I\!N$

$$\widetilde{ev}_n^{P_{Y_\alpha}}(f,t_i) = f(t_i) = ev(f,t_i)$$

so that

$$F\left(ev(f,t_i),\ 0 \le t_i \le t,\ i \in I\!N\right) = F\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,t_i),\ 0 \le t_i \le t,\ i \in I\!N\right)$$

and consequently that

$$I_{D_t^{(n)}}(f) = I_{D_t^{(n)}}\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot)\right)$$

which proves the first assertion of the lemma.

The same reason (and the definition of $\tilde{s}_n$) yields that

$$\tilde{s}_n(f,t) = \tilde{s}_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),t\right).$$

Finally, when $t < T_n(f)$ and $f \in \tilde{D}[0,T]$

$$\begin{aligned}
\widetilde{ev}_n^{P_{Y_\alpha}}(f,t) &= ev^{P_{Y_\alpha}}(f,t) \\
&= \alpha \int_0^t \tilde{s}(f,x)\,\beta_1(dx) + \tilde{B}_\alpha^{Y_\alpha}(f,t) \\
&= \alpha \int_0^t \tilde{s}_n(f,x)\,\beta_1(dx) + \tilde{B}_\alpha^{Y_\alpha}(f,t) \\
&= \alpha \int_0^t \tilde{s}_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),x\right)\beta_1(dx) + \tilde{B}_\alpha^{Y_\alpha}(f,t)
\end{aligned}$$

and when $t \ge T_n(f)$

$$\widetilde{ev}_n^{P_{Y_\alpha}}(f,t) = ev^{P_{Y_\alpha}}(f,t) - \alpha \int_{T_n}^t \tilde{s}(f,x)\,\beta_1(dx)$$

$$
\begin{aligned}
&= \alpha \int_0^{T_n} \tilde{s}(f,x)\,\beta_1(dx) + \tilde{B}_\alpha^{Y_\alpha}(f,t) \\
&= \alpha \int_0^{t} \tilde{s}_n(f,x)\,\beta_1(dx) + \tilde{B}_\alpha^{Y_\alpha}(f,t) \\
&= \alpha \int_0^{t} \tilde{s}_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),x\right)\beta_1(dx) + \tilde{B}_\alpha^{Y_\alpha}(f,t).
\end{aligned}
$$

q.e.d.

Define now $\widetilde{\Psi}_n : \tilde{D}[0,T] \longrightarrow I\!R$ by the relation

$$
\begin{aligned}
\ln\left[\widetilde{\Psi}_n(f)\right] &= -\sqrt{\alpha}\int_0^T \tilde{s}_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),x\right)\tilde{B}_1^{Y_\alpha}(f,dx) \\
&\quad -\frac{\alpha}{2}\int_0^T \tilde{s}_n^2\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),x\right)\beta_1(dx).
\end{aligned}
$$

Then, since by definition $\int_0^T \tilde{s}_n^2(f,x)\,\beta_1(dx) \leq n$,

Lemma 6

$$E_{\tilde{P}_{Y_\alpha}}\left[\widetilde{\Psi}_n\right] = 1.$$

Lemma 7

For $f \in D[0,T]$, let Ψ be defined by

$$\ln\left[\Psi(f)\right] = -\sqrt{\alpha}\int_0^T s(f,x)\,B_1^{Y_\alpha}(f,dx) - \frac{\alpha}{2}\int_0^T s^2(f,x)\,\beta_1(dx)$$

and let $\widetilde{\Psi}$ denote the restriction of Ψ to $\tilde{D}[0,T]$ $\left(\widetilde{\Psi} = \Psi \circ J\right)$. Then

$$\lim_{n\to\infty}\widetilde{\Psi}_n(f) = \widetilde{\Psi}(f)$$

in probability, with respect to $\tilde{P}_{Y_\alpha}$.

Proof: For $(f,t) \in [\![0, T_n]\!]$

$$\widetilde{ev}_n^{P_{Y_\alpha}}(f,t) = f(t)$$

so that, for almost every $f \in \tilde{D}[0,T]$, with respect to $\tilde{P}_{Y_\alpha}$,

$$\begin{aligned}
\lim_{n\to\infty} \int_0^T \tilde{s}_n^2\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot), x\right)\beta_1(dx) &= \lim_{n\to\infty}\int_0^T I_{[\![0,T_n]\!]}(f,x)\,\tilde{s}^2(f,x)\,\beta_1(dx) \\
&= \int_0^T \tilde{s}^2(f,x)\,\beta_1(dx)
\end{aligned}$$

by monotone convergence. Furthermore, for almost every $f \in \tilde{D}[0,T]$, with respect to $\tilde{P}_{Y_\alpha}$, for n large enough $T_n(f) = T$, so that for that same f, for n large enough,

$$\sup_{0\le t\le T}\left\{|\tilde{s}(f,t)|\, I_{]\!]T_n,T]\!]}(f,t)\right\} = 0.$$

Consequently

$$\lim_n \tilde{P}_{Y_\alpha}\left(f \in \tilde{D}[0,T] : \sup_{0\le t\le T}\left\{|\tilde{s}(f,t)|\, I_{]\!]T_n,T]\!]}(f,x)\right\} > \epsilon\right) = 0$$

and therefore (continuity of the integral [17, 5.5.3, p. 98]), if

$$\mathcal{J}_n(f,t) = \int_0^t \tilde{s}_n(f,x)\,\tilde{B}_1^{Y_\alpha}(f,dx) - \int_0^t \tilde{s}(f,t)\,\tilde{B}_1^{Y_\alpha}(f,dx)$$

then

$$\lim_n \tilde{P}_{Y_\alpha}\left(f \in \tilde{D}[0,T] : \sup_{0\le t\le T}|\mathcal{J}_n(f,t)| > \epsilon\right) = 0.$$

q.e.d.

Lemma 8

Let the probability measure $\tilde{Q}_n^{Y_\alpha}$ be defined on $\mathcal{D}$ by the following relation:

$$\tilde{Q}_n^{Y_\alpha} = \tilde{P}_{Y_\alpha} \circ \left[\widetilde{ev}_n^{P_{Y_\alpha}}\right]^{-1}.$$

Then, for $A \in \mathcal{D}_{T_n}$,

$$\tilde{Q}_n^{Y_\alpha}(A) = \tilde{P}_{Y_\alpha}\left(\tilde{D}[0,T] \cap A\right) = P_{Y_\alpha}(A).$$

Proof: First, it can be shown, as in the continuous case [17, 2.2.6, p35], that $\mathcal{D}_{T_n} = \sigma\left(ev^{T_n}(\cdot, t),\ t \in [0,T]\right)$. Let $\theta_n :\ D[0,T] \longrightarrow D[0,T]$ be defined by the relation

$$ev(\theta_n(f), t) = ev^{T_n}(f,t) = f(t \wedge T_n(f)).$$

Let then $f_0 \in D[0,T]$ be fixed but arbitrary, and set $t_0 = T_n(f_0)$. If $t \leq t_0$, then

$$ev(f_0, t) = f_0(t) = f_0(t \wedge T_n(f_0)) = ev^{T_n}(f_0, t) = ev(\theta_n(f_0), t).$$

Thus, for every ψ adapted to $\mathcal{D}_{t_0}$, $\psi(f_0) = \psi(\theta_n(f_0))$. In particular,

$$I_{\{T_n \leq t_0\}}(\theta_n(f_0)) = I_{\{T_n \leq t_0\}}(f_0) = 1.$$

Consequently, for every ϕ adapted to $\mathcal{D}_{T_n}$,

$$\phi(\theta_n(f_0)) = \phi(\theta_n(f_0))\, I_{\{T_n \leq t_0\}}(\theta_n(f_0)).$$

But $\phi I_{\{T_n \leq t_0\}}$ is adapted to $\mathcal{D}_{t_0}$, so that

$$\phi(\theta_n(f_0)) = \phi(f_0)\, I_{\{T_n \leq t_0\}}(f_0) = \phi(f_0).$$

As ϕ is adapted to $\mathcal{D}$ it has, for fixed, measurable F and $t_i \in [0,T]$, $i \in I\!N$, the following representation:

$$\phi(f) = F(ev(f, t_i),\ 0 \leq t_i \leq T,\ i \in I\!N).$$

Using the relation $\phi(f) = \phi(\theta_n(f))$, valid for $f \in \mathcal{D}_{T_n}$

$$\begin{aligned} \phi(f) = \phi(\theta_n(f)) &= F(ev(f \circ \theta_n, t_i),\ 0 \leq t_i \leq T,\ i \in I\!N) \\ &= F\left(ev^{T_n}(f, t_i),\ 0 \leq t_i \leq T,\ i \in I\!N\right) \end{aligned}$$

which is adapted to $\sigma\left(ev^{T_n}(\cdot,t),\ t\in[0,T]\right)$. This establishes that $\mathcal{D}_{T_n}$ is contained in $\sigma\left(ev^{T_n}(\cdot,t),\ t\in[0,T]\right)$. The reverse inclusion is obtained by noting that ev is continuous to the right, so that [17, p. 41] $ev^{T_n}(\cdot,t)$ is adapted to $\mathcal{D}_{t\wedge T_n}$, and thus that

$$\sigma\left(ev^{T_n}(\cdot,t),\ t\in[0,T]\right)\subseteq\sigma\left(\cup_{t\in[0,T]}\mathcal{D}_{t\wedge T_n}\right)\subseteq\mathcal{D}_{T_n}.$$

Finally, for B Borel in $I\!R$ and $A=\left\{f\in D[0,T]:\ ev^{T_n}(f,t)\in B\right\}$,

$$\begin{aligned}\tilde{Q}_n^{Y_\alpha}(A) &= \tilde{P}_{Y_\alpha}\left(f\in\tilde{D}[0,T]:\ \widetilde{ev}_n^{Y_\alpha}(f,\cdot)\in A\right)\\ &= \tilde{P}_{Y_\alpha}\left(f\in\tilde{D}[0,T]:\ \widetilde{ev}_n^{Y_\alpha}(f,t\wedge T_n(f))\in B\right)\\ &= \tilde{P}_{Y_\alpha}\left(f\in\tilde{D}[0,T]:\ ev_n^{T}(f,t)\in B\right)\\ &= P_{Y_\alpha}(A).\end{aligned}$$

The proof is then complete with a monotone class argument. *q.e.d.*

Lemma 9

The assumptions are those of Proposition 11. The sequence $\left\{\tilde{\Psi}_n,\ n\in I\!N\right\}$ is then uniformly integrable for $\tilde{P}_{Y_\alpha}$.

Proof: Let $\sigma_t^{\tilde{P}_{Y_\alpha}}\left(\widetilde{ev}_n^{P_{Y_\alpha}}\right)$ denote the σ-field generated by $\left\{\widetilde{ev}_n^{P_{Y_\alpha}}(\cdot,s),\ s\le t\right\}$ and the sets of $\widetilde{\mathcal{D}}_t$ which have measure zero for $\tilde{P}_{Y_\alpha}$. By Lemma 5, the following holds almost surely with respect to $\tilde{P}_{Y_\alpha}$, for $t\in[0,T]$ fixed but arbitrary:

$$\widetilde{ev}_n^{P_{Y_\alpha}}(f,t)=\alpha\int_0^t\tilde{s}_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),x\right)\beta_1(dx)+\tilde{B}_\alpha^{Y_\alpha}(f,t).$$

$\tilde{B}_\alpha^{Y_\alpha}(\cdot,t)$ is thus adapted to $\sigma_t^{\tilde{P}_{Y_\alpha}}\left(\widetilde{ev}_n^{P_{Y_\alpha}}\right)$.

The setup is now as follows. The underlying probability is $\tilde{P}_{Y_\alpha}$. $\widetilde{ev}_n^{P_{Y_\alpha}}$ is a process with paths in $D[0,T]$. $\tilde{B}_\alpha^{Y_\alpha}$ is a process for which **A1** holds. As $\tilde{B}_\alpha^{Y_\alpha}(\cdot,t)$ is adapted to

$\sigma_t^{\tilde{P}_{Y_\alpha}}\left(\widetilde{ev}_n^{P_{Y_\alpha}}\right)$, it follows from Proposition 8 that there is a process $\widetilde{B}_{\alpha,n}^{Y_\alpha}$ which factors $\widetilde{B}_\alpha^{Y_\alpha}$ through $\widetilde{ev}_n^{P_{Y_\alpha}}$:

$$\widetilde{B}_\alpha^{Y_\alpha}(f,t) = \widetilde{B}_{\alpha,n}^{Y_\alpha}\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),t\right)$$

almost surely with respect to $\tilde{P}^{Y_\alpha}$, and for which **A1** holds, with respect to the probability measure $\tilde{Q}_n^{Y_\alpha}$ defined in Lemma 8. Then set, for $f \in D[0,T]$, almost surely with respect to $\tilde{Q}_n^{Y_\alpha}$,

$$\ln[\Phi_n(f)] = -\sqrt{\alpha}\int_0^T \tilde{s}_n(f,x)\,\widetilde{B}_{1,n}^{Y_\alpha}(f,dx) - \frac{\alpha}{2}\int_0^T \tilde{s}_n^2(f,x)\,\beta_1(dx).$$

By Proposition 10, almost surely with respect to $\tilde{P}_{Y_\alpha}$,

$$\tilde{\Psi}_n(f) = \Phi_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot)\right).$$

Furthermore, the equation

$$\widetilde{ev}_n^{P_{Y_\alpha}}(f,t) = \alpha\int_0^t \tilde{s}_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),x\right)\beta_1(dx) + \widetilde{B}_\alpha^{Y_\alpha}(f,t)$$

can be rewritten as

$$\widetilde{ev}_n^{P_{Y_\alpha}}(f,t) = \alpha\int_0^t \tilde{s}_n\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),x\right)\beta_1(dx) + \widetilde{B}_{\alpha,n}^{Y_\alpha}\left(\widetilde{ev}_n^{P_{Y_\alpha}}(f,\cdot),t\right)$$

which yields

$$ev^{\tilde{Q}_n^{Y_\alpha}}(f,t) = \alpha\int_0^t \tilde{s}_n(f,x)\,\beta_1(dx) + \widetilde{B}_{\alpha,n}^{Y_\alpha}(f,t)$$

almost surely with respect to $\tilde{Q}_n^{Y_\alpha}$. Applying Lemma 6, it follows that

$$E_{\tilde{Q}_n^{Y_\alpha}}[\Phi_n] = E_{\tilde{P}_{Y_\alpha}}\left[\Phi_n \circ \widetilde{ev}_n^{P_{Y_\alpha}}\right] = E_{\tilde{P}_{Y_\alpha}}\left[\tilde{\Psi}_n\right] = 1.$$

The two relations

$$ev^{\tilde{Q}_n^{Y_\alpha}}(f,t) = \alpha\int_0^t \tilde{s}_n(f,x)\,\beta_1(dx) + \widetilde{B}_{\alpha,n}^{Y_\alpha}(f,t)$$

and

$$E_{\tilde{Q}_n^{Y_\alpha}}[\Phi_n] = 1$$

together with Proposition 7, ensure that $\tilde{Q}_n^{Y_\alpha}$ and P_{B_α} are mutually absolutely continuous and that, almost surely with respect to $\tilde{Q}_n^{Y_\alpha}$,

$$\frac{dP_{B_\alpha}}{d\tilde{Q}_n^{Y_\alpha}}(f) = E_{\tilde{Q}_n^{Y_\alpha}}\left[\Phi_n \mid \underline{ev}^{\tilde{Q}_n^{Y_\alpha}} = f\right] = \Phi_n(f).$$

But, according to Theorem 1 (item d), Φ_n has, with respect to P_{B_α} the following equivalent representation for some Poisson process $B_2^{Y_\alpha,B_\alpha}$:

$$\begin{aligned}
\ln[\Phi_n](f) &= -\int_0^T \tilde{s}_n(f,x)\, ev^{P_{B_\alpha}}(f,dx) \\
&\quad + \frac{\alpha}{2}\int_0^T \tilde{s}_n^2(f,x)\,\beta_1(dx) \\
&\quad + \sqrt{1-\alpha}\int_0^T \tilde{s}_n(f,x)\,\tilde{B}_2^{Y_\alpha,B_\alpha}(f,dx).
\end{aligned}$$

Define the following stochastic processes

$$\begin{aligned}
M_n(f,t) &= \int_0^T \tilde{s}_n(f,x)\, ev^{P_{B_\alpha}}(f,dx) \\
N_n(f,t) &= \int_0^T \tilde{s}_n(f,x)\,\tilde{B}_2^{Y_\alpha,B_\alpha}(f,dx) \\
V_n(f,t) &= \int_0^T \tilde{s}_n^2(f,x)\,\beta_1(dx) \\
W_n(f,t) &= -M_n(f,t) + \sqrt{1-\alpha}\,N_n(f,t) + \frac{\alpha}{2}V_n(f,t)
\end{aligned}$$

and let $K > 0$ denote an arbitrary constant. Then

$$\int_{\{\tilde{\Psi}_n > K\}} \tilde{\Psi}_n(f)\,\tilde{P}_{Y_\alpha}(df) = \int_{\{\Phi_n > K\}} \Phi_n(f)\,\tilde{Q}_n^{Y_\alpha}(df) = P_{B_\alpha}(\Phi_n > K).$$

But

$$\begin{aligned}
P_{B_\alpha}(\Phi_n > K) &= P_{B_\alpha}(f \in D[0,T] :\ W_n(f,T) > \ln[K]) \\
&\leq P_{B_\alpha}\left(f \in D[0,T] :\ |M_n(f,T)| > \frac{\ln[K]}{3}\right)
\end{aligned}$$

$$+P_{B_\alpha}\left(f\in D[0,T]:\ |N_n(f,T)|>\frac{\ln[K]}{3\sqrt{1-\alpha}}\right)$$
$$+P_{B_\alpha}\left(f\in D[0,T]:\ V_n(f,T)>\frac{2\ln[K]}{3\alpha}\right).$$

Now,

$$\begin{aligned}
P_{B_\alpha}(f\in D[0,T] &:& |M_n(f,T)|\frac{\ln[K]}{3}\Big) \\
&=& P\left(\omega\in\Omega:\ \left|\int_0^T \tilde{s}_n(B_\alpha(\omega,\cdot),x)\,B_\alpha(\omega,dx)\right|\frac{\ln[K]}{3}\right) \\
&\le& P\left(\omega\in\Omega:\ \sqrt{\alpha}\left|\int_0^T \tilde{s}_n(B_\alpha(\omega,\cdot),x)\,B_1(\omega,dx)\right|\frac{\ln[K]}{6}\right) \\
&& +P\left(\omega\in\Omega:\ \sqrt{1-\alpha}\int_0^T |\tilde{s}_n|(B_\alpha(\omega,\cdot),x)\,B_2(\omega,dx)\frac{\ln[K]}{12}\right) \\
&& +P\left(\omega\in\Omega:\ \sqrt{1-\alpha}\int_0^T |\tilde{s}_n|(B_\alpha(\omega,\cdot),x)\,\beta_2(dx)>\frac{\ln[K]}{12}\right)
\end{aligned}$$

and, since, for a continuous local martingale M, and constants $\alpha>0$ and $K>0$ [18, 2.83. Lemma, p.19][16]

$$P(\omega\in\Omega:\ |M(\omega,t)|>\alpha)\le P(\omega\in\Omega:\ \langle M\rangle(\omega,t)>K)+2e^{-\frac{\alpha^2}{2K}}$$

one has, for $L>0$,

$$\begin{aligned}
P(\omega\in\Omega &:& \sqrt{\alpha}\left|\int_0^T \tilde{s}_n(B_\alpha(\omega,\cdot),x)\,B_1(\omega,dx)\right|>\frac{\ln[K]}{6}\Big) \\
&\le& P\left(\omega\in\Omega:\ \int_0^T \tilde{s}_n^2(B_\alpha(\omega,\cdot),x)\,\beta_1(dx)>L\right) \\
&& +2\exp\left\{-\frac{\left[\frac{\ln[K]}{6}\right]^2}{2L}\right\}.
\end{aligned}$$

[16]See also the remark that follows the proof.

Choosing $L = \ln[K]$, the exponential term becomes $K^{-\frac{1}{72}}$. Furthermore, as

$$P\left(\omega \in \Omega : \int_0^T \tilde{s}^2 (B_\alpha(\omega, \cdot), x) \beta_1(dx) < \infty\right) = 1$$

it follows that

$$\lim_{K\to\infty} P\left(\omega \in \Omega : \sqrt{\alpha}\left|\int_0^T \tilde{s}_n (B_\alpha(\omega, \cdot), x) B_1(\omega, dx)\right| > \frac{\ln[K]}{6}\right) = 0$$

independently of n. Now, if τ_p denotes the time at which jump number p of the Poisson process B_2 occurs, since $|\{p \in I\!N : \tau_p(\omega) \leq T\}| < \infty$, for any $\omega \in \Omega$

$$\int_0^T |\tilde{s}| (B_\alpha(\omega, \cdot), x) B_2(\omega, dx) = \sum_{\tau_p \leq T} |\tilde{s}| (B_\alpha(\omega, \cdot), \tau_p(\omega)) < \infty$$

from which it follows that

$$\lim_{K\to\infty} P\left(\omega \in \Omega : \sqrt{1-\alpha}\int_0^T |\tilde{s}_n| (B_\alpha(\omega, \cdot), x) B_2(\omega, dx) > \frac{\ln[K]}{12}\right) = 0$$

independently of n. Finally, by assumption,

$$P\left(\omega \in \Omega : \int_0^T |\tilde{s}| (B_\alpha(\omega, \cdot), x) \beta_2(dx) < \infty\right) = 1$$

so that

$$\lim_{K\to\infty} P\left(\omega \in \Omega : \sqrt{1-\alpha}\int_0^T |\tilde{s}_n| (B_\alpha(\omega, \cdot), x) \beta_2(dx) > \frac{\ln[K]}{12}\right) = 0$$

independently of n. Consequently,

$$\lim_{K\to\infty} P_{B_\alpha}\left(f \in D[0,T] : |M_n(f,T)| > \frac{\ln[K]}{3}\right) = 0$$

independently of n. Since $\tilde{Q}_n^{Y_\alpha}$ and P_{B_α} are mutually absolutely continuous, stochastic integrals with respect to these probabilities are indistinguishable [17, p.245], and thus

$$P_{B_\alpha}\left(f \in D[0,T] : |N_n(f,T)| > \frac{\ln[K]}{3\sqrt{1-\alpha}}\right) = \tilde{Q}_n^{Y_\alpha}\left(f \in D[0,T] : |N_n(f,T)| > \frac{\ln[K]}{3\sqrt{1-\alpha}}\right).$$

As N_n is adapted to $\mathcal{D}_{T_n}$, by Lemma 8,

$$\begin{aligned}
\tilde{Q}_n^{Y_\alpha}\left(f \in D[0,T]\right. &: \left.|N_n(f,T)| > \frac{\ln[K]}{3\sqrt{1-\alpha}}\right) \\
&= P_{Y_\alpha}\left(f \in D[0,T] : |N_n(f,T)| > \frac{\ln[K]}{3\sqrt{1-\alpha}}\right) \\
&= P\left(\omega \in \Omega : \left|\int_0^T \tilde{s}_n(Y_\alpha(\omega,\cdot),x)\,\tilde{B}_2(\omega,dx)\right| > \frac{\ln[K]}{3\sqrt{1-\alpha}}\right) \\
&\leq P\left(\omega \in \Omega : \int_0^T |\tilde{s}|(Y_\alpha(\omega,\cdot),x)\,B_2(\omega,dx) > \frac{\ln[K]}{6\sqrt{1-\alpha}}\right) \\
&\quad + P\left(\omega \in \Omega : \int_0^T |\tilde{s}|(Y_\alpha(\omega,\cdot),x)\,\beta_2(\omega,dx) > \frac{\ln[K]}{6\sqrt{1-\alpha}}\right).
\end{aligned}$$

Consequently, as above,

$$\lim_{K\to\infty} P_{B_\alpha}\left(f \in D[0,T] : |N_n(f,T)| > \frac{\ln[K]}{3\sqrt{1-\alpha}}\right) = 0$$

independently of n. The term containing V_n similarly has a limit that vanishes. Lemma 9 is thus proved. *q.e.d.*

Remark: If M is a local martingale, null at the origin, and such that its jumps are, almost surely, uniformly bounded ($|\Delta M| \leq \mu < \infty$), almost surely, then[17]

$$P(|M_t| > K) \leq P\left(2\varphi\left(\mu\frac{K}{L}\right)[M]_t > L\right) + 2e^{-\frac{K^2}{2L}}$$

where

$$\varphi(x) = -\frac{x + \ln(1-x)_+}{x^2}.$$

When M is continuous, $\mu = 0$, and this inequality allows the assumptions on the integrability of s with respect to β_2 to be bypassed. Thus, even for s's with bounded

[17] The proof for the continuous case ($\mu = 0$), can be found in [18, 2.83, p.19]. The proof given here is similar.

jumps, there is no obvious extension of the method that works for the continuous case.

The proof of the inequality goes as follows. For $\nu > 0$

$$\begin{aligned} P(|M_t| > K) &= P(\{M_t > K\} \cap \{\nu[M]_t \leq L\}) \\ &\quad + P(\{-M_t > K\} \cap \{\nu[M]_t \leq L\}) \\ &\quad + P(\nu[M]_t > L). \end{aligned}$$

Fix $\lambda > 0$. Then, for arguments of M_t and $[M]_t$ in the appropriate set, $\lambda M_t > \lambda K$ and $\frac{\lambda^2}{2}\nu[M]_t \leq \frac{\lambda^2}{2}L$, so that

$$\lambda M_t - \frac{\lambda^2}{2}\nu[M]_t > \lambda K - \frac{\lambda^2}{2}L.$$

Consequently,

$$\begin{aligned} P(\{M_t > K\} \cap \{\nu[M]_t \leq L\}) &\leq P\left(\lambda M_t - \frac{\lambda^2}{2}\nu[M]_t > \lambda K - \frac{\lambda^2}{2}L\right) \\ &= P\left(e^{\lambda M_t - \frac{\lambda^2}{2}\nu[M]_t} > e^{\lambda K - \frac{\lambda^2}{2}L}\right). \end{aligned}$$

But, when $\tilde{M} = \lambda M$, the former inequality can be written in the form:

$$\tilde{M}_t - \frac{\nu}{2}\left[\tilde{M}\right]_t > \lambda K - \frac{\lambda^2}{2}L.$$

φ is strictly positive and increasing on $]-\infty, 1[$, and infinite and positive on $[1, \infty[$. Choosing for ν the value $\nu = 2\varphi(\lambda\mu)$, it follows that [28, Lemma 23.19, p.449] $e^{\tilde{M}_t - \frac{\nu}{2}[\tilde{M}]_t}$ is a supermartingale. Using Doob's inequality, it follows that

$$\begin{aligned} P(\{M_t > K\} \cap \{\nu[M]_t \leq L\}) &\leq P\left(\lambda M_t - \frac{\lambda^2}{2}\nu[M]_t > \lambda K - \frac{\lambda^2}{2}L\right) \\ &\leq e^{-\lambda K + \frac{\lambda^2}{2}L} E\left[e^{\tilde{M}_0 - \frac{\nu}{2}[\tilde{M}]_0}\right] \\ &= e^{-\lambda K + \frac{\lambda^2}{2}L}. \end{aligned}$$

The minimum of $\psi(\lambda) = -\lambda K + \frac{\lambda^2}{2} L$ is achieved for $\lambda_{\min} = \frac{K}{L}$, and then $\psi(\lambda_{\min}) = -\frac{K^2}{L}$. The value of ν is then

$$\nu_{\min} = 2\varphi(\lambda_{\min}\mu) = 2\varphi\left(\mu \frac{K}{L}\right).$$

The same calculation yields the same bound for the probability involving $-M$.

Lemma 10

Let $(\Omega, \mathcal{A}, P)$ and $(\Omega, \mathcal{A}, Q)$ denote two probability spaces, and assume that $\Omega_0 \in \mathcal{A}$ is such that $P(\Omega_0) = Q(\Omega_0) = 1$. Define

$$\mathcal{A}_0 = \mathcal{A} \cap \Omega_0 \textit{ and, for } A \in \mathcal{A},\ A_0 = A \cap \Omega_0.$$

Set finally

$$P_0(A_0) = P(A \cap \Omega_0) \textit{ and } Q_0(A_0) = Q(A \cap \Omega_0).$$

Then, whenever P_0 and Q_0 are mutually absolutely continuous, so are P and Q and furthermore, almost surely, with respect to P and Q,

$$\frac{dQ}{dP}(\omega) = \begin{cases} \frac{dQ_0}{dP_0}(\omega) & \textit{if } \omega \in \Omega_0 \\ 0 & \textit{if } \omega \notin \Omega_0. \end{cases}$$

Proof: Let $J_0 : \Omega_0 \longrightarrow \Omega$ be defined by $J_0(\omega) = \omega$. For $A \in \mathcal{A}$, $J_0^{-1}(A) = A \cap \Omega_0$, so that J_0 is measurable for $\mathcal{A}_0$ and $\mathcal{A}$. Thus, for $A \in \mathcal{A}$,

$$P_0 \circ J_0^{-1}(A) = P_0(A \cap \Omega_0) = P(A \cap \Omega_0) = P(A).$$

Define, for $\omega \in \Omega$,

$$f(\omega) = \begin{cases} \frac{dQ_0}{dP_0}(\omega) & \text{if } \omega \in \Omega_0 \\ 0 & \text{if } \omega \notin \Omega_0. \end{cases}$$

As $\frac{dQ_0}{dP_0}$ is measurable for $\mathcal{A}_0$, and thus for $\mathcal{A}$, and since $f = I_{\Omega_0} \frac{dQ_0}{dP_0}$, f is measurable for $\mathcal{A}$. Furthermore, for $A \in \mathcal{A}$,

$$\int_A f\, dP = \int_{A_0} [f \circ J_0]\, dP_0 = \int_{A_0} \frac{dQ_0}{dP_0} dP_0 = Q_0(A_0) = Q(A).$$

Thus $f = \frac{dQ}{dP}$. Mutual absolute continuity holds since $\frac{dQ_0}{dP_0} > 0$, almost surely with respect to P_0. *q.e.d.*

Epilogue to Proposition 11

Lemmas 7 and 8 yield that

$$\lim_{n\to\infty} \widetilde{\Psi}_n(f) = \widetilde{\Psi}(f),$$

in $L_1\left[\tilde{P}_{Y_\alpha}\right]$. From Lemma 6, $E_{\tilde{P}_{Y_\alpha}}\left[\widetilde{\Psi}\right] = 1$. But (Proposition 7), if $\tilde{P}_{B_\alpha}$ is the restriction of P_{B_α} to $\tilde{D}[0,T]$ also produced by $\tilde{B}_\alpha^{Y_\alpha}$, then $\tilde{P}_{Y_\alpha}$ and $\tilde{P}_{B_\alpha}$ are mutually absolutely continuous. So, by Lemma 11, P_{Y_α} and P_{B_α} are mutually absolutely continuous. Furthermore $E_{P_{Y_\alpha}}[\Psi] = 1$.

Corollary 6

If $\beta_2 = \beta_1$ or if, almost surely, $S(\omega,\cdot) \in H(N_\alpha)$, Lemma 8 is true without the integrability conditions on s with respect to β_2, since then for $i = 1, 2$

$$\left\{\int_0^t |s(x)|\,\beta_i(dx)\right\}^2 \le \beta_i([0,T]) \int_0^T s^2(x)\,\beta_i(dx).$$

But then, to be true, Proposition 11 does not require those same conditions either.

Proposition 12

It is assumed that **A0**, **A1**, **A4**, *and* **A5** *hold, that s is predictable and that both*

$$P_{B_\alpha}\left(f \in D[0,T] : \int_0^T |s|(f,x)\,\beta_2(dx) < \infty\right) = 1$$

and

$$P_{Y_\alpha}\left(f \in D[0,T] : \int_0^T |s|(f,x)\,\beta_2(dx) < \infty\right) = 1$$

hold.

Then P_{Y_α} is absolutely continuous with respect to P_{B_α} and almost surely with respect to P_{Y_α},

$$\begin{aligned}\ln\left[\frac{dP_{Y_\alpha}}{dP_{B_\alpha}}(f)\right] &= \int_0^T \tilde{s}(f,x)\, ev^{P_{Y_\alpha}}(f,dx)\\ &\quad -\frac{\alpha}{2}\int_0^T \tilde{s}^2(f,x)\,\beta_1(dx)\\ &\quad -\sqrt{1-\alpha}\int_0^T \tilde{s}(f,x)\,\tilde{B}_2^{Y_\alpha}(f,dx).\end{aligned}$$

Proof: Absolute continuity comes from the Corollary to Proposition 7. Let f belong to $D[0,T]$ and

$$\begin{aligned}\ln[\Phi_n(f)] &= \int_0^T \tilde{s}_n(f,x)\, ev^{P_{B_\alpha}}(f,dx)\\ &\quad -\frac{\alpha}{2}\int_0^T \tilde{s}_n^2(f,x)\,\beta_1(dx)\\ &\quad -\sqrt{1-\alpha}\int_0^T \tilde{s}_n(f,x)\,\tilde{B}_2^{Y_\alpha,B_\alpha}(f,dx).\end{aligned}$$

Let T_n be the stopping time of the previous proposition, and set

$$C_n = \{f \in D[0,T] : \ T_n(f) = T\}.$$

Then, for $A \in \mathcal{D}$, $A \cap C_n$ belongs to $\mathcal{D}_{T_n}$ [31, 56.1, p.189], and by Lemma 8,

$$\tilde{Q}_n^{Y_\alpha}(A \cap C_n) = \tilde{P}_{Y_\alpha}(A \cap C_n).$$

As $P_{Y_\alpha}\left(\tilde{D}[0,T]\right) = 1$, $\lim_n P_{Y_\alpha}(C_n) = 1$, $\tilde{Q}_n^{Y_\alpha}$ and P_{B_α} are mutually absolutely continuous, and almost surely with respect to P_{B_α},

$$\frac{d\tilde{Q}_n^{Y_\alpha}}{dP_{B_\alpha}} = \Phi_n.$$

Then

$$\begin{aligned}
P_{Y_\alpha}(A) &= \lim_n P_{Y_\alpha}(A \cap C_n) \\
&= \lim_n P_{Y_\alpha}\left(A \cap \tilde{D}[0,T] \cap C_n\right) \\
&= \lim_n \tilde{P}_{Y_\alpha}(A \cap C_n) \\
&= \lim_n \tilde{Q}_n^{Y_\alpha}(A \cap C_n) \\
&= \lim_n \int_{A \cap C_n} (f) \frac{d\tilde{Q}_n^{Y_\alpha}}{dP_{B_\alpha}}(f)\, P_{B_\alpha}(df) \\
&= \lim_n \int_A I_{C_n} \Phi_n(f)\, P_{B_\alpha}(df).
\end{aligned}$$

Let now

$$\begin{aligned}
\ln[\Phi(f)] &= \int_0^T \tilde{s}(f,x)\, ev^{P_{Y_\alpha}}(f,dx) \\
&\quad -\frac{\alpha}{2}\int_0^T \tilde{s}^2(f,x)\,\beta_1(dx) \\
&\quad -\sqrt{1-\alpha}\int_0^T \tilde{s}(f,x)\,\tilde{B}_2^{Y_\alpha}(f,dx).
\end{aligned}$$

The proof then proceeds as follows. First the sequence of localized Radon-Nikodým derivatives $\{I_{C_n}\Phi_n,\ n \in I\!N\}$ is shown to converge in probability for P_{B_α}, and then it is shown to be uniformly integrable, still with respect to P_{B_α}. It must then converge in $L_1[P_{B_\alpha}]$ towards an integrable limit. Since P_{Y_α} is absolutely continuous with respect to P_{B_α}, the limit of the $\{I_{C_n}\Phi_n,\ n \in I\!N\}$ exists also with respect to P_{Y_α}, and it has the same value. But in that case, the limit can be identified: it is Φ.

When $T_n(f) = T$, $\int_0^T \tilde{s}^2(f,x)\,\beta_1(dx) \leq n$. Consequently, letting $C = \tilde{D}[0,T]$, and using the fact that on C_n, $I_C = 1$:

$$I_{C_n}(f)\,\Phi_n(f) = I_{C_n}(f)\, e^{I_C(f)\ln[\Phi_n(f)]}.$$

The following definitions will shorten some unwieldy expressions:

$$M_n(f,t) = \int_0^t \tilde{s}_n(f,x)\, ev^{P_{B_\alpha}}(f,dx)$$

$$
\begin{aligned}
\tilde{s}_{n,p}(f,t) &= \tilde{s}_n(f,t) - \tilde{s}_{n+p}(f,t) \\
M_{n,p}(f,t) &= \int_0^t \tilde{s}_{n,p}(f,x)\, ev^{P_{B_\alpha}}(f,dx) \\
M_{n,p}^{(1)}(\omega,t) &= \int_0^t \tilde{s}_{n,p}(B_\alpha(\omega,\cdot),x)\, B_1(\omega,dx) \\
M_{n,p}^{(2)}(\omega,t) &= \int_0^t \left|\tilde{s}_{n,p}(B_\alpha(\omega,\cdot),x)\right| B_2(\omega,dx) \\
M_{n,p}^{(3)}(\omega,t) &= \int_0^t \left|\tilde{s}_{n,p}(B_\alpha(\omega,\cdot),x)\right| \beta_2(dx).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
P_{B_\alpha}(f \in D[0,T] \quad &: \quad I_C(f)\left|M_n(f,T) - M_{n+p}(f,T)\right| > K) \\
&= P_{B_\alpha}(f \in D[0,T] : I_C(f)\left|M_{n,p}(f,T)\right| > K) \\
&\leq P\left(\omega \in \Omega : \sqrt{\alpha}\, I_C(B_\alpha(\omega,\cdot))\left|M_{n,p}^{(1)}(\omega,T)\right| > \frac{K}{3}\right) \\
&\quad + P\left(\omega \in \Omega : \sqrt{1-\alpha}\, I_C(B_\alpha(\omega,\cdot))\, M_{n,p}^{(2)}(\omega,T) > \frac{K}{3}\right) \\
&\quad + P\left(\omega \in \Omega : \sqrt{1-\alpha}\, I_C(B_\alpha(\omega,\cdot))\, M_{n,p}^{(3)}(\omega,T) > \frac{K}{3}\right).
\end{aligned}
$$

By the inequality from [18, 2.84, p.19],

$$
P\left(\omega \in \Omega : \sqrt{\alpha}\, I_C(B_\alpha(\omega,\cdot))\left|M_{n,p}^{(1)}(\omega,T)\right| > \frac{K}{3}\right)
$$

is dominated by

$$
P\left(\omega \in \Omega : \alpha\, I_C(B_\alpha(\omega,\cdot))\, \langle M_{n,p}^{(1)}\rangle(\omega,T) > L\right) + 2\exp\left\{-\frac{K^2}{18L}\right\}.
$$

But, with respect to P,

$$
\langle M_{n,p}^{(1)}\rangle(\omega,T) = \int_0^T I_{]\!]T_n(B_\alpha(\omega,\cdot)),T_{n+p}(B_\alpha(\omega,\cdot))]\!]}(\omega,x)\, \tilde{s}^2(B_\alpha(\omega,\cdot),x)\, \beta_1(dx)
$$

and since, for $B_\alpha(\omega,\cdot) \in C = \tilde{D}[0,T]$,

$$\int_0^T \tilde{s}^2 (B_\alpha(\omega,\cdot), x)\, \beta_1(dx) < \infty$$

then

$$\lim_{n,p\to\infty} P\left(\omega \in \Omega : \sqrt{\alpha}\, I_C(B_\alpha(\omega,\cdot)) \left|M_{n,p}^{(1)}(\omega,T)\right| > \frac{K}{3}\right) = 0.$$

Given the assumptions on the integrability of $|\, s\,|$, a similar argument yields that

$$\lim_{n,p\to\infty} P\left(\omega \in \Omega : \sqrt{1-\alpha}\, I_C(B_\alpha(\omega,\cdot))\, M_{n,p}^{(2)}(\omega,T) > \frac{K}{3}\right) = 0$$

and that

$$\lim_{n,p\to\infty} P\left(\omega \in \Omega : \sqrt{1-\alpha}\, I_C(B_\alpha(\omega,\cdot))\, M_{n,p}^{(3)}(\omega,T) > \frac{K}{3}\right) = 0.$$

Thus, with respect to P_{B_α}, the sequence

$$\left\{ I_C(f) \int_0^T \tilde{s}_n(f,x)\, ev^{P_{B_\alpha}}(f,dx),\ n \in I\!N \right\}$$

has a limit in probability, which will be denoted $J_{B_\alpha}(f)$.

Now, for $f \in \tilde{D}[0,T]$,

$$\lim_{n\to\infty} \int_0^T \tilde{s}_n^2(f,x)\, \beta_1(dx) = \int_0^T \tilde{s}^2(f,x)\, \beta_1(dx) < \infty$$

and, for $I_C(f) \int_0^T \tilde{s}_n(f,x)\, \tilde{B}_2^{Y_\alpha,B_\alpha}(f,dx)$, the arguments already given are repeated. As, trivially, almost surely with respect to P_{B_α}, $\lim_n I_{C_n} = I_C$,

$$\begin{aligned} P_{B_\alpha} - \lim_n \{I_{C_n}(f) \ln[\Phi_n(f)]\} &= J_{B_\alpha}(f) \\ &\quad -\frac{\alpha}{2} I_C(f) \int_0^T \tilde{s}^2(f,x)\, \beta_1(dx) \\ &\quad -\sqrt{1-\alpha}\, I_C(f) \int_0^T \tilde{s}(f,x)\, \tilde{B}_2^{Y_\alpha,B_\alpha}(f,dx). \end{aligned}$$

The exponential of this limit will be denoted $\Phi^{P_{B_\alpha}}$.

As P_{Y_α} is absolutely continuous with respect to P_{B_α}, on one hand,

$$\int_0^T \tilde{s}_n(f,x)\, ev^{P_{Y_\alpha}}(f,dx) = \int_0^T \tilde{s}_n(f,x)\, ev^{P_{B_\alpha}}(f,dx)$$

and, on the other, as seen in the proof of Theorem 1,

$$\int_0^T \tilde{s}_n(f,x)\, \tilde{B}_2^{Y_\alpha,B_\alpha}(f,dx) = \int_0^T \tilde{s}_n(f,x)\, \tilde{B}_2^{Y_\alpha}(f,dx)$$

so that, with respect to P_{Y_α}, Φ_n has the following representation:

$$\begin{aligned}\ln[\Phi_n(f)] &= \int_0^T \tilde{s}_n(f,x)\, ev^{P_{Y_\alpha}}(f,dx) \\ &\quad -\frac{\alpha}{2}\int_0^T \tilde{s}_n^2(f,x)\,\beta_1(dx) \\ &\quad -\sqrt{1-\alpha}\int_0^T \tilde{s}_n(f,x)\,\tilde{B}_2^{Y_\alpha}(f,dx)\,.\end{aligned}$$

But the assumptions made, in particular **A4**, now imply that the limit in probability, with respect to P_{Y_α}, of the sequence $\{\Phi_n,\ n\in I\!N\}$ is Φ. So, with respect to P_{Y_α}, $\Phi^{P_{B_\alpha}} = \Phi$.

To finish the proof, it must be confirmed that the sequence $\{I_{C_n}\Phi_n,\ n\in I\!N\}$ is uniformly integrable with respect to P_{B_α}, which ensures that

$$\lim_{n\to\infty} E_{P_{B_\alpha}}[I_{C_n}\Phi_n] = E_{P_{B_\alpha}}\left[\Phi^{P_{B_\alpha}}\right].$$

But, since $\tilde{Q}_n^{Y_\alpha}$ and P_{B_α} are mutually absolutely continuous, as seen in the proof of Lemma 9, and since $\Phi_n = \frac{d\tilde{Q}_n^{Y_\alpha}}{dP_{B_\alpha}}$ is one of the Radon-Nikodým derivatives, setting

$$\begin{aligned}\tilde{D}_n &= \{f\in D[0,T]:\ I_{C_n}(f)\,\Phi_n(f) > K\}\,,\\ D_n &= \{f\in D[0,T]:\ \Phi_n(f) > K\}\,,\end{aligned}$$

gives

$$\begin{aligned}\int_{\tilde{D}_n} I_{C_n}(f)\,\Phi_n(f)\,P_{B_\alpha}(df) &\le \int_{D_n} \Phi_n(f)\,P_{B_\alpha}(df)\\ &= \tilde{Q}_n^{Y_\alpha}(D_n)\\ &= \tilde{P}_{Y_\alpha}\circ\left[\underline{\widetilde{ev}}_n^{P_{Y_\alpha}}\right]^{-1}(D_n)\\ &= \tilde{P}_{Y_\alpha}\left(f\in\tilde{D}[0,T]:\ \Phi_n\circ\underline{\widetilde{ev}}_n^{P_{Y_\alpha}}(f)>K\right).\end{aligned}$$

But on $[\![0,T_n]\!]$, $\widetilde{ev}_n^{P_{Y_\alpha}}=ev$, so that

$$\begin{aligned}\tilde{P}_{Y_\alpha}\left(f\in\tilde{D}[0,T]\right. &:\ \left.\Phi_n\circ\underline{\widetilde{ev}}_n^{P_{Y_\alpha}}(f)>K\right)\\ &= \tilde{P}_{Y_\alpha}\left(f\in\tilde{D}[0,T]:\ \Phi_n(f)>K\right)\\ &= P_{Y_\alpha}\left(f\in D[0,T]:\ \{f\in D[0,T]:\ \Phi_n(f)>K\}\cap\tilde{D}[0,T]\right).\end{aligned}$$

Now, assumptions **A0, A1**, **A4** and **A5** yield Proposition 9, therefore

$$ev^{P_{Y_\alpha}}(f,t)=\alpha\int_0^t s(f,x)\,\beta_1(dx)+B_\alpha^{Y_\alpha}(f,t).$$

So, using the representation of Φ_n with respect to P_{Y_α},

$$\begin{aligned}\int_{\tilde{D}_n} I_{C_n}(f)\,\Phi_n(f)\,P_{B_\alpha}(df) \le\ & \tilde{P}_{Y_\alpha}\left(\sqrt{\alpha}\int_0^T \tilde{s}_n(f,x)\,\tilde{B}_1^{Y_\alpha}(f,dx)\right.\\ &\left.+\frac{\alpha}{2}\int_0^T \tilde{s}_n^2(f,x)\,\beta_1(dx)>\ln[K]\right).\end{aligned}$$

The right hand side goes to zero as in previous arguments. *q.e.d.*

Corollary 7

When $\beta_2 = \beta_1$, or when, almost surely, $S(\omega, \cdot) \in H(N_\alpha)$, the integrability assumptions on s with respect to β_2 of Proposition 12 are no longer necessary, as the argument given in the Corollary to Proposition 11 is still valid.

Corollary 8

Given assumptions **A0**, **A1**, **A4** *and* **A5**, *assumption* **A6** *is necessary and sufficient for mutual absolute continuity of P_{B_α} and P_{Y_α}.*

4.6.2 Weak solution of a stochastic differential equation

The innovations representation of the signal-plus-noise process, within the adopted RKHS framework, requires the seemingly unrelated, preliminary results that follow. Their reason for being presented here will emerge in the next section, when the existence and the form of the likelihood for the filtered processes will be addressed. Further, the results of Proposition 14 and 15 can potentially be used for extracting the signal from noise when the likelihood ratio is known.

A weak solution of the equation

$$Y_\alpha(\omega, t) = \alpha \int_0^t s(Y_\alpha(\omega, \cdot), x)\, \beta_1(dx) + B_\alpha(\omega, t)$$

is a triple $\{B_1^w, B_2^w, P^w\}$ such that

1. P^w is a probability measure on $\mathcal{D}$ such that, with respect to it,
   (a) B_1^w is a generalized Brownian motion, adapted to $\underline{\mathcal{D}}$, with variance $V_{P^w}[B_1^w(\cdot, t)] = \beta_1(t)$;
   (b) B_2^w is a Poisson process, adapted to $\underline{\mathcal{D}}$, for which $E_{P^w}[B_2^w(\cdot, t)] = \beta_2(t)$;

(c) B_1^w and B_2^w are independent.

2. and for fixed $t \in [0,T]$, almost surely with respect to P^w,

$$ev^{P^w}(f,t) = \alpha \int_0^T s(f,x)\,\beta_1(dx) + B_\alpha^w(f,t)$$

where

$$\begin{aligned} B_\alpha^w(f,t) &= \sqrt{\alpha}\; B_1^w(f,t) + \sqrt{1-\alpha}\; \tilde{B}_2^w(f,t) \\ \tilde{B}_2^w(f,t) &= B_2^w(f,t) - \beta_2(t). \end{aligned}$$

Lemma 11

Let B_α be a process satisfying **A1**. *The process $ev^{P_{B_\alpha}}$ has then, with respect to P_{B_α}, the representation*

$$ev^{P_{B_\alpha}} = \sqrt{\alpha}\; B_1^{ev} + \sqrt{1-\alpha}\; \tilde{B}_2^{ev}$$

where

$$P_{B_\alpha} \circ [\underline{B}_1^{ev}]^{-1} = P \circ \underline{B}_1^{-1} \text{ and } P_{B_\alpha} \circ [\underline{B}_2^{ev}]^{-1} = P \circ \underline{B}_2^{-1}$$

and $\tilde{B}_2^{ev} = B_2^{ev} - \beta_2$, for some probability space $(\Omega, \mathcal{A}, P)$.

Proof: First, given P_{B_α} it can always, without restriction, be assumed that it is a measure induced from a $(\Omega, \mathcal{A}, P)$ space by a generalized Brownian motion B_1 and an independent Poisson process B_2 summed to give the process $B_\alpha = \sqrt{\alpha}B_1 + \sqrt{1-\alpha}\tilde{B}_2$, as in assumption **A1**. The process B_2^{ev} is then defined by the equality:

$$B_2^{ev}(f,t) = \frac{1}{\sqrt{1-\alpha}} \sum_{u \le t} \left\{ \Delta ev^{P_{B_\alpha}} \right\}(f,u).$$

For fixed $0 \le t_1 < \cdots < t_n \le T$, and a Borel set $G \in I\!R^n$, let

$$G_D = \{ f \in D[0,T] : \; (B_2^{ev}(f,t_1), \ldots, B_2^{ev}(f,t_n)) \in G \}.$$

Let $G_D^{\Omega} = \underline{B}_{\alpha}^{-1}(G_D)$. If $\omega \in G_D^{\Omega}$, then

$$\left(\frac{1}{\sqrt{1-\alpha}} \sum_{u \le t_1} \{\Delta B_\alpha\}(\omega, u), \ldots, \frac{1}{\sqrt{1-\alpha}} \sum_{u \le t_n} \{\Delta B_\alpha\}(\omega, u)\right) \in G$$

that is

$$(B_2(\omega, t_1), \ldots, B_2(\omega, t_n)) \in G.$$

B_2^{ev} is thus, with respect to P_{B_α}, a Poisson process such that

$$E_{P_{B_\alpha}}[B_2^{ev}(\cdot, t)] = \beta_2(t).$$

Similarly, it can be shown that, with respect to P_{B_α}, B_1^{ev}, defined by

$$B_1^{ev} = \frac{1}{\sqrt{\alpha}} \left\{ev^{P_{B_\alpha}} - \sqrt{1-\alpha}\ \left(B_2^{ev} - \beta_2\right)\right\}$$

is a generalized Brownian motion such that

$$E_{P_{B_\alpha}}[B_2^{ev}(\cdot, t)] = \beta_2(t).$$

q.e.d.

Corollary 9

$$\sigma_t^{\circ}\left(ev^{P_{B_\alpha}}\right) = \sigma_t^{\circ}\left(B_1^{ev}\right) \vee \sigma_t^{\circ}\left(B_2^{ev}\right).$$

Proposition 13

Let s be progressively measurable for $\mathcal{D}$, and assume that, for every $f \in D[0,T]$,

$$\int_0^T s^2(f, x)\, \beta_1(dx) < \infty \ \text{and} \ \int_0^T |s|(f, x)\, \beta_2(dx) < \infty.$$

With the notation of Lemma 11, define for almost every $f \in D[0,T]$, with respect to P_{B_α},

$$\ln[\Phi(f)] = \sqrt{\alpha}\int_0^T s(f,x)\, B_1^{ev}(f,dx) - \frac{\alpha}{2}\int_0^T s^2(f,x)\,\beta_1(dx).$$

Then,

$$Y_\alpha(\omega,t) = \alpha\int_0^t s(Y_\alpha(\omega,\cdot),x)\,\beta_1(dx) + B_\alpha(\omega,t)$$

has a weak solution if, and only if $E_{P_{B_\alpha}}[\Phi] = 1$, in which case the solution is unique.

Proof: Suppose first that $E_{P_{B_\alpha}}[\Phi] = 1$. Let then P^w be defined, as a probability, by the relation $dP^w = \Phi dP_{B_\alpha}$. Also define B_α^w by

$$B_\alpha^w(f,t) = \alpha\int_0^t \{-s\}(f,x)\,\beta_1(dx) + ev^{P_{B_\alpha}}(f,t).$$

As Φ can be written in the form

$$\ln[\Phi(f)] = -\sqrt{\alpha}\int_0^T \{-s\}(f,x)\, B_1^{ev}(f,dx) - \frac{\alpha}{2}\int_0^T \{-s\}^2(f,x)\,\beta_1(dx)$$

Girsanov's theorem (Proposition 6) is applied to obtain that

$$P^w \circ [\underline{B_\alpha^w}]^{-1} = P_{B_\alpha} \circ \left[\underline{ev}^{P_{B_\alpha}}\right]^{-1} = P_{B_\alpha}.$$

As furthermore,

$$ev^{P^w}(f,t) = \alpha\int_0^t s(f,x)\,\beta_1(dx) + B_\alpha^w(f,t)$$

then there exists a weak solution since, for instance using Lemma 11, for G, a Borel set of $I\!R^n$, $0 \le t_1 < t_2 < t_3 < \cdots < t_n \le T$,

$$G_D = \{f \in D[0,T] : (f(t_1), f(t_2), f(t_3), \ldots, f(t_n)) \in G\}$$

and

$$B_2^w(f,t) = \sum_{u \le t}\{\Delta B_\alpha^w(f,t)\}$$

then

$$P^{w}\left(f\in D\left[0,T\right]:\ B_2^{w}\left(f,\cdot\right)\in G_D\right)=P_{B_\alpha}\left(f\in D\left[0,T\right]:\ B_2^{ev}\left(f,\cdot\right)\in G_D\right).$$

Suppose now that a weak solution exists. Then, by definition,

$$ev^{P^w}\left(f,t\right)=\alpha\int_0^t s\left(f,x\right)\beta_1\left(dx\right)+B_\alpha^{w}\left(f,t\right)$$

which can be rewritten in the form

$$ev^{P^w}\left(f,t\right)=\alpha\int_0^t s\left(ev^{P^w}\left(f,\cdot\right),x\right)\beta_1\left(dx\right)+B_\alpha^{w}\left(f,t\right).$$

Proposition 11 can then be applied to get that P^w and $P_{B_\alpha^w}$ are mutually absolutely continuous, and that, almost surely, with respect to $P_{B_\alpha^w}$,

$$\begin{aligned}\ln\left[\frac{dP^w}{dP_{B_\alpha^w}}\right](f) \;=\; & \int_0^T s\left(f,x\right)ev^{P_{B_\alpha^w}}\left(f,dx\right)\\ & -\frac{\alpha}{2}\int_0^T s^2\left(f,x\right)\beta_1\left(dx\right)\\ & -\sqrt{1-\alpha}\int_0^T s\left(f,x\right)\tilde{B}_2^{ev^{P^w},B_\alpha^w}\left(f,dx\right)\end{aligned}$$

where $\tilde{B}_2^{ev^{P^w},B_\alpha^w}$ is the representation of $\tilde{B}_2^{w}$, with respect to $P_{B_\alpha^w}(\equiv P_{B_\alpha})$. Furthermore, with respect to $P_{B_\alpha^w}$ (Lemma 11)

$$ev^{P_{B_\alpha^w}}=\sqrt{\alpha}B_1^{ev}+\sqrt{1-\alpha}\tilde{B}_2^{ev}.$$

Consequently

$$\begin{aligned}E_{P_{B_\alpha^w}}\left[\ln\left[\frac{dP^w}{dP_{B_\alpha^w}}\right]-\ln\left[\Phi\right]\right]^2 \;=\; & (1-\alpha)\,E_{P_{B_\alpha^w}}\left[\int_0^T s\left(f,x\right)B_2^{ev}\left(f,dx\right)\right.\\ & \left.-\int_0^T s\left(f,x\right)B_2^{ev^{P^w},B_\alpha^w}\left(f,dx\right)\right]^2.\end{aligned}$$

Now the evaluation map ev is a semimartingale with respect to P^w as well as with respect to $P_{B_\alpha^w}$. As these two probability measures are mutually absolutely continuous,

$[ev]^{P^w} = [ev]^{P_{B^w_\alpha}}$. As $[ev]^{P^w} = B^w_2$ and $[ev]^{P_{B^w_\alpha}} = B^{ev}_2$, and taking into account the fact that $B^w_2 = \tilde{B}_2^{ev^{P^w}, B^w_\alpha}$ (proof of Theorem 1), then $E_{P_{B_\alpha}}[\Phi] = 1$.

Suppose now that a second solution $\left(B^{\tilde{w}}_1, B^{\tilde{w}}_2, P^{\tilde{w}}\right)$ exists. Since

$$\frac{dP^{\tilde{w}}}{dP_{B_\alpha}} = \Phi,$$

$P^{\tilde{w}} = P^w$. *q.e.d.*

Corollary 10

Proposition 13 will be true whenever $\beta_1 = \beta_2$, *or* $S(\omega, \cdot) \in H(N_\alpha)$, *for every* $\omega \in \Omega$.

Corollary 11

If it is assumed, in Proposition 13, that only

$$P_{B_\alpha}\left(f \in D[0,T] : \int_0^T s^2(f,x)\,\beta_1(dx) < \infty\right) = 1,$$

and that

$$P_{B_\alpha}\left(f \in D[0,T] : \int_0^T |s|(f,x)\,\beta_2(dx) < \infty\right) = 1,$$

hold there is still a solution, but it cannot be claimed any longer that it is unique.

Lemma 12

Let $(\Omega, \mathcal{A}, P)$ be a probability space, and let $\underline{\mathcal{B}}^{(1)}$ and $\underline{\mathcal{B}}^{(2)}$ be, with respect to P, two independent filtrations of $\mathcal{A}$. Set

$$\mathcal{B}_t = \mathcal{B}_t^{(1)} \vee \mathcal{B}_t^{(2)} \text{ and } \underline{\mathcal{B}} = \{\mathcal{B}_t,\ t \in [0,T]\}.$$

Then, if M is a martingale for $\underline{\mathcal{B}}^{(1)}$, it is also a martingale for $\underline{\mathcal{B}}$.

Proof: $\mathcal{B}_t$ is generated by sets of the form

$$B = B^{(1)} \cap B^{(2)},\ B^{(1)} \in \mathcal{B}_t^{(1)},\ B^{(2)} \in \mathcal{B}_t^{(2)}.$$

If now $u < v$, and $B^{(1)} \in \mathcal{B}_u^{(1)},\ B^{(2)} \in \mathcal{B}_u^{(2)}$,

$$\begin{aligned}
\int_{B^{(1)} \cap B^{(2)}} M(\omega, v)\, P(d\omega) &= P\left(B^{(2)}\right) \int_{B^{(1)}} M(\omega, v)\, P(d\omega) \\
&= P\left(B^{(2)}\right) \int_{B^{(1)}} M(\omega, u)\, P(d\omega) \\
&= \int_{B^{(1)} \cap B^{(2)}} M(\omega, u)\, P(d\omega)).
\end{aligned}$$

The proof ends with a monotone class argument. *q.e.d.*

Proposition 14

Suppose Y_α is a process, defined on $(\Omega, \mathcal{A}, P)$, adapted to $\underline{\mathcal{A}}$, with paths in $D[0,T]$, such that P_{Y_α} and P_{B_α} are mutually absolutely continuous. When $\beta_1 = \beta_2 \equiv \beta$, the following can be found:

a. a process s, defined on $(D[0,T], \mathcal{D}, P_{Y_\alpha})$, predictable for $\underline{\mathcal{D}}$;

b. a zero-mean, generalized Brownian motion B_1 and a generalized Poisson process B_2, defined on $(\Omega, \mathcal{A}, P)$, adapted to $\underline{\sigma}^\circ(Y_\alpha)$, with

$$V[B_1(\cdot, t)] = \beta(t) \text{ and } E[B_2(\cdot, t)] = \beta(t),$$

such that, for $B_\alpha = \sqrt{\alpha}B_1 + \sqrt{1-\alpha}\tilde{B}_2$ *and for* $t \in [0,T]$ *fixed but arbitrary, almost surely, with respect to* P,

$$Y_\alpha(\omega,t) = \alpha \int_0^t s(Y_\alpha(\omega,\cdot),x)\,\beta(dx) + B_\alpha(\omega,t)$$

with

$$P_{B_\alpha}\left(f \in D[0,T] : \int_0^T s^2(f,x)\,\beta(dx) < \infty\right) = 1$$

and

$$P_{Y_\alpha}\left(f \in D[0,T] : \int_0^T s^2(f,x)\,\beta(dx) < \infty\right) = 1.$$

Proof: By Lemma 11, $ev^{P_{B_\alpha}} = \sqrt{\alpha}\; B_1^{ev} + \sqrt{1-\alpha}\; \tilde{B}_2^{ev}$. Let

$$\mathcal{B}_t^{(1)} = \sigma_t^{\circ}\left(B_1^{ev}\right), \text{ and } \underline{\mathcal{B}}^{(1)} = \left\{\mathcal{B}_t^{(1)},\ t \in [0,T]\right\}.$$

$\underline{\mathcal{B}}^{(1)}$ is a Brownian filtration.

Consider now the martingale L defined for $\underline{\mathcal{B}}^{(1)}$ as

$$L(f,t) = E_{P_{B_\alpha}}\left[\frac{dP_{Y_\alpha}}{dP_{B_\alpha}} \mid \mathcal{B}_t^{(1)}\right].$$

It has a modification [17, 9.7.5, p.241] $\tilde{L}$ which is continuous to the right and has continuous paths, almost surely, with respect to P_{B_α}. $\tilde{L}$ has then the representation [17, 9.7.4, p.239]

$$\tilde{L}(f,t) = 1 + \sqrt{\alpha}\int_0^t s(f,x)\,B_1^{ev}(f,dx)$$

where s is predictable for $\underline{\mathcal{B}}^{(1)}$. Furthermore

$$P_{B_\alpha}\left(f \in D[0,T] : \int_0^T s^2(f,x)\,\beta(dx) < \infty\right) = 1.$$

Let

$$\tilde{T}(f) = \inf\left\{t \in [0,T] : \left[\tilde{L}(f,t) = 0\right] \text{ or } \left[\tilde{L}(f,t-) = 0\right]\right\}.$$

On $\left[\left[\tilde{T}, T\right]\right]$, the paths of $\tilde{L}$ are, almost surely with respect to P_{B_α}, equal to zero. However, because P_{B_α} and P_{Y_α} are mutually absolutely continuous, $\tilde{L}(f,T) > 0$, almost surely, with respect to P_{B_α}. Consequently,

$$P_{B_\alpha}\left(f \in D[0,T] : \inf_{t\in[0,T]} \tilde{L}(f,t) > 0\right) = 1.$$

The expression $\ln\left[\tilde{L}(f,t)\right]$ makes sense, almost surely, with respect to P_{B_α}, and Itô's formula then yields:

$$\ln\left[\tilde{L}(f,t)\right] = \sqrt{\alpha}\int_0^t \frac{s(f,x)}{\tilde{L}(f,x)} B_1^{ev}(f,dx) - \frac{\alpha}{2}\int_0^t \left(\frac{s(f,x)}{\tilde{L}(f,x)}\right)^2 \beta(dx),$$

that is

$$\tilde{L}(f,t) = e^{\sqrt{\alpha}\int_0^t \frac{s(f,x)}{\tilde{L}(f,x)} B_1^{ev}(f,dx) - \frac{\alpha}{2}\int_0^t \left(\frac{s(f,x)}{\tilde{L}(f,x)}\right)^2 \beta(dx)}.$$

Then set

$$\tilde{s}(f,t) = \frac{s(f,x)}{\tilde{L}(f,x)}.$$

Since

$$\begin{aligned}\int_0^T \tilde{s}^2(f,x)\,\beta(dx) &= \int_0^T \left(\frac{s(f,x)}{\tilde{L}(f,x)}\right)^2 \beta(dx)\\ &\leq \frac{1}{\inf_{t\in[0,T]} \tilde{L}^2(f,t)} \int_0^T s^2(f,x)\,\beta_1(dx)\end{aligned}$$

then

$$P_{B_\alpha}\left(f \in D[0,T] : \int_0^T \tilde{s}^2(f,x)\,\beta(dx) < \infty\right) = 1$$

so that also

$$P_{Y_\alpha}\left(f \in D[0,T] : \int_0^T \tilde{s}^2(f,x)\,\beta(dx) < \infty\right) = 1.$$

Finally, $E_{P_{B_\alpha}}\left[\tilde{L}(\cdot,T)\right] = 1$. Consequently, there exists a weak solution to the "formal"[18] equation

$$Y_\alpha(\omega,t) = \alpha \int_0^T \tilde{s}\left(Y_\alpha(\omega,\cdot),x\right)\beta(dx) + B_\alpha(\omega,t).$$

By the Corollary to Lemma 11 and Lemma 12, $\tilde{L}$ is, with respect to P_{B_α}, a martingale for $\underline{\mathcal{D}}$. On $(D[0,T],\mathcal{D})$, and for the filtration $\underline{\mathcal{D}}$ let us define

$$P_\alpha^w(df) = \tilde{L}(f,T)\,P_{B_\alpha}(df)$$

and, with respect to P_α^w,

$$B_\alpha^w(f,t) = -\alpha \int_0^t \tilde{s}(f,x)\,\beta(dx) + ev^{P_\alpha^w}(f,t).$$

By Girsanov's theorem

$$P_\alpha^w \circ \left[\underline{B}_\alpha^w\right]^{-1} = P_{B_\alpha}.$$

Finally, $\tilde{L}(\cdot,T)$ is a version of $\frac{dP_{Y_\alpha}}{dP_{B_\alpha}}$ as it is a martingale for $\underline{\mathcal{D}}$ (Lemma 12). Consequently $P_\alpha^w = P_{Y_\alpha}$. Then set

$$B_\alpha^{Y_\alpha} = B_\alpha^w \circ \underline{Y}_\alpha^{-1}.$$

q.e.d.

Proposition 15

Suppose Y_α is a process, defined on $(\Omega,\mathcal{A},P)$, adapted to $\underline{\mathcal{A}}$, with paths in $D[0,T]$, such that P_{Y_α} is absolutely continuous with respect to P_{B_α}. When $\beta_1 = \beta_2 \equiv \beta$, the following can be found:

a. a process s, defined on $(D[0,T],\mathcal{D},P_{Y_\alpha})$, progressively measurable for $\underline{\mathcal{D}}$;

[18] Note that the B_α of the "formal" equation is not the same as the B_α of the proposition's conclusion.

b. a zero-mean, generalized Brownian motion B_1 and a generalized Poisson process B_2, defined on $(\Omega, \mathcal{A}, P)$, adapted to $\underline{\sigma}^{\circ}(Y_\alpha)$, with

$$V[B_1(\cdot,t)] = \beta(t) \quad \textit{and} \quad E[B_2(\cdot,t)] = \beta(t)$$

such that, for $B_\alpha = \sqrt{\alpha} B_1 + \sqrt{1-\alpha} \tilde{B}_2$, and, for $t \in [0,T]$ fixed but arbitrary, almost surely, with respect to P

$$Y_\alpha(\omega,t) = \alpha \int_0^t s(Y_\alpha(\omega,\cdot), x)\,\beta(dx) + B_\alpha(\omega,t)$$

with

$$P_{Y_\alpha}\left(f \in D[0,T] : \int_0^T s^2(f,x)\,\beta(dx) < \infty\right) = 1.$$

Proof: As in Proposition 14,

$$\tilde{L}(f,t) = 1 + \sqrt{\alpha} \int_0^t s(f,x)\, B_1^{ev}(f,dx)$$

with

$$P_{B_\alpha}\left(f \in D[0,T] : \int_0^T s^2(f,x)\,\beta(dx) < \infty\right) = 1.$$

But now $\tilde{L}$ can equal zero, therefore define

$$T_n(f) = \begin{cases} \inf\left\{t \in [0,T] : \tilde{L}(f,t) < \frac{1}{n}\right\} & \text{if } \left\{t \in [0,T] : \tilde{L}(f,t) < \frac{1}{n}\right\} \neq \emptyset \\ T & \text{if } \left\{t \in [0,T] : \tilde{L}(f,t) < \frac{1}{n}\right\} = \emptyset. \end{cases}$$

If $B^{(1)} \in \mathcal{B}^{(1)}_{t \wedge T_n}$, then

$$\begin{aligned} P_{Y_\alpha}\left(B^{(1)}\right) &= \int_{B^{(1)}} \tilde{L}(f,T)\, P_{B_\alpha}(df) \\ &= \int_{B^{(1)}} E\left[\tilde{L}(\cdot,T) \mid \mathcal{B}^{(1)}_{t\wedge T_n}\right] P_{B_\alpha}(df) \\ &= \int_{B^{(1)}} \tilde{L}(f, t\wedge T_n)\, P_{B_\alpha}(df). \end{aligned}$$

Thus, on $\mathcal{B}^{(1)}_{t\wedge T_n}$, $P_{Y_\alpha}(df) = \tilde{L}(f, t \wedge T_n) P_{B_\alpha}(df)$. But, as $\tilde{L}(\cdot, t \wedge T_n) \geq \frac{1}{n}$,

$$P_{B_\alpha}(df) = \frac{P_{Y_\alpha}(df)}{\tilde{L}(f, t \wedge T_n)}$$

still on $\mathcal{B}^{(1)}_{t\wedge T_n}$, so that, since $D[0, T]$ belongs to $\mathcal{B}^{(1)}_{t\wedge T_n}$,

$$E_{P_{Y_\alpha}}\left[\frac{1}{\tilde{L}(\cdot, t \wedge T_n)}\right] = P_{B_\alpha}(D[0, T]) = 1.$$

The sequence $\{T_n,\ n \in I\!N\}$ is increasing and bounded. It thus has a limit, denoted $\lim_n T_n$, which is a stopping time. As $\tilde{L}$ is continuous, almost surely, with respect to P_{Y_α}, by Fatou's lemma

$$\begin{aligned} E_{P_{Y_\alpha}}\left[\frac{1}{\tilde{L}(\cdot, t \wedge \lim_n T_n)}\right] &= E_{P_{Y_\alpha}}\left[\liminf_n \left\{\frac{1}{\tilde{L}(\cdot, t \wedge T_n)}\right\}\right] \\ &\leq \liminf_n E_{P_{Y_\alpha}}\left[\frac{1}{\tilde{L}(\cdot, t \wedge T_n)}\right] \\ &= 1 \end{aligned}$$

that is,

$$E_{P_{Y_\alpha}}\left[\frac{1}{\tilde{L}(\cdot, t \wedge \lim_n T_n)}\right] \leq 1.$$

As $\tilde{L}(\cdot, \lim_n T_n) = 0$, almost surely with respect to P_{Y_α}, necessarily $\lim_n T_n = T$, almost surely with respect to P_{Y_α}. Furthermore, as

$$\int_0^{T_n} \left\{\frac{s(f, x)}{\tilde{L}(f, x)}\right\}^2 \beta(dx) \leq n^2 \|s(f, \cdot)\|^2_{L_2[\beta]}$$

it follows that

$$\begin{aligned} 1 &= P_{Y_\alpha}\left(f \in D[0, T] :\ \|s(f, \cdot)\|^2_{L_2[\beta]} < \infty\right) \\ &\leq P_{Y_\alpha}\left(f \in D[0, T] :\ \int_0^{T_n} \left\{\frac{s(f, x)}{\tilde{L}(f, x)}\right\}^2 \beta(dx) < \infty\right). \end{aligned}$$

Consequently,

$$P_{Y_\alpha}\left(f \in D[0,T] : \int_0^T \left\{\frac{s(f,x)}{\tilde{L}(f,x)}\right\}^2 \beta(dx) < \infty\right) = 1.$$

As $I_{[\![0,T_n]\!]}\frac{s}{\tilde{L}}$ is in $L_2[\beta]$, almost surely with respect to P_{B_α}, the process $\widetilde{B}_{\alpha,n}$ can legitimately be defined on $(D[0,T], \mathcal{D}, P_{B_\alpha})$ and for the filtration $\underline{\mathcal{D}}$ by the following relation:

$$\widetilde{B}_{\alpha,n}(f,t) = -\alpha \int_0^t I_{[\![0,T_n]\!]}(f,x)\ \frac{s(f,x)}{\tilde{L}(f,x)}\beta(dx) + ev^{P_{B_\alpha}}(f,t).$$

$\tilde{L}^{T_n}$ is a martingale for the filtration $\underline{\mathcal{B}}^{(1)}$, and thus, by Lemma 13, for the filtration $\underline{\mathcal{D}}$, define on $\mathcal{D}_t$, a probability Q_n by setting

$$Q_n(df) = \tilde{L}(f, t \wedge T_n)\, P_{B_\alpha}(df).$$

Then it must be shown that, on $(D[0,T], \mathcal{D}, Q_n)$, $\widetilde{B}_{\alpha,n}$ is martingale for $\underline{\mathcal{D}}$ such that

$$Q_n \circ \underline{\widetilde{B}}_{\alpha,n}^{-1} = P_{B_\alpha}.$$

But, almost surely with respect to P_{B_α},

$$\tilde{L}(f, t \wedge T_n) \geq \frac{1}{n}$$

so that $\ln\left[\tilde{L}(f, t\wedge T_n)\right]$ can be computed and consequently Itô's formula can be applied to obtain, almost surely with respect to P_{B_α}, the following equality:

$$\begin{aligned}\ln\left[\tilde{L}(f,t\wedge T_n)\right] &= \sqrt{\alpha}\int_0^t I_{[0,T_n]}(f,x)\ \frac{s(f,x)}{\tilde{L}(f,x)} B_1(f,dx) \\ &\quad -\frac{\alpha}{2}\int_0^t I_{[\![0,T_n]\!]}(f,x)\ \left\{\frac{s(f,x)}{\tilde{L}(f,x)}\right\}^2 \beta(dx).\end{aligned}$$

But

$$E_{P_{B_\alpha}}\left[\tilde{L}(\cdot, t\wedge T_n)\right] = 1$$

because of the martingale property of $\tilde{L}(\cdot, t \wedge T_n)$ for $\underline{\mathcal{D}}$ and P_{B_α}. Therefore the Girsanov's theorem can be invoked to assert that, for the base $(D[0,T], \mathcal{D}, Q_n)$ and the filtration $\underline{\mathcal{D}}$,

$$Q_n \circ \underline{\widetilde{B}}_{\alpha,n}^{-1} = P_{B_\alpha}.$$

Now $\widetilde{B}_{\alpha,n+1}^{T_n} = \widetilde{B}_{\alpha,n}$, again for the base $(D[0,T], \mathcal{D}, P_{B_\alpha})$ and the filtration $\underline{\mathcal{D}}$, the process

$$\widetilde{B}_\alpha(f,t) = -\alpha \int_0^t I_{[\![0, \lim_n T_n]\!]}(f,x) \ \frac{s(f,x)}{\tilde{L}(f,x)} \beta(dx) + ev^{P_{B_\alpha}}(f,t)$$

can be defined.

Since $\lim_n T_n = T$, almost surely with respect to P_{Y_α}, and that P_{Y_α} is absolutely continuous with respect to P_{B_α}, then almost surely with respect to P_{Y_α},

$$\widetilde{B}_\alpha(f,t) = -\alpha \int_0^t \frac{s(f,x)}{\tilde{L}(f,x)} \beta(dx) + ev^{P_{Y_\alpha}}(f,t).$$

Finally, it is necessary to check that, for the base $(D[0,T], \mathcal{D}, P_{Y_\alpha})$ and the filtration $\underline{\mathcal{D}}$,

$$P_{Y_\alpha} \circ \underline{\widetilde{B}}_\alpha^{-1} = P_{B_\alpha}.$$

To that end, note that

$$\tilde{L}(f, t \wedge T_n) \ \widetilde{B}_\alpha(f, t \wedge T_n) = \tilde{L}(f, t \wedge T_n) \ \widetilde{B}_{\alpha,n}(f, t \wedge T_n).$$

But, on the base $(D[0,T], \mathcal{D}, Q_n)$ and for the filtration $\underline{\mathcal{D}}$, $\widetilde{B}_{\alpha,n}$ is a martingale, and $Q_n(df) = \tilde{L}(f, t \wedge T_n) P_{B_\alpha}(df)$, so that $\tilde{L}(\cdot, \cdot \wedge T_n) \ \widetilde{B}_\alpha(\cdot, \cdot \wedge T_n)$ is a martingale on the base $(D[0,T], \mathcal{D}, P_{B_\alpha})$ and for the filtration $\underline{\mathcal{D}}$, and consequently a martingale on the base $(D[0,T), \mathcal{D}, Q_n)$ for the same filtration. Since $\underline{\mathcal{B}}^{(1)} \subseteq \underline{\mathcal{D}}$, T_n is a stopping time for $\underline{\mathcal{D}}$. Since, $\mathcal{D}_{t \wedge T_n} = \mathcal{B}_{t \wedge T_n}^{(1)} \vee \mathcal{B}_{t \wedge T_n}^{(2)}$ and $Q_{n|\mathcal{D}_{t \wedge T_n}} = P_{Y_\alpha|\mathcal{D}_{t \wedge T_n}}$, it follows that

$$P_{Y_\alpha}(df) = \tilde{L}(f, t \wedge T_n) P_{B_\alpha}(df).$$

$\widetilde{B}_\alpha$ is thus a local martingale on the base $(D[0,T], \mathcal{D}, P_{Y_\alpha})$ for the filtration $\underline{\mathcal{D}}$.

Finally it must be shown that $\widetilde{B}_\alpha$ has, with respect to P_{Y_α} the same law as B_α with respect to P. But, for scalars $\theta_1, \ldots, \theta_p$, forming the vector $\underline{\theta}_p$, and times $0 \leq t_1, \ldots, t_p \leq T$,

$$\begin{aligned} E_{P_{Y_\alpha}}\left[e^{i\langle \underline{\theta}_p, \underline{\widetilde{B}}_\alpha^{(p)}\rangle_{I\!R^p}}\right] &= \lim_n E_{P_{Y_\alpha}}\left[e^{i\langle \underline{\theta}_p, \underline{\widetilde{B}}_{\alpha,n}^{(p)}\rangle_{I\!R^p}}\right] \\ &= \lim_n E_{Q_n}\left[e^{i\langle \underline{\theta}_p, \underline{\widetilde{B}}_{\alpha,n}^{(p)}\rangle_{I\!R^p}}\right] \\ &= E_P\left[e^{i\langle \underline{\theta}_p, \underline{\widetilde{B}}^{(p)}\rangle_{I\!R^p}}\right] \end{aligned}$$

where $\underline{\widetilde{B}}_\alpha^{(p)}$, $\underline{\widetilde{B}}_{\alpha,n}^{(p)}$ and $\underline{\widetilde{B}}^{(p)}$ are vectors with respective components $\widetilde{B}_\alpha(\cdot, t_i)$, $\widetilde{B}_{\alpha,n}(\cdot, t_i)$ and $B_\alpha(t_i)$, for $0 \leq t_i \leq T$, $1 \leq i \leq p$. *q.e.d.*

Corollary 12

It is assumed that **A0, A1** *and* **A2** *hold. Then the following innovations representation is valid almost surely with respect to* P *for* $t \in [0,T]$:

$$Y_\alpha(\omega, t) = \int_0^t \tilde{s}(Y_\alpha(\omega, \cdot), x)\,\beta_1(dx) + B_\alpha^{Y_\alpha}(\omega, t)$$

where

a. $\tilde{s}$ *is defined on* $(D[0,T], \mathcal{D}, P_{Y_\alpha})$, *and is progressively measurable for* $\underline{\mathcal{D}}$;

b. $B_\alpha^{Y_\alpha}$ *is defined on* $(\Omega, \mathcal{A}, P)$, *adapted to* $\underline{\sigma}^\circ(Y_\alpha)$;

c. $P \circ \left[\underline{B}_\alpha^{Y_\alpha}\right]^{-1} = P_{B_\alpha}$.

5.0 ABSOLUTE CONTINUITY AND LIKELIHOOD FOR P_{N_α} AND P_{X_α}.

The previous section derived explicit expressions for the likelihood ratio of the probability laws of unfiltered processes B_α and Y_α. That is interesting as a separate part but it is also a step in achieving results about the existence and the form of the likelihood ratios for the probability laws of the filtered processes N_α and X_α. Hence, these formulae are obtained by means of the conditional law of B_α given N_α derived as a functional on a "defiltering" or inversion process.

5.1 THE INVERSION PROCESS M

The Cramér-Hida representation says intuitively that the paths of B_α and N_α are, probabilistically, in one-to-one correspondence. The mathematical expression for this intuition is the process M whose definition and properties follow.

Terms whose definitions are omitted are those of sections 3 and 4. $I_{[0,t]}$ denotes the indicator of the interval $[0,t]$. The basic probability space is

$$(L_2[0,T], \mathcal{B}(L_2[0,T]), P_{N_\alpha}).$$

For $t \in [0,T]$ fixed but arbitrary, the following variables are considered on $L_2[0,T] \times [0,T]$:

$$M_i(f,t) = \frac{1}{\lambda_i}\langle U\left[I_{[0,t]}\right], e_i\rangle_{L_2[0,T]}\langle f, e_i\rangle_{L_2[0,T]}.$$

Then $E_{P_{N_\alpha}}[M_i(\cdot,t)] = 0$, and that

$$\begin{aligned} E_{P_{N_\alpha}}[M_i(\cdot,t)\,M_j(\cdot,t)] &= \frac{1}{\lambda_i\lambda_j}\langle U\left[I_{[0,t]}\right], e_i\rangle_{L_2[0,T]}\ \langle U\left[I_{[0,t]}\right], e_j\rangle_{L_2[0,T]} \\ &\quad \times\ E_{P_{N_\alpha}}\left[\langle f, e_i\rangle_{L_2[0,T]}\langle f, e_j\rangle_{L_2[0,T]}\right] \\ &= \delta_{i,j}\ \langle I_{[0,t]}, J[e_i]\rangle_{L_2[\beta_\alpha]}\ \langle I_{[0,t]}, J[e_j]\rangle_{L_2[\beta_\alpha]}. \end{aligned}$$

Lemma 13

The family $\{J[e_i],\ i \in I\!N\}$ *is a complete orthonormal set in* $L_2[\beta_\alpha]$.

Proof: Let f be arbitrary in $L_2[\beta_\alpha]$, and suppose that

$$\langle f, J[e_i]\rangle_{L_2[\beta_\alpha]} = 0,\ i \in I\!N.$$

Then $J^\star[f]$ is orthogonal to K (the closure of the range of the square root of the covariance operator), that is $J^\star[f] \in \mathcal{N}\left(R_\alpha^{\frac{1}{2}}\right)$, which means that $U[f] = 0$. However, it has already been established (Proposition 2) that the only possibility in $L_2[\beta_\alpha]$ is $f = 0$. Finally,

$$\langle J[e_i], J[e_j]\rangle_{L_2[\beta_\alpha]} = \langle e_i, J^\star J[e_j]\rangle_{L_2[0,T]}.$$

But $J^\star J$ is the projection onto the closure of the range of $R_\alpha^{\frac{1}{2}}$, so that

$$\langle e_i, J^\star J[e_j]\rangle_{L_2[0,T]} = \langle e_i, e_j\rangle_{L_2[0,T]}.$$

q.e.d.

Corollary 13

a. $\sum_{i=1}^{\infty} E_{P_{N_\alpha}}\left[M_i^2(\cdot, t)\right] = \left\|I_{[0,t]}\right\|^2_{L_2[\beta_\alpha]} = \beta_\alpha(t).$

b. *For* $t \in [0,T]$ *fixed but arbitrary, the series* $\sum_{i=1}^{\infty} M_i(f,t)$ *converges almost surely, with respect to* P_{N_α}, *and in* $L_2[P_{N_\alpha}]$.

The following notation will be used

$$M^{(n)}(f,t) = \sum_{i=1}^{n} M_i(f,t),\ M(f,t) = \sum_{i=1}^{\infty} M_i(f,t).$$

Lemma 14

For $(i,t) \in I\!N \times [0,T]$ *fixed but arbitrary,*

$$E\left[B_\alpha(\cdot,t)\ \langle N_\alpha(\cdot,\cdot), e_i\rangle_{L_2[0,T]}\right] = \langle U\left[I_{[0,t]}\right], e_i\rangle_{L_2[0,T]}.$$

Proof:

$$\begin{aligned}
&E\left[B_\alpha(\cdot,t)\ \langle N_\alpha(\cdot,\cdot), e_i\rangle_{L_2[0,T]}\right] \\
&\quad= E\left[B_\alpha(\cdot,t)\int_0^T e_i(x)\,dx\int_0^T F(x,u)\,B_\alpha(\cdot,du)\right] \\
&\quad= \int_0^T e_i(x)\,dx \\
&\qquad\times E\left[\left\{\int_0^T I_{[0,t]}(u)\,B_\alpha(\cdot,du)\right\}\left\{\int_0^T F(x,v)\,B_\alpha(\cdot,dv)\right\}\right] \\
&\quad= \int_0^T e_i(x)\ dx\int_0^T F(x,u)\,I_{[0,t]}(u)\,\beta_\alpha(du) \\
&\quad= \int_0^T e_i(x)\ U\left[I_{[0,t]}\right](x)\,dx \\
&\quad= \langle U\left[I_{[0,t]}\right], e_i\rangle_{L_2[0,T]}.
\end{aligned}$$

q.e.d.

Corollary 14

For $t \in [0,T]$ *fixed but arbitrary,*

$$E\left[\{M(N_\alpha(\cdot,\cdot),t) - B_\alpha(\cdot,t)\}^2\right] = 0.$$

Proof:

$$\begin{aligned}
E\left[\{M(N_\alpha(\cdot,\cdot),t) - B_\alpha(\cdot,t)\}^2\right] &= E_{P_{N_\alpha}}\left[M^2(\cdot,t)\right] \\
&- 2\,E\left[M(N_\alpha(\cdot,\cdot),t)\ B_\alpha(\cdot,t)\right] \\
&+ E\left[B_\alpha^2(\cdot,t)\right].
\end{aligned}$$

It is already known that $E_{P_{N_\alpha}}\left[M^2(\cdot,t)\right] = E\left[B_\alpha^2(\cdot,t)\right] = \beta_\alpha(t)$. But, using Lemma 14,

$$\begin{aligned}
&E\left[M\left(N_\alpha(\cdot,\cdot),t\right)\ B_\alpha(\cdot,t)\right]\\
&\quad= \lim_n \sum_{i=1}^{n} \frac{1}{\lambda_i} \langle U\left[I_{[0,t]}\right], e_i\rangle_{L_2[0,T]}\ E\left[B_\alpha(\cdot,t)\langle N_\alpha(\cdot,\cdot), e_i\rangle_{L_2[0,T]}\right]\\
&\quad= \lim_n \sum_{i=1}^{n} \frac{1}{\lambda_i} \langle U\left[I_{[0,t]}\right], e_i\rangle^2_{L_2[0,T]}\\
&\quad= \left\|I_{[0,t]}\right\|^2_{L_2[\beta_\alpha]}\\
&\quad= \beta_\alpha(t).
\end{aligned}$$

q.e.d.

As an immediate consequence of the above, the following proposition holds.

Proposition 16

Let $0 < t_1 < \cdots < t_n \le T$, and $\theta_1, \ldots, \theta_n$, be arbitrary constants. Then,

a. $E_{P_{N_\alpha}}\left[e^{i\sum_{j=1}^{n}\theta_j M(\cdot,t_j)}\right] = E\left[e^{i\sum_{j=1}^{n}\theta_j B_\alpha(\cdot,t_j)}\right]$.

b. M has, with respect to P_{N_α}, independent increments.

c. For $0 < s < t \le T$, $E_{P_{N_\alpha}}\left[M(\cdot,s)\,M(\cdot,t)\right] = \beta_\alpha(s\wedge t)$.

d. For $0 < s < t \le T$, $E_{P_{N_\alpha}}\left[\{M(\cdot,t) - M(\cdot,s)\}^2\right] = \beta_\alpha(t) - \beta_\alpha(s)$.

Corollary 15

Let $t \in [0,T]$ be fixed, but arbitrary, and let $\mathcal{M}_t^\circ$ be the σ-algebra generated by $\{M(\cdot,s),\ s \le t\}$, on $L_2[0,T]$. Then, with respect to P_{N_α}, M is a square integrable martingale for $\underline{\mathcal{M}}^\circ = \{\mathcal{M}_t^\circ,\ t \in [0,T]\}$.

Proposition 17

The process M is separable.

Proof: Let T_c denote a countable subset of $[0, T]$. M is, with respect to P_{N_α}, a zero-mean, square integrable martingale, so is its restriction to T_c. There is thus [17, 3.2.1, p.49] a measurable subset N of $L_2[0, T]$, such that $P_{N_\alpha}(N) = 0$, and, for $f \in N^c$, and any monotone sequence $\{t_n,\ n \in \mathbb{N}\} \subseteq T_c$, the sequence $\{M(f, t_n),\ n \in \mathbb{N}\}$ is convergent in $\overline{\mathbb{R}}$. However, since the sequence $\{M(\cdot, t_n),\ n \in \mathbb{N}\}$ also converges in $L_2[P_{N_\alpha}]$, if $\lim_n t_n = t$, the limit in $L_2[P_{N_\alpha}]$ of $\{M(\cdot, t_n),\ n \in \mathbb{N}\}$ is $M(\cdot, t)$. Consequently, for $f \in N^c$,

$$\lim_n M(f, t_n) = M(f, t).$$

q.e.d.

Corollary 16

With respect to P_{N_α}, the paths of M almost surely belong to $D[0, T]$.

Proof: Separability of M and the fact that it is a martingale yield the following:

$$\begin{aligned}
P(\omega \in \Omega \quad &: \quad \sup_{t \in [0,T]} |M(N_\alpha(\omega, \cdot), t) - B_\alpha(\omega, t)| > \epsilon\Big) \\
&= P\left(\omega \in \Omega : \sup_{t \in T_c} |M(N_\alpha(\omega, \cdot), t) - B_\alpha(\omega, t)| > \epsilon\right) \\
&\leq \frac{1}{\epsilon^2} E\left[\left\{\sup_{t \in T_c} |M(N_\alpha(\omega, \cdot), t) - B_\alpha(\omega, t)|\right\}^2\right] \\
&\leq \frac{4}{\epsilon^2} E\left[\{M(N_\alpha(\omega, \cdot), T) - B_\alpha(\omega, T)\}^2\right] = 0.
\end{aligned}$$

q.e.d.

5.2 THE CONDITIONAL LAW OF B_α GIVEN N_α

When the one-to-one correspondence between the filtered and unfiltered processes holds in $L_2[P]$, it is possible to express the relationship between B_α and N_α. In the Proposition 18 this relationship will be expressed in terms of the conditional probability law of the unfiltered process B_α when the filtered process N_α is given.

Proposition 18

The assumptions are those of Section 3.1. Then B_α has, with respect to N_α, a regular conditional law which is a point mass located at M.

Proof: Let $F \subseteq D[0,T]$, and $G \subseteq L_2[0,T]$ be measurable subsets. Then

$$\begin{aligned} P(\omega \in \Omega \quad &: \quad B_\alpha(\omega,\cdot) \in F,\ N_\alpha(\omega,\cdot) \in G) \\ &= \quad P(\omega \in \Omega :\ M(N_\alpha(\omega,\cdot),\cdot) \in F,\ N_\alpha(\omega,\cdot) \in G). \end{aligned}$$

In $L_2[P]$, for $t \in [0,T]$ fixed but arbitrary,

$$B_\alpha(\cdot,t) = M(N_\alpha(\cdot,\cdot),t).$$

This equality is obviously true whenever

$$\begin{gathered} 0 \le t_1 < \cdots < t_p \le T, \\ B_i \in \mathcal{B}[I\!R],\ 1 \le i \le p, \\ F = \left\{ f \in D[0,T] :\ ev_{t_1}(f) \in B_1,\ \ldots\ , ev_{t_p}(f) \in B_p \right\}, \\ \{g_1,\ \ldots\ , g_q\} \subseteq L_2[0,T], \\ \tilde{B}_j \in \mathcal{B}[I\!R],\ 1 \le j \le q, \\ G = \left\{ g \in L_2[0,T] :\ \langle g, g_1\rangle_{L_2[0,T]} \in \tilde{B}_1,\ \ldots\ , \langle g, g_q\rangle_{L_2[0,T]} \in \tilde{B}_q \right\}. \end{gathered}$$

As such sets generate the corresponding σ-algebras, the equality is true in general. But then

$$\begin{aligned} P(\omega \in \Omega \quad &: \quad M(N_\alpha(\omega,\cdot),\cdot) \in F,\ N_\alpha(\omega,\cdot) \in G) \\ &= \int_G P_{N_\alpha}(dg)\, P(M \circ N_\alpha \in F \mid N_\alpha = g) \\ &= \int_G P_{N_\alpha}(dg)\, E[I_F(M \circ N_\alpha) \mid N_\alpha = g] \\ &= \int_G P_{N_\alpha}(dg)\, I_F(M(g)). \end{aligned}$$

q.e.d.

Corollary 17

$$E_{P_{N_\alpha}}\left[\frac{dP_{Y_\alpha}}{dP_{B_\alpha}} \mid N_\alpha = g\right] = \frac{dP_{Y_\alpha}}{dP_{B_\alpha}}(M(g)).$$

5.3 EXISTENCE AND FORM OF THE LIKELIHOOD

The objective of the calculus undertaken so far is reached in Theorem 2. In few words, the content of this theorem is:

a. The absolute continuity of the probability law of the signal-plus-noise with respect to the probability law of the noise holds under minimal assumptions.

b. When, with respect to the probability law of the noise, the norm of the transmitted signal in the reproducing kernel Hilbert space of the noise is finite, the mutual absolute continuity holds. Then the likelihood ratio exists and is expressed in explicit form.

Theorem 2

Fix $\alpha = \frac{1}{2}$ and write B for B_α, N for N_α, and Y for Y_α. Other notation is as already encountered. Assume then that

$$N(\omega, t) = \int_0^T F(t, x) \, B(\omega, dx)$$

where

I. *assumptions* **A0** *and* **A1** *are valid for B with $\beta_1 = \beta_2 = \beta$,*

II. *F is a non-anticipative ($F(t,x) = 0$, for $x > t$), measurable function, defined on $[0,T] \times [0,T]$, whose equivalence classes generate $L_2[\beta]$,*

III. *$S(\omega, \cdot) \in H(N)$, almost surely, with respect to P.*

The following statements are then valid.

a. *P_{S+N} is absolutely continuous with respect to P_N.*

b.
$$\frac{dP_{S+N}}{dP_N}(f) = \tilde{\Lambda} \circ M(f) \tag{2}$$

where, for $f \in \mathcal{L}_2[0,T]$, M is the process

$$M(f,t) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \langle UI_{[0,t]}, e_k \rangle_{L_2[0,T]} \langle f, e_k \rangle_{L_2[0,T]}.$$

c. *With respect to P_Y, and for $f \in D[0,T]$, $\tilde{\Lambda}$ has the representation*

$$\begin{aligned}\ln\left[\tilde{\Lambda}(f)\right] &= \int_0^T s(f,x)\, ev^{P_Y}(f,dx) \\ &-\frac{1}{4}\int_0^T s^2(f,x)\,\beta(dx) \\ &-\frac{1}{\sqrt{2}}\int_0^T s(f,x)\,\tilde{B}_2^Y(f,dx) \qquad (3)\end{aligned}$$

with $\tilde{B}_2^Y$, a Poisson martingale, independent of B_1^Y, and s, the predictable process resulting from the RKHS condition of assumption III.

d. *With respect to P_B, $\tilde{\Lambda}$ can be approximated by the sequence $I_{C_n}\Phi_n$, where $C_n = \{f \in D[0,T] : T_n(f) = T\}$, T_n is the stopping time of Proposition 11, and Φ_n is given by the following expression, which must be interpreted as that of (c)*

$$\begin{aligned}\ln\left[\Phi_n(f)\right] &= \int_0^T \tilde{s}_n(f,x)\, ev^{P_B}(f,dx) \\ &-\frac{1}{4}\int_0^T \tilde{s}_n^2(f,x)\,\beta(dx) \\ &-\frac{1}{\sqrt{2}}\int_0^T \tilde{s}_n(f,x)\,\tilde{B}_2^{Y,B}.\end{aligned}$$

e. *If it can be assumed that*

$$P_N\left(f \in D[0,T] : \int_0^T s^2(M(f,\cdot),x)\,\beta(dx) < \infty\right) = 1,$$

then P_{S+N} and P_N are mutually absolutely continuous, and mutatis mutandis, *the likelihood formula of (c) holds with respect to $P_B = P_{\underline{N}\circ\underline{M}^{-1}}$. A sufficient condition for that, in terms of S, is*

$$E\left[\exp\left\{\frac{1}{2}\|S(\cdot,\cdot)\|^2_{H(N)}\right\}\right] < \infty.$$

Proof: In order to prove (a) and (b) it is enough to note the following. Assumption III, in conjunction with [12, Thm 3, Step 3, p.170] means that, for some appropriate s,

$$P\left(\omega \in \Omega : \int_0^T s^2(\omega,x)\,\beta(dx) < \infty\right) = 1.$$

The Corollary to Proposition 7 then yields that P_Y is absolutely continuous with respect to P_B, and then, from Proposition 15, it follows that Y has a stochastic integral representation. The specific form of the likelihood follows then from Proposition 12.

Now, as *mutatis mutandis* [12, Thm 1, p.163]

$$\underline{N} = \Phi \circ \underline{B} \text{ and } \underline{S+N} = \Phi \circ \underline{Y}$$

for any Borel set A of $L_2[0,T]$,

$$\begin{aligned}
P_{S+N}(A) &= P_Y\left(\Phi^{-1}(A)\right) \\
&= \int_{D[0,T]} I_A(\Phi(f)) \frac{dP_Y}{dP_B}(f)\, P_B(df) \\
&= \int_{\Omega} I_A(\underline{N}(\omega)) \frac{dP_Y}{dP_B}(\underline{B}(\omega))\, P(d\omega) \\
&= \int_A E\left[\frac{dP_Y}{dP_B} \mid \underline{N} = f\right] P_N(df).
\end{aligned}$$

But, because (Proposition 18) the law of B given N is a regular conditional probability, with mass concentrated at $\underline{M}$,

$$E\left[\frac{dP_Y}{dP_B} \mid \underline{N} = f\right] = \int_{D[0,T]} \frac{dP_Y}{dP_B}(g)\, P_{B|\underline{N}=f}(dg) = \frac{dP_Y}{dP_B}(\underline{M}(f)).$$

Point (c) was derived in Theorem 1 but it is relevant here. Also, the convergence result in point (d) was proven in Proposition 12. Finally, (e) is arrived at by the direct application of the corollary of Proposition 11. *q.e.d.*

6.0 CONCLUDING DISCUSSION

A summary description of the results as well as the relevance of the proposed signal and noise models to sonar techniques are presented in this section. It is also noted that there are similarities between the underwater acoustic channel and the mobile communication channel, which may enable these results to be applied in that environment, as well.

6.1 CONTEXT OF APPLICABILITY

The detection problem of interest was formulated in the Introduction by means of the hypotheses test given by relation (1). Fig. 1.1 gives a general picture of the approach to the problem. In fact, when the detector is chosen to be based on a likelihood ratio, in order to obtain a rigorous solution the following four operations have to be successfully accomplished.

A. Establish the existence of the likelihood ratio. Technically this means that the absolute continuity of P_{S+N} with respect to P_N has to be proved.

B. Derive explicitly the likelihood ratio, when it exists, as a functional Λ, computable for each received signal and without knowing which of the P_{S+N} or P_N regimes are applicable.

C. Determine the threshold Λ_0 required for decision (see fig. 1.1), when the functional Λ is available. A Λ_0 is associated with every predefined probability of false alarm δ and can be obtained from the equation

$$\delta = P_N\left(f \in L_2[0,T] : \Lambda(f) > \Lambda_0\right). \tag{4}$$

Also, for every Λ_0 the probability of detection $1-\eta$ is obtained from the relation

$$\eta = P_{S+N}\left(f \in L_2[0,T] : \Lambda(f) \leq \Lambda_0\right). \tag{5}$$

The quality of detection is quantified then by the *receiver operating characteristic* obtained by plotting the probabilities of detection $1-\eta$ versus the probabilities of false alarm δ.

D. Find a discretisation for which the likelihood ratio satisfies (4) and (5). Assuming that the received signal is observed in discrete form, for example $f(t_1), f(t_2), \ldots f(t_n)$, it has to be checked that approximations Λ_n of Λ provide

$$P_N\left(f \in L_2[0,T] : \Lambda_n\left(f(t_1), f(t_2), \ldots, f(t_n)\right) > \Lambda_0^{(n)}\right) \approx \delta$$

and

$$P_{S+N}\left(f \in L_2[0,T] : \Lambda_n\left(f(t_1), f(t_2), \ldots, f(t_n)\right) > \Lambda_0^n\right) \approx 1 - \eta$$

where $\Lambda_0^{(n)}$ is the value of the threshold obtained when Λ is replaced by its approximation Λ_n in relation (4).

While the results of points (C) and (D) in the previous description may be strongly dependent on the particular features of the detection problem, the answers to the points (A) and (B) require a theoretical approach only. The objective of this report was to provide the mathematical tools in order to be able to fulfill operations (A) and (B). In this context, Theorem 2 says that when the noise is modeled as the superposition of filtered Gaussian and Poisson components, then

a. P_{S+N} is absolutely continuous with respect to P_N if the signal's finite energy condition

$$P\left(\omega \in \Omega : \int_0^T s^2(\omega, x)\beta(dx) < \infty\right) = 1 \tag{6}$$

holds, where $\beta(t)$ is the variance of the noise [19].

b. The functional Λ having the required properties from (B) above is obtained by means of the relations (2) and (3) if, in addition, P_N is absolutely continuous with respect to P_{S+N}. A sufficient condition for that is

$$P_N\left(f \in D[0,T] : \int_0^T s^2(M(f,\cdot), x)\beta(dx) < \infty\right) = 1$$

where M is the inversion process defined in section 5.1 [20]. This condition is satisfied if

[19] In most practical situations (6) reduces to

$$P\left(\omega \in \Omega : \int_0^T s^2(\omega, x)dx < \infty\right) = 1.$$

[20] Which can be thought of as a whitening filter.

$$E\left[\exp\left\{\frac{1}{2}\parallel S(\cdot,\cdot)\parallel^2_{H(N)}\right\}\right] = E\left[\exp\left\{\frac{1}{2}\int_0^T s^2(\cdot,x)\beta(dx)\right\}\right] < \infty.$$

Condition (6) is generally satisfied for the common types of signals met in practice. The main steps of the algorithm to perform for the computation of the functional Λ are described below. Note that this algorithm requires knowledge of

a. the span of time T available for observation, i.e. the number of discrete samples;

b. the unfiltered noise variance $\beta : [0,T] \to I\!R_+$;

c. the causal filter $F : [0,T] \times [0,T] \to I\!R$;

d. the signal $s : \Omega \times [0,T] \to I\!R$. [21]

The received signal is assumed to be a continuous waveform $f(t)$ such that

$$\int_0^T f^2(x)dx < \infty.$$

The algorithm consists of the following steps:

Step 1. Compute the noise covariance

$$C_N(t,\tau) = \int_0^{t\wedge\tau} F(t,x)F(\tau,x)\beta(dx).$$

Step 2. Compute the eigenvalues λ_i, $1 \leq m$, and the orthonormal eigenvectors e_i, $1 \leq m$, (m can be finite or infinite) of the covariance operator associated with $C_N(t,\tau)$.

Step 3. Approximate the inversion process $M(f,t)$ by

$$M_n(f,t) = \sum_{i=1}^{n} M^{(i)}(f,t)$$

[21] In some applications the general parameters given by the type of modulation are known and the specific information may be estimated in parallel with the detection.

where $n \leq m$,

$$M^{(i)}(f,t) = \frac{1}{\lambda_i}\langle UI_{[0,t]}, e_i\rangle_{L_2[0,1]}\langle f, e_i\rangle_{L_2[0,1]}$$

and

$$UI_{[0,t]}(\tau) = \int_0^{t\wedge\tau} F(\tau,x)dx.$$

Step 4. Check that

$$E\left[\exp\left\{\frac{1}{2}\int_0^T s^2(\cdot,x)\beta(dx)\right\}\right]$$

is finite.

Step 5. If the answer at the previous step is positive then compute the functional $\tilde{\Lambda}$ giving the likelihood ratio for the unfiltered processes

$$\begin{aligned}\left[\tilde{\Lambda}(f)\right] &= \exp\left\{\int_0^T s(f,x)\, ev(f,dx)\right.\\ &\quad -\frac{1}{4}\int_0^T s^2(f,x)\,\beta(dx)\\ &\quad \left.-\frac{1}{\sqrt{2}}\int_0^T s(f,x)\left(B_2(f,dx)-\beta(dx)\right)\right\}.\end{aligned}$$

Numerical implementation of this algorithm has to be performed in conjunction with solutions for the operations (C) and (D).

6.2 CONNECTION BETWEEN THE THEORY OF LIKELIHOOD RATIO DETECTION ON FILTERED GAUSSIAN PLUS POISSON NOISE AND APPLICATIONS FROM SONAR

The present model was developed as an approach to the requirements met in active sonar. The active sonar is a bistatic system: the source and the receiver are located at separate points[22](see fig. 6.1). Basically, active sonar works as follows: the source injects an acoustic signal into the underwater channel with the objective of detecting the existence of a target by observing the signal at the receiver. Independent of the target's presence, the injected signal is distorted by the underwater channel. This consists of surface, bottom and volume scatterers. The velocities of the surface and volume scatterers are assumed to be random variables, as are the amplitude of their returns. The distribution of volume scatterers is assumed to be inhomogeneous. Volume scattering is produced by thermal layers, biological sources and suspended particles. As a result of these scatterers, the channel produces spreading in time, frequency and angle. The Doppler effect is present because of the relative motions among the source, the target and the medium [32]. The large values of the time delay spread give rise to a *frequency selective fading channel.* In addition to the fading, the signal is distorted by the echoes due to returns from surface, volume and bottom scatterers. These form an additional component of the noise, called *reverberation noise.* In fig. 6.1 the fading effect, the reverberation and the background noise at the receiver are represented. The transmitted pulse is spread by the channel and may undergo other changes as a result of the interaction with a contact, an object which represents in fact the potential target to be detected. The reverberation and the fading are coexisting phenomena.

The active sonar system is said to be a *reverberation limited environment* because the reverberation component dominates the background noise. Since the background noise exists equally in the presence or absence of the target, the detection model does not consider it.

As a consequence of the random fluctuations in the submarine environment described above, the signal observed at the receiver may be modeled as an oscillation process defined as

$$\xi(t) = \sum_k \gamma_k e^{iu_k t} \qquad (7)$$

where γ_k are random variables. Hence, this is the superposition of oscillations with frequency $\frac{u_k}{2\pi}$. The parameters of the transmitted signal may be components of γ_k and

[22]In a monostatic system, as is the case for passive sonar, the transmitter and the receiver share the same sensors.

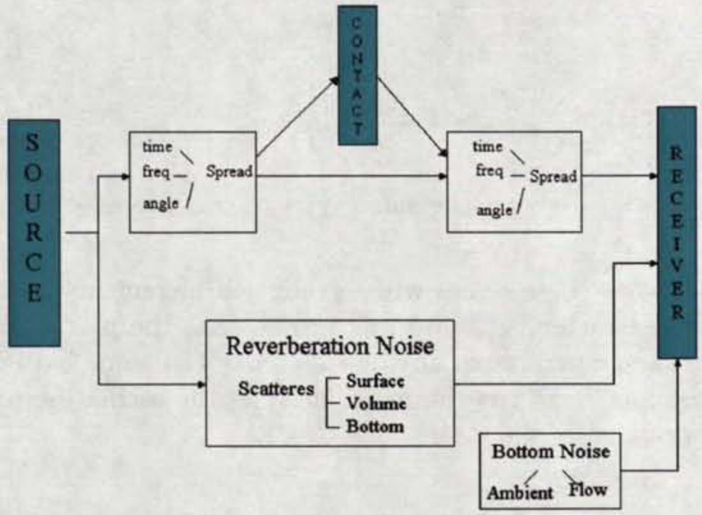

Figure 6.1: Active sonar diagram: reverberation limited environment when reverberation noise surpasses the bottom noise.

u_k.

For underwater propagation of the acoustic wave the scatterers are not homogeneous, rather they are close enough together to interact [33]. This fact, associated with the reverberation aspect, leads to the assumption that in relation (7) giving the oscillation process, the random variables $(\gamma_k)_k$ are correlated, i.e.

$$E(\gamma_k \gamma_j) = g_{kj} < \infty.$$

Then $\xi(t)$ is not a stationary process and cannot be studied by means of the linear theory of random processes, as Fourier transforms of an orthogonal stochastic measure [34]. $\xi(t)$ is then a particular case of an *harmonizable process* [35][36]. As a second order process, the signal observed at the receiver and modeled by means of the oscillation process $\xi(t)$ can be represented by means of the Cramér-Hida decomposition [10]. That means that it can be seen as a superposition of stochastic integrals with respect to stochastic processes with orthogonal increments $B_k(t), 1 \leq k \leq K$ [23] in the

[23] K is called the *Cramér-Hida multiplicity* of $\xi(t)$

form given by

$$\xi(t) = \sum_{k=1}^{K} \int_0^t F_k(t,x) dB_k(x) \tag{8}$$

where $F_k(t,s)$ are applications satisfying the same type of conditions as $F(t,s)$ in Theorem 2.

A particular class of stochastic processes with orthogonal increments is the set of processes with independent increments. Itô [37] proved that the processes with independent increments are generated essentially by the sum of Gaussian and Poisson processes. Hence, if we assume $K = 1$ is the multiplicity of the oscillation process $\xi(t)$ then $\xi(t)$ may be represented in the form

$$\xi(t) = \int_0^t F(t,x) dB(x) \tag{9}$$

where the right side of the previous relation is exactly the process $N(t)$ considered in this report. This is a kind of non-message bearing or "non-intelligent" noise.

If a target is present on the channel then one of the fluctuations modeled by the oscillation process has a particular behaviour. It is smoother than the other oscillations: it has an "intelligent"[38] character. Then the signal observed at the receiver has one component outstanding in the oscillation process model. This component is modeled by a stochastic process $s(t)$ which includes the information carried by the target, in the form

$$\xi(t) = \int_0^t F(t,x) \left[s(x)\beta(dx) + dB(x) \right]. \tag{10}$$

The same factor $F(t,x)$ multiplies both the noise and the "signal" $s(t)$ as a consequence of the fact that the injected signal is their common root.

Hence, the detection problem consists of determining for a given observed signal at the receiver which one of the relation (9) or (10) applies. As emphasized in the Introduction, the Neyman-Pearson criterion is suitable for sonar detection because of its optimality (the probability of detection is maximized for a fixed probability of false alarm). Hence, the likelihood approach proposed in this report is relevant for underwater detection problems.

Now $\xi(t)$ can replace $X(t)$ in the hypotheses test (1) in the Introduction and the theory presented here can be used to derive an appropriate detector.

The channel modeling used for sonar applications does not differ in essence from that used in mobile communications [39] because whenever a narrowband signal is received from a scattering medium a fading phenomenon occurs. In addition to the distortion produced by fading, signals on a wireless channel may be affected by interference, a phenomenon for which the uncorrelation assumption is not appropriate. This situation can be modelled, as for the case of the reverberation phenomenon, by the model proposed here.

6.3 MAIN CONTRIBUTIONS

As the title emphasizes, this document is a theoretical development of likelihood ratio detection. From mathematical point of view the new part consists of tailoring stochastic calculus for second order processes as they arise from the Cramér-Hida decomposition, when a jump process component is present. Continuing the ideas of [12],[8],[13],[7], instead of working on an abstract underlying probability space the problem here is modeled directly on the space of simple paths. This aspect brings valuable results but at the same time involves restrictions which were avoided with the expense of a significant amount of technicalities.

The main result of the calculus developed here is the derivation of explicit formulae for the likelihood ratios for filtered signals and noise, expressed by relations (2) and (3) in Theorem 2. The usefulness of this is as follows:

a. The effect of the communication channel is modeled by the Cramér-Hida framework, as a causal transformation corresponding to the time variant systems arising in real applications.

b. The new feature of the model is the impulsive noise component, represented by a filtered Poisson process. This fits some types of "non-intelligent" noise [38] arising in communication systems as interference which is incoherent relative to the transmitted signal.

c. The "noise" may not be statistically independent from the transmitted signal. Thus the model can be used to describe phenomena such as reverberation or interference.

d. The mathematical derivation leads to a likelihood ratio formula with no dependence on the noise paths. The detector based on $\frac{dP_{S+N}}{dP_N}$ is a functional on the received path only. This point is crucial for the applicability of the theory.

Some auxiliary results obtained here also deserve attention:

a. The inversion process M, defined in section 5.1, enables the effect of the channel to be removed, in a statistical sense. Its original construction comes from [12].

b. When the likelihood ratio does not exist, an approximation is provided in Theorem 2 d.

c. Proposition 14 and 15 may be applied for an inverse problem: when the likelihood ratio is known, they yield a method for extracting the transmitted signal from noise.

d. The likelihood ratio computation may serve to solve further estimation problems by Bayesian or maximum likelihood methods, as described in [40].

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