# STUDY INTO IMPROVED SPECTRUM EFFICIENCY FOR FDMA/TDMA TRANSMISSION IN MOBILE SATELLITE AND MOBILE ENVIRONMENTS 

Contract Final Report<br>Contract U6800-6-3505<br>March 1997

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# STUDY INTO IMPROVED SPECTRUM EFFICIENCY FOR FDMATTDMA TRANSMISSION IN MOBILE SATELLITE AND MOBILE ENVIRONMENTS: CONTRACT OVERVIEW 

W F McGee<br>Contract U6800-6-3505<br>19 March 1997

The purpose of this note is to provide a short overview of the contract.
The overall project was intended to examine the feasibility of spectrally efficient communication using VSB communication. There were three basic goals:

## 1. Provide support for design of the downlink path.

Four filter designs have been supplied, one based on half-band filter designs, a minimum/maximum phase design, one on Lawton decomposition, and a linear pfiasé design. Assistance in MATLAB programming was also provided.

Reports include
1.1 Study Into Improved Spectrum Efficiency For FDMA/TDMA Transmission In Mobile And Mobile Satellite Environments: 1a. Filter Designs For Polyphase Network
1.2. Study Into Improved Spectrum Efficiency For FDMA/TDMA Transmission In Mobile And Mobile Satellite Environments:1b. Polyphase Realization Of VSB Filter Banks Using Spectral Factorization Into Symmetrical Complex Functions: Lawton's Factorization
1.3. Phasing Filterbanks
1.4. Prototype Filters for VSB Filter Banks Derived from Half-Band Filters.
$\cdot 1.5$. Estimating Performance Degradation Of Phase And Timing Error•On VSB Filterbanks
1.6. Phasing VSB Filter Banks

There have been several CRC publications based on this assistance.'These may be obtained from the first author.

M Sablatash, W F McGee and J Lodge Bandwidth-On-Demand Multiple Access Communications System IDesign Combining Wavelet IPacket Trees and DFT Polyphase Filter Banks, NJIT March 1997. I
M Sablatash, W F McGee and J Lodgé, Transmitter and Receiver Filter IBank Designs for Bandwidth on Demand Multiple Access Communications based on Combining Wavelet Packet Filter Bank Trees and DFT Polyphase Synthesis and Analysis Filter Banks, CRC Report VPCS \#33/96
2. Design uplink path. This work is ongoing.

The basic goal has been to determine system options that will tolerate adjacent channel operation that is improperly phased. and timed. If properly phased and timed, performance would be as in the downlink direction.

A basic summary of the problem has been reported.

### 2.1 Communication With Unsynchronized FDM Transmitters Using Maximally Decimated Filter Banks

First, a variety of filter designs were evaluated, both by examining the pulse response, and estimating the error rate. The results may be summarized as follows. There is little crosstalk between channels that are not adjacent. The crosstalk between adjacent channels is that of a pulse through a narrowband filter, hence long, and with fixed area. The delay of the adjacent channel interference is determined by whether or not the crosstalk has gone through a minimum or a maximum phase network. Only a linear phase design provided crosstalk that was uniform from channel to channel; oddly enough, the Lawton design, which has symmetrical but complex pulse responses, also exhibited this lead/lag phenomenon.
2.2 Study Into Improved Spectrum Efficiency For FDMA/TDMA Transmission In Mobile Satellite And Mobile Environments: 3a. Basic. Properties Of VSB Filterbanks
2.3 Study Into Improved Spectrum Efficiency For FDMA/TDMA Transmission In Mobile Satellite And Mobile Environments: 3b. Zero Forcing, Minimum MSE And Decision Feedback Multi-User Receivers
2.4 Study Into Improved Spectrum Efficiency For FDMÁTDMA Transmission In Mobile Satellite And Mobile Environments: 3c. Effect Of Linear Phase Prototypes On Filterbank Crosstalk'

An approximation to error rate based on steepest descent has been developed.

### 2.5 Error Rate Approximation

A variety of equalizers has been evaluated. The minimum mean squared error equalizer minimizes the perturbations to the signal due to noise, intersymbol interference in the same channel, and adjacent channel interference. The decision feedback equalizer uses previously received data in the same channel to subtract tails of pulses. The vector decision feedback equalizers also uses the information about the data received in the adjacent channels. The design and evaluation of the vector feedback equalizer required a program to perform the Bauer factorization of positive definite Toeplitz matrices.

### 2.6 Program Bauer: Spectral Factorization Of Matrix Polynomials

"Since operation with a lack of phasing would allow QAM triansmisision, the performance of QAM systems was made. The conclusion of a partial study is that QAM performance is marginally superior, but the optimum equalizers to combat adjacent ehannel interference require broadband equalizers, rather that the equalizers for VSB which span a channel and its two adjacent channels only.

```
\because:
```

2.7 Study Into Improved Spectrum Efficenency For FDMA/TDMA Transmission In Mobile Satellite And Mobile Environments:3d. Zero Forcing, Ḿinimum MSE And Decision Feedback Multi-User QAM Receivers

Since spectrum coding puts nulls in the power spectrum, the theory of spectral coding for this application has been developed. The idea Here is that the adjacent channel interference in VSB systems considered may, with the worst data sequence, completely close the data eye, if at an equal level. The spectral coding should result in a worst case interference equal to twice the peak of the crossstalk pulse, not the area, and this peak may be reduced by reducing the excess bandwidth of the transition band between channels.
2.8 Study Into Improved Spectrum Efficiency For FDMA/TDMA Transmission In Mobile Satellite And Mobile Environments: 3e. Minimum MSE And Decision Feedback Multi-User Receivers Using Spectrum Control

The modifications to error rate calculations have been detailed.

### 2.9 Modified Duobinary Error Rate Calculation

The previous studies amount, in the theory of multi user communication, to implementation of meansquared error decorrelating receivers. The next phase of the work is to investigate in detail iterative decoding strategies. These involve selecting the received channel with the largest signal level, decoding it, and using this information to reduce the interference into the adjacent channels, and to reiterate this process. This will occur in conjunction with error control coding which may be used in an iterative decoding strategy to feed back decoding decisions that render certain transmitted data patterns to be more likely.

As preliminary work, we examined relevant work on Intersymbol interference, and summarized it.

### 2.10 Review Of Intersymbol Interference Mitigation

A detailed examination of the calculations used in manipulating matrices for MLSE has been summarized.

### 2.11 Maximum Likelihood Intersymbol Interference Receivers

A review of CDMA multi-user detectors has been prepared.
2.12 Review Of Multi-User CDMA

As preliminary work on this aspect, we examined the factorization of correlation matrices, and prepared a comprehensive summary.

### 2.13 Correlation Matrices And Sequences: A Survey

We attempted to summarize Markov detection, because of background reading of iterative decoding of product codes.
2.14 Markov Detection

## 3. Consider filterbanks for spectrum ́́monitoring.

This is unrelated to the previous two projects. The short study resulted in a report and a presentation of the recommendations on 18 March 1997. The use of polyphase FIR Nyquist or root-Nyquist filters was argued.
3.1 Polyphase Filters For Communicationis EW Systems

## , 4. Miscellaneous

Some sideroads were explored.
4.1 An Vestigially Analytic Wavelet
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# STUDY INTO IMPROVED SPECTRUM EFFICIENCY FOR FDMA/TDMA TRANSMISSION IN MOBILE AND MOBILE SATELLITE ENVIRONMENTS: 

## 1a. FILTER DESIGNS FOR POLYPHASE NETWORK

W F McGee<br>Contract U6800-6-1604

19/03/97 11:53 AM


#### Abstract

This report and an associated report on Lawton decomposition are submitted towards milestone 1 of the contract.

Three filters have been designed. They meet the same basic requirements. All are 8 -band root-Nyquist filters with 40 dB attenuation in the stopband, and with 25 percent excess bandwidth. One filter, F88_8, with 88 coefficients, is a minimum phase filter with equiripple stopband, one filter is the same filter but in the Lawton decomposition, and one filter, G6, is linear phase and has 96 coefficients. We have not been able to determine the equiripple linear phase polynomial meeting these requirements.


## DESIGN TECHNIQUE

## Equiripple stopband filter

The equiripple stopband filters that work with the Nyquist filter itself are the easiest to design and we use a variation of Samueli's method[1]. This consists of finding a Remez equiripple approximation that uses the points 0 (not -e!) and eas the extremum values. One the Remez approximation is complete, the zeros of the resulting polynomial in $\mathrm{w}=\mathrm{z}+\mathrm{z}^{-1}$ are found. The zeros on the unit circle are, by design, double, and known. Thus, only a very few zeros are left to be determined. The polynomłals for the root-Nyquist filters are found by multiplying factors corresponding to each zero. For the minimum phase polynomial, the zeros are those inside the unit circle and one of each pair on the unit circle. For the Lawton decomposition, the zeros are one of each pair on the unit circle, and those zeros that are inside the unit circle in the upper half plane and outside the unit circle in the lower half plane. As in our previous designs, the only critical point in the design is to randomize the order of the zegro factors when forming the product polynomials. We have found that using the DOS SORT command and sorting on the fourth digit is sufficient. $\because$.

The linear phase filter G6 is designed using a variant of Jain and Crochiere's method[2]. The stopband energy is minimized, subject to a linearized set of equations imposing the Nyquist criterion. Since it is the coefficients of the polynomial itself used in the design, there is no need for subsequent zero-finding or spectral factorization.

## REFERENCES

1. H. Samueli, 'On the Design of Optimal Equiripple FIR Digital Filters for Data Transmission Applications', IEEE Trans Circuits and Systems, Vol. 35, No. 12, Dec. 1988.
2. V. K. Jain and R. Crochiere, 'Quadrature Mirror Filter Design in the Time Domain', IEEE Trans Acoustics, Speech and Signal Processing', Vol. ASSP-32, April 1984, pp. 353-361.
$1.0000000000000 \mathrm{E}+0000$ $1.61823816025980 \mathrm{E}+0000$ $2.75725070143926 \mathrm{E}+0000$ $4.23765368648018 \mathrm{E}+0000$ $6.02474781466944 \mathrm{E}+0000$ $8.03622653380794 \mathrm{E}+0000$ $1.01428365007125 \mathrm{E}+0001$ $1.21761320164196 \mathrm{E}+0001$ $1.39432700075629 \mathrm{E}+0001$ $1.52476417442110 \mathrm{E}+0001$ $1.59130444845834 \mathrm{E}+0001$ $1.58082544813400 \mathrm{E}+0001$ $1.48684414183357 \mathrm{E}+0001$ $1.31099754404797 \mathrm{E}+0001$ $1.06358518429227 \mathrm{E}+0001$ $7.63013198882190 \mathrm{E}+0000$ $4.34131936534654 \mathrm{E}+0000$ $1.05623472563835 \mathrm{E}+0000$ $-1.93253381418263 \mathrm{E}+0000$ $-4.36140256349403 \mathrm{E}+0000$ $-6.02804212276844 \mathrm{E}+0000$ $-6.81652319922357 \mathrm{E}+0000$ $-6.71164600815125 \mathrm{E}+0000$ $-5.80020887682843 \mathrm{E}+0000$ $-4.25882636501843 \mathrm{E}+0000$ $-2.32986737604750 \mathrm{E}+0000$ $-2.88843174734262 \mathrm{E}-0001$ $1.59214682897403 \mathrm{E}+0000$ $3.07971538706617 \mathrm{E}+0000$ $4.00823871114362 \mathrm{E}+0000$ $4.29862714002227 \mathrm{E}+0000$ $3.96435913780564 \mathrm{E}+0000$ $3.10426053637859 \mathrm{E}+0000$ 1.88350315778043E+0000 $5.06027995667962 \mathrm{E}-0001$ -8.17202502692844E-0001 -1.89923186421550E+0000 $-2.60281403997851 \mathrm{E}+0000$ $-2.85719089500247 \mathrm{E}+0000$ $-2.66386822722237 E+0000$ $-2.09111423072150 \mathrm{E}+0000$ $-1.25867303383170 \mathrm{E}+0000$ -3.15640287789661E-0001 $5.84635608734211 \mathrm{E}-0001$ $1.30851063902361 \mathrm{E}+0000$ $1.76116787798889 \mathrm{E}+0000$ $1.89821058627549 \mathrm{E}+0000$ $1.72840974004139 \mathrm{E}+0000$ $1.30780331979459 \mathrm{E}+0000$ $7.26733828917106 \mathrm{E}-0001$ $9.24513708599137 \mathrm{E}-0002$ $-4.89551272644754 \mathrm{E}-0001$ -9.32327684756014E-0001 $-1.17979227504980 \mathrm{E}+0000$ $-1.21290323238222 \mathrm{E}+0000$ $-1.04920913277061 \mathrm{E}+0000$ $-7.36406041998196 E-0001$ $-3.41506225688045 \mathrm{E}-0001$

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Filter G6
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-4. $37880235656367 \mathrm{E}-0003$
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The Lawton decomposition filter is defined in another report.
FIGURES
2. Minimum phase filter pulse response.
3. Nyquist filter time response.
4. Frequency response of equiripple filter F88_8.
5. Frequency response of linear phase filter G6.

Pulse response of root-Nyquist filter


Nyquist filter response


Frequency response of filter



# STUDY INTO IMPROVED SPECTRUM EFFICIENCY FOR FDMA/TDMA TRANSMISSION IN MOBLE AND MOBILE SATELLITE ENVIRONMENTS: 

# 1b. POLYPHASE REALIZATION OF VSB FILTER BANKS <br> USING SPECTRAL FACTORIZATION INTO SYMMETRICAL COMPLEX FUNCTIONS: LAWTON'S FACTORIZATION 

W F McGee<br>Contract U6800-6-1604

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#### Abstract

The design of filters for pulse transmission is most efficiently done using spectral decomposition. Ordinarily this leads to a minimum/maximum-phase real-coefficient polynomial decomposition. Both must be used as prototypes in a transmit filterbank for optimal VSB communication. Lawton has given a factorization into two complex symmetrical functions. This implies that the same prototype filter may be used in the transmit filterbank. This leads to the following features -relatively easy to design for equiripple or weighted response. -more efficient realization than min/max designs. -symmetrical responses.


## Introduction

Linear phase prototype filters are popular in digital signal processing using filterbanks. One of the reasons is that the transmit filters and the receive filters are based on one prototype filter, and the filters themselves may be realized with the Bellanger Polyphase realization using the FFT. But the length of the filters is longer than needed to meet stopband filtering requirements; as a rough guide the length of the linear phase filters is about 25 percent more than of a minimum/maximum phase factorization of the Nyquist filter. Unfortunately, the minimum/maximum phase decomposition requires that both prototype filters be used in both the transmit filters and the receiver filters: The minimum phase factor is used for half the filters and the maximum phase for the other half. This, in turn, means that the number of filter coefficient multiplications is twice that required if the filters were the same. It is true that there are savings in the DFT calculation, but for long filters, almost all the time is taken in multiplications with filter coefficients[3].

Recently Lawton[1] presented another factorization of a Nyquist polynomial that results in symmetrical filters, albeit with complex coefficients. Thus the number of coefficients would be the same as that of a minimum/maximum phase decomposition. And; it turns out, only one filter type need be used for the transmitter and one for the receiver; the Bellanger Polyphase realization is allowable, and the number of filter multiplications is thereby' halved over that of the minimum/maximum phase realizations.

Therefore, one may conclude that the use of the Lawton decomposition results in a considerable computational saving, of the order of 25 percent, over the use of linear phase prototype filters, and 50 percent over the use of minimum/maximum phase decomposition.

This report presents the relevant theory. An appendix gives an example of a Lawton root-Nyquist filter for an 8 -band filterbank. Another Appendix contains a set of simple MATLAB programs that we used to verify the theory. Filter responses for this 8 -band, 25 -percent excess bandwidth, $40-\mathrm{dB}$ stopband loss design are given.

An M-band VSB filter bank may be based on a prototype Nyquist Filter N(z) which is factored into two root-Nyquist filters using the relationship

$$
N(z)=H(z) G^{*}(z)
$$

The polynomial $\mathrm{G}^{*}(\mathrm{z})$ has coefficients that are the complex conjugate of those of $\mathrm{G}(\mathrm{z})$. Lawton provided a factorization in which G and H are complex, but with the following properties

$$
H(z)=G(z)=H(1 / z)
$$

These filters are all non-causal, they are made causal as follows

$$
P(z)=z^{-(L-1) / 2} H(z)
$$

where $P(z)$ is causal and $L-1$ is the order of the polynomial $P(z)$ which has $L$ coefficients, commonly called the length of the filter in the DSP literature.

A polynomial is specified by its zeros and an arbitrary constant which we take to be real, and, for this discussion, equal to unity. Thus if the zeros of $P(z)$ are $z_{i}$,

$$
P(z)=\prod_{i=1}^{L-1}\left(1-z^{-1} z_{i}\right)
$$

A polynomial $\mathrm{P}(\mathrm{z})$ will be complex symmetrical if, when $1 / \mathrm{z}_{\mathrm{i}}$ is distinct from zi , it is also a zero. $\mathrm{P}(\mathrm{z})$ may have an arbitrary number of zeros at $\mathrm{z}=-1$; since it is to be a lowpass filter it will not have zeros at $\mathrm{z}=1$.

The theory of VSB filterbanks using Lawton polynomials is as follows. The M transmit filters have M real input sequences at a rate $2 / \mathrm{M}$ applied to them. The filter outputs are summed, and this signal is then applied .to a set of $M$ receive filters, the real outputs of which as sampled at the rate $2 / \mathrm{M}$. The transmit filters and receive filters are frequency-shifted versions of a prototype filter $\dot{\mathrm{P}}(\mathrm{z})$ and $\mathrm{P}^{*}(z)$, with appropriate phasing; the frequency shifts are multiples of $1 / M_{3}$.
The overall system is linear, and, neglecting the output sampling, time invariant. Thus the pulse response contains all system information. Since the filterbanks in question have selective filters, it is only necessary to consider, for a given transmitter input;, the response at three receive filter outputs; that of the receiver at the same frequency, and the two adjacent. .

In a well designed system the real part of the pulse response at the same frequency will be nonzero at the sampling instant, and will be zero at all times that are displaced from it by multiples of $\mathrm{M} / 2$, the inverse of the sample rate. With the filter's under question, the sampling instant is L-1 sample times after the pulse is applied at the input. Similarly, in the adjacent channels, the real part of the pulse response in those channels should vanish at the main sample time and at samples displaced from it by multiples of $M / 2$.

In what follows we will assume that the length $L$ is a multiple of $M$; this makes the polyphase filters all be the same length. This assumption may be removed, but does not appear to lead to great system savings, unless low delay is desired.

We claim that if the transmit filters $T_{k}(z)$ and the receive filters $R_{k}(z)$ are represented by

$$
\begin{aligned}
& T_{k}(z)=e^{j \phi_{k}} W^{(k+1 / 2)(L-1) / 2} P\left(W^{k+1 / 2} z\right) \\
& R_{k}(z)=e^{-j \phi_{k}} W^{(k+1 / 2)(L-1) / 2} P *\left(W^{k+1 / 2} z\right)
\end{aligned}
$$

where

$$
\phi_{k+1}-\phi_{k}=\text { odd multiple of } \pi / 2
$$

a well-designed system will result.
Consider transmission to the receiver mate. The response is $T_{k}(z) R_{k}(z)$ and is given by

$$
T_{k}(z) R_{k}(z)=W^{(k+1 / 2)(L-1)} P\left(W^{k+1 / 2} z\right) P *\left(W^{k+1 / 2} z\right)
$$

But, by design, $\mathrm{P}(\mathrm{z}) \mathrm{P}^{*}(\mathrm{z})$ is a Nyquist polynomial with real coefficients $\mathrm{n}_{\mathrm{k}}$ and these coefficients vanish at samples $L-1+s M$, where $s$ is not zero. Consequently the coefficient of $\mathrm{z}^{-(\mathrm{L}-1+\mathrm{rM} / 2)}$ is

$$
n_{L-1+r M / 2} W^{-(k+1 / 2)(L-1+r M / 2)} W^{(k+1 / 2)(L-1)}=n_{L-1+r M / 2} W^{-(k+1 / 2)(r M / 2)}=n_{L-1+r M / 2} e^{j 2 \pi(k+1 / 2)(r M / 2 M)}
$$

which vanishes when $r$ is even and nonzero by Nyquist design, and is imaginary when $r$ is odd; thus there in no intrachannel intersymbol interference.

For an adjacent channel the response is, typically, the real part of $T_{k+1}(z) R_{k}(z)$. This is given by

$$
T_{k+1}(z) R_{k}(z)=e^{j\left(\phi_{k+1}-\phi_{k}\right)} W^{(k+1)(L-1)} P\left(W^{k+1+1 / 2} z\right) P^{*}\left(W^{k+1 / 2} z\right)
$$

If the zeros of $\mathrm{P}\left(\mathrm{W}^{k+3 / 2} \mathrm{z}\right)$ and $\mathrm{P}^{*}\left(\mathrm{~W}^{\mathrm{k}+1 / 2} \mathrm{z}\right)$ are examined, it will be détermined that they are symmetrical about a line half-way between the two center frequencies, and thus may be represented by $C\left(W^{k+1} z\right)$ where $C(z)$ has real coefficients $c_{k}$. The response at samples that are displaced by $\mathrm{rM} / 2$ from the center sampling point L-1, $r$ an integer, is thus

$$
\begin{aligned}
& c_{L-1+r M / 2} e^{j\left(\phi_{k+1}^{\left.-\phi_{k}\right)}\right.} W^{(k+1)(L-1)} W^{-(k+1)(L-1+r M / 2)} \\
& =c_{L-1+r M / 2} e^{j\left(\phi_{k+1}-\phi_{k}\right)} W^{-(k+1)(r M / 2)}= \pm c_{L-1+r M / 2} e^{j\left(\phi_{k+1} \cdot \dot{\phi}_{k}\right)}
\end{aligned}
$$

and the real part vanishes if the phase didférences are an odd multiple of $\pi / 2$.

All that remains of the design procedure is to write down the expressions for the transmitter and receiver, introduce the polyphase components, and identify the components.

Recall that

$$
\begin{gathered}
T_{k}(z)=e^{j \phi_{k}} W^{(k+1 / 2)(L-1) / 2} P\left(W^{k+1 / 2} z\right) \\
R_{k}(z)=e^{-j \phi_{k}} W^{(k+1 / 2)(L-1) / 2} P *\left(W^{k+1 / 2} z\right) \\
\phi_{k+1}-\phi_{k}=\text { odd multiple of } \pi / 2
\end{gathered}
$$

First, however, we will make some assumptions about filter lengths to simplify the discussion; these assumptions are not hard to remove. The filter length $L$ is usually a multiple of $M$. Secondly, the phase shifts are often chosen[2] as

$$
\phi_{k}=(k+1 / 2)(2(L / M)+1) \pi / 2
$$

Thus,

$$
\begin{aligned}
& T_{k}(z)=W^{-(k+1 / 2)(M / 2+1) / 2} P\left(W^{k+1 / 2} z\right) \\
& R_{k}(z)=W^{(k+1 / 2)(M / 2-1) / 2} P *\left(W^{k+1 / 2} z\right)
\end{aligned}
$$

The claimed efficiencies follow because when the phase components of $P\left(W^{1 / 2} z\right)$ are introduced, both the transmitter and receiver only involve the multiplication, of only one of the prototype filter coefficients for every input set of data.

## MORE EFFICIENT REALIZATIONS

There are two cases, one involving the DFT with elements $\mathrm{e}^{\mathrm{j} 2 \pi \mathrm{kr} / \mathrm{M}}$, and the second involving the odd-time odd-frequency DFT with elements $\mathrm{e}^{\mathrm{j} 2 \pi(\mathrm{k}+1 / 2)(\mathrm{r}+1 / 2) \mathrm{M}}$. For the DFT , probably the simplest approach is to perform a polyphase expansion on $\mathrm{A}(\mathrm{z})=\mathrm{W}^{(\mathrm{M} / 2+1) / 4} \mathrm{P}\left(\mathrm{W}^{1 / 2} \mathrm{z}\right)$,

$$
A(z)=\sum_{r=0}^{M-1} z^{-k} A_{r}\left(z^{M}\right)
$$

Then the output from the transmitter is

$$
\therefore \sum_{r=0}^{M-1} z^{-r} A_{r}\left(z^{M}\right) \sum_{k=0}^{M-1} W^{-r k} W^{-k(M / 2+1) / 2} X_{k}\left(z^{M / 2}\right)
$$

These expression may be simplified to

$$
\sum_{r=0}^{M-1} z^{-r} A_{r}\left(z^{M}\right) \sum_{k=0}^{M-1} W^{-(r+(M / 2+1) / 2) k} X_{k}\left(z^{M / 2}\right)
$$

If the integer $\mathrm{M} / 2+1$ is an even integer (case DFT ) then may be accomplished by taking the DFT of the inputs, choosing the $\mathrm{r}+(\mathrm{M} / 2+1) / 2$ as inputto the $r$ polyphase filter, and selecting the output that is desired. Otherwise, is necessary to multiply the Minputs by a phasing factor; we call this case OODFT for odd-time odd-frequency discrete Fourier transform.

Using the DFT leads to the following diagram.


Figure 1 Transmitter. $\dot{M}$ real input sequences arè applied at rate $2 / \mathrm{M}$ to an IFFT transformer, resulting in $M$ complex sequences at rate $2 / \mathrm{M}$. These are then filtered by the M polyphase filters, and the output is obtained by summing the filter outputs properly. The sampling operation may be move to just before the delay lines. If $M / 2$ is even, then a further phasing is necessary at the input.

For case DFT (M/2 odd) the inputs are, in fact, real sequences, the DFT may be done as a DFT of order $\mathrm{M} / 2$ on the complex signals $\mathrm{X}_{2 \mathrm{i}}+\mathrm{j} \mathrm{X}_{2 i+1}$. Also the output chain of delays may be expressed as a chain of length $\mathrm{M} / 2$ by incorporating delays $\mathrm{z}^{-\mathrm{M} / 2}$ in the structure itself.

The receiver takes the real part of the output of the filters when the input signal is applied. This is

$$
R_{k}(z)=W^{(k+1 / 2)(M / 2-1) / 2} P *\left(W_{,}^{k+1 / 2} z\right)
$$

Probably the simplest approach is to take the polyphase expansion of $\mathrm{B}(\mathrm{z})=\mathrm{W}^{(\mathrm{M} / 2-1) / 2} \mathrm{P} *\left(\mathrm{~W}^{1 / 2} \mathrm{z}\right)$,
so

$$
\therefore B(z)=\sum_{r=0}^{M-1} z^{-r} B_{r}\left(z^{M}\right)
$$

$$
R_{k}(z)=\sum_{r=0}^{M-1} z^{-r} W^{-r k} W^{k(M / 2-1)} B_{r}\left(z^{M}\right)=\sum_{r=0}^{M-1} z^{-r} B_{r}\left(z^{M}\right) W^{k(M / 2-r-1)}
$$

leading to the following block diagram for the receiver.


Figure 2. Receiver. A complex input sequence is applied at the left. It passes through a delay line with M taps, is then fed through M polyphase filters with complex coefficients, and applied to the DFT indicated. The output consists of the real parts of the DFT sampled at rate $2 / \mathrm{M}$. The sampling operation may be moved back to the output of the delay lines. When $M / 2$ is an even integer, further phasing is required at the output.

The usual simplifications may be applied here as well. Since, for the .DFT case (M/2 odd) the outputs are real, the DFT may be replaced with a DFT of order M/2 and appropriate real and imaginary parts of the output taken, and the input delay chain of length $M$ may be replaced with one of length $M / 2$ by incorporating delays of length $\mathrm{M} / 2$ within the filtering structure.

## REDUCTION IN ORDER OF DFTS DUE TO REAL INPUTS AND OUTPUTS

The following material appears in elementary textbooks and we include it here for completeness.

- The replacement of the IDFT proceeds by noting that if a real input vector $\mathbf{x}$ with components $\mathrm{x}_{\mathrm{k}}$ is Fourier transformed to a vector U with components $\mathrm{U}_{\mathrm{k}}$, then $\mathrm{U}_{\mathrm{M}-\mathrm{k}}=\mathrm{U}^{*} \mathrm{k}$, and a real vector y corresponds to transform $\cdot V$, then $w=x+j y$ corresponds to $W$, then $W_{k}=X_{k}+j Y_{k}$ and $W_{M-k}{ }^{*}=X_{k}-j Y_{k}$, thereby
$2 \mathrm{X}_{\mathrm{k}}=\mathrm{W}_{\mathrm{k}}+\mathrm{W}_{\mathrm{M}-\mathrm{k}}^{*}$ and $\mathrm{j} 2 \mathrm{Y}_{\mathrm{k}}=\mathrm{W}_{\mathrm{k}}-\mathrm{W}^{*}{ }_{\mathrm{M}-\mathrm{k}}$.
On the other hand we are interested in the M -point transform of the M -vector which we write as the $\mathrm{M} / 2$ transform of the even and odd numbered components:

$$
\begin{aligned}
& Y_{k}=Y^{*}{ }_{M-k}=\sum_{r=0}^{M-1} W^{-r k} x_{r}=\sum_{r=0}^{M / 2-1} e^{j 2 \pi / k /(M / 2)} x_{2 r}+e^{j 2 \pi / / M} \sum_{r=0}^{M / 2-1} e^{j 2 \pi r k /(M / 2)}= \\
& {\left[W_{k}+W^{*}{ }_{M / 2-k}+e^{j 2 \pi k / M}\left(W_{k}-W^{*}{ }_{M / 2-k}\right)\right] / 2}
\end{aligned}
$$

where

$$
W_{k}=\sum_{r=0}^{M / 2-1}\left(x_{2 r}+j x_{2 r+1}\right) e^{j 2 \pi k /(M / 2)}=W_{M / 2+k}
$$

is the $M / 2$-point IDFT of the complex input sequence $\mathrm{x}_{2 \mathrm{r}}+\mathrm{j} \mathrm{x}_{2 r+1}$
On the other hand, if only the real output of a DFT is desired, then, since

$$
x_{r}+j y_{r}=\sum_{k=0}^{M-1} Y_{k} e^{-j 2 \pi \pi k / M}
$$

then

$$
\begin{aligned}
& 2 x_{2 r}=\sum_{k=0}^{M-1}\left(Y_{k} e^{-j 2 \pi 2 r k / M}+Y_{k} e^{j 2 \pi 2 r k / M}\right) \\
& =\sum_{k=0}^{M-1} e^{-j 2 \pi \pi k /(M / 2)}\left(Y_{k}+Y^{*}{ }_{M-k}\right) \\
& =\sum_{k=0}^{M / 2-1} e^{-j 2 \pi \pi k /(M / 2)}\left(Y_{k}+Y^{*}{ }_{M-k}+Y_{M / 2+k}+Y^{*}{ }_{M / 2-k}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& 2 j x_{2 r+1}=j \sum_{k=0}^{M-1}\left(Y_{k} e^{-j 2 \pi(2 r+1) k / M /}+Y_{k}^{*} e^{j 2 \pi(2 r+1) k / M}\right) \\
& =j \sum_{k=0}^{M-1} e^{-j 2 \pi k /(M / 2)} e^{-j 2 \pi k / M}\left(Y_{k}+Y^{*}{ }_{M-k}\right) \\
& =j \sum_{k=0}^{M / 2-1} e^{-j 2 \pi r k /(M / 2)} e^{-j 2 \pi k / M}\left(Y_{k}+Y_{M-k}^{*}-Y_{M / 2+k}-Y^{*}{ }_{M / 2-k}\right)
\end{aligned}
$$

so

$$
\begin{aligned}
& 2\left(x_{2 r}+j x_{2 r+1}\right) \\
& =\sum_{k=0}^{M / 2-1} e^{-j 2 \pi r k /(M / 2)}\left[\left(1+j e^{-j 2 \pi k / M}\right)\left(Y_{k}+Y^{*}{ }_{M-k}\right)+\left(1-j e^{-j 2 \pi k / M}\right)\left(Y_{M / 2+k}+Y_{M / 2-k}\right)\right]
\end{aligned}
$$

which is an M/2-point DFT.
Simplifying the transmitter using a delay line of length $M / 2$ is obtained by noting that the output is

$$
\begin{aligned}
& \sum_{r=0}^{M-1} z^{-r} A_{r}\left(z^{M}\right) V_{r}\left(z^{M / 2}\right) \text { where } \\
& V_{r}\left(z^{M / 2}\right)=\sum_{k=0}^{M-1} W^{-(r+M / 2+1) k} X_{k}\left(z^{M / 2}\right)
\end{aligned}
$$

and the output may be rewritten as .

$$
\begin{aligned}
& \sum_{r=0}^{M-1} z^{-r} A_{r}\left(z^{M}\right) V_{r}\left(z^{M / 2}\right) \\
& =\sum_{r=0}^{M / 2-1} z^{-r}\left[A_{r}\left(z^{M}\right) V_{r}\left(z^{M / 2}\right)+z^{-M / 2} \mathrm{~A}_{\mathrm{r}+\mathrm{M} / 2} \mathrm{~V}_{\mathrm{r}+\mathrm{M} / 2}\left(\mathrm{z}^{\mathrm{M} / 2}\right)\right]
\end{aligned}
$$

Using the M/2-point DFT appears to leads to a QAM system; a complex input sequence $z_{r}=\mathrm{x}_{2 \mathrm{r}}+\mathrm{j} \mathrm{x}_{2 \mathrm{r}+1}$ is applied at rate $2 / \mathrm{M}$ and recovered at the output. But ordinarily with a QAM system, the transmitted signal is also analytic; i.e., it is not a function of $z^{*}$ r. Practically, this means that if an input sinusoid is applied, that frequency and its image will also appear. The potential for the image frequency to occur is because the operation of conjugation is performed in the description above. This is important because many system functions depend on the complex signal being analytic; an example is the usual adaptive equalizer.

For example, the transmitted signal is

$$
\begin{aligned}
& X_{2 r}\left(z^{M / 2}\right) T_{2 r}(z)+X_{2 r+1}\left(z^{M / 2}\right) T_{2 r+1}(z) \\
& \frac{Z_{r}\left(z^{M / 2}\right)+Z_{r}^{*}\left(z^{M / 2}\right)}{2} T_{2 r}(z)+\frac{Z_{r}\left(z^{M / 2}\right)-Z_{r}^{*}\left(z^{M / 2}\right)}{2 j} T_{2 r+1}(z) \\
& =Z_{r}\left(z^{M / 2}\right) \frac{T_{2 r}(z)-j T_{2 r+1}(z)}{2}+Z_{r r}^{*}\left(z^{M / 2}\right) \frac{T_{2 r}(z)+j T_{2 r+1}(z)}{2}
\end{aligned}
$$

which shows explicitly that the image signal $Z^{*}\left(z^{M / 2}\right)$ is being transmitted.
Thus, I would not call this QAM:

## USE OF THE OODFT

When the quantity $\mathrm{M} / 2$ is even, I . e. M/4 is an integer, there are two approaches that may be made. We sketch the possibilities here; they have not been fully explored. One approach is to note that in this case the input sequences may be staggered, and the phase difference between adjacent channels is not required; all the $\varphi_{\mathrm{k}}$ may be zero. The second approach is to perform a polyphase expansion directly on $\mathrm{P}(\mathrm{z})$, and then the transmitted signal may be represented

$$
\begin{aligned}
& \sum_{k=0}^{M-1} X_{k}\left(z^{M / 2}\right) T_{k}(z)=\sum_{k=0}^{M-1} X_{k}\left(z^{M / 2}\right) W^{-(k+1 / 2)(M / 2+1) / 2} P\left(W^{k+1 / 2} z\right) \\
& =\sum_{r=0}^{M-1} z^{-r} P_{r}\left(-z^{M}\right) \sum_{k=0}^{M-1} W^{-r(k+1 / 2)} W^{-(k+1 / 2)(M / 2+1) / 2} X_{k}\left(z^{M / 2}\right) \\
& =\sum_{r=0}^{M-1} z^{-r} P_{r}\left(-z^{M}\right) \sum_{k=0}^{M-1} W^{-(k+1 / 2)\left(r+1 / 2+M^{\prime} / 4\right)} X_{k}\left(z^{M / 2}\right)
\end{aligned}
$$

This is seen as the odd-time odd-frequency DFT[4] of the input sequence, with the $\mathrm{r}+\mathrm{M} / 4$ th output applied to the filter $\mathrm{P}_{\mathrm{r}}\left(-z_{\mathrm{M}}\right)$ and applied, with a delay of r , to the output. We recall some properties of the OODFT considered as a matrix $O$. The matrix $O$ has elements

$$
O_{r, k}=W^{-(r+1 / 2)(k+1 / 2)}=O_{k, r}
$$

and satisfies

$$
\begin{aligned}
& O_{r+M, k}=W^{-(r+1 / 2)(k+1 / 2)-M(k+1 / 2)}=-W^{-(r+1 / 2)(k+1 / 2))}=-O_{r, k} \\
& O_{r+M / 2, k}=W^{-(r+1 / 2)(k+1 / 2)-M(k+1 / 2) / 2}=O_{r, k} e^{j 2 \pi(k+1 / 2) / 2}=(-1)^{k} j O_{r, k}
\end{aligned}
$$

$$
O_{r, s}^{2}=\sum_{k=0}^{M-1} W^{-(r+1 / 2)(k+1 / 2)} W^{-(k+1 / 2)(s+1 / 2)}=\sum_{k=0}^{M-1} W^{-(s+r+1)(k+1 / 2)}
$$

and the sum on $k$ is zero unless $r+s+1$ is equal to $M$ in which case the sum is $-M$, thus

$$
O^{2}=-M J
$$

whereas

$$
O O_{r, s}^{H}=\sum_{k=0}^{N-1} W^{-(r+1 / 2)(k+1 / 2)+(k+1 / 2)(s+1 / 2)}=\sum_{k=0}^{N-1} W^{-(r-s)(k+1 / 2)}
$$

thus $\mathrm{OO}^{\mathrm{H}}=\mathrm{MI}$, where I is the identity matrix. Thus $O / \sqrt{M}$ is unitary.

If $\Lambda$ is a diagonal matrix; then $\mathrm{O} \Lambda \mathrm{OH}$ is not quite circulant, since although
$O \Lambda O_{r, s}^{H}=\sum_{k=0}^{M-1} W^{-(r+1 / 2)(k+1 / 2)} \Lambda_{k, k} W^{(s+1 / 2)(k+1 / 2)}=\sum_{k=0}^{M-1} \Lambda_{k, k} W^{-(r-s)(k+1 / 2)}$.
depends on r-s, and is thus the same for every element parallel to the diagonal, when wraparound occurs, there is a change of sign.

These are related to the DCT-IV transforms of order $\mathrm{N}=\mathrm{M} / 2$

$$
\begin{aligned}
C_{r, k} & =\sqrt{\frac{2}{N}} \cos \left(\frac{\pi}{N}(k+1 / 2)(r+1 / 2)\right) \\
S_{r, k} & =\sqrt{\frac{2}{N}} \sin \left(\frac{\pi}{N}(k+1 / 2)(r+1 / 2)\right)
\end{aligned}
$$

The OODFT is given by

$$
\begin{aligned}
& X(k)=\frac{1}{M} \sum_{n=0}^{M-1} x(n) W^{(k+1 / 2)(n+1 / 2)} \\
& \therefore(n)=\sum_{k=0}^{M-1} X(k) W^{-(k+1 / 2)(n+1 / 2)} \\
& x,
\end{aligned}
$$

If the numbers $\mathrm{X}(\mathrm{k})$ are real, then the numbers $\mathrm{X}(\mathrm{n})$ satisfy

$$
x(M-1-n)=-x^{*}(n)
$$

and an OODFT may be used to find the OODFT of two real vectors, the same as the DFT.
Finally, it is to be noted that there are fast OODFT transforms, both DIF and DIT.

## FURTHER WORK

Further simplifications in the signal processing may be possible. Choosing the length of the filters to be different than a multiple of $M$ may be useful, for example. A comparison with the extensive works of

Gopinath and Burrus, who deal with a similar structure, may be useful. Perfect reconstruction filters of this type have not been studied. The whole theory should be implemented in a set of program similar to Malvar[2].

## CONCLUSION

The Lawton factorization leads to a pair of transfer function each of which is symmetrical, and the transmit and receiver filters are frequency translates of only one filter. This leads to a significant reduction in the number of multiplies that must be made.

## APPENDIX 1

Real and imaginary parts of coefficients of an 87-th order Lawton polynomial $P(z)$ achieving a 43-dB stopband loss in a 25 -percent excess bandwidth 8-band filterbank.

$$
\begin{array}{rr}
1.00000000000000 \mathrm{E}+0000 & 0.00000000000000 \mathrm{E}+0000 \\
6.02634201281453 \mathrm{E}-0001^{\prime} & -1.73308749421065 \mathrm{E}-0001 \\
6.50770138305990 \mathrm{E}-0001 & -4.43028339509892 \mathrm{E}-0.001 \\
.5 .68503743634005 \mathrm{E}-0001 & -8.06825982961425 \mathrm{E}-0001 \\
3.10811148147989 \mathrm{E}-0001 & -1.23999149967188 \mathrm{E}+0000 \\
-1.52367563784534 \mathrm{E}-0001 & -1.69233315188185 \mathrm{E}+0000 \\
-8.27544224516866 \mathrm{E}-0001 & -2.08924316060963 \mathrm{E}+0000 \\
-1.69112761412608 \mathrm{E}+0000 & -2.33770405264433 \mathrm{E}+0000 \\
-2.68629946269709 \mathrm{E}+0000 & -2.33720684778200 \mathrm{E}+0000 \\
-3.72435100047456 \mathrm{E}+0000 & -1.99461831624773 \mathrm{E}+0000 \\
-4.69092136051541 \mathrm{E}+0000 & -1.24112773464388 \mathrm{E}+0000 \\
-5.45685456648806 \mathrm{E}+0000 & -4.87011711252875 \mathrm{E}-0002 \\
-5.89264088209232 \mathrm{E}+0000 & 1.55686510941870 \mathrm{E}+0000 \\
-5.88475368230399 \mathrm{E}+0000 & 3.48905855101805 \mathrm{E}+0000 \\
-5.35174950667316 \mathrm{E}+0000 & 5.60204142389873 \mathrm{E}+0000 \\
-4.25785314280447 \mathrm{E}+0000 & 7.70131354124089 \mathrm{E}+0000 \\
-2.62194321626213 \mathrm{E}+0000 & 9.56321292569113 \mathrm{E}+0000 \\
-5.20375083669207 \mathrm{E}-0001 & 1.09612997598607 \mathrm{E}+0001 \\
1.91713949367004 \mathrm{E}+0000 & 1.16963134599259 \mathrm{E}+0001 \\
4.51844237437369 \mathrm{E}+0000 & 1.16254573223672 \mathrm{E}+0001 \\
7.08552945036195 \mathrm{E}+0000 & 1.06864520856758 \mathrm{E}+0001 \\
9.41492536145766 \mathrm{E}+0000 & 8.91221948677361 \mathrm{E}+0000 \\
1.13193600552482 \mathrm{E}+0001 & 6.43319940483572 \mathrm{E}+0000 \\
1.26480522663350 \mathrm{E}+0001 & 3.46603124675669 \mathrm{E}+0000 \\
1.33030761142224 \mathrm{E}+0001 & 2.89394235061912 \mathrm{E}+0001 \\
1.32497771219642 \mathrm{E}+0001 & -2.79012653749097 \mathrm{E}+0000 \\
1.252002169620095 \mathrm{E}+0001 & -5.47754597982280 \mathrm{E}+0000 \\
1.12079504350214 \mathrm{E}+0001 & -7.52866683303737 \mathrm{E}+0000 \\
9.45893087582937 \mathrm{E}+0000 & -8.78305004837776 \mathrm{E}+0000 \\
7.45314676287339 \mathrm{E}+0000 & -9.18557588374261 \mathrm{E}+0000 \\
5.38597041384117 \mathrm{E}+0000 & -8.79305266663689 \mathrm{E}+0000 \\
3.44744203460105 \mathrm{E}+0000 & -7.76447990719038 \mathrm{E}+0000 \\
1.80327930613551 \mathrm{E}+0000 & -6.33627367321365 \mathrm{E}+0000 \\
5.79381003132657 \mathrm{E}-0001 & -4.78590737772589 \mathrm{E}+0000 \\
-1.48612703832777 \mathrm{E}-0001 & -3.38974131165848 \mathrm{E}+0000 \\
-3.60013635630387 \mathrm{E}-0001 & -2.38123469553518 \mathrm{E}+0000 \\
-8.77982530200664 \mathrm{E}-0002 & -1.91658020643768 \mathrm{E}+0000 \\
5.87894449670849 \mathrm{E}-0001 & -2.05261751727577 \mathrm{E}+0000 \\
1.5507047609697 \mathrm{E}+0000 & -2.74122894054465 \mathrm{E}+0000 \\
2.66168908691052 \mathrm{E}+0000 & -3.84031046999130 \mathrm{E}+0000 \\
3.77459238146163 \mathrm{E}+0000 & -5.14024678312375 \mathrm{E}+0000 \\
4.75074832540598 \mathrm{E}+0000 & -6.40079201212804 \mathrm{E}+0000 \\
5.47289811975366 \mathrm{E}+0000 & -7.39282659083567 \mathrm{E}+0000 \\
5.85617614047588 \mathrm{E}+0000 & -7.93768236494214 \mathrm{E}+0000 \\
5.85618860991805 \mathrm{E}+0000 & -7.93767568165784 \mathrm{E}+0000 \\
5.47288652920018 \mathrm{E}+0000 & -7.39283162356135 \mathrm{E}+0000 \\
4.75075837550812 \mathrm{E}+0000 & -6.40078994833277 \mathrm{E}+0000
\end{array}
$$

```
    3.77458417944626E+0000
    2.66169549404792E+0000
    1.55069985004573E+0000
    5.87898227186675E-0001
-8.78011640265987E-0002
-3.60011492514568E-0001
-1.48614046876179E-0001
    5.79381488854228E-0001
    1.80327965173330E+0000
    3.44744101594852E+0000
    5.38597183940840E+0000
    7.45314522820101E+0000
    9.45893227039002E+0000
    1.12079493305102E+0001
    1.25200224661555E+0001
    1.32497766516082E+0001
    1.33030763597247E+0001
    1.26480521650670E+0001
    1.13193600779609E+0001.;
    9.41492537313545E+0000
    7.08552942898205E+0000
    4.51844239422947E+0000
    1.91713947883487E+0000
-5.2037507.3867829E-0001
-2.62194322219755E+0000
-4.25785313948548E+0000
-5.35174950838414E+0000
-5.88475368150392E+0000
-5.89264088242702E+0000
-5.45685456636677E+0000
-4.69.092136055271E+0000
-3.72.435100046566E+0000
-2.68629946269863E+0000
-1.69112761412604E+0000
-8.27544224516811E-0001
-1.52367563784541E-0001
    3.10811148148011E-0001
    5.68503743634027E-0001
    6.50770138306008E=0001
    6.02634201281470E-0001
    9.999999999999.98E-0001
```

$-5.14024514400275 \mathrm{E}+0000$ $-3.84031584511743 \mathrm{E}+0000$ -2.74122045678967E+0000 $-2.05262800041196 \mathrm{E}+0000$ $-1.91656904488342 \mathrm{E}+0000$ $-2.38124529241214 E+0000$ $-3.38973220851781 \mathrm{E}+0000$ $-4.78591449482749 \mathrm{E}+0000$ $-6.33626860409012 \mathrm{E}+0000$ $-7.76448318601540 E+0000$ $-8.79305075600213 \mathrm{E}+0000$ $-9.18557686979046 \mathrm{E}+0000$ $-8.78304961395778 \mathrm{E}+0000$ $-7.52866698119001 \mathrm{E}+0000$ $-5.47754595506901 E+0000$ $-2.79012652378293 E+0000$ $2.89394218636269 \mathrm{E}-0001$ $3.46603125623450 \mathrm{E}+0000$ $6.43319940156883 \mathrm{E}+0000$ $8.91221948681571 \mathrm{E}+0000$ $1.06864520868263 \mathrm{E}+0001$ $1.16254573209018 \mathrm{E}+0001$ $1.16963134614286 \mathrm{E}+0001$ $1.09612997584975 \mathrm{E}+0001$ $9.56321292676915 \mathrm{E}+0000$ $7.70131354052050 \mathrm{E}+0000$ $5.60204142430150 \mathrm{E}+0000$ $3.48905855083330 \mathrm{E}+0000$ $1.55686510948667 \mathrm{E}+0000$ $-4.87011711439881 E-0002$ $-1.24112773464097 \mathrm{E}+0000$ $-1.99461831624745 \mathrm{E}+0000$ $-2.33720684778252 \mathrm{E}+0000$ $-2.33770405264418 \mathrm{E}+0000$ $-2.08924316060974 \mathrm{E}+0000$ $-1.69233315188186 \mathrm{E}+0000$ $-1.23999149967190 \mathrm{E}+0000$ -8.06825982961436E-0001 $-4.43028339509899 \mathrm{E}-0001$ $-1.73308749421068 \mathrm{E}-0001$ $-1.19262238973405 \mathrm{E}-0015$
\% makefilters: makes a set of transmit and receive filters load lawton $\%$ a mat-file with the coefficients
[transmit(i,:), receive(i, : )]=filters(lawton,i-1, 8)

```

\section*{makefilt.m}
for \(i=1: 8\)
end

\section*{filters.m}
function[transmit, receive] \(=\) filters ( \(a, k, M\) )
for \(i=1\) : length \((a)\)
transmit \((\mathrm{j})=\exp (\mathrm{j} *(\mathrm{pi} / \mathrm{M}) *(2 * \mathrm{k}+1) *(\mathrm{~m} / 4.0+0.5)) * a(\mathrm{i}) * \exp (j *(\mathrm{pi} / \mathrm{m}) *(\mathrm{i}-1) *(2 * \mathrm{k}+1)) ;\) end
for \(i=1:\) length (a)
receive \((\mathrm{i})=\exp (\mathrm{j} *(\mathrm{pi} / \mathrm{M}) *(2 * \mathrm{k}+1) *(-\mathrm{M} / 4.0+0.5)) * \operatorname{conj}(\mathrm{a}(\mathrm{i})) * \exp (\mathrm{j} *(\mathrm{pi} / \mathrm{M}) *(\mathrm{i}-1) *(2 * \mathrm{k}+1)) ;\) end
typical session
makefilt
plot(conv(transmit(2,:),receive(3,:)))

\section*{REFERENCES}
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\section*{FIGURES}
1. Frequency response of filter.

Frequency response of filter


\section*{PHASING FILTERBANKS}

W F McGee
19/03/97 11:52 AM
The purpose of this note is to document the phasing of filter banks.

There are two cases that commonly occur. In the first, which occurs with Lawton banks, the transmit filters are all frequency translates of one another, and the receive filters are their conjugates [i.e., sequence reversed, conjugated.]. If the phase of the transmit filters is \(\tau_{i}\) and the corresponding receive filter is \(-\rho_{i}\), then the transfer function of the main channel is
\[
e^{j\left(\tau_{i}-\rho_{i}\right)} H_{i}(z) H_{i^{*}}(z)
\]
and of the crosstalk channel is
\[
e^{j n\left(\tau_{i+1}-\rho_{i}\right)} \dot{H}_{i+1}(z) H_{i^{*}}(z)
\]
where
\[
\begin{aligned}
& H_{i}(z)=H\left(W^{i+1 / 2} z\right) \\
& H_{i^{*}}(z)=H_{*}\left(W^{i+1 / 2} z\right)
\end{aligned}
\]

The main coefficient is the \(L\) th, which is generally the order of \(H\) or \(H_{*}\). Thus, \(H\) is of length \(L+1\), order \(L\), \(\mathrm{HH}_{*}\) is of order 2 L , and the middle coefficient is the Lth.

The Lt coefficient of the main channel is
\[
e^{j\left(\tau_{i}-\rho_{i}\right)} n_{L} z^{-L}\left(W^{i+1 / 2}\right)^{-L}
\]
. where \(n_{L}\) is real, and this is to be real. This restricts the phase differences.
On the other hand, for the crosstalk transfer functions, the Lith coefficient is
and this is to be imaginary.
Consequently the requirement is
\[
\begin{aligned}
& \tau_{i .}-\rho_{i}+\frac{2 \pi}{M}(i+1 / 2) L=K_{i} \pi \\
& \tau_{i+1}-\rho_{i}+\frac{2 \pi}{M}(i+1) L=\left(2 U_{i}+1\right) \frac{\pi}{2} \\
& \tau_{i-1}-\rho_{i}+\frac{2 \pi}{M} i L=\left(2 L_{i}+1\right) \frac{\pi}{2}
\end{aligned}
\]

The difference of the first two equations is
\[
\tau_{i+1}-\tau_{i}=\frac{\pi}{2}+\left(U_{i}-\dot{K_{i}}\right) \pi-\frac{2 \pi}{M} \frac{L}{2}
\]
which is satisfied by
\[
\tau_{i}=\frac{\pi}{2}(i+1 / 2)(1-2 L / M), \text { among many solutions. }
\]

This means that the transmitters are
\[
e^{j(i+1 / 2) \frac{\pi}{2}} W^{(i+1 / 2) L / 2} H\left(W^{i+1 / 2} z\right)
\]

With this, the formula for the receive phase shift is
\[
\rho_{i}=\frac{\pi}{2}(i+1 / 2)(1+2 L / M)
\]

The receive filters are
\[
e^{-j(i+1 / 2) \frac{\pi}{2}} W^{(i+1 / 2) L / 2} H_{*}\left(W^{i+1 / 2} z\right)
\]

Another common filter pair involves minimum-maximum phase pairs. Here \(\mathrm{H}(\mathrm{z})\) the prototype is real. The transmitting filters are respectively
\[
\left\{e^{j \tau_{0}} H\left(W^{1 / 2} z\right), e^{j \tau_{1}} H_{*}\left(W^{3 / 2} z\right), \ldots\right\}
\]
and the receive filters are
\[
\left\{e^{-\rho \tau_{0}} \cdot H_{*}\left(W^{1 / 2} z\right), e^{-j \rho \tau_{1}} H\left(W^{3 / 2} z\right), \Lambda\right\}
\]

The main channel responses are
\[
e^{j\left(t_{i}-\dot{p}_{i}\right)} H\left(W^{i+1 / 2} z\right) H_{*}\left(W^{i+1 / 2} z\right)
\]
and the crosstalk channels are
\[
e^{j\left(\tau_{i+1}-\rho_{i}^{\prime}\right)} H_{*}\left(W^{i+1+1 / 2} z\right) H_{*}\left(W^{i+1 / 2} z\right)=e^{J\left(\tau_{i+1}-\rho_{i}\right)} C\left(W^{i+1} z\right)
\]

But these requirements will result in the same equations for transmit and receive phase, and therefore the same solutions are valid.

\section*{Prototype Filters for VSB Filter Banks Derived from Half-Band Filters W F McGee \\ 24 September, 1996}

The purpose of this note is to document the use of half-band filters for prototype M-band filters.
An M-band prototype Nyquist filter \(N(z)\) must meet the requirements
\[
\sum_{r=0}^{M / 2-1} N_{M}\left(W^{2 r} z\right)+N_{M}^{*}\left(W^{2 r+1} z\right)=M
\]

Here W is an M-th root of unity
\[
W=e^{-j 2 \pi / M}
\]
and \(N^{*}(z)\) is \(N(z)\) with conjugated coefficients.
As a first set of prototype filters based on the half-band filters \(N_{i, i}(z)\) [we use the notation \(N_{i, i}\) instead of \(N_{2, i}\) to save space] satisfying
\[
N_{, i}(z)+N_{, i}(-z)=2
\]
let
\[
N_{M}(z)=\prod_{i=0}^{\log _{2} M-1} N_{, i}\left(z^{2^{i}}\right)=N_{, 0}(z) N_{, 1}\left(z^{2}\right) N_{, 2}\left(z^{4}\right) \Lambda
\]

The general proof that this construction will work is not difficult, but we give a demonstration for \(M=4\). The requirement in that case is
\[
\begin{aligned}
& N_{, 0}(z) N_{, 1}\left(z^{2}\right)+N_{, 0}(-j z) N_{, 1}\left(-z^{2}\right)+N_{, 0}(-z) N_{, 1}\left(z^{2}\right)+N_{.0}(j z) N_{, 1}\left(-z^{2}\right)= \\
& {\left[N_{, 0}(z)+N_{, 0}(z)\right] N_{, 1}\left(z^{2}\right)+\left[N_{.0}(-j z)+N_{, 0}(j z)\right] N_{, 1}\left(-z^{2}\right)=} \\
& 2\left[N_{, 1}\left(z^{2}\right)+N_{, 1}\left(-z^{2}\right)\right]=4 .
\end{aligned}
\]

The general proof follows by noting that all the polynomials are real and
\[
\therefore N_{M}(z)=N_{M / 2}(z) N\left(z^{M / 2}\right)
\]
and the Nyquist criterion is satisfied since •
\[
\begin{aligned}
& \sum_{k=0}^{M-1} N_{M}\left(W^{k} z\right)_{\mathrm{I}}=\sum_{k=0}^{M-1} N_{M \cdot 2}\left(W^{k} z\right) N_{;}\left((-1)^{k} z^{M / 2}\right) \\
& =\sum_{k=0}^{M / 2-1} N_{M / 2}\left(W^{2 k} z\right) N_{\cdot}\left(z^{M / 2}\right)+\sum_{k=0}^{M / 2-1} N_{M / 2}\left(W^{2 k} W z\right) N_{,}\left(-z^{M / 2}\right) \\
& =\frac{M}{2} N_{\cdot}\left(z^{M / 2}\right)+N_{\cdot}\left(-z^{M / 2}\right) \\
& =M
\end{aligned}
\]

Another choice of prototype is
\[
N_{M}\left(W^{1 / 2} z\right)=N_{.0}(-j z) N_{.0}(z) N_{, 1}\left(z^{2}\right) \Lambda
\]
i.e.,
\[
N_{M}(z)=N_{.0}\left(-W^{-1 / 2} j z\right) N_{, 0}\left(W^{-1 / 2} z\right) N_{11}\left(W^{-1} z^{2}\right) \Lambda
\]

For \(\mathrm{M}=2, \mathrm{~W}=-1, \mathrm{~N}_{2}(\mathrm{z})=\mathrm{N}_{0}(\mathrm{z})\), as expected.
For \(M=4, W=-j\),
\[
N_{4}(z)=N_{.0}\left(W^{1 / 2} z\right) N_{, 0}\left(W^{-1 / 2} z\right)=N_{4}^{*}(z)
\]
and the Nyquist criterion is
\[
\begin{aligned}
& N_{, 0}\left(W^{1 / 2} z\right) N_{.0}\left(W^{-1 / 2} z\right)+N_{, 0}\left(W^{3 / 2} z\right) N_{.0}\left(W^{1 / 2} z\right) \\
& +N_{, 0}\left(W^{5 / 2} z\right) N_{.0}\left(W^{3 / 2} z\right)+N_{.0}\left(W^{7 / 2} z\right) N_{.0}\left(W^{5 / 2} z\right) \\
& =N_{, 0}\left(W^{1 / 2} z\right) N_{.0}\left(-W^{3 / 2} z\right)+N_{.0}\left(W^{3 / 2} z\right) N_{.0}\left(W^{1 / 2} z\right) \\
& +N_{, 0}\left(-W^{1 / 2} z\right) N_{, 0}\left(W^{3 / 2} z\right)+N_{, 0}\left(-W^{3 / 2} z\right) N_{, 0}\left(-W^{1 / 2} z\right) \\
& =2\left[N_{, 0}\left(W^{1 / 2} z\right)+N_{.0}\left(-W^{1 / 2} z\right)\right]=4
\end{aligned}
\]

For \(\mathrm{M}=8\) the construction yields
\[
\begin{aligned}
& N_{8}(z)=N_{, 0}\left(-W^{-1 / 2} j z\right) N_{, 0}\left(W^{-1 / 2} z\right) N_{, 1}\left(W^{-1} z^{2}\right) \\
& =N_{, 0}\left(W^{3 / 2} z\right) N_{, 0}\left(-W^{7 / 2} z\right) N_{, 1}\left(-W^{3} z^{2}\right) \\
& N^{*}(z)=N_{, 0}\left(W^{-3 / 2} z\right) N_{, 0}\left(-W^{-7 / 2} z\right) N_{, 1}\left(-W^{-3} z^{2}\right) \\
& =N_{.0}\left(-W^{5 / 2} z\right) N_{, 0}\left(W^{1 / 2} z\right) N_{, 1}\left(W^{2}\right)
\end{aligned}
\]
and the Nyquist criterion is satisfied, since
\[
\begin{aligned}
& N_{, 0}\left(W^{3 / 2} z\right) N_{, 0}\left(-W^{7 / 2} z\right) N_{, 1}\left(-W^{3} z^{2}\right) \\
& +N_{, 0}\left(W^{7 / 2} z\right) N_{, 0}\left(-W^{11 / 2} z\right) N_{, 1}\left(-W^{7} z^{2}\right) \\
& +N_{, 0}\left(W^{1 / 2} z\right) N_{, 0}\left(-W^{15 / 2} z\right) N_{, 1}\left(-W^{11} z^{2}\right) \\
& +N_{, 0}\left(W^{15 / 2} z\right) N_{, 0}\left(-W^{19 / 2} z\right) N_{, 1}\left(-W^{15} z^{2}\right) \\
& +N_{, 0}\left(W^{-1 / 2} z\right) N_{, 0}\left(W^{3 / 2} z\right) N_{, 1}\left(W^{3} z^{2}\right) \\
& +N_{, 0}\left(W^{-5 / 2} z\right) N_{, 0}\left(W^{7 / 2} z\right) N_{, 1}\left(W^{7} z^{2}\right) \\
& +N_{, 0}\left(W^{-9 / 2} z\right) N_{, 0}\left(W^{11 / 2} z\right) N_{, 1}\left(W^{11} z^{2}\right) \\
& +N_{, 0}\left(W^{-13 / 2} z\right) N_{, 0}\left(W^{15 / 2} z\right) N_{, 1}\left(W^{15} z^{2}\right) \\
& =N_{, 0}\left(W^{3 / 2} z\right) N_{, 0}\left(-W^{7 / 2} z\right) N_{, 1}\left(-W^{3} z^{2}\right) \\
& +N_{, 0}\left(W^{7 / 2} z\right) N_{, 0}\left(W^{3 / 2} z\right) N_{, 1}\left(W^{3} z^{2}\right) \\
& +N_{, 0}\left(-W^{3 / 2} z\right) N_{, 0}\left(W^{7 / 2} z\right) N_{, 1}\left(-W^{3} z^{2}\right) \\
& +N_{, 0}\left(-W^{7 / 2} z\right) N_{, 0}\left(-W^{3 / 2} z\right) N_{, 1}\left(W^{3} z^{2}\right) \\
& +N_{, 0}\left(-W^{7 / 2} z\right) N_{, 0}\left(W^{3 / 2} z\right) N_{, 1}\left(W^{3} z^{2}\right) \\
& +N_{, 0}\left(-W^{3 / 2} z\right) N_{.0}\left(W^{7 / 2} z\right) N_{, 1}\left(-W^{3} z^{2}\right) \\
& +N_{, 0}\left(W_{\because / 2}^{7 / 2} z\right) N_{, 0}\left(-W^{3 / 2} z\right) N_{, 1}\left(W^{3} z^{2}\right) \\
& +N_{, 0}\left(W^{3 / 2} z\right) N_{, 0}\left(-W^{7 / 2} z\right) N_{, 1}\left(-W^{3} z^{2}\right) \\
& =N_{, 1}\left(-W^{3} z^{2}\right)\left\{N_{, 0}\left(W^{3 / 2} z\right) N_{, 0}\left(-W^{7 / 2} z\right)+N_{, 0}\left(-W^{3 / 2} z\right) N_{, 0}\left(W^{7 / 2} z\right)\right. \\
& \left.+N_{, 0}\left(-W^{3 / 2} z\right) N_{, 0}\left(W^{7 / 2} z\right)+N_{, 0}\left(W^{3 / 2} z\right) N_{, 0}\left(-W^{7 / 2} z\right)\right\} \\
& +N_{, 1}\left(W^{3} z^{2}\right)\left\{N_{, 0}\left(W^{7 / 2} z\right) N_{, 0}\left(W^{3 / 2} z\right)+N_{, 0}\left(-W^{7 / 2} z\right) N_{, 0}\left(-W^{3 / 2} z\right)\right. \\
& \left.+N_{, 0}\left(-W^{7 / 2} z\right) N_{, 0}\left(W^{3 / 2} \dot{z}\right)+N_{, 0}\left(W^{7 / 2} z\right) N_{.0}\left(-W^{3 / 2} z\right)\right\} \\
& =4\left(N_{, 1}\left(-W^{3} z^{2}\right)+N_{3 x}\left(W^{3} z^{2}\right)\right)=8
\end{aligned}
\]

The general proof for the second construction follows by noting that
\[
\begin{aligned}
& N_{M / 2}(z)=N_{.0}\left(-j W_{M / 2}^{-1 / 2} z\right) N_{.0}\left(W_{M / 2}^{1 / 2} z\right) N_{, 1}\left(W_{M / 2}^{-1} z^{2}\right) \Lambda \\
& N_{M}(z)=\dot{N}_{.0}\left(-j W_{M}^{-1 / 2} z\right) N_{.0}\left(W_{M}^{-1 / 2} z\right) N_{, 1}\left(W_{M}^{-1} z^{2}\right) \Lambda \\
= & N_{M / 2}\left(W_{M}^{1 / 2} z\right) N_{,}\left(\left(W_{M}^{-1 / 2} z\right)^{M / 4}\right) \\
& N^{*}{ }_{M}\left(W_{M} z\right)=N^{*}{ }_{M / 2}\left(W_{M}^{-1 / 2} W_{M} z\right) N_{,}\left(\left(W_{M}^{1 / 2} W_{M} z\right)^{M / 4}\right) \\
= & N^{*}{ }_{M / 2}\left(W_{M}^{1 / 2} z\right) N_{( }\left(-\left(W_{M}^{-1 / 2} z\right)^{M / 4}\right)
\end{aligned}
\]
and the Nyquist criterion is satisfied since
\[
\begin{aligned}
& \sum_{k=0}^{M / 2-1} N_{M}\left(W^{2 k} z\right)+\sum_{k=0}^{M / 2-1} N^{*}{ }_{M}\left(W^{2 k+1} z\right) \\
& =\sum_{k=0}^{M / 2-1} N_{M / 2}\left(W_{M}^{1 / 2} W^{2 k} z\right) N_{,}\left(\left(W_{M}^{-1 / 2} W^{2 k} z\right)^{M / 4}\right)+\sum_{k=0}^{M / 2-1} N^{*}{ }_{M / 2}\left(W_{M}^{1 / 2} W^{2 k} z\right) N_{,}\left(-\left(W_{M}^{-1 / 2} W^{2 k} z\right)^{M / 4}\right) \\
& =\sum_{k=0}^{M / 2-1} N_{M / 2}\left(W_{M}^{1 / 2} W^{2 k} z\right) N_{( }\left((-1)^{k}\left(W_{M}^{-1 / 2} z\right)^{M / 4}\right)+\sum_{k=0}^{M / 2-1} N^{*}{ }_{M / 2}\left(W_{M}^{1 / 2} W^{2 k} z\right) N_{,}\left(-(-1)^{k}\left(W_{M}^{-1 / 2} z\right)^{M / 4}\right) \\
& =N_{,}\left(\left(W_{M}^{-1 / 2} z\right)^{M / 4}\right)\left[\sum_{k=0}^{M / 4-1} N_{M / 2}\left(W_{M}^{1 / 2} W^{4 k} z\right)+\sum_{k=0}^{M / 4-1} N^{*}{ }_{M / 2}\left(W_{M}^{1 / 2} W^{4 k+2} z\right)\right] \\
& +N\left(-\left(W_{M}^{-1 / 2} z\right)^{M / 4}\right)\left[\sum_{k=0}^{M / 2-1} N_{M / 2}\left(W_{M}^{1 / 2} W^{4 k+2} z\right)+\sum_{k=0}^{M / 2-1} N^{*}{ }_{M / 2}\left(W_{M}^{1 / 2} W^{4 k} z\right)\right] \\
& =\frac{M}{2} N_{\cdot}\left(\left(W_{M}^{-1 / 2} z\right)^{M / 4}\right)+N_{( }\left(-\left(W_{M}^{-1 / 2} z\right)^{M / 4}\right)=M
\end{aligned}
\]

\section*{Which choice is better?}

There does not appear to be a great difference in length. Pick \(\mathrm{M}=8\) as an example. The first choice requires three filters whose orders could be, approximately, \(\mathrm{L}, \mathrm{L} / 2\) and \(\mathrm{L} / 4\). The order of the product would be \(\mathrm{L}+\mathrm{L}+\mathrm{L}=3 \mathrm{~L}\). For the second method, the filter lengths would be \(\mathrm{L}, \mathrm{L}\) and \(\mathrm{L} / 2\), and the overall order would be approximately \(\mathrm{L}+\mathrm{L}+\mathrm{L}=3 \mathrm{~L}\).

\title{
ESTIMATING PERFORMANCE DEGRADATION \\ OF PHASE AND TIMING ERROR ON VSB FILTERBANKS
}

\author{
W F McGee
}

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The purpose of this note is to document calculations concerning the degradation in performance of VSB filterbanks due to phase and timing errors. The report shows that, for the filters with high stopband loss, system performance is'determined by a knowledge of the envelope function, i.e., the pulse response of the cascade of two root-Nyquist filters, and a crosstalk function, which is the response of two root-Nyquist filters displaced in freqency by \(+/\) - the center of the lowest frequency baseband VSB channel. Formulas are presented for the eye opening. The performance equivalence of stagger-QAM is shown.

If there is timing offset, then a phase shift should be applied to the received carrier so that VSB zero crossings are maintained.

A computer program to allow estimates of error rate is included, making use of these concepts.

\section*{A. MULTICHANNEL VSB SYSTEMS}

A multichannel VSB communication system consists of a transmitter with M real binary ( \(+1,-1\) ) inputs \(\mathrm{Q}_{\mathrm{k}}\left(\mathrm{z}^{\mathrm{M} / 2}\right)\) at rate \(\mathrm{M} / 2\) applied to appropriately phased root-Nyquist filters centered at an appropriate center frequency producing a complex output, and the sum of all the bandpass filter outputs is added together to form the channel sequence. [We do not consider the mapping from this sequence to a real radio signal, but assume perfect QAM communications.] The receiver consists of a bank of filters matched to the transmit filters, and a detector which takes the real part of the receive filter outputs and examines the sign of this sequence.

Such a system is linear. In this report we use time \(t\) in the formulas for responses, but the responses are, in fact, sampled responses.

The center frequencies for a VSB system arè \((k+1 / 2) / M\), for \(k=0 \ldots, M-1\).
The system design is based entirely on ágair of root-Nyquist filters \(\mathrm{G}(\mathrm{z})\) and \(\mathrm{H}(\mathrm{z})\) such that \(\mathrm{N}(\mathrm{z})=\mathrm{G}(\mathrm{z}) \mathrm{H}(\mathrm{z})\) is a Nyquist filter; \(\mathrm{N}(\mathrm{z})\) has equally spaced zeros, and the spacing is M samples.

Since there are \(M\) channels and \(M(M-1)\) possible crosstalk paths it might be thought that the calculation of system performance is very complicated. But is we assume that there is no crosstalk between channels that are not adjacent, then only the channel itself, and the two adjacent channels, and the noise, will affect performance. This is the case with the filters that we are using. In any case, with such high stopband losses the assumption that the interference from these channels is like Gaussian noise is undoubtedly useful, since the transfer function is not very regular..

The system, then, is characterized by two response functions. The first is a Nyquist response \(m(t)\) representing the overall response of a cascade of the two baseband prototype M-band root-Nyquist filters. We assume that \(\mathrm{m}(\mathrm{t})\) is centered so that the primary sample time is \(\mathrm{t}=0\), and the \(\mathrm{m}(\mathrm{km})=0\) for k not zero [the Nyquist property]. Further, we assume that \(m(t)\) is real.

The half-Nyquist transfer functions are \(\mathrm{H}(\mathrm{z})\) and \(\mathrm{H}^{*}(1 / \mathrm{z})\), and \(\mathrm{m}(\mathrm{t})\) is the transform of \(\mathrm{H}(\mathrm{z}) \mathrm{H}^{*}(1 / z)\). For our study, we assume that \(\mathrm{H}(\mathrm{z})\) has real coefficients, so that \(\mathrm{H}^{*}(\mathrm{z})=\mathrm{H}(\mathrm{z})\). The function \(\mathrm{m}(\mathrm{t})\) is called the envelope function.

Due to the properties of VSB systems，the pulse response in the VSB channel with center frequency \(1 / 2 \mathrm{M}\) is
\[
m(t) \cos \left(\frac{2 \pi}{M} \frac{1}{2} t\right)
\]

This response has additional zeros（i．e．，more than \(m(t)\) itself）at odd multiples of \(M / 2\) ，allowing VSB pulse communication at a rate \(2 / \mathrm{M}\) ．

The imaginary part of the pulse response in this channel is
\[
m(t) \sin \left(\frac{2 \pi}{M} \frac{1}{2} t\right)
\]
and the imaginary part has a zero at \(\mathrm{t}=0\) as well as times kM ；there is a double zero here．
The second response characterizing system performance is one related to crosstalk．The crosstalk in a VSB system with narrowband filters is then determined by the transfer function between adjacent channels．For VSB，the crosstalk between the two channels bordering 0－frequency is proportional to that of a narrow lowpass filter with transfer function \(\mathrm{H}\left(\mathrm{W}^{1 / 2} \mathrm{z}\right) \mathrm{H}^{*}\left(\mathrm{~W}^{-1 / 2} \mathrm{z}\right)\) which is symmetric in frequency about 0 － frequency，and has a real pulse response \(c(t)\) ．Here \(W\) is the Mth root of unity，
\[
W=e^{-j 2 \pi / M}
\]

Due to the phasing of VSB channels，the crosstalk pulse response between the two channels is \(\mathrm{jc}(\mathrm{t})\) ．The real part of the crosstalk vanishes at all times，in particular at time that are multiples of \(M / 2\) ，thereby eliminating the effects of crosstalk from adjacent this adjacent channel which is sending only real baseband symbols．

The crosstalk of the channel above the low－frequency channel（with positive frequency）is that of a narrow－ band bandpass filter centered at the frequency \(1 / \mathrm{M}\) and is

and the real part vanishes at integer multiples of times \(M / 2\) ．

\section*{B. PHASE OFFSET}

Suppose that there is a phase error of \(\theta\), but no timing error. Then the pulse response becomes
\[
\operatorname{Re}\left[m(t) e^{j \frac{2 \pi}{M} \frac{1}{2} t} e^{-j \theta}\right]
\]
which at sample times \(\mathrm{kM} / 2\) is
\[
m(k M / 2) \cos \left(\frac{2 \pi \mathrm{kM}}{4 \mathrm{M}}-\theta\right)
\]
and for \(\mathrm{k}=0\) this is \(\mathrm{m}(0) \cos (\theta)\), vanishes for k otherwise even, and is \(\mathrm{m}(\mathrm{kM} / 2) \sin (\theta)\), to within a sign, for k odd.

The crosstalk from the negative frequency channel is
\[
\operatorname{Re}\left(j c(t) e^{-j \theta}\right)=c(t) \sin (\theta)
\]
and from the other channel is
\[
\operatorname{Re}\left(j c(t) e^{j 2 \pi / M} e^{-j \theta}\right)=c(t) \sin \left(\frac{2 \pi t}{M}-\theta\right)
\]
and for times that are multiples of \(\mathrm{M} / 2\), this too is proportional, to within a \(\operatorname{sign}\), \(\operatorname{to} \mathrm{c}(\mathrm{t}) \sin (\theta)\).

\section*{C. TIMING OFFSET AND PHASE OFFSET}

Here, the samples are not taken at time \(t=0\), but are offset to a sample time \(t\), assumed small; the pulses are still sent at multiples of times \(\mathrm{kM} / 2\).

The sampled pulse response indicating intersymbol interference is then
\[
\begin{aligned}
& \operatorname{Re}\left[m(t+k M / 2) e^{j \frac{2 \pi 1}{M 2}(t+k M / 2)} e^{-j \theta}\right] \\
& =m(t+k M / 2) \cos \left(\frac{2 \pi}{2 M} t-\theta+k \frac{\pi}{2}\right) \\
& =m(t+k M / 2) \cos \left(\frac{2 \pi}{2 M} t-\theta\right) \text { for } \mathrm{k} \text { even } \\
& = \pm m(t+k M / 2) \sin \left(\frac{2 \pi}{2 M} t-\theta\right) \text { for } \mathrm{k} \text { odd }
\end{aligned}
\]

The crosstalk from the negative frequency channel is
\[
\operatorname{Re}\left[j c(t+k M / 2) e^{-j \theta}\right]=c(t+k M / 2) \sin (\theta)
\]
and the crosstalk from the other channel is
\[
\operatorname{Re}\left[j c(t+k M / 2) e^{j \frac{2 \pi}{M}(t+k M / 2)} e^{-j \theta}\right]= \pm c(t+k M / 2) \sin \left(\frac{2 \pi}{M} t-\theta\right)
\]

There seem to be two possible choices of \(\theta\) that would be best.
In the first, set \(\theta=0\). Then there is no crosstalk from the negative frequency channel, the crosstalk from the other channel is
the main signal is
\[
\pm c(t+k M / 2) \sin \left(\frac{2 \pi}{M} t\right)
\]
\[
m(t) \cos \left(\frac{2 \pi t}{2 M}\right)
\]
the interference from the other symbols is
\[
\begin{aligned}
& m(t+k M / 2) \cos \left(\frac{2 \pi t}{2 M}\right) \text { for } \mathrm{k} \text { even } \\
& m(t+k M / 2) \sin \left(\frac{2 \pi t}{2 M}\right) \text { for } \mathrm{k} \text { odd }
\end{aligned}
\]

The other choice would be
\[
\theta=\frac{2 \pi t}{2 M}
\]

The main signal is \(m(t)\), the intrachannel interference comes only from even numbered symbols, and is
\[
\cdots(t+k M / 2) \mathrm{k} \text { even }
\]
and the interference from the two adjacent channels is
\[
\therefore \quad+
\]
\[
+c(t+k M / 2) \sin (\theta)
\]

It is difficult to analyze these two cases, but note that when the noise is large it is only necessary to compare the mean squared errors of the two. When the phase shift is kept to 0 , the mean squared error is
\[
\begin{aligned}
& \operatorname{MSE}(\theta=0)= \sum_{k \pm 0, k \text { even }} m^{2}(t+k M / 2) \cos ^{2}\left(\frac{2 \pi t}{2 M}\right) \\
&+\sum_{k \text { odd }} m^{2}(t+k M / 2) \sin ^{2}\left(\frac{2 \pi t}{2 M}\right)+\sum_{\text {all }} c^{2}(t+k M / 2) \sin ^{2}\left(2 \frac{2 \pi t}{2 M}\right) \\
& \text { signal power }(\theta=0)=m^{2}(t) \cos ^{2}\left(\frac{2 \pi t}{2 M}\right) \\
& \operatorname{MSE}\left(\theta=\frac{2 \pi t}{2 M}\right)= \sum_{k \pm 0, k \text { even }} m^{2}(t+k M / 2)+2 \sum_{\text {all }} c^{2}(t+k M / 2) \sin ^{2}\left(\frac{2 \pi t}{2 M}\right) \\
& \text { signal power }\left(\theta=\frac{2 \pi t}{2 M}\right)=m^{2}(t)
\end{aligned}
\]

Since the contribution to the mean squared error has no contributions from the odd-numbered (and usually larger) odd symbols in the channel, and the crosstalk from the adjacent channels has been reduced by a factor of 2 for small timing errors, it appears that the best choice of phase offset and timing phase will be the second, that is
\[
\theta=\frac{2 \pi t}{2 M}
\]

\section*{D. STAGGERED QAM}

For completeness, we include an analysis of a staggered QAM system with the same prototype filters and show that the performance of the dc channel is the same as the previous VSB channel with delay tracked by the phase.

Recall that in an SQAM filterbank the I and \(Q\) channels are staggered, with symbols being transmitted on the I channel at multiples of times M , and on the Q channels at odd multiples of times \(\mathrm{M} / 2\).

The pulse response is \(m(t)\), which, as noted, is real.
With SQAM the crosstalk filters are located at frequencies \(+/-1 / 2 \mathrm{M}\), and therefore the pulse response from a symbol in the I channel of the positive frequency adjacent channel is
\[
j c(t) e^{j \frac{2 \pi}{2 M} t}
\]
and from the lower channel is
\[
j c(t) e^{j \frac{2 \pi}{2 M} t}
\]

With \(c(t)\) real, \(m(t)\) real and Nyquist, there is no intersymbol or interchannel interference.

\section*{D1. SQAM PHASE SHIFT}

The response in the I-channel to a symbol in the I channel is
\[
\operatorname{Re}\left(m(t) e^{-j \theta}\right)=m(t) \cos (\theta)
\]
which vanishes for times kM because \(\mathrm{m}(\mathrm{kM})\) is zero, and the response to a symbol from the Q channel is
\[
\operatorname{Re}\left(j m(t) e^{-j \theta}\right)=m(t) \sin (\theta)
\]

The crosstalk from the positive frequency adjacent channel I to I is
\[
\operatorname{Re}\left(j c(t) e^{j \frac{2 \pi}{2 M} t} e^{-j \theta}= \pm c(t) \sin \left(\frac{2 \pi}{2 M} t-\theta\right)\right.
\]
which at times \(\mathrm{t}=\mathrm{kM}\) is
\[
\pm c(t) \sin (\theta)
\]
and from Q to I is
\[
\pm c(t) \cos \left(\frac{2 \pi}{2 M} t-\theta\right)
\]
which at odd sample times ( \(2 \mathrm{k}+1\) ) \(\mathrm{M} / 2\) is also
\[
\pm c(t) \sin (\theta)
\]

Thus, the interference is identical to the VSB interference.

\section*{D2. SQAM TIMING AND PHASE SHIFT}

For time and phase shift, the intrachannel responses are
\[
\begin{aligned}
& m(t+k M) \cos (\theta) \\
& m(t+(2 k+1) M / 2) \sin (\theta)
\end{aligned}
\]
for signals from the \(I\) and \(Q\) channels respectively.
The interchannel interference is
\(\operatorname{Re}\left(j c(t+k M) e^{j \frac{2 \pi}{2 M}(t+k M)} e^{-j \theta}= \pm c(t+k M) \sin \left(\frac{2 \pi}{2 M}(t+k M)-\theta\right)= \pm c(t+k M) \sin \left(\frac{2 \pi}{2 M} t-\theta\right)\right.\)
from \(I\) to \(I\) and from \(Q\) to \(I\) is
\[
\begin{aligned}
& \operatorname{Re}\left(c(t+(2 K+1) M / 2) e^{j \frac{2 \pi}{2 M}(t+(k+1 / 2) M)} e^{-j \theta}=\right. \\
& \pm c(t+(2 K+1) M / 2) \cos \left(\frac{2 \pi}{2 M}(t+(k+1 / 2) M)-\theta\right)= \\
& \pm c(t+(2 K+1) M / 2) \sin \left(\frac{2 \pi}{2 M} t-\theta\right)
\end{aligned}
\]
and from the negative frequency channel the response is the same with a sign reversal of \(\theta\), that is,
\[
\begin{gathered}
\pm c(t+k M) \sin \left(\frac{2 \pi}{2 M} t+\theta\right) \\
\pm c(t+(2 K+1) M / 2) \sin \left(\frac{2 \pi}{2 M} t+\theta\right)
\end{gathered}
\]

These are identical to the results of VSB with the proviso that these refer to the non-zero \(\theta\).

\section*{E. EYE OPENING}

The opening is the signal less all possible interference, and is, therefore
\[
\begin{aligned}
& m(t) \cos \theta \\
& -\sum_{k \neq 0}|m(t+k M) \| \cos \theta| \\
& -\sum_{k}|m(t+(2 k+1) M / 2)| \sin \theta \mid \\
& -\sum_{k}|c(t+k M / 2)|\left|\sin \left(\frac{2 \pi t}{2 M}-\theta\right)\right| \\
& -\sum_{k}|c(t+k M / 2)|\left|\sin \left(\frac{2 \pi t}{2 M}+\theta\right)\right|: \ldots \\
& =m(t) \cos \theta \\
& -\sum_{k \neq 0}|m(t+k M)| \cos \theta \mid \\
& -\sum_{k}|m(t+(2 k+1) M / 2)| \sin \theta \mid \\
& \left.\left.-2 \sum_{k}|c(t+k M / 2)| \max \left(\left\lvert\, \sin \left(\frac{2 \pi t}{2 M}\right) \cos \theta\right.\right)|,| \cos \left(\frac{2 \pi t}{2 M}\right) \sin \theta\right) \mid\right)
\end{aligned}
\]

\section*{F. OTHER CHANNELS \({ }^{\prime}\)}

We showed that a timing offset of t required an associated phase shift of
\[
\theta=\frac{2 \pi}{M} \frac{1}{2} t
\]
for the first channel in a VSB system, and that, this meant that the response to odd numbered symbols was zero. This is because多:
\[
\sin \left(\frac{2 \pi}{M} \frac{1}{2}\left(t+(2 l+1) \frac{M}{2}\right)-\frac{2 \pi}{M} \frac{1}{2} t\right)=0
\]
and it is clear that a phase shift of
\[
\frac{2 \pi}{M}(k+1 / 2) t
\]
would effect the same of the kth channel.
But this may also be accomplished for the kth channel with the earlier phase shift, since
\[
\begin{aligned}
& \sin \left(\frac{2 \pi}{M}\left(k+\frac{1}{2}\right)\left(t+(2 l+1) \frac{M}{2}\right)-\frac{2 \pi}{M} \frac{1}{2} t\right) \\
& \sin \left(\frac{2 \pi}{M}\left(k+\frac{1}{2}\right)\left(t+(2 l+1) \frac{M}{2}\right)-\frac{2 \pi}{M}\left(k+\frac{1}{2}\right) t\right)= \\
& \sin \left(\frac{\pi}{2}(2 k+1)(2 l+1)\right)=0
\end{aligned}
\]

\section*{CONCLUSION}

The basic formulas to calculate the error rate and eye opening for VSB and SQAM filter banks have been derived. They depend on two functions, \(m(t)\), the prototype Nyquist channel response, and \(c(t)\), the prototype crosstalk function. A program using these formulas have been written, and the results agree, more, or less, with simulations done by M. Sablatash, for a slightly different set of filters.

\section*{ACKNOWLEDGEMENTS}

John Lodge proposed this problem and solution. Mike Sablatash provided simulation results.

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\section*{APPENDIX}

\section*{ESTIMATION OF ERROR RÁTES}

Here is a program to estimate error rates, given the prototype filter for a 32 -channel system.

\section*{\%TIMING.M}
\%timing calculates predicted error rate for VSB channels \%with time offset \(t\) and phase offset theta
load mp32
\(\therefore\) 蔡
samplerate \(=16\);
\(\mathrm{j}=\mathrm{sqrt}(-1)\);
\%calculate the channel response for \(S Q A M m(t)\)
\(\mathrm{m}=\operatorname{conv}(\mathrm{mp} 32, \mathrm{fliplr}(\operatorname{mp} 32))\);
\%calculate the crosstalk function \(c(t)\)
h=zeros(size(mp32));
for \(\mathrm{kk}=1\) : length (mp32)
\(\mathrm{h}(\mathrm{kk})=\operatorname{mp} 32(\mathrm{kk}) * \exp (\mathrm{j} * 2 * \mathrm{pi} *(\mathrm{kk}-1) /(2 * 32))\);
end
\(c=r e a l(\operatorname{conv}(h, \operatorname{conj}(h)))\);
hold off
for thetai=0:0
thetai
theta=2*pi*thetai/360;
for offset=0:8
offset
```

    a=[cos(theta)*sample(m,32,offset)
    sin(theta)*sample(m,32,offset+samplerate) ...
sample(c,16,offset)*sin((2*pi*offset/(2*32))-theta)
sample(c,16,offset)*sin((2*pi*offset/(2*32))+theta)];
[u v]=max(a);
save=a(1);
a(1)=abs(u);
a(v)=save;
a=real(a);
d=[];
ploterr
hold on
end %offset
gtext(str(thetai))
end %theta
hold off
j=sqrt(-1);
i=sqrt(-1);

```

\section*{\%ploterr.m}
```

if d==[]
d=2:2:30;
end
pe=zeros(1, length(d));
sigma=zeros(1,length(d));
for k=1:length(d)
[pe(k) sigma(k)]=err_rate(a,d(k));
end
sigma=-8.68*log(sigma);
for i=2:length(sigma)
if sigma(i)<sigma(i-1)
sigma(i)=sigma(i-1);
pe(i)=pe(i-1);
end
end
semilogy(sigma,pe);
xlabel('SNR [dB]');
ylabel('Pe');
grid;
function [pe,sigma]=err rate(a,d)
% ISI, calculates error rate
% a(1) contains the peak,. the rest is ISI
% d is a parameter usualiy %jetween 2 and 30
s=a(1);
elnot=-d*a(1);
eldouble=0;
for i=2:length(a)
s=s-a(i)*tanh(d*a(i));
c=abs(d*a(i));
temp=a(i)/cosh(c);
elnot=elnot+c-log(2)+iog(1+exp (-2*c));
eldouble=eldouble+temp*temp;
end
if s>0
s=squrt(s/d);
elnot=elnot+(d*d*s*s/2);
eldouble=eldouble+s*s;
pe=0.5*erfcx(d*sqrt(eldouble)/sqrt(2))*exp(elnot);
sigma=s;
sigma=sqrt(2)*sigma;
else
pe=0.5;
sigma=100000;

```
end
Sample output from program with input the filter MP32.MAT, the 32-channel filter derived from the MPEG coding standard.

\(\therefore\)

pe.doc

\author{
Phasing VSB Filter Banks \\ W F McGee \\ 25 September, 1996
}

The purpose of this note is to record the phasings required for VSB filter banks. Quasi-perfect VSB filter banks use a pair of transfer functions \(H(z)\) and \(G(z)\) where \(H(z) G^{*}(z)\) forms an M-band VSB-Nyquist filter. We assume that these transfer functions are not necessarily causal, but suppose, for the sake of argument that \(\mathrm{H}(\mathrm{z})\) is causal, of order N , and finally assume that H and \(\mathrm{G}^{*}\) form a minimum-maximum phase pair, that is \(G^{*}(z)=H^{*}(1 / z)\), or \(G(z)=H(1 / z)\).

\section*{Basic result}

The transfer functions of the transmit filters for an M-band VSB filter bank may consist of M/2 transfer functions, \(H\left(W^{2 r+1 / 2} z\right)\) where \(r\) ranges from 0 to \(M / 2-1\), and the \(M / 2\) transfer functions \(j G\left(W^{2 r-1 / 2} z\right)\) where \(r\) has the same range. The receive filters are correspondingly \(\mathrm{G}^{*}\left(\mathrm{~W}^{2 \mathrm{r}+1 / 2} \mathrm{z}\right)\) and \(-\mathrm{jH} *\left(\mathrm{~W}^{2 \mathrm{r}-1 / 2} \mathrm{z}\right)\). If we set \(A(z)=H\left(W^{1 / 2} z\right)\) be the transfer function of the first filter mentioned, then the transfer functions are \(A\left(W^{2 r} z\right)\) and
\[
j G\left(W^{-1 / 2} z\right)=j H\left(1 / W^{-1 / 2} z\right)=j H\left(W^{1 / 2} / z\right)=j A(1 / z)
\]
and its frequency translates by \(\mathrm{W}^{2 \mathrm{r}}\).

\section*{Restoring causality.}

Focus attention on the first two ( \(\mathrm{r}=0\) ) transfer functions; the same result applies to all. Since \(\mathrm{H}(\mathrm{z})\) is causal it will not be affected. To make \(G(z)\) causal we must multiply it by \(z^{-D}\) where \(D\) is equal or larger than the degree of H . Also, system performance will not be affected if \(D\) is a multiple of \(\mathrm{M} / 2\), since this is the rate that the input sequence appears. Thus, for simplicity, choose \(D\) to be the smallest multiple of \(M / 2\) that is larger than \(N\); that is,
\[
D=\frac{M}{2}\left(1+\left[N \operatorname{div}(M / 2)^{\prime}\right]\right)
\]
if N is not divisible by \(\mathrm{M} / 2\) and \(\mathrm{D}=\mathrm{N}\) if N is divisible by \(\mathrm{M} / 2\).
This may all be accomplished simply by padding \(\mathrm{H}(\mathrm{z})\) with zeros to bring its order up to a multiple of \(\mathrm{M} / 2\).
Polyphase filtering

Finally, we have to realize the pair of Mritransfer functions \(A\left(W^{2 r} z\right)\) and \(j B\left(W^{2 r} z\right)\) where
\(B(z)=z^{-D} A(1 / z)\) as specified. Each may be realized by the polyphase expansion of \(A(z)\) and \(B(z)\) of order M/2. Thus if
\[
\begin{aligned}
& A(z)=\sum_{k=0}^{M / 2-1} z^{-k} A_{k}\left(z^{M / 2}\right) \\
& B(z)=\sum_{k=0}^{M / 2-1} z^{-k} B_{k}\left(z^{M / 2}\right)
\end{aligned}
\]
then
\[
A\left(W^{2 r} z\right)=\sum_{k=0}^{M / 2-1} W^{-2 r k} z^{-k} A_{k}\left(z^{M / 2}\right)=\sum_{k=0}^{M / 2-1} e^{j 2 \pi r k /(M / 2)} z^{-k} A_{k}\left(z^{M / 2}\right)
\]
and
\[
B\left(W^{2 r} z\right)=\sum_{k=0}^{M / 2-1} e^{j 2 \pi r k /(M / 2)} z^{-k} B_{k}\left(z^{M / 2}\right)
\]

If we include the \(M / 2\) real input sequences \(X_{i}\left(z^{M / 2}\right)\) and \(Y_{i}\left(z^{M / 2}\right)\) in the description then the output is simply
\[
\sum_{k=0}^{M / 2-1} z^{-k}\left\{A_{k}\left(z^{M / 2}\right) \sum_{r=0}^{M / 2-1} e^{j 2 \pi r k /(M / 2)} X_{r}\left(z^{M / 2}\right)+B_{k}\left(z^{M / 2}\right) \sum_{r=0}^{M / 2-1} e^{j 2 \pi \pi k /(M / 2)} j Y_{r}\left(z^{M / 2}\right)\right\}
\]

The calculation may be further simplified by noting that the DFT \(U_{k}\) and \(V_{k}\) of two real vectors with components \(X_{r}\) and \(Y_{r}\) may be obtained on one DFT of the complex vector with components \(X_{r}+j Y_{r}\) resulting in \(\mathrm{W}_{\mathrm{k}}\), since ,
\[
\begin{aligned}
& U_{k}=\left(W_{k}+W_{*}^{*}\right. \\
& j V k=\left(W_{k}-W_{12-k}^{*}\right) / 2
\end{aligned}
\]

Thus, since we have defined the sequence \(\mathrm{W}_{\mathrm{k}}\left(\mathrm{z}^{\mathrm{M} / 2}\right)\) by
\[
W_{k}\left(z^{M / 2}\right)=\sum_{r=0}^{M / 2-1} e^{j 2 \pi k r /(M / 2)}\left[X_{r}\left(z^{M / 2}\right)+j Y_{r}\left(z^{M / 2}\right)\right]
\]
then the output may be written in the equivalent forms
\[
\left.\sum_{k=0}^{M / 2-1} z^{-k}\left\{\frac{A_{k}\left(z^{M / 2}\right)+B_{k}\left(z^{M / 2}\right)}{2 \vdots} W_{k}\left(z^{M / 2}\right)\right]+\left[\frac{A_{k}\left(z^{M / 2}\right)-B_{k}\left(z^{M / 2}\right)}{2} W_{M / 2-k}^{*}\left(z^{M / 2}\right)\right]\right\}
\]
and many others.

\section*{COMMUNICATION WITH UNSYNCHRONIZED FDM TRANSMITTERS USING MAXIMALLY DECIMATED FILTER BANKS}

Maximally decimated filterbanks[1] achieve very efficient communications, approaching efficiencies of 100 per cent. But it requires that users be synchronized. We examine methods to achieve comparable performance with users having different delays. We are primarily concerned with the uplink of transmission.

A second problem in these multi-user communication systems is the near-far problem in which the users are not received with the same signal levels. These variations may arise from multipath fading, or because nearer transmitters inherently achieve less attenuation.

\section*{1. PREVIOUS APPROACHES TO THIS PROBLEM}
a. Frequency orthogonal signaling.

The most common approach is to use filter banks thatare not 100 -percent spectrally efficient, but which do allow the users to be unsynchronized in time. European work[2-5] on transmultiplexers for Satellite Communications is typical. The idea is use a set of bandpass Nyquist filters that are notoverlapping in frequency at all. The efficiency then depends on the sharpness of the cutoff of the filters. The depth of the stopband depends on the tolerances to interference, and must include tolerances for near-far deviations. Efficient realizations using tree filterbanks is in the references cited.
channel spectrum


Figure 1. The use of non-overlapping spectrums means that there is little interference between users. But the receiver must them operate on each channel separately.
b. OFDM


The second approach [6] is to use orthogonal frequency division multiplexing, OFDM. Each user may be allowed to use QAM signaling. The interesting extension of the technique using the DFT is to use a synchronous extension. This means that the in-channel pulse response is held at its maximum for a fraction of a symbol time. The interchannel interference is held to be zero over the same fraction of time. The efficiency is decreased by the fraction of time that the response is held.

time

\section*{c: OFDM inchannel response}
dotted: no synchronous extension solid: synchronous extension


Figure 2. OFDM with synchronous extension has a guard time which decreases the signaling rate but allows a wide tolerance to timing errors, both inchannel and interchannel. But the time frequencies must still be accurate. Figure a illustrates inchannel responses for general systems of the type we consider, and figure \(b\) the interchannel response. Note that there is little room for timing error. Figure c, on the other hand, has a nariow pulse with a flat top, and Figure dillustrates that there is a significant amount of time in which the interchannel interference is small.

OFDM is therefore useful for quasi-syrighronous communication, in which users transmit synchronously with an error tolerance equal to half the hold time.
c. Orthogonal Multiplexed S-QAM or VSB systems.

The systems of this type have a long history[7-. They are mainly intended for applications in which a given channel is covered with a set of evenly spaced carriers. The modulation on each carrier is synchronous, but the number of levels, carrier level, etc. may be varied to cope with channels responses that are not flat with frequency, or for which the noise is not flat. These systems are the subject of our study.

\section*{CDMA}

There has been a great flurry of activity for CDMA for mobile radio systems, because such systems are quite efficient for cellular radio systems. CDMA channel responses, including the matched filters in the receivers, have responses rather different from those of narrowband systems shown above.


\section*{a: CDMA inchannel response}


\section*{b: CDMA interchannel response}

Figure CDMA systems use a set of transmit and receive matched filters, just like the two cases previously considered. But the filters are derived from pseudo-noise sequences, and produce a sharp localized pulse response in time, but which has small, but not negligible, noise-like responses, at other times.

Conventional CDMA systems operate on a per-channel basis, and treat the interference from other users as noise. The research in Multi-user CDMA systems has been to design a receiver that makes use of information about other user's received data in deciding about the data of a particular user. There are three review papers [14-17]. The background theory appears in several papers[18-24].

Applications have been made of these ideas to wavelet packet bases systems at MIT [25-30] in which nonorthogonal waveforms are used.

Because of the great commercial interest in CDMA, many additional studies have appeared, many of which \(\qquad\) have to do with realizing receivers that approximate the ideal multi-user receiver, which, even with a Viterbi receiver, is usually judged too complex to build. The theory has been extended to sub-optimal receivers[]. Among these are linear minimum mean square receivers[]. Adaptive Systems have been studied []. Engineering efficiency has been studied. Applications have been made to other systems like O-FDM, and trellis coded modulation. Finally, improvements in delay tracking have been made.

\section*{ACKNOWLEDGMENTS}

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\title{
STUDY INTO IMPROVED SPECTRUM EFFICIENCY FOR FDMA/TDMA TRANSMISSION IN MOBILE SATELLITE AND MOBILE ENVIRONMENTS:
}

\title{
3a. BASIC PROPERTIES OF VSB FILTERBANKS \\ Revision 1
}

\author{
W F McGee \\ Contract: U6800-6-1604
}
14. March 1997

\begin{abstract}
This report is associated with milestone 3 of the contract. It is a more-or-less completely revised version issued in preliminary form in December 1996. It examines the basic waveforms in VSB filterbanks previously derived, and the 'elementary' analysis of 'adjac̣ent channel interference, without using any multiuser detection properties. Thus, the purpose of the report is to clarify the problem.
\end{abstract}

The methodology has been to examine the pulse waveforms, compare with Saltzberg's analysis, and an error rate analysis of a simple signed-bit receiver using a characteristic function approximation to the error rate.

The conclusions are as follows:
1. When adjacent channels are not aligned in phase, there is a serious degradation in performance. With the adjacent channels at the same signal level as the channel under study, the worst combination of data symbols on one adjacent channel alone will close the eye opening.
2. The sensitivity to timing phase is large, but the use of the quadrature channel should permit phasing, and manipulation of the real signal should allow timing to be extracted.
3. Modified duobinary coding of waveforms does not reduce adjacent channel interference significantly. However, if the bandwidth of the overlap regien is reduced significantly, modified duobinary coding offers a significant reduction in sensitivity to adjacent channel interference.
4. It is likely preferable that adjacent channel crosstalk that is symmetrical is advișable, which implies that linear phase, or almost linear phase, transmit and receive filters be used.
\[
z
\]

\section*{INTRODUCTION}

The focus of this study is the filterbank-based communication system recently documented by Sablatash, McGee and Lodge[6]. It is a 32-channel bank, in which two stages of two-band filtering is followed by an 8 -band polyphase filterbank. The purpose of this note is to record calculations on basic filterbanks[5], to give some idea of the elementary signal processing that is involved. After reviewing Saltzberg's analysis, we give the pulse responses of our system, and deduce sensitivity measures. Finally, we present results on systems using modified duobinary coding. An appendix presents our reasons for preferring mean-squared error ( \(\mathrm{L}_{2}\) norm) rather than sum of absolute values ( \(\mathrm{L}_{1}\) norm) used by Saltzberg.

\section*{SALTZBERG'S ANALYSIS}

Filterbanks using vestigial single sideband with orthogonal multiplexing were introduced by R.W. Chang[1] in 1966, and were analyzed in detail by B. R Saltzberg[2] in 1967. These are the same filterbanks, in essence, that we are considering for use; Saltzberg used offset QAM which is equivalent to the Vestigial Sideband modulation that we are studying.. The filters are normalized so that the response is always equal to one if the channel is sampled at the pulse peak.

Saltzberg did not use error rate as a criterion for his analysis, since in those days it was thought to be too difficult to analyze or to simulate. Instead, he used eye closure as the error criterion. The eye closure criterion considers the effect of the worst possible transmitted data sequence, and ranks one system as better than another if the sensitivity to the worst data sequence is less. Saltzberg was particularly interested in the voiceband modem channel, subject as it is to delay and gain slopes.

Saltzberg considered two ideal filters, a 100 -percent and a 50 -percent raised cosine channel. In these terms, our filters correspond to 100 -percent raised cosine filters since the pulse spectrum becomes insignificant at the next carrier frequency. Finally, Saltzberg analyzed the staggered QAM system, which we know is equivalent to the VSB systems that we are studying.

Six types of distortion were analyzed, and these are presented in the following table.
D1 intersymbol interference from the in-phase channel
D2 intersymbol interference from the quadrature chạnnel,.:
D3 crosstalk from the in-phase part of the upper adjacent interference
D4 crosstalk from the quadrature part of the upper adjacent channel.
D5 crosstalk from the in-phase part of the lower-adjacent channel
D6 crosstalk from the quadrature.lower adjacent channel.
In the VSB context, these may be considered as follows
D1 intersymbol interference from even numbered time slots
D2 intersymbol interference from odd-numbered time slots
D3 crosstalk from pulses in even numbered time slots in the upper adjacent channel
D4 crosstalk from pulses.in odd-numbered time slots in the upper adjacent channel
D5 crosstalk from pulses in even numbered time slots in the lower adjacent channel D6 crosstalk from pulses in odd-numbered time slots in the lower adjacent channel.

Saltzberg uses as a distortion measure either D1+D2, which he calls the single channel distortion, or \(\mathrm{D} 1+\mathrm{D} 2+\mathrm{D} 3+\mathrm{D} 4+\mathrm{D} 5+\mathrm{D} 6\), the total distortion. When a 100 percent root raised cosine filter is used rather than a 50 percent root raised cosine filter, he found that the effect of carrier phase offset was less with the 100 -percent raised cosine filter due to the selower decay of the waveform, but that the total distortion was about the same, because it is swamped by the adjacent channel distortion.

When broadband distortions that become mores severe as the bandwidth is increased, such a linear delay distortion (i.e: parabolic phase), Saltzbery determined that the 50 -percent raised cosine pulses were less sensitive because the bandwidth was smaller, and the pulse spectrum was not a subject to the distortion that the broader band 100 -percent raised cosine pulses were subjected to.

We do not repeat Saltzberg's analysis, but we do present a comparable analysis.

\section*{ANALYSIS}

Figure 1 gives the real pulse response of the first channel[5], end to end. Note that there is no intersymbol interference at timing instants of 16 samples apart, and that there is the typical ringing effect present in narrow band filters. The response is symmetrical. This figure is also the average channel response conditioned on a +1 being transmitted. Although the response of all the channels is different, the samples channel responses are identical.

Figure 2 presents the pulse response of the imaginary part of the channel response, which indicates sensitivity to in-channel misadjustment in phase. This is the Hilbert transform of the real part of Figure 1, and the sharp zero crossing at the sampling point is useful for timing recovery and phase recovery. As the
theory predicts, the interference is zero at even sample points, and is a maximum at odd sampling points. As well, the maximum interference has a magnitude about 0.68 .

Figure 3 presents that absolute value of the pulse response of this channel; it shows the characteristics of an 32-band Nyquist filter, i.e., zeros every 32 samples.

Saltzberg presents in his Figure 3 the peak distortion as a function of channel timing misadjustment. At the decision point, the eye width is very narrow, about 24 percent Also, the signal peak is about 3.2 times the pulse peak.

The pulse response from the channel that is adjacent, i.e. channel 32, is presented in Figure 4. [Other responses to other channels are similar, but this response is purely imaginary, and easier to comprehend.] As designed, the real part (Figure 4) is almost identically zero.

But the imaginary part[Figure 5] is not zero. The response is characterized by a ramp down and up. The rate of change will depend on the bandwidth of the transition,band between the two filters. But the area is fixed by the design of the filters, since, with the normalization noted, the area under this curve must be about 16 . Since an interfering data stream occurs every 16 samples, this means that the peak interference from that adjacent channel, given the worst data sequence, is unity. This means, approximately, that, given a sequence of \(150 / 16=9\) sample pulses there is a probability that, if the phasing between the adjacent channels is at its worst, that the 'eye opening' will be completely closed, which means that, due to this one interferer alone, the error rate will be worse that the probability of this sequence occurring times 0.5 .

If there are two interfering channels, the peak interference will be twice as large.

\section*{MODIFIED DUOBINARY FILTERING}

When there are phase misadjustments in the adjacent channels, the triangular-shaped pulse response of the crosstalk indicated is not very useful; it is better to have a response which is flat, not triangular. This may be accomplished by passing the triangle signal through a digital differentiator. The useful filter would be the transfer function corresponding to modified duobinary filtering[3],
\[
M(z)=0.5-0.5 z^{-32}
\]
which has 32 zeros equally spaced around the unit circle, at the crossover frequéncy of the adjacent channels. Since the slope of the crosstalkfuñiction is approximately \(0.25 / 50\), or 0.005 per sample, the difference indicated will result in a pulse of \(32 * 0.005 / 2\), or about 0.08 with the bandwidth indicated. This is shown in Figure 9.

This means that the duobinary filtering has reduced the peak crosstalk from an adjacent channel under phase misalignment by more than \(20^{\prime} \log (0.25 / 0.08) \mathrm{dB}\), about 10 .

The main channel is, of course, also affected by the duobinary coding. and instead of the single peak of Figure 1, we have a doublet of Figure 7. The ripples of the response have been reduced in level, since the sidelobes have been added so that they tend to cancel. The imaginary part, Figure 8, is now symmetrical.

When a binary signal is applied to a duobinary channel, a three-level signal results, and the magnitudes are \(+1,0\), and -1 . Unfortunately, this means that the sensitivity has been reduced by 6 dB , since the distance to the decision threshold is reduced from 1 to 0.5 . The sensitivity to other fluctuations has been similarly reduced. However, the noise is reduced by \(3 \mathrm{~dB}[\mathrm{M}(\mathrm{z})\) is always less than unity].

Let us consider the effect of adjacent channel bandwidth. As we have argued, the peak interference is proportional to the area under Figure 5, and is fixed at unity. On the other hand, the peak of Figure 5 is
proportional to the adjacent channel bandwidth; the narrower the band, the smaller and wider the adjacent channel interference.

With duobinary coding, the peak interference with the worst data sequence is the area under the absolute value of Figure 9. This is just twice the area under each lobe, but this area is simply the peak of the pulse of Figure 5 , that is, \(2 \times 0.25\) or 0.5 . But since the distance to the threshold in duobinary is reduced from 1 to 0.5 this means that the adjacent channel will also completely close the eye under duobinary coding with these bandwidths. This would indicate that duobinary coding would be advantageous only is the bandwidth is reduced from 100 percent used, to 25 or 10 percent, thereby resulting in an increase in the filter lengths.

Since the duobinary coding introduces correlation in the decoded signal, it is possible to demonstrate[4] that a Viterbi sequence detector will recover 3 dB in performance, thereby providing the same performance, in noise, as binary signaling. However, the use of a modified duobinary filter in the receiver has introduced correlation in the noise samples, and the design of the Viterbi receiver will be complicated by this.

\section*{ERROR RATE ANALYSIS: N厅 CONVOLUTIONAL CODING}

Although much can be seen from an examination of pulse response, calculations have also been made of error rates for our system. scenarios. Here we study the impact of allowing the adjacent channels to be at their worst phase offsets. Instead of system simulation we have approximated the error rate using a steepest descent approximation, which at this stage of the investigation is warranted.

The error rate calculations are summarized in one plot, Figure 10. This shows the error rate as a function is signal to noise ratio for four systems. The first, labeled base, is for a single channel in which that adjacent channels are phased properly.

The line labeled 'equal level' shows the effect of adjacent channel interference from both adjacent channels at the same level as the receiver; the graph indicated an asymptotic error rate floor of about \(4 \times 10^{-2}\).

The line labeled 'base duobinary' shows the typical 3-dB degradation of a duobinary system in which the adjacent channels are properly phased.

The final result is labeled 'duobinary equal level' and shows that the error rate is even poorer than the base system for signal to noise ratios of interest. This decrease is due almost entirely to the effect that the ' 0 ' level with three level coding has two adjacent thresholds, whereas the ' 1 ' of binary and duobinary coding have only one adjacent threshold; as indicated previously, both the base and the duobinary system have eyes that are closed by one adjacent carrier of the same level but with the worst phase offset.

\section*{CONCLUSION}

The pulse responses have been examined. The inchannel response demonstrates the Nyquist property of no intersymbol interference. The sensitivity to mistiming appears to be large, as Saltzberg showed. The quadrature response is the Hilbert transform of the channel pulse, and may be used for phasing. The real crosstalk is zero, as expected, but the quadrature crosstalk signal is fixed in area, so the peak interference is always large; this suggests sensitivity to phase misalignments in the adjacent channel. On the other hand, the width is so large that timing misalignments between channels do not appear to be important. This sensitivity to phasing misadjustments is also shown when the error rate is determined.

The use of modified duobinary coding does not reduce the sensitivity to the adjacent when 100 -percent raised cosine filters are used.

The next phase of the work examines multi-user correlative receivers and the impact of spectral coding..
1. The real part of the response in channel to a transmitted pulse in channel 1. Since this is a VSB Nyquist filter response, the zero crossings occur every 16 samples as required. These responses are really sequences, which have been plotted with straight lines joining the sequence values.
2. The imaginary part of the response in channel 1 to a transmitted pulse in channel 1 . This is the Hilbert transform of Figure 1. The sharp transition is useful for time and phase adjustment.
3. The absolute value of the response in channel 1 to a transmitted pulse in channel 1 . The zero crossings occur every 32 samples, since this filter is an 32-band QAM filter.
4. The real part of the crosstalk response in channel 1 to a pulse in channel 8 . This is essentially zero, as predicted[ note the multiplier ].
5. The imaginary part of the crosstalk from channel 32 -into a receiver for channel 1 . The narrower the transition bandwidth of these filters, the wider and less high these crosstalk responses. But the area is fixed. The other significant feature of the curve is the shift. The pulse of Figure 1 is centered at sequence 244, whereas the crosstalk is centered at about sample 275 . This is because the transmit and receive filters are both minimum phase. If we consider crosstalk through two maximum phase networks, the pulse shape is advanced.
6. Real part of the pulse response for channel 1 transmitter and channel 1 receiver when the duobinary filter is inserted. In comparison to Figure 1, there are two lobes. But also, the ripples are somewhat smaller; this effect is more pronounced for smaller bandwidths.
7. Imaginary part of the pulse response for channel 1 transmitter and channel 1 receiver when the modified duobinary filter is inserted. In comparison to Figure 2, there are three lobes, and the response is symmetrical.
8. The real part of the crosstalk with duobinary coding. The real part of the crosstalk is essentially zero.
9. The imaginary part of the crosstalk when a modified duobinary filter is inserted into the signal stream. Although the response has been reduced from the peak of Figure 5, from 0.25 to 0.08 , the absolute value of the area is about 0.5 , thereby making the worst sequence in the adjacent channel capable of closing the eye opening.
10. Probability of error vs. SNR. Four cases are presented -base- the standard plot of binary signaling through matched filters,. -equal level-binary signaling, adjacent channels at equal level, worth phasing. -base duobinary- typical Per/SNR duobinary plot. -duobinary equal level- the same as equal level, but with duobinary coding. With these system bandwidths, there is no advantage to duobinary coding.

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APPENDIX 1 Basis for elementary calculations of crosstalk.
Since the crosstalk from channel 32 to channel 1 is purely imaginary and appears, from the plots, to be essentially triangular and of one sign, the worst interference comes from a stream of +1 s , and is effectively then \(1 / 16\) the sum of the coefficients. If the crosstalk transfer function is
\[
C(z)=\sum c_{k} z^{-k}
\]
then \(1 / 16\) the sum of the coefficients is \(C(1)\), i.e., \(C\) evaluated at zero frequency, which we have argued has magnitude \(M / 2\), where \(M\) is the number of channels. In our case, \(M=32\), so the peak interference is 1 . This is a general result since, for general \(M\), the sampling rate is \(2 / \mathrm{M}\), and so this sum is always unity. This means that one adjacent channel will close the eye, if there is 90 -percent phase misadjustment.

To evaluate the interference as noise we have to determine
\[
\frac{2}{M} \sum\left|c_{k}\right|^{2}
\]
and this is
\[
\frac{2}{M} \int_{0}^{2 \pi}\left|C\left(e^{j \omega}\right)\right|^{2} d \omega / 2 \pi
\]
and this integral is approximately given fy the formula ( \(2 / \mathrm{M}\) ) (height)(half-width) of \(\mathrm{C}(\omega)\). But the height is \((\mathrm{M} / 2)^{2}\), so the sum of squares is approximately ( \(\mathrm{M} / 2\) )(half-width). Thus, reducing the transition bandwidths of the filters will reduced the mean square interference from adjacent channels, but will not affect the peak interference.

Appendix 2

\section*{PLACEMENT OF DUOBINARY FILTER}

One of the issues in duobinary filtering is where to place the filter. The are two possibilities, before or after the noise source. The two possibilities do not affect the signal, or the interference, but will affect the noise.
a: Basic Nyquist system
noise

b: Duobinary filter before noise source
noise


\section*{c: Duobinary filter after nolse source:}


In placement \(b\), the received carrier power is reduced by \(1 / 2\) because of the filter compared to \(a\), the base system, and therefore for the same carrier power, we may increase the transmit level in by 3 dB , producing a received signal 3 dB larger. Of course, this is overcome by the 6 dB penalty in SNR because of 3-level reception.

In placement c , the carrier power is the same, but the noise power at the decision is reduced 3 dB by the filter. Therefore the noise performance is the same is \(b\) with its carrier power increased by 3 dB . The disadvantage of c is that the noise is correlated from bit-to-bit whereas in a : and b :, this is not the case.
-If peak transmitted power is the relevant parameter, as it would beif the output stage is to be driven as hard as possible, then duobinary coding is not as attractive, since it does not send any energy half the time, and increasing the transmitted power by 3 dB may not be feasible.

Finally it should be mentioned that transmit duobinary filtering may be done digitally; there is no need for an actual filter.

\section*{APPENDICES}

The MATLAB program m-files used in the calculations are appended. The filter coefficients are from report [6]..
```

%MAKEPLOT.M
%makeplots
% makes the plots for report 3a
%call mkfilt first
b=sample(imag(conv(transmit(32,:),receive(1,:))),16,0);
a=[1 b b];
d=0.2:0.2:3.0;
ploterr
gtext('equal level'); 僦..;
hold on
title('Error rates with adjacent channels equal level, 90 degrees phase error');
a=[1]
d=2:2:20;
ploterr;
gtext('base');
w=[[0.5 0-0.5];
c=conv(w,b);
a=[0.5 c c];
d=[];
plotduo
gtext('Duobinary:equal level');
a=[0.5];
gtext('base duobinary');
plotduo
hold off

```
```

\%ERR_RATE.M
function [pe,sigma]=err_rate(a,d)
\% ISI, calculates error rate
$\% \mathrm{a}(1)$ contains the peak, the rest is ISI
$\% \mathrm{~d}$ is a parameter usually between 2 and 30
$\mathrm{s}=\mathrm{a}(1)$;
elnot=-d*a(1);
eldouble=0;
for $\mathrm{j}=2$ :length $(\mathrm{a})$
$\mathrm{s}=\mathrm{s}-\mathrm{a}(\mathrm{i}) * \tanh (\mathrm{~d} * \mathrm{a}(\mathrm{i}))$;
$\mathrm{c}=\mathrm{abs}(\mathrm{d} * \mathrm{a}(\mathrm{i}))$;
temp $=\mathrm{a}(\mathrm{i}) / \cosh (\mathrm{c})$;
elnot=elnot $+c-\log (2)+\log (1+\exp (-2 * c))$;
eldouble=eldouble+temp*temp;
end
if $s>0$
$\mathrm{s}=\mathrm{sqrt}(\mathrm{s} / \mathrm{d})$;
elnot=elnot $+\left(d^{*} d^{*} s^{*}\right.$ s/2);
eldouble=eldouble+s*s;
pe=0.5*erfcx(d*sqrt(eldouble)/sqrt(2))*exp(elnot);
sigma $=$ sqrt $(2) *$ s;
else
pe=0.5;
sigma $=100000$;
end

```
```

%PLOTCHAN.M
z=-conv(transmit(1,:),receive(1,:));
plot(real(z));
grid
title('Channel 1 response: Real part');
xlabel('Sample every }16\mathrm{ samples');
pause
print
plot(imag(z));
grid
title('Channel }1\mathrm{ response: Imaginary part');
xlabel('Sample every 16 samples');
pause
print
plot(abs(z));
grid
title('Channel 1 response: absolute value');
xlabel('Sample every }16\mathrm{ samples');
pause
print
y=conv(transmit(32,:), receive(1,:));
plot(real(y));
grid
xlabel('Sample every 16 samples');
title('Crosstalk 32->1 Real part');
pause
print
plot(imag(y));
grid
xlabel('Sample every 16 samples');
title('Crosstalk 32->1 Imaginary part');
pause
print
w=[0.50000000000000000000000000000000-0.5];
zd=conv(z,w);
plot(real(zd)); . . . . .
grid
title('Channel 1 duobinary response: Real'part');
xlabel('Sample every 16 samples');
pause
print
plot(imag(zd));
grid
title('Channel 1 duobinary response: Imaginary part');
xlabel('Sample every 16 samples');
pause
print
yd=conv(y,w);
plot(real(yd));
grid
title('Duobinary Crosstalk 32->1 Real part');
xlabel('Sample every 16 samples');
pause
print

```
```

plot(imag(yd));
grid
title('Duobinary Crosstalk 32->1 Imaginary part');
xlabel('Sample every 16 samples');
pause
print

```
```

%FILTERT.M
function[transmit,receive]=filtert(a,k,M)
j=sqrt(-1);
b=conj(flipir(a));
for i=1:length(a)
transmit(i)=exp(j*(pi/2)*(k+0.5)*(1-2*(length(a)-1)/M))*a(i)*exp(j*(pi/M)*(i-1)**(2*k+1));
end
for i=1:length(a)
receive(i)=-exp(-j*(pi/2)*(k+0.5)*(1+2*(length(a)-1)/M))*b(i)*exp(j*(pi/M)*(i-1)*(2*k+1));
end

```
```

%MKFILT.M
clear all;
% makefilters: makes a set of transmitter and receive filters
load f88_8
f88_8=f88_8';
lawton=f88_8;
sum=0;
for i=1:length(lawton)
sum=sum+lawton(i)*conj(lawton(i));
end
sum=sqrt(sum);
lawton=lawton/sum;
for i=1:2:8
[transmit(i,:),receive(i,:)]=filtert(lawton,i-1,8);
end
for i=2:2:8
[transmit(i,:),receive(i,:)]=filtert(fliplr(lawton),i-1,8);
end
load mv4
load mv8
% the high pass filters
hmv4=mv4;
hmv8=mv8;
for i=1:2:length(hmv4)
hmv4(i)=-hmv4(i);
end;
for i=1:2:length(hmv8)
hmv8(i)=-hmv8(i);
end;
% hmv4 and hmv8 are the high pass versions
m4=zeros(1,1:(1+8*(length(mv4)-1)));
m8=zeros(1,1:(1+4*(length(mv8)-1)));
hm4=m4;
hm8=m8;
for i=1:length(mv4)
hm4(1+8*(i-1))=hmv4(i);
m4(1+8*(i-1))=mv4(i);
end
for i=1:length(mv8)
hm8(1+4*(i-1))=lmv8(i);
m8(1+4*(i-1))=mv8(i);
end
% construct the transmit and receive filters for the leaves of the tree
tr=[];
re=[];
tr=[ conv(m4,m8)];
re=[conv(fliplr(m4),fliplr(m8))];
tr=[tr;conv(fliplr(hm4),m8)];
re=[re;conv(hm4,fliplr(m8))];
tr=[tr;conv(fliplr(hm4),fliplr(hm8))];
re=[re;conv(hm4,hm8)];

```
```

tr=[tr;conv(m4,fliplr(hm8))];
re=[re;conv(fliplr(m4),hm8)];
% now make the thirty-two filters
t=[];
r=[];
for i=1:2:8
for j=1:4
t=[t;conv(transmit(i,:),tr(j,:))];
r=[r;conv(receive(i,:),re(j,:))];
end
for j=1:4
t=[t;conv(transmit(i+1,:),tr(5-j,:))];
r=[r;conv(receive(i+1,:),re(5-j,:))];
end
end
transmit=[];
receive=[];
tr=[];
re=[];
% scale
transmit=2*t;
receive=2*r;
t=[];
r=[];
i=sqrt(-1);
j=sqrt(-1);

```
```

%PLOTERR.M
if d==[]
d=2:2:30;
end
pe=zeros(1,length(d));
sigma=zeros(1, length(d));
for k=1:length(d)
[pe(k) sigma(k)]=err_rate(a,d(k));
end
sigma=-8.68*log(sigma);
for i=2:length(sigma)
if sigma(i)<sigma(i-1)
sigma(i)=sigma(i-1);
pe(i)=pe(i-1);
end
end
semilogy(sigma,pe);
xlabel('SNR [dB]');
ylabel('Pe');
grid;

```
```

%PLOTDUO.M
% normally a(1) will be about 0.5
if d==[]
d=2:2:30;
end
pe=zeros(1,length(d));
sigma=zeros(1,length(d));
for k=1:length(d)
[pe(k) sigma(k)]=err_rate(a,d(k));
end
sigma=-8.68*log(sigma)-3.0;
for i=2:length(sigma)
if sigma(i)<sigma(i-1)
sigma(i)=sigma(i-1);
pe(i)=pe(i-1);
end
end
semilogy(sigma,1.5*pe);
xlabel('SNR [dB]');
ylabel('Pe');
grid;

```

Channel 1 response: Real part


Channel 1 response: Imaginary part


\section*{2}

Channel 1 response: absolute value


\section*{3}


Crosstalk 32->1 Imaginary part


\section*{5}

Channel 1 duobinary response: Real part


Channel 1 duobinary response: Imaginary part


\[
8
\]

Duobinary Crosstalk 32->1 Imaginary part


Error rates with adjacent channels equal level, 90 degrees phase error


10
\(00000000000021 \mathrm{E}+0000\)
\(40278134399483 \mathrm{E}-0001\) -. \(21721039217445 \mathrm{E}-0001\) -6. \(40278134399483 \mathrm{E}-0001\) \(00000000000021 \mathrm{E}+0000\)
\(-7.21721039217445 \mathrm{E}-0001\)
\(00000000000021 \mathrm{E}+0000\)
\(0.00000000000000 \mathrm{E}+0000\) \(27918989578463 \mathrm{E}-0001\) \(65509918457476 \mathrm{E}-0001\) \(05745509635903 E+0000\) \(0.00000000000000 \mathrm{E}+0000\) \(14236874072934 \mathrm{E}+0000\) \(0.00000000000000 \mathrm{E}+0000\) \(00000000000000 \mathrm{E}+0000\) - \(4.63194208113660 \mathrm{E}-0002\) \(-4.17607227614819 \mathrm{E}-0002\) \(24067091227403 \mathrm{E}+0000\) \(.00000000000000 \mathrm{E}+0000\) \(-1.26823425105153 \mathrm{E}+0000\) \(.00000000000000 \mathrm{E}+0000\)
0. \(00000000000000 \mathrm{E}+0000\) \(.47569296961566 \mathrm{E}-0002\)
\(.63823555869763 \mathrm{E}-0002\)
\(-1.02356107114780 \mathrm{E}+0000\) \(.00000000000000 \mathrm{E}+0000\)
\(.68348752776696 \mathrm{E}-0001\)
. \(00000000000000 \mathrm{E}+0000\)
\(00000000000000 \mathrm{E}+0000^{\circ}\) \(-1.40948454943345 \mathrm{E}-0003\) \(49027580105406 \mathrm{E}-0006\)
\(.74165210262871 \mathrm{E}-0001\)
\(.00000000000000 \mathrm{E}+0000\) 0.0
\(.88392118940505 \mathrm{E}-0001\) 0
\(.00000000000000 \mathrm{E}+0000\)

\(00000000000000 E+0000\)
\(-1.17358105451601 \mathrm{E}-0006\) \(43716214615755 \mathrm{E}-0006\) \(04094810272267 \mathrm{E}-0003\) \(0.00000000000000 E+0000\)
\(34751142644831 \mathrm{E}-0003\)
\(0.00000000000000 \mathrm{E}+0000\)
\(00000000000000 \mathrm{E}+0000\) \(04756930047549 \mathrm{E}-0006\) \(9.20303551106275 \mathrm{E}-0007\) \(72412351434512 \mathrm{E}-0004\) \(00000000000000 \mathrm{E}+0000\)
\(7.29764730407603 \mathrm{E}-0004\) 0
\(.00000000000000 \mathrm{E}+0000\)
\(0.00000000000000 \mathrm{E}+0000\) - \(11091302922995 \mathrm{E}-0008\)
. \(17081552371984 \mathrm{E}-0007\) \(.37042698562900 \mathrm{E}-0005\) \(0.00000000000000 \mathrm{E}+0000\)
\(48016968008226 \mathrm{E}-0005\) \(0.00000000000000 \mathrm{E}+0000\)
\(.00000000000000 \mathrm{E}+0000\) \(-4.10536543860894 \mathrm{E}-0010\) 1. \(13793247968309 \mathrm{E}-0008\) \(.85741688646044 \mathrm{E}-0006\) \(.00000000000000 \mathrm{E}+0000\)
2. \(13535236634116 \mathrm{E}-0006\)

0
\(.00000000000000 \mathrm{E}+0000\)
\(0.00000000000000 \mathrm{E}+0000\) \(.29784319301628 \mathrm{E}-0010\) \(.38000733351422 \mathrm{E}-0009\) \(-3.24301668906853 \mathrm{E}-0008\) . \(00000000000000 \mathrm{E}+0000\) 0
\(.24388184446862 \mathrm{E}-0007\) 0.0
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    295115
3387e-06
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10514 e-05
00130127
0279385
.00029771
    000146091
    74529e-07
    16796e-07
1. \(13756 \mathrm{e}-06\)
    00433409
    \(7417 e-05\)
    \(7543 e-05\)
    .000976957
    1097e-05
    54797e-05
    35871e-09
4.70746e-07
    0273e-07
    000113436
2. \(35577 \mathrm{e}-06\)
    \(.07065 e-06\)
    00303829
    715e-06
-4.13227e-06
    72677e-09
    00619e-08
    9543e-08
-9.86603e-06
    82186e-07
    4907e-07
    6762e-05
\(2.9543 e-07\)
    53706e-07
    6903e-10
    70227e-10
5.17875e-09
- \(77253 \mathrm{e}-07\) :
    46048e-08
    02202e-09
3.97213e-07
Э \(32157 \mathrm{e}-08\)
    94777e-08
    10852e-10
\(-4.89375 \mathrm{e}-11\)
    52529e-10
    \(35635 e-08\)
-1.02927e-08
8. \(66809 \mathrm{e}-09\)
    5241e-07
    26287e-08
\(-1.57695 e-08\)
\(3.07623 e-11\)
    \(4933 e-11\)
    10954 e-10
\(-3.6903 e-08\)
    .21712e-09
    35773e-10
    \(00299 e-07\)
9.39615e-09
    5. 52571e-09
    \(.56826 e-12\)
    38107e-13
\(-4.83386 e-11\)
    \(.8942 \mathrm{e}-09\)
    \(.11149 e-09\)
    -1933e-09

\title{
STUDY INTO IMPROVED SPECTRUM EFFICIENCY FOR FDMA/TDMA TRANSMISSION IN MOBILE SATELLITE AND MOBILE ENVIRONMENTS:
}

\section*{3c. EFFECT OF LINEAR PHASE PROTOTYPES ON FILTERBANK CROSSTALK}

\author{
W F McGee
}

Contract: U6800-6-1604
14 March 1997

The calculations of the report [1] have been extended to include a linear phase filter. The particular design used is the MPEG 64-channel linear phase baseband filter, the coefficients of which were supplied by Seymour Shlien. [Although not immediately apparent, these coefficients are 17-bit integers.] After decimating the coefficient array, the same calculations as documented in the previous report have been made.

The frequency response of the filters is shown in Figure 1; the stopband loss is more than adequate for this application. The crosstalk between channels is shown in Figure 2. The shapes are now symmetrical compared to the earlier report, and somewhat broader, and thus, not as high.

The pulse response with a mean-squared error equalizer is shown in Figure 3. The zero crossings have been kept, more or less, and the interchannel crosstalk has the downward blip which was evidenced in the earlier study. The frequency response of this filter is contrasted to the response of a matched filter in Figure 4.

Decision feedback, as expected, retards the pulse response, Figure 5, while maintaining the zero crossings and decreasing the adjacent channel crosstalk before the main sample. The frequency response for this decision feedback equalizer is shown in Figure 6.

The coefficients of the filter are included.; there are 256 coefficients, with mirror symmetry.
The performance of the various equalizers has been characterized às in the previous report, in a table.
TABLE 1 Mean squared error for VSB communication system with adjacent channels at equal level, 90 -degrees phase shifted, with various equalizers. There is no channel coding.'
\begin{tabular}{|l|c|c|c|}
\hline \multicolumn{1}{|c|}{ System } & \begin{tabular}{c} 
RKative Level \\
of adjacent \\
channels \\
\((\mathrm{dB})\)
\end{tabular} & \begin{tabular}{c} 
Relative phase of \\
adjacent \\
channels \\
(degrees)
\end{tabular} & \begin{tabular}{c} 
Mean Squared \\
Error \\
(Unbiased)
\end{tabular} \\
\hline Baseline & 0 & 0 & \(0.2(7 \mathrm{~dB})\) \\
\hline Matched Filter alone & 0 & 90 & \(0.436(3.6 \mathrm{~dB})\) \\
\hline \begin{tabular}{l} 
Minimum Mean Squared Error \\
Equalizer
\end{tabular} & 0 & 90 & \(0.295(5.3 \mathrm{~dB})\) \\
\hline \begin{tabular}{l} 
Single Decision Feedback \\
(Main channel only)
\end{tabular} & 0 & 90 & \(0.267(5.7 \mathrm{~dB})\) \\
\hline \begin{tabular}{l} 
Vector Decision Feedback \\
(Main+Adjacent Channels)
\end{tabular} & 0 & 90 & \(0.238(6.2 \mathrm{~dB})\) \\
\hline
\end{tabular}

\section*{CONCLUSION}

This linear phase filter improves performance.

\section*{Figures}
1. Frequency response of the transmit ands receive filters for 32 -channel communication system using filters MP32.
2. The absolute value of the sample crosstalk between all channels. There is no observable crosstalk between non-adjacent channels; adjacent channel crosstalk is smaller and symmetrical.
3. Response of the main channel and adjacent channel crosstalk when a minimum mean squared error receiver is implemented. The signal to noise ratio is 7 dB . The adjacent channel crosstalk is reduced.
4. Frequency response of the minimum mean squared error equalizer.
5. Pulse response and adjacent channel responses with a decision feedback equalizer. These equalizers assume that the in-channel response after the center may be eliminated by decision feedback. The adjacent channel crosstalk is also deduced a bit.
```

%

```
6. The equalizer frequency response corresponding to the pulse responses of Fig. 5.

\section*{ACKNOWLEDGMENT}

Seymour Shlien for the coefficients of the MPEG filters.

\section*{REFERENCES}
1. W F McGee "Study Into Improved Spectrum Efficiency For FDMA/TDMA Transmission In Mobile Satellite And Mobile Environments: 3b. Zero Forcing, Minimum MSE And Decision Feedback Multi-User Receivers, Jan 22, 1997, Contract: U6800-6-1604




Receive MSE equalizer with/without interference



5

ted 15:01 24 Jan 97
:oefficents for filter MP32
\(6974344700016660 e-006\) \(6974344700016660 e-006\) \(-2.6974344700016660 \mathrm{e}-006\) - \(-3948689400033320 e-006\) \(3948689400033320 e-006\) \(.0921266333833290 \mathrm{e}-006\) \(-1.0789561103385000 e-005\) \(3486995573386660 e-005\) \(6184430043388330 \mathrm{e}-005\) -1.8881864513390000e-005 \(-2.4276556676771660 e-005\) . \(9671425616774990 \mathrm{e}-005\) \(.7763552250158320 e-005\) \(-4.5855678883541650 e-005\) \(-5.6645416763548320 e-005\) \(.0132412336934980 e-005\) \(.3619231133699980 e-005\) - \(1.0250109564709000 e-004\) \(.2138296016048000 e-004\) \(.4296208236725000 \mathrm{e}-004\) \(.6993607351402330 e-004\) \(-1.9690971110755320 e-004\) 2. \(2927927830081660 \mathrm{e}-004\) 2.6164707772786320e-004 \(2.9941186741437320 e-004\) \(-3.3717488933466660 e-004\) 3.7493791125495990e-004 \(4.1539854442187660 e-004\) \(4.5585917758879310 \mathrm{e}-004\) \(-4.9362396727530310 \mathrm{e}-004\) \(5.2868937794897320 \mathrm{e}-004\) \(5.6105894514223640 e-004\) -5.8803328984225310e-004 \(-6.0691480080239970 e-004\) \(6.1500586677605310 \mathrm{e}-004\) 6.1230825552942970e-004 5.9612435581590650e-004 \(5.6105894514223640 \mathrm{e}-004\) \(-5.0980786698882660 e-004\) \(-4.3967527787526980 e-004\) \(-3.4257011182791320 e-004\) \(-2.2388405580756990 e-004\) \(-7.8224538970318300 \mathrm{e}-005\) \(9.7106226707086640 e-005\) \(2.9941186741437320 \mathrm{e}-004\) \(5.3138698919559650 \mathrm{e}-004\) \(7.9303406692345970 \mathrm{e}-004\) \(1.0816554893513400 \mathrm{e}-003\) \(1.3999488677258600 \mathrm{e}-003\) \(1.7398213683071500 \mathrm{e}-003\) \(2.1012729910952090 e-003\) \(2.4789032102981420 \mathrm{e}-003\) \(2.8700391633963590 \mathrm{e}-003\) \(3.2638444434817320 \mathrm{e}-003\) 3. \(6576674012292730 \mathrm{e}-003\) \(4.0407069850551860 \mathrm{e}-003\) \(4.4075538303364920 e-003\) \(4.7447198808400760 e-003\) \(5.0441264449557990 \mathrm{e}-003\) \(5.2922854664505450 \mathrm{e}-003\) \(5.4811182537141780 e-003\) \(5.5971013951892480 \mathrm{e}-003\) \(5.6294691946162950 \mathrm{e}-003\) 5. \(5647335957622010 e-003\) \(5.3947982293546650 \mathrm{e}-003\) \(5.1061750391605680 \mathrm{e}-003\) \(4.6907676559076080 \mathrm{e}-003\)
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9.6135192724275220e-002 \(6173830093395490 \mathrm{e}-002\) \(6306689401863750 e-002\) 6. \(6609607820374700 \mathrm{e}-002\) 5. \(7155063763811420 \mathrm{e}-002\) \(.8010939454893660 e-002\) \(.9241758360529500 e-002\) \(3.0898608924380440 \mathrm{e}-002\) \(3035761569297920 e-002\) \(5693450654373150 e-002\) . \(9014100073703300 \mathrm{e}-003\) \(2.6920074277165190 \mathrm{e}-003\) - \(2.9185820237058460 e-003\) \(9168526130014810 e-003\) . \(2302822017832550 e-002\) \(-1.6073785555887570 \mathrm{e}-002\) \(.9243195928075320 e-002\) 2. \(1827316583589070 e-002\) \(2.3845044943284920 e-002\) \(-2.5325902702981100 e-002\) 2. \(6299588335117490 \mathrm{e}-002\) \(2.6798628738080640 \mathrm{e}-002\) \(2.6863328981610400 \mathrm{e}-002\) \(-2.6536999338014940 e-002\) 2. \(5854464801762480 e-002\) \(2.4867344146379460 e-002\) \(-2.3615765664984100 e-002\) \(-2.2142862853262910 e-002\) \(2.0492122760145690 e-002\) \(1.8706502104697310 e-002\) \(-1.6823671984994790 e-002\) \(-1.4881533308723320 e-002\) -1.2915140879959280e-002 \(-1.0959531825116870 e-002\) \(-9.0389775743508200 e-003\) \(-7.1831766001010340 e-003\) \(-5.4109732902371050 e-003\) \(-3.7439874255887850 e-003\) \(-2.1956893847270260 e-003\) \(7.7685016720993640 e-004\) \(4.9901918976854980 e-004\) \(1.6319257572733000 \mathrm{e}-003\) \(2.6164707772786320 \mathrm{e}-003\) \(3.4526595530831990 e-003\) \(4.1404973879856480 \mathrm{e}-003\) \(4.6907676559076080 e-003\) \(5.1061750391605680 e-003\) \(5.3947982293546650 e-003\) \(5.5647335957622010 e-003\) \(5.6294691946162950 \mathrm{e}-003\) \(5.5971013951892480 e-003\) \(5.4811182537141780 e-003\) \(5.2922854664505450 \mathrm{e}-003\) \(5.0441264449557990 \mathrm{e}-003\) \(4.7447198808400760 \mathrm{e}-003\) \(4.4075538303364920 \mathrm{e}-003\) \(4.0407069850551860 e-003\) 3. \(6576674012292730 e-003\) 3. \(2638444434817320 e-003\) \(2.8700391633963590 \mathrm{e}-003\) \(2.4789032102981420 e-003\)
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} \(-5.3948689400033320 e-006\) [-2.6974344700016660e-006 \(-2.6974344700016660 e-006\) \(-2.6974344700016660 e-006\)

\section*{APPENDIX 3}

\section*{ERROR RATE APPROXIMATION}

We review the approximation of error rates for intersymbol and interchannel interference-limited signals in Gaussian noise.

Given a signal x and (real) Gaussian noise n of variance \(\sigma\) forming a signal s , the probability density function is the Fourier transform of the characteristic function \(\Phi(\omega)\)
\[
p(s)=\int_{-\infty}^{\infty} \Phi_{s}(\omega) e^{-j \omega s} \frac{d \omega}{2 \pi}
\]
and the probability that \(s\) is less than 0 is
\[
\int_{-\infty}^{0} p(s) d s=\int_{-\infty}^{\infty} \frac{\Phi_{s}(\omega)}{-j \omega} \frac{d \omega}{2 \pi}=\int_{-\infty}^{\infty} \frac{\Phi_{x}(\omega) e^{-\omega^{2} \sigma^{2} / 2}}{-j \omega} \frac{d \omega}{2 \pi}=\int_{-\infty}^{\infty} \frac{E\left(e^{j \omega x}\right) e^{-\omega^{2} \sigma^{2} / 2}}{-j \omega} \frac{d \omega}{2 \pi} .
\]

Intersymbol interference is typically the sum of a large number of terms that are linearly independent and proportional to signaling symbols \(b_{k}\) that are, typically, +1 and -1 , equally probable, and a central response \(a_{0} b_{0}\) for which \(a_{0}\) is positive and \(b_{0}\) is +1 and fixed. It makes no difference whether the interfering symbols are in the same or an adjacent channel. In this case that characteristic function is
\[
\Phi_{x}(\omega)=e^{j \omega x_{0}} \prod \cos \left(\omega a_{k}\right)
\]
and the problem of approximating the error rate is to approximate the integral
\[
\int_{-\infty}^{\infty} e^{j \omega x_{0}} e^{-w^{2} \sigma^{2} / 2} \prod_{k \neq 0} \cos \left(\omega a_{k}\right) \frac{d \omega}{-2 \pi j \omega}
\]

The best method appears to be to use the method of stationary phase by changing the line of integration from along the real \(\omega\) axis to a line parallel to the real axis but displaced by a distance \(d\), i.e., replace \(\omega\) by \(\omega+\mathrm{jd}\), and to choose d to make the logarithm of the product of the terms not including \(1 / \omega\) stationary, i.e., have a vanishing first derivative, along thée real d axis. Then replace the logarithm with only the constant and second power in d, and perform the now-trivial integration.

Thus define
\[
\begin{aligned}
& L(j \omega)^{\prime}=j \omega a_{0}+(j \omega)^{2} \sigma^{2} / 2+\sum \ln \left(\cosh \left(j \omega a_{k}\right)\right) \\
& L(-d)=-d a_{0}+d^{2} \sigma^{2} / 2+\sum \ln \left(\cosh \left(d a_{k}\right)\right) \\
& \approx L\left(-d_{0}\right)+\left(d-d_{0}\right)^{2} L_{0} / 2=L_{0}+\left(d-d_{0}\right)^{2} L_{0}^{\prime \prime} / 2
\end{aligned}
\]
where \(\mathrm{d}_{0}\) is chosen to satisfy \(\mathrm{L}^{\prime}\left(-\mathrm{d}_{0}\right)=0\).
Then the error rate is approximated by
\[
\begin{equation*}
P E \approx Q\left(d_{0} \sqrt{L_{0}^{\prime \prime}}\right) e^{d_{0}^{2} L_{0} / 2} e^{L_{0}} \tag{*}
\end{equation*}
\]

For example, if there is no intersymbol interference, the optimum value of \(-\mathrm{d}_{0}\) is obtained by finding the stationary point of
\[
L=-d a_{0}+d^{2} \sigma^{2} / 2
\]
which is at \(d=a_{0} / \sigma^{2}\) with the second derivative equal to \(\sigma^{2}\). The approximation is then \(\mathrm{Q}\left(\mathrm{a}_{0} / \sigma\right)\) which is an exact result.

For non-zero intersymbol interference, it is necessary to solve the equation
\[
0=-a_{0}+d \sigma^{2}+\sum_{k} a_{k} \tanh \left(d a_{k}\right)
\]
i.e.
\[
\begin{equation*}
d \sigma^{2}=a_{0}-\sum_{k} a_{k} \tanh _{r, t}\left(d a_{k}\right) \tag{**}
\end{equation*}
\]
and evaluate
\[
\begin{equation*}
L^{\prime \prime}=\sigma^{2}+\sum_{k} \frac{a_{k}^{2}}{\cosh ^{2}\left(d a_{k}\right)} \tag{}
\end{equation*}
\]

To solve ** explicitly for d, a common approach is to assume a value of d , calculate the right hand side RHS, and then update d to \(\left(\mathrm{d}+\mathrm{RHS} / \sigma^{2}\right) / 2\). It is probably just as effective to use a Newton method, since the second derivative is needed at the next step.

If solving the nonlinear equation \(* *\) appears daunting, another approach is simply to assume a variety of values of \(d\), calculate \(\sigma\) from the equation \({ }^{*}, L^{\prime \prime}\) from ( \({ }^{* *}\) ). In this connection there are two asymptotic values of interest.

When \(d\) is very large, we may replace \(\tanh \left(\mathrm{da}_{\mathrm{k}}\right)\) by \(\operatorname{sign}\left(\mathrm{a}_{\mathrm{k}}\right)\), thus *
\[
\begin{align*}
& \sigma^{2}=\frac{a_{0}-\sum\left|a_{k}\right|}{d}, \\
& \dot{L}^{\prime}=\sigma^{2}, \\
& L_{0}=-d\left(a_{0}-\sum\left|a_{k}\right|\right)-K \ln (2)+d^{2} \sigma^{2} / 2
\end{align*}
\]
and the error rate approximation is
\[
P E \approx \frac{1}{2^{K}} Q\left(\frac{a_{0}-\sum_{k=1}^{K}\left|a_{k}\right|}{\sigma}\right)
\]
which states that when the noise is very small, the error rate is determined by the one data sequence in \(2^{\mathrm{M}}\) which makes the signal most sensitive to error.

On the other hand, when \(\mathbf{d}\) is small, the resultant equations are
\[
\begin{aligned}
& d=\frac{a_{0}}{\Sigma^{2}} \\
& \Sigma^{2}=\sigma^{2}+\sum a_{k}^{2} \\
& L^{\prime \prime}=\Sigma^{2} \\
& L_{0}=-d a_{0}+d^{2} \Sigma^{2} / 2 \\
& P E \approx Q\left(\frac{a_{0}}{\Sigma}\right)
\end{aligned}
\]

This is ordinarily interpreted as meaning that when the noise is large, the error may be determined by considering the noise to be increased by the sum of squares of the interference, and this augmented noise power be used to approximate the error rate.

The calculation has been implemented in two C++ progratns, attached, one of which accepts the noise power as input as well as the intersymbol interference terms, and other accepts the parameter d .

For bandpass systems the noise \(\sigma^{2}\) is replaced by \(2 \sigma^{2 s}\), since the noise power is the sum of the \(I\) and \(Q\) noise, but only the I (or Q ) noise influences the error rate in a synchronous system.
```

// pe1.cpp -- given sigma, finds pe
\#include <iostream.h>
\#include <math.h>
\#include <conio.h>
// given sigma, computes pe
double error(double a[10],int k,double sigma);
double pi=3.14159;
double q(double x);
int main(void)
{
clrscr;
double a[10]={1,0,0,0,0,0,0,0,0,0,0};
double sigma;
for (int k=0;k<20;k++) ..;';
( `
sigma=exp(-k/8.68);:
cout << "\n" << ermor(a,10, sigma) << ". " << k;
}
return 0;
}
double error(double a[10], int k,double sigma)
// d is fixed, returms error rate
{
double elnot, eldouble}=0.0
double d;
double delta;
double c,temp;
int i;
int counter=0;
d=sigma*sigma;
for (i=0;i<k;i++) d=d+a[i]*a[i];
d=1.0/d;
while ((fabs (delta/d)>0.00001)\&\& ((counter)<30))
{
counter++; delta=1.0;

```
```

        for (i=0;i<k;i++)delta=delta-a[i]*tanh(d*a[i]);
        delta=0.5*((delta/(sigma*sigma))-d);
        d=d+delta;
    }
    elnot=-d;
    for (i=0;i<k;i++)
    {
            c=fabs(d*a[i]);
            if (c>500) temp=0;
            else temp=a[i]/cosh(c);
            elnot=elnot+c-log(2)+\operatorname{log}(1+\operatorname{exp}(-2*c));
            eldouble=eldouble+temp*temp;
    }
    elnot=(d*d*sigma*sigma/2)+elnot;
    eldouble=eldouble+sigma*sigma;
    double pe=q(d*sqrt(eldouble))*exp((d*d*eldouble/2)+elnot);
    return pe;
    }
double q(double x) // error function
// from Numerical Recipes in C page 176
{
double t,z,ans;
z=fabs(x/sqrt(2));
t=1.0/(1.0+0.5*z);
ans=0.5*t*exp(-z*z-
1.26551223+t*(1.00002368+t*(0.37409196+t*(0.09678418+
t*(-0.18628806+t* (0.27886807+t* (-1.13520398+t*(1.48851587+
t*(-0.82215223+t*0.17087277)))))))));
return x>=0 ? ans : 1.0-ans;
}
// pe2.cpp given values of d, calculates error rate given sigma, a[k]
// assume main signal is unity
\#include <iostream.h>
\#include <math.h>
\#include <conio.h>
double error(double a[10],int k,double sigma);
double q(double x); // errior function
double pi=3.14159;
int main(void)
{
clrscr; .䘫'
double a[10]={0,0,0,0,0,0,0,0,0,0};
double d=0.1;
for (int i=0;i<10;i++)
{
error(a,10,d);
d=2*d;
}
return 0;
}
double error(double a[10],int k,double d)
// d is fixed, returns error rate
{
double elnot, eldouble=0.0;
double sigma=1.0;
double c,temp;
elnot=-d;
for (int i=0;i<k;i++)
{

```
```

                sigma=sigma-a[i]*tanh(d*a[i]);
                c=fabs(d*a[i]);
                if (c>500) temp=0;
                else temp=a[i]/cosh(c);
                elnot=elnot+c-\operatorname{log}(2)+\operatorname{log}(1+\operatorname{exp}(-2*C));
                eldouble=eldouble+temp*temp;
    }
    sigma=sqrt(sigma/d);
    elnot=(d*d*sigma*sigma/2)+elnot;
    eldouble=eldouble+sigma*sigma;
    double pe=q(d*sqrt(eldouble))*exp((d*d*eldouble/2)+elnot);
    cout << "\n" << d << " " << sigma << " snr = " << -8.68*log(sigma) <<
    " Error rate = " << pe;
return pe;
}
double q(double x) // error function
// from Numerical Recipes in C page,176
{
double t,z,ans;
z=fabs(x/sqrt(2));
t=1.0/(1.0+0.5*z);
ans=0.5*t*exp(-z*z-
1.26551223+t*(1.00002368+t**0.37409196+t*(0.09678418+
t*(-0.18628806+t*(0.27886807+t* (-1.13520398+t*(1.48851587+
t*(-0.82215223+t*0.17087277)))))))));
return x>=0 ? ans : 1.0-ans;
}

```

\title{
PROGRAM BAUER: SPECTRAL FACTORIZATION OF MATRIX POLYNOMIALS
}

\author{
W F McGee
}

01/13/97 4:55 PM
The purpose of this note is comment the program bauer.cpp.

\section*{USAGE}

\section*{An input file INPUT.DTA}
contains the following
line 1 -order of matrices \(A i, i>=0\)
line 2 number of matrices
line 3 maximum number of iterations
line 3 and so on elements of matrices by row and colimni.e. 11,12, etc.
An output file RESULTS.DTA will contain the following
line 1 -order of matrices Bi
line 2 number of matrices
line 3 etc. elements of the matrices Bi .

\section*{DETAILS}

Kazanjian[1] has presented a FORTRAN program to factor matrix polynomials using Bauer's method. By this we mean factoring a matrix the elements of which are which are powers of \(z\) and \(z^{-1}\), into the product of a matrix whose polynomial elements are powers of \(z\) and one which is powers of \(z^{-1}\).

In addition we demand that the original matrix, when \(z\) is replaced by \(\exp (j \omega)\), is positive definite.
If \(A\) is the original matrix, it may be represented as
\[
\mathbf{A}=\cdots \mathbf{A}_{-k} z^{k}+\cdots \mathbf{A}_{-1} z+\mathbf{A}_{0}+\mathbf{A}_{1} z^{-1}+\cdots+\mathbf{A}_{k} z^{-k}+\cdots
\]

We impose the requirement that \(\mathbf{A}_{-k}=\mathbf{A}_{\mathbf{k}}{ }^{\mathrm{H}}\), where \({ }^{\mathrm{H}}\) indicates Hermitian transposition. The number of terms is necessarily finite (for computation), and only, the positive elements need be considered.

The matrix spectral factorization means to write \(\mathbf{A}=\mathbf{B}_{+}\left(z^{-1}\right) \mathbf{B}_{-}(z)\) where \(\mathbf{B}_{+}\left(\mathrm{e}^{-\mathrm{j} \omega}\right)^{\mathbf{H}}=\mathbf{B} \cdot\left(\mathrm{e}^{\mathrm{j} \omega}\right)\). Here
\[
\begin{aligned}
& \mathbf{B}_{+}=\mathbf{B}_{0}+\mathbf{B}_{1} z^{-1}+\mathbf{B}_{2} z^{-2}+\cdots \\
& \mathbf{B}_{-}=\mathbf{B}_{0}^{H}+\mathbf{B}_{1}^{H} z+\mathbf{B}_{2}^{H} z^{2}+\cdots
\end{aligned}
\]

The Bauer factorization is iterative, In it, the matrix Tm is formed. which an mxm Hermitian Toeplitz matrix with each elements a matrix,
\[
\mathbf{T}_{i, j}=\mathbf{A}_{i-j}
\]

The matrix \(\mathbf{T m}\) is factored by Cholesky decomposition into a product of two matrices \(\mathbf{T m}=\mathbf{L m L} \mathbf{m}^{\mathrm{H}}\). the Hermitian conjugates of each other, which are respectively lower and upper diagonal. The elements B are found as the limit for large \(m\) of the last row of \(L\).

The Cholesky decomposition is more familiar as the method of completing squares. The first step is to derive an iterative method to determine Lm.

We have
\[
\mathbf{T}_{m+1}=\left(\begin{array}{cc}
\mathbf{L}_{m} \mathbf{L}_{m}^{H} & \mathbf{A}_{1}^{m} \\
\mathbf{A}_{1}^{m^{H}} & \mathbf{A}_{0}
\end{array}\right)
\]
which may be written
\[
\mathbf{T}_{m+1}=\left(\begin{array}{cc}
\mathbf{L}_{m} & \mathbf{0} \\
\mathbf{A}_{1}^{m^{H}} \mathbf{L}_{m}^{H^{-1}} & \mathbf{L}^{(m)}
\end{array}\right)\left(\begin{array}{cc}
\mathbf{L}_{m}^{H} & \mathbf{L}_{m}^{-1} \mathbf{A}_{1}^{m} \\
\mathbf{0} & \mathbf{L}^{(m) H}
\end{array}\right)
\]
where
\[
\mathbf{L}^{(m)} \mathbf{L}^{(m) H}=\mathbf{A}_{0}-\mathbf{A}_{1}^{m H} \mathbf{L}_{m}^{H^{-1}} \mathbf{L}_{m}^{-1} \mathbf{A}_{1}^{m}
\]

The vector of matrices \(\mathrm{A}_{1}{ }^{\mathrm{m}}\) is
\[
\mathbf{A}_{1}^{m}=\left(\begin{array}{c}
\mathbf{A}_{m} \\
\mathbf{A}_{m-1} \\
\cdots \\
\mathbf{A}_{1}
\end{array}\right)
\]
and \(L(m)\) is the Cholesky decomposition of the matrix indicated.
The calculation \(\mathbf{L}_{m}{ }^{-1} A_{1}{ }^{m}\) is easily performed because \(L_{m}\) is lower triangular. Thus if we let \(B\) be the result, we have
i.e.
\[
\mathbf{A}_{1}^{m}=\mathbf{L}_{m} \mathbf{B}_{1}^{m}
\]
\[
\begin{aligned}
& \mathbf{A}_{m, 1}=\mathbf{L}_{m, 11} \mathbf{B}_{m} \\
& A_{m-1}=\mathbf{L}_{m, 21} \mathbf{B}_{m}+\mathbf{L}_{m, 22} \mathbf{B}_{m-1} \\
& \because \\
& \text { thus } \\
& \mathbf{B}_{m}=\mathbf{L}_{m, 11}^{-1} \mathbf{A}_{m} \\
& \mathbf{B}_{m-1}=\mathbf{L}_{m, 22}^{-1}\left(\mathbf{A}_{m-1}-\mathbf{L}_{m, 21} \mathbf{B}_{m}\right)
\end{aligned}
\]
and calculations such as
\[
\mathbf{B}_{m}=\mathbf{L}_{m, 11}^{-1} \mathbf{A}_{m}
\]
may be easily done also by back-substitution since each diagonal element \(\mathbf{L}_{m, 11}\) is itself lower triangular.
After a while we run out of coefficients for expanding the matrix Tm and have to start substituting zeros. In other words, Tm becomes banded. The resultant lower diagonal matrices then also become banded.

Examining the effect of introducing zeros for \(\mathbf{A}_{\mathrm{m}}\) indicates that, for example, only the elements of the lower triangular matrix \(L\) with indices larger than 2 are used. Since we are only interested in the limiting values of the elements so calculated, this means that the whole procedure may be accomplished with only \(\mathrm{m}^{*}(\mathrm{~m}+1) / 2\) , matrices of \(L\), where \(m\) is the highest positive order +1 in the matrix polynomial \(\mathbf{A}(z)\), and for the next iteration the matrices are shifted up and back to make room for a new bottom row of \(L\) to be calculated. In the actual algorithm we actually start with the matrix all zero matrices, and gradually build up \(L\) from the lower right corner until it is full, and then iterate using the principles given here.

Instead of an \(m\) by \(m\) array of matrices, Kazanjian uses a linear array of \(m(m+1) / 2\) matrices, and keep tracks of the differences in order to affect the required matrix multiplications.

By considering the limiting behaviour of
\[
\left.\left(\begin{array}{llll}
1 & z^{-1} & z^{-2} & \cdots
\end{array}\right) \mathbf{T}_{m}\left(\begin{array}{c}
1 \\
z \\
z^{2} \\
\cdots
\end{array}\right)=\left(\mathbf{E}_{m}^{H}(1) \quad z^{-1}: \dot{z}^{-2} \quad \ddots \cdot\right)\right){ }^{H} \mathbf{L}_{m}^{H}\left(\begin{array}{c}
1 \\
z \\
z^{2} \\
\cdots
\end{array}\right)
\]
we can argue (and Kazanjian proved) that the bottom row of the \(L\) matrix, in the limit, gives the matrices \(B\) required for the spectral factorization.

\section*{REFERENCE}
1. Nerses N. Kazanjian, "Bauer-type Factorization of Positive Matrices and the Theory of Matrix Polynomials Orthogonal.on the Unit Circle", Ph.D. Thesis, Polytechnic Institute of Brooklyn, 1977.

\section*{APPENDIX}
```

\#include <fstream.h>
\#include <math.h>
\#include <stdio.h>
// bauer factorization of positive definite matrices
// reference Appendix 1 page 69, Kazanjian thesis
int main()
{
cout << "BAUER calculates matrix spectral factorization\n";
cout << "Usage: input file is called 'input.dta'\n";
cout << "order of matrices, number of matrices, number of iterations\n";
cout << "coefficients of matrices\n";
cout << "Output is in file 'results.dta'\n";
ifstream fin("input.dta");
int n,m,lc;
fin >> n;
fin >> m;
fin >> lc;
int m2=(m* (m+1))/2;
cout << n<< " " << m<< " " << m2<<"\n";
double*** a=new double**[m];
for (int i=0;i<n;i++)
{
a[i]=new double*[n];
for (int j=0;j<n;j++)a[i][j]=new double[n];
}
double***b=new double**[m2];
for ( i=0;i<n;i++)
{
b[i]=new double*[n];
for (int j=0;j<n;j++) b[i][j]=new double[n];
}
for (int' k=0;k<m;k++)
for (int i=0;i<n;i++)
for (int j=0;j<n;j++)
{ 嫁,"
fin >> a[n-i-1][n-j-1][k]; ";
cout << a[n-i-1][n-j-1][k]<<<."\n";
}
fin.close();
cout << n << " " <<,m<< " " << m2<< "\n";
double sum;
int kl,k2,k3,k4;
// zero the working matrix
cout << "Zero the working matrix\n";
for (i=1;i<=n;i++)
for (int j=1;j<=n;j++)
for (int k=1;k<=m2;k++)
{
b[i-1][j-1][k-1]=0; }
int nc=0; // iteration counter
double trace1=0.0; // termination devices
double trace2=0.0;
// update the matrices based on the square root

```
```

    // 25
    iterate: if (nc>0)
{
k2=0;
if (m>1) for (int k=1;k<=m-1;k++)
{
k}3=\textrm{k}2+1
kl=m+1-k;
k2+=k1;
k4=k-1;
if (b[0][0][k3-1]!=0.0) // check if pivot is zero
for (int j=1;j<=n;j+i)
for (int i=1;i<=n;i++)
{
sum=0.0;
if ((k!=1) || (j!=1))
if (k!=1)
int 12=0;
int 14=0;
for (int ell=l;ell<=k4;ell++)
{
int 13=k+14;
int ll=m+1-ell;
12=12+11;
14=12-ell;
for (init ii=1;ii< =n;ii++)
sum=sum+b[i-1][ii-1][12-1]*b[j-1][ii-1][13-1];
}
}
if (j!=1)
for (int ii=1;ii<=(j-1);ii++)
sum=sum+b[i-1][ii-1][k2-1]*b[j-1][ii-1][k3-1];
}
}
b[i-1][j-1][k2-1]=(a[i,-1][j-1][kl-1]-sum)/b[j-1][j-1][k3-1];
}
}

```

```

    }
    }
    // square root
for (i=1;i<=n;i++)
{
sum=0.0;
k2=0;
if ((m-1)>0) for (int k=1;k<= (m-1); k++)
{
kI=m+1-k;
k2+=k1;
for (int j=1;j<=n;j++)
sum=sum+b[i-1][j-1][k2-1]*b[i-1][j-1][k2-1];
}
if (i!=1)
for (int j=1;j<=(i-1);j++) sum=sum+b[i-1][j-1][m2-1]*b[i-1][j-1][m2-1];
}
b[i-1][i-1][m2-1]=sqrt(a[i-1][i-1][0]-sum);
if (i<n)

```
```

    {
    for (int j=i+1;j<=n;j++)
    {
        sum=0.0;
        int k2=0;
        if (m>1) for (int k=1;k<=(m-1);k++)
        {
            kl=m+1-k;
                k2+=kl;
                for (int ell=1;ell<=n;ell++) sum+=b[i-1][ell-1][k2-1]*b[j-1][ell-1][k2-
    1];
} // 160
if (i>1)
for (int ell=1;ell<<(i-1);ell++)
sum+=b[i-1][e11-1][m2-1]*b[j-1][ell-1][m2-1];
} //180
b[j-1][i-1][m2-1]=(a[j-1][i-1][0]-sum)/b[i-1][i-1][m2-1];
} // 150
} // 110
} // 110
for (i=1,trace2=0;i< =n;i++)
{
trace2+=b[i-1][i-1][m2-1];
}
cout << "Trace difference " << fabs(trace2-tracel) << "\n";
if (((fabs(trace2-tracel)>1.0e-9)\&\& (nc<lc)))
tracel=trace2;
nc+=1;
k2=0;
if (m>1) for (int k=1;k<=(m-1); k++)
{
kl=m+1-k;
k3 =k2+1;
k2+=kl;
k4=k2-1;
for (int ell=k3;ell<<k4; ell+++)
for (int i=1;i<=n;i++)
for (int j=1;j<=n;j++)
b[i-1][j-1][el1-1]=b[i-1][j-1][el1+k1-1];
}
goto iterate;
}
else
{
// output routines ,
cout << "Number of Cyles"" << nc << "\n";
cout << "coefficients of the spectral factor of K\n";
k2=m2;
for (int k=1;k<=m;k++)
{
kl=k-1;
k2=k2-k1;
cout << "a!" << (k-1) <<" l\n";
for (int i=1;i<=n;i++)
{
for (int j=1;j<=n;j++)
{
if (a[n-i][n-j][k-1]<0) printf(" %6.4e " ,a[n-i][n-j][k-1]);
else printf(" +%6.4e " ,a[n-i][n~j][k-1]);}
cout << "\n";

```
```

        }
        cout.precision(15);
        k2=m2;
        ofstream fout("results.dta");
        fout << n << "\n";
        fout << m << "\n";
        for ( k=1;k<=m;k++)
    {
        k1=k-1;
        k2=k2-k1;
        cout << "b[ " << (k-1) << " ]\n";
        for (int i=1;i<=n;i++)
        {
            for (int j=1;j<=n;j+i)
            {
            fout << b[n-j][n-i][k2-1] << "\n";;
                if (b[n-j][n-i][k2-1]<0)priritf(" . %6.4e " ,b[n-j][n-i][k2-1]);
            else printf(" +%6.4e ",b[n-j][n-i][k2-1]);
            }cout << "\n";
        }
        }
        fout.close();
        goto exit;
    }
    exit:
for (i=0;i<n;i++)
for (int j=0;j<n;j++)delete[n]b[i][j];
delete[n]b[i];
}
delete[m2]b;
for (i=0;i<n;i++).
{
for (int j=0;j<n;j++)delete[n] a[i][j];
delete[n]a[i];
}
delete[m]a;
return 0;
}
INPUT, DTA
3
2
10
1
-1
O
-1
6
0.5 1
0
0.5
1
I
0
0
-1
0
-0.5
0

```
RESULTS.DTA
3
```

2
0.808466
0
-0.351098
2.3716
0
0.108569
0.405186
0.973226
0.183115
-0.183115
-0.0915577
0.421656
-0.421556
-0.210828
0
0

```

\title{
STUDY INTO IMPROVED SPECTRUM EFFICIENCY FOR FDMA/TDMA TRANSMISSION IN MOBILE SATELLITE AND MOBILE ENVIRONMENTS:
}

\title{
3b. ZERO FORCING, MINIMUM MSE AND DECISION FEEDBACK MULTI-USER RECEIVERS
}

\author{
W F McGee \\ Contract: U6800-6-1604 \\ 14 March 1997
}

\begin{abstract}
This report is a progress report of work towards milestone 3 of this contract. We investigate the decorrelating receiver for multi-user systems when there are only two adjacent channel users.
\end{abstract}

A MATLAB program to evaluate MSE equalizers has been made and is enclosed. Results indicate that a MSE equalizer tends to increase the cutoff of the charinel filters, which is consistent with our earlier studies.

A decision feedback design based on whitening the output from a MSE equalizer has been designed and tested.

The theoretical foundations for the study have been made and are presented herein.
For a system with a signal to noise ratio of \(7 \mathrm{~dB}, 90\)-degree phase shifting in the adjacent channels reduced to signal to mean-squared error to 2.45 dB , a minimum mean squared error equalizer increased this to 4.47 dB , decision feedback of the channel itself raised theratio of signal to mean squared error to 4.99 dB , and feedback of the decicision from the adjacent channels raised the signal to mean squared error to about \(G \cdot G\) dB.

These results are encouraging and deserve verification.
The contents of the report are as follows. In section 1 we argue that VSB filterbanks are very sensitive to poor phasing between adjacent channels. In section 2 we present the theory of the minimum mean squared error receiver, and give an example of the use "of a MATLAB program in section 3. Section 4 gives the theory of decision feedback equalization:when only the decoded data from the channel itself is used, and section 5 presents results of the MATLAB implementation of the calculations. Section 6 gives the theory for vector feedback, in which decisions are fed back from the channel and the two adjacent channels. Section 7 gives some numerical results of this the ofy which is based on the theory of matrix spectral factorization. Section 8 reviews the theory associated with equalizers with a finite number of taps, but these have not yet been implemented. Section 9 contains the summary of the results.

One appendix discussed some details of the theory, another some properties of the correlation matrices, and an attached report describes the C++ matrix factorization program BAUER.EXE that we have developed.

\section*{1. THERE IS NO ZERO-FORCING LINEAR RECEIVER FOR IMPROPERLY PHASED MAXIMALLY DECCMATED VSB SIGNALS}

There is an inherent and fundamental problem with the use of maximally decimated VSB filter banks when there is not phase synchronization between the adjacent channels. There is no zero-forcing equalizer. The argument is as follows. Consider a VSB channel at zero frequency over the positive frequencies and another over the negative frequencies, but overlapping the first at 0 frequency, and consider the transmitted sequence consisting of all \(\mathbf{+ 1}\) s. The output from both transmit filters is a dc signal. If the two channels are separated by 90 -degrees phase shift, then when the real part is taken, the dc signals may be separated. But if there were a linear zero forcing equalizer that could handle arbitrary phase shifts, that equalizer cannot eliminate the dc signal from the adjacent channel without eliminating the dc signal from the channel itself.

Thus, there is no zero-forcing equalizer; an equalizer which will eliminate adjacent channel interference from an adjacent channel that has its phase set improperly, in the absence of noise, without affect the channel itself.

Consequently there is no zero-forcing decorrelator.

\section*{2. MSE LINEAR RECEIVER FOR MAXIMALLY DECIMATED VSB SIGNALS}

Three data streams are presented to filters with responses \(\mathrm{H}_{0}(\omega), \mathrm{H}_{1}(\omega), \mathrm{H}_{2}(\omega)\), added together. Noise is added to the combination, and an equalizer \(\mathrm{E}(\omega)\) to minimize the mean square difference between the real part of its output and the input sequence to the filter H1i( \(\omega\) ).


Figure 2.1 Binary \((+/-1)\) data is presented to the input equalizers. These include channel gain, phase and delay, on a per channel basis. After an equalizer \(\mathrm{E}(\omega)\) we try to determine the digits sent through the equalizer H1. The filters without the channel gain, phase, and delay, are assumed to be those of adjacent channels of a VSB multi-channel communication system with no intersymbol or interchannel interference, and with restricted bandwidth. If there were no channel delay or phase shift, the performance would be optimum with an equalizer matched to the transmitted pulse shaping filter H 1 .
"This is a classic problem[1] in noise theory. The mean squared error is"
\[
\begin{aligned}
& M S E=\int_{-\infty}^{-\infty} N(f)|E(f)|^{2} d f \\
& +\int_{-1 / 2 T}^{1 / 2 T} \left\lvert\, \frac{1}{2 T} \sum_{m} H_{1}(f+m / T) E(f+m / T)+H_{1}^{*}(-f+m / T) E^{*}(-f+m / T)-1^{2} d f T\right. \\
& +\int_{-1 / 2 T}^{1 / 2 T}\left|\frac{1}{2 T} \sum_{m} H_{0}(f+m / T) E(f+m / T)+H_{0}^{*}(-f+m / T) E^{*}(-f+m / T)\right|^{2} d f T \\
& +\int_{-1 / 2 T}^{1 / 2 T}\left|\frac{1}{2 T} \sum_{m} H_{2}(f+m / T) E(f+m / T)+H_{2}^{*}(-f+m / T) E^{*}(-f+m / T)\right|^{2} d f T
\end{aligned}
\]
and the problem posed is to determine \(\mathrm{E}(\mathrm{f})\) to minimize the mean squared error MSE. The first term is the noise passing through the equalizer, which, for our purposes, we assume white with spectral density \(\mathrm{N}_{0}\), the second term represents the intersymbol interference in the channel under study, and the other two terms represent the crosstalk from the adjacent channels.

Before we start, it would appear reasonable that the receive filter be matched to the transmitted signal filter \(\mathrm{H}_{1}(\mathrm{f})\), and in particular, it will not pass frequencies beyond the bandedge of its filters.

The minimization is done by taking partial derivatives with respect to \(\mathrm{E}^{*}(\mathrm{f})\) and results in the equation
\[
\begin{aligned}
& 0=N_{0} E(f) \\
& +H_{1}^{*}(f)\left[\frac{1}{2 T} \sum_{m} H_{1}(f+m / T) E(f+m / T)+H_{1} *(-f+m / T) E^{*}(-f+m / T)-1\right] \\
& +H_{0} *(f)\left[\frac{1}{2 T} \sum_{m} H_{0}(f+m / T) E(f+m / T)+H_{0} *(-f+m / T) E^{*}(-f+m / T)\right] \\
& +H_{2} *(f)\left[\frac{1}{2 T} \sum_{m} H_{2}(f+m / T) E(f+m / T)+H_{2} *(-f+m / T) E *(-f+m / T)\right]
\end{aligned}
\]

Each of the expression in square brackets is periodic in \(f\) with period \(1 / T\), and we represent them by \(\lambda_{1}(f)\), \(\lambda_{0}(f)\) and \(\lambda_{2}(f)\), and also satisfies \(\left.\lambda_{i}{ }^{*}(-f)=\lambda_{i} f\right)\). These equation may be written
\[
\begin{aligned}
& 0=N_{0} E(f)+H_{1} *(f) \lambda_{1}(f)+H_{0} *(f) \lambda_{0}(f)+H_{2} *(f) \lambda_{2}(f) \\
& \lambda_{1}(f)=\frac{1}{2 T} \sum_{m} H_{1}(f+m / T) E(f+m / T)+H_{1} *(-f+m / T) E *(-f+m / T)-1 \\
& \lambda_{0}(f)=\frac{1}{2 T} \sum_{m} H_{0}(f+m / T) E(f+m / T)+H_{0} *(-f+m / T) E *(-f+m / T) \\
& \lambda_{2}(f)=\frac{1}{2 T} \sum_{m} H_{2}(f+m / T) E(f+m / T)+H_{2} *(-f+m / T) E *(-f+m / T)
\end{aligned}
\]

The equalizer \(E(f)\) is obtained by substituting the first equation in the three following, resulting in three equations for the three unknowns \(\lambda_{i}(f)\); and then putting these equations back into the first to solve for \(E(f)\).

There is another important result that is obtained from the expression fort \(E(f)\). Since the parameters \(\lambda_{i}(f)\) are periodic in \(f\), they may be realized with (perhaps infinitely long) FIR filters. Thus, when samples are taken at the equalizer output, this is equivalent to sampling the outputs of the three matched filters and passing them to FIR filters.

These three equations are of the form
\[
\begin{aligned}
& \lambda_{1}(f)=-G_{1,1} \lambda_{1}(f)-G_{1,0}(f) \lambda_{0}(f)-G_{1,2}(f) \lambda_{2}(f)-1 \\
& \lambda_{0}(f)=-G_{0,1} \lambda_{1}(f)-G_{0,0}(f) \lambda_{0}(f)-G_{0,2}(f) \lambda_{2}(f) \\
& \lambda_{2}(f)=-G_{2,1} \lambda_{1}(f)-G_{2,0}(f) \lambda_{0}(f)-G_{2,2}(f) \lambda_{2}(f)
\end{aligned}
\]
i.e.,
\[
\left(\begin{array}{c}
-1 \\
0 \\
0
\end{array}\right)=\left(\begin{array}{ccc}
1+G_{1,1} & G_{1,0} & G_{1,2} \\
G_{0,1} & 1+G_{0,0} & G_{0,2} \\
G_{2,1} & G_{2,0} & 1+G_{2,2}
\end{array}\right)\left(\begin{array}{l}
\lambda_{1}(f) \\
\lambda_{0}(f) \\
\lambda_{2}(f)
\end{array}\right)
\]
where
\[
\begin{aligned}
& G_{i, j}(f)=\frac{1}{2 T N_{0}} \sum H_{i}(f+m / T) H_{j}^{*}(f+m / T)+H_{i} *(-f+m / T) H_{j}(-f+m / T) \\
& =G_{j, i} *(f)=G_{i, j} *(-f)
\end{aligned}
\]

But, because of the properties of the filters, the elements whose indices are separated by 2 or more are zero. Thus, \(\mathrm{G}_{0,2}=\mathrm{G}_{2,0}=0\). [If the problem had been written in the frequency-ordered way, the matrix would be a bordered diagonal matrix, i.e., tridiagonal.] Also, because of the filters, there will only be one term in the sum for most of the elements, except perhaps for the main diagonal term. In any case, the equations are
\[
\left(\begin{array}{c}
-1 \\
0 \\
0
\end{array}\right)=\left(\begin{array}{ccc}
1+G_{1,1} & G_{1,0} & G_{1,2} \\
G_{0,1} & 1+G_{0,0} & 0 \\
G_{2,1} & 0 & 1+G_{2,2}
\end{array}\right)\left(\begin{array}{l}
\lambda_{1}(f) \\
\lambda_{0}(f) \\
\lambda_{2}(f)
\end{array}\right)
\]

These equations may be solved and result in
\[
\left(\begin{array}{l}
\lambda_{1} \\
\lambda_{0} \\
\lambda_{2}
\end{array}\right)=\frac{1}{\Delta}\left(\begin{array}{ccc}
\left(1+G_{00}\right)\left(1+G_{22}\right) & -G_{10}\left(1+G_{22}\right) & -G_{12}\left(1+G_{00}\right) \\
-G_{01}\left(1+G_{22}\right) & \left(1+G_{11}\right)\left(1+G_{22}\right), & G_{0,1} G_{1,2} \\
-G_{21}\left(1+G_{00}\right) & G_{2,1} G_{1,0} & \left(1+G_{11}\right)\left(1+G_{00}\right)
\end{array}\right)\left(\begin{array}{c}
-1 \\
0 \\
0
\end{array}\right)
\]
and the determinant
is real and positive.
\[
\Delta=\left(1+G_{11}\right)\left(1+G_{00}\right)\left(1+\dot{G}_{22}\right)-G_{01} G_{10}\left(1+G_{22}\right)-G_{21} G_{12}\left(1+G_{00}\right)
\]
\(\therefore 8+\)
Also, if the transmitting filters are root-Nyquist filters, the diagonal terms are a constant equal to \(1+2 \mathrm{~T} / \mathrm{N}_{0}\). In any case, we can solve exactly, with the result
\[
\begin{aligned}
& \lambda_{0}=-\frac{\lambda_{1} G_{0,1}}{1+G_{0,0}} \\
& \lambda_{2}=-\frac{\lambda_{1} G_{2.1}}{1+G_{2,2}}
\end{aligned}
\]

Solving for \(\lambda_{1}(f)\) and substituting back to find \(E(f)\) we have
\[
E(f)=\frac{H_{1} *(f)-H_{0} *(f) \frac{G_{0,1}}{1+G_{0,0}}-H_{2} *(f) \frac{G_{2,1}}{1+G_{2,2}}}{N_{0}\left[1+G_{1,1}-\frac{\left|G_{0,1}(f)\right|^{2}}{1+G_{0,0}}-\frac{\left|G_{2,1}(f)\right|^{2}}{1+G_{2,2}}\right]}
\]

The transfer function from the input to the sampled output is the sampled real part of \(H_{1}(f) E(f)\) which is
\[
\begin{aligned}
& \operatorname{sampled}\left\{\operatorname{Re}\left[H_{1}(f) E(f)\right]\right\}= \\
& \operatorname{sampled}\left\{\operatorname { R e } \left[\frac{\left.\left.H_{1}(f) H_{1}^{* *}(f)-H_{1}(f) H_{0} *(f) \frac{G_{0,1}}{1+G_{0,0}}-H_{1}(f) H_{2} *(f) \frac{G_{2,1}}{1+G_{2,2}}\right]\right\}}{N_{0}\left[1+G_{1,1}-\frac{\left|G_{0,1}(f)\right|^{2}}{1+G_{0,0}}-\frac{\left|G_{2,1}(f)\right|^{2}}{1+G_{2,2}}\right]}\right.\right. \\
& =\frac{\left[G_{1,1}-\frac{\left|G_{0,1}(f)\right|^{2}}{1+G_{0,0}}-\frac{\left|G_{2,1}(f)\right|^{2}}{1+G_{2,2}}\right]}{\left[1+G_{1,1}-\frac{\left|G_{0,1}(f)\right|^{2}}{1+G_{0,0}}-\frac{\left|G_{2,1}(f)\right|^{2}}{1+G_{2,2}}\right]} \\
& \text { sampled }\left\{\left[H_{0}(f) E(f)\right]\right\} \\
& =\operatorname{sampled}\left\{\frac{H_{0}(f) H_{1} *(f)-H_{0}(f) H_{0} *(f) \frac{G_{0,1}}{1+G_{0,0}}}{N_{0}\left[1+G_{1,1}-\frac{\left|G_{0,1}(f)\right|^{2}}{1+G_{0,0}}-\frac{\left|G_{2,1}(f)\right|^{2}}{1+G_{2,2}}\right]}\right. \\
& \left.=\frac{G_{01}(f)-G_{00} \frac{G_{0,1}}{1+G_{0,0}}}{\left[1+G_{1,1}-\frac{\left|G_{0,1}(f)\right|^{2}}{\left.\frac{\mid G_{2,1}}{}(f)\right|^{2}}\right.} 1\right] . \\
& =\frac{\frac{G_{0,1}}{1+G_{0,0}}}{\left[1+G_{1,1}-\frac{\left|G_{0,1}(f)\right|^{2}}{1+G_{0,0}}-\frac{\left|G_{2,1}(f)\right|^{2}}{1+G_{2,2}}\right]} \\
& \operatorname{sampled}\left\{\left[H_{2}(f) E(f)\right]\right\}=\frac{\frac{G_{2,1}}{1+G_{2,2}}}{\left[1+G_{1,1}-\frac{\left|G_{0,1}(f)\right|^{2}}{1+G_{0,0}}-\frac{\left|G_{2,1}(f)\right|^{2}}{1+G_{2,2}}\right]}
\end{aligned}
\]

An explicit calculation of the mean squared error is tedious, and the details are in an appendix. The result is
\[
M S E=\int_{-1 / 2 T}^{1 / 2 T} \frac{d f T}{\left[1+G_{1,1}-\frac{\left|G_{0,1}(f)\right|^{2}}{1+G_{0,0}}-\frac{\left|G_{2,1}(f)\right|^{2}}{1+G_{2,2}}\right]}
\]
and this also puts the power spectrum of the error in view; it is not flat. This is the so-called biased MSE[3,4], and is related to the unbiased MSEU by
\[
\frac{1}{M S E U}=\frac{1}{M S E}-1
\]

When the adjacent channels are in phase and the filters are half-Nyquist, satisfying
\[
\frac{1}{2 T} \sum\left|H_{i}(f+m / T)\right|^{2}+\left|H_{j}(-f+m / T)\right|^{2}=1
\]
then \(G_{0,1}\) and \(G_{2,1}\) both vanish. For example, the terms in \(G_{0,1}\) that are significant near zero frequency are
\[
H_{0}(f) H_{1} *(f)+H_{0} *(-f) H_{1}(-f)
\]
which vanishes if \(\mathrm{H}_{0}(\mathrm{f})=\mathrm{jH}_{1}(-\mathrm{f})\). On the other hand, if
\[
H_{0}(f)=j e^{j \theta} H_{1}(-f)
\]
then this sum is
\[
H_{0}(f) H_{1} *(f)\left[1-e^{-j 2 \theta}\right]=2 j \sin (\theta) H_{0}(f) H_{1} *(f)
\]

Then the equalizer is
\[
\therefore \underset{\prime}{2} E(f)=\frac{H_{1} *(f)}{N_{0}+1}
\]
the usual matched filter.
Consider the crosstalk between channel 1 and channel 0 , and suppose, as is common, that \(\mathrm{H}_{0}(\mathrm{f})=\mathrm{jH}(\mathrm{H}(\mathrm{f})\). Then the expression for \(\mathrm{G}_{1,0}\) is
\[
G_{1,0}(f)=\frac{1}{2 T N_{0}} \sum-j H_{1}(f+m / T) H_{1}^{*}(-f+m / T)+j H_{1} *(-f+m / T) H_{1}(f+m / T)=0
\]

On the other, for the worst phasing, \(\mathrm{H}_{0}(\mathrm{f})=\mathrm{H}_{1}(-\mathrm{f})\), and
\(G_{1,0}(f)=\frac{1}{2 T N_{0}} \sum H_{1}(f+m / T) H_{1} *(-f+m / T)+\dot{H}_{1} *(-f+m / T) H_{1}(f+m / T)\)
\(=\frac{1}{T N_{0}} \sum H_{1}(f+m / T) H_{1} *(-f+m / T)\)
The time has probably come to draw some pictures.

.Figure 2.2 Typical frequency responses. In a are plotted the responses of the transmit and receive filters, showing the normalization that results in a received pulse of height unity. The crossover frequency represents a loss of 3 dB , as indicated. Part b shows the response of the overall channels. The bandwidth is \(1 / 2 \mathrm{~T}\), the height is 2 T and the area is unity as required. The third figure indicates the response of the adjacent channel filters. These are narrow, but have a fixed height, \(T\).

We now examine the equalizer frequencyresponse in detail.
Looking first at the denominator in the band of interest, this denominator is proportional to
\[
1+\frac{1}{N_{0}}-\frac{\left|G_{0,1}(f)\right|^{2}}{1+\frac{1}{N_{0}}}-\frac{\left|G_{2,1}(f)\right|^{2}}{1+\frac{1}{N_{0}}}
\]

At 0 frequency, the fourth term is zero. \(\mathrm{G}_{0,1}\) is equal to \(\left(1 / \mathrm{N}_{0}\right)\) for the worst case phasing. Thus the denominator at zero frequency is equal to
\[
\frac{N_{0}+2}{N_{0}+1}
\]

At the frequency \(1 / 4 \mathrm{~T}\), the center of the filter H 1 , the denominator is just \(1+1 / \mathrm{N}_{0}=\left(\mathrm{N}_{0}+1\right) / \mathrm{N}_{0}\). The denominator appears to be periodic with period \(1 / 2 \mathrm{~T}\). If we include the factor \(\mathrm{N}_{0}\), this means that the transfer function is multiplied by an expression that varies from
\(\frac{1}{N_{0}+1}\) at the filter center frequency to \(\frac{N_{0}+1}{N_{0}\left(N_{0}+2\right)}\) at 0 and \(1 / 2 \mathrm{~T}\).

In the worst case, the numerator is
\[
H_{1} *-\frac{H_{0} * G_{0,1}}{1+1 / N_{0}}-\frac{H_{2} * G_{2,1}}{1+1 / N_{0}}
\]
which in this frequency band is equal to
\[
H_{1} *\left[1-\frac{\left|H_{0}(f)\right|^{2}}{T\left(N_{0}+1\right)}-\frac{\left|H_{2}(f)\right|^{2}}{T\left(N_{0}+1\right)}\right]
\]

The second factor is equal to 1 at the filter center frequency and \(\mathrm{N}_{0} /\left(\mathrm{N}_{0}+1\right)\) at 0 frequency.
Considering both terms together, the net effect is to multiply the matched filter \(\mathrm{H}_{1}{ }^{*}(\mathrm{f})\) by a factor which is
\[
1 /\left(N_{0}+2\right)
\]
at the bands edges at 0 and \(1 / 2 \mathrm{~T}\) and
\[
1 /\left(N_{0}+1\right)
\]
at the filter center frequency.

\section*{3. CALCULATION OF MSE••}

For actually computing MSE, a MATLAB program has been constructed, and a listing is enclosed.
Some figures are included for equalizers of a channel with a very sharp cutoff. Fig.3.1 is the pulse response of the transmit/receiver filter for one chingnel. The pair form a Nyquist filter with 100 -percent excess bandwidth. Fig. 3.2 is the absolute value of all the crosstalk responses. Most are very small, but the adjacent channel responses are large, as expected. There are three typical responses, and their mirrors. In the next few figures we build up the equalizer response for the first channel. First the denominator in Fig. 3.3 The pulse response of the denominator is shown in Fig. 3.4. For the numerator, . Fig 3.5 is the first term in the numerator of the equalizer transfer function. The second term, Fig. 3.6, and the third term, Fig. 3.7, represent crosstalk from the adjacent channels which the equalizer will try to reduce. Fig. 3.8 gives all the numerator. The overall equalizer response is in Fig. 3.9, which we compare to the equalizer design realized when there is no interference; this is simply the matched filter to the transmitted pulse. The pulse responses are shown in Fig. 3.10.

\section*{4. DECISION FEEDBACK EQUALIZATION}

The improvements that result with the use of data decisions is a complicated problem, and has been solved by Kavehrad and Salz[2]. But first, we consider a simpler approach, in which only that decoded data from the channel itself is used, and this approach is based on the use of a prediction filter to whiten the error sequence resulting from the MSE equalizer.

As we derived, the error sequence from the linear minimum MSE equalizer has the spectrum
\[
\operatorname{MSE}(f)=\frac{T}{\left[1+G_{1,1}-\frac{\left|G_{0,1}(f)\right|^{2}}{1+G_{0,0}}-\frac{\left|G_{2,1}(f)\right|^{2}}{1+G_{2,2}}\right]} .
\]

A spectral factorization of the denominator is of the form
\[
\left[1+G_{1,1}-\frac{\left|G_{0,1}(f)\right|^{2}}{1+G_{0,0}}-\frac{\left|G_{2,1}(f)\right|^{2}}{1+G_{2,2}}\right]=A_{0}\left(1+B_{+}\right)\left(1+B_{-}\right)=A_{0} \prod_{\mid z_{i}<1}\left(1-z^{-1} z_{i}\right) \prod_{\mid z_{i}>1}\left(1-z / z_{i}\right)
\]

Thus, \(1+B_{+}\)is causal and \(1+B\). is anti-causal. If the output of the MSE equalizer is followed with the equalizer ( \(1+B_{+}\)) the resulting sequence will have an error sequence that is white and the mean squared error is \(1 / \mathrm{A}_{0}\). This is equivalent to the following DFE structure


Figure 4.1 Decision Feedback Receiver. The causal filter B+ whitens the error sequence from the non-DFE equalizer \(\mathrm{E}(\mathrm{f})\) making the sequence spectrum white. The decision feedback removes the correlated part of the signal that results.

This is the receiver that minimizes
\[
\begin{aligned}
& . M S E=\int_{-\infty}^{\infty} N(f)|E(f)|^{2} d f \\
& +\int_{-1 / 2 T}^{1 / 2 T}\left|\frac{1}{2 T} \sum_{m} H_{1}(f+m / T) E(f+m / T)+H_{1}^{*}(-f+m / T) E^{*}(-f+m / T)-\left(1+B_{+}(f)\right)\right|^{2} d f T_{1} \\
& +\int_{-1 / 2 T}^{1 / 2 T}\left|\frac{1}{2 T} \sum_{m} H_{0}(f+m / T) E(f+m / T)+H_{0} *(-f+m / T) E^{*}(-f+m / T)\right|^{2} d f T \\
& +\int_{-1 / 2 T}^{1 / 2 T} 1 \frac{1}{2 T} \sum_{m} H_{2}(f+m / T) E(f+m / T)+\left.H_{2} *(-f+m / T) E^{*}(-f+m / T)\right|^{2} d f T
\end{aligned}
\]

The solution to these equations has already been derived, and is
\[
\left(\begin{array}{l}
\lambda_{1} \\
\lambda_{0} \\
\lambda_{2}
\end{array}\right)=\frac{1}{\Delta}\left(\begin{array}{ccc}
\left(1+G_{00}\right)\left(1+G_{22}\right) & -G_{10}\left(1+G_{22}\right) & -G_{12}\left(1+G_{00}\right) \\
-G_{01}\left(1+G_{22}\right) & \left(1+G_{11}\right)\left(1+G_{22}\right) & G_{0,1} G_{1,2} \\
-G_{21}\left(1+G_{00}\right) & G_{2,1} G_{1,0} & \left(1+G_{11}\right)\left(1+G_{00}\right)
\end{array}\right)\left(\begin{array}{c}
-\left(1+B_{+}\right) \\
0 \\
0
\end{array}\right)
\]
with the further requirement that \(\lambda_{1}\) have only positive exponents of \(z\), i.e. \(\lambda_{1}\) is anti-causal, and
\[
\Delta=\left(1+G_{11}\right)\left(1+G_{00}\right)\left(1+G_{22}\right)-G_{01} G_{10}\left(1+G_{22}\right)-G_{21} G_{12}\left(1+G_{00}\right) .
\]

But since
\[
\lambda_{1}=\frac{-\left(1+B_{+}\right)}{\Delta /\left[\left(1+G_{00}\right)\left(1+G_{22}\right)\right]}
\]
this means that \(1+B_{+}\)is equal to the spectral factor of the denominator which is causal, and \(\lambda_{I}\) is the anticausal remainder when these factors are canceled. In the appendix we show that the MSE is given by the average of \(-\lambda_{1}{ }^{*}\). If the'spectral factorization is
\[
\Delta /\left[\left(1+G_{00}\right)\left(1+G_{22}\right)\right]=\alpha\left(1+B_{+}\right)\left(1+B_{-}\right)
\]
then \(-\lambda_{1} *\) is \(1 /\left(\alpha\left(1+B_{+}\right)\right)\)and the integral for the MSE is simply \(1 / \alpha\). The factor \(\alpha\) is found, finally, by using Jensen's theorem concerning the logarithm of analytic functions.

The mean squared error for the decision feedback receiver is less than that of the MSE receiver[1] and is equal to
\[
M S E_{D F E}=\exp \left(-\int_{-1 / 2 T}^{1 / 2 T} \ln \left(1+G_{1,1}-\frac{\left|G_{0,1}(f)\right|^{2}}{1+G_{0,0}}-\frac{\left|G_{2,1}(f)\right|^{2}}{1+G_{2,2}}\right) d f T\right)
\]

This expression may be derived without spectral factorization, but is a useful check on the factorization.
However, we could also apply feedback from all the past received digits to further reduce the error, and this is the subject of vector feedback.

\section*{5. EXAMPLE}

The design of a single feedback equalizer using the above theory have been done using the filters of the 32 channel VSB communication system deecribed by Sablatash, McGee and Lodge[7]. The calculations are done by the MATLAB program EQUAL1.M attached. The program first calculates the transmit and receive filters based on a prototype filter design. These are then adjusted by the relative gains and phases of the adjacent channels; we usually assume eqtál Tevel and 90 -degrees phase shift. The minimum mean squared error equalizer is then computed, and its mean squared error calculated. The response to a pulse in the channel and the two adjacent channels is plotted in Fig. 3.10, and the frequency response is shown in Fig. 3.10. The required spectral factorization using the zeros of the denominator are then found, and those inside the unit circle used to define \(B_{+}\). The decision feedback equalizer is computed, along with the mean squared error for the decision feedback'receiver. The pulse responses are again displayed, in Fig. 5.1, and the channel frequency responses shown in Fig. 5.2. Observing the pulse responses, the decision feedback equalizer tends to keep the precursor response with good zero crossing, but the response after the center sample is allowed to vary, which is, of course, what should happen. In Fig. 5.3 we give the positive part of the spectral factorization of the denominator. This, convolved with its mirror image, gives the denominator, to within a scale factor.

The calculation was repeated with an interchange of the transmitter and receiver filters, which tends to make the crosstalk response the mirror image. This had minimal impact, except the equalized crosstalk was a mirror image of the other case.

\section*{6. VECTOR FEEDBACK}

With vector feedback we assume that we have access to all the previous received digits of all the channels.


Figure 6.1 Vector Feedback. In Figure 4.1 only the decoded output from the main channel is used for reducing mean squared error. In this model, all the relevant adjacent channel decoded data are used.

With a mean-squared error design criterion, the goal is to minimize
\[
\begin{aligned}
& M S E=\int_{-\infty}^{\infty} N(f)|E(f)|^{2} d f \\
& +\int_{-1 / 2 T}^{1 / 2 T} \frac{1}{2 T} \sum_{m} H_{1}(f+m / T) E(f+m / T)+H_{\mathrm{I}} *(-f+m / T) E^{*}(-f+m / T)-\left.\left(1+B_{+}(f)\right)\right|^{2} d f T \\
& +\int_{-1 / 2 T}^{1 / 2 T} \frac{1}{2 T} \sum_{m} H_{0}(f+m / T) E(f+m / T)+H_{0} *(-f+m / T) E^{*}(-f+m / T)-\left.C_{+}(f)\right|^{2} d f T \\
& ++\int_{-1 / 2 T}^{1 / 2 T} \frac{1}{2 T} \sum_{m} H_{2}(f+m / T) E(f+m / T)+H_{2} *(-f+m / T) E^{*}(-f+m / T)-\left.D_{+}(f)\right|^{2} d f T
\end{aligned}
\]
\[
\text { where } \mathrm{B}+, \mathrm{C}+\text { and } \mathrm{D}+\text { are causal and have only positive exponents of } \mathrm{z}^{-1} \text { and thus the unknowns are the }
\] (real) numbers \(\mathrm{b}_{\mathrm{k}}, \mathrm{c}_{\mathrm{k}}\) and \(\mathrm{d}_{\mathrm{k}}\), where
\[
\begin{aligned}
& B_{+}=\sum_{k=1}^{\infty} b_{k} z^{-k} \\
& C_{+}=\sum_{k=1}^{\infty} c_{k} z^{-k} \\
& D_{+}=\sum_{k=1}^{\infty} d_{k} z^{-k}
\end{aligned}
\]

When the partial derivative with respect to the equalizer \(E\) is taken we find that
\[
0=N_{0} E(f)+H_{1}^{*}(f) \lambda_{1}(f)+H_{0}^{*}(f) \lambda_{0}(f)+H_{2} *(f) \lambda_{2}(f)
\]
where
\[
\begin{aligned}
& \left.\lambda_{1}(f)=\frac{1}{2 T} \sum_{m}\left[H_{1}(f+m / T) E(f+m / T)+H_{1} *(-f+m / T) E^{*}(-f+m / T)\right]-\left(1+B_{+}(f)\right)\right] \\
& \lambda_{0}(f)=\frac{1}{2 T} \sum_{m}\left[H_{0}(f+m / T) E(f+m / T)+H_{0} *(-f+m / T) E^{*}(-f+m / T)-C_{+}(f)\right] \\
& \left.\lambda_{2}(f)=\frac{1}{2 T} \sum_{m}\left[H_{2}(f+m / T) E(f+m / T)+H_{2} *(-f+m / T) E^{*}(-f+m / T)\right]-D_{+}(f)\right]
\end{aligned}
\]

This is the same as before, except the forms of \(\lambda_{1}\) and \(\lambda_{2}\) are different The partial derivatives with respect to the coefficients of the polynomials \(\mathrm{B}+\mathrm{C}+\) and \(\mathrm{D}+\) lead to the requirement that \(\lambda_{1}, \lambda_{2}\), and \(\lambda_{0}\) have no terms that are powers of \(\mathrm{z}^{-1}\). As before, we may substitute for \(E\) and obtain a set of equations for \(\lambda_{1}, \lambda_{2}\), and \(\lambda_{0}\). These are
\[
\left(\begin{array}{c}
-1-B_{+} \\
-C_{+} \\
-D_{+}
\end{array}\right)=\left(\begin{array}{ccc}
1+G_{1,1} & G_{1,0} & G_{1,2} \\
G_{0,1} & 1+G_{0,0} & 0 \\
G_{2,1} & 0 & 1+G_{2,2}
\end{array}\right)\left(\begin{array}{c}
\lambda_{1}(f) \\
\lambda_{0}(f) \\
\lambda_{2}(f)
\end{array}\right)
\]

The solution to these equations is
\[
\left(\begin{array}{l}
\lambda_{1} \\
\lambda_{0} \\
\lambda_{2}
\end{array}\right)=\frac{1}{\Delta}\left(\begin{array}{ccc}
\left(1+G_{00}\right)\left(1+G_{22}\right) & -G_{10}\left(1+G_{22}\right) & -G_{12}\left(1+G_{00}\right) \\
-G_{01}\left(1+G_{22}\right) & \left(1+G_{11}\right)\left(1+G_{22}\right) & G_{0,1} G_{1,2} \\
-G_{21}\left(1+G_{00}\right) & G_{2,1} G_{1,0} & \left(1+G_{11}\right)\left(1+G_{00}\right)
\end{array}\right)\left(\begin{array}{c}
-\left(1+B_{+}\right) \\
-C_{+} \\
-D_{+}
\end{array}\right)
\]
where
\[
\Delta=\left(1+G_{11}\right)\left(1+G_{00}\right)\left(1+G_{22}\right)-G_{01} G_{10}\left(1+G_{22}\right)-G_{21} G_{12}\left(1+G_{00}\right)
\]

We rewrite these equations to put the fact of the tridiagonality of the matrix in evidence.
\[
\left(\begin{array}{c}
-C_{+} \\
-1-B_{+} \\
-D_{+}
\end{array}\right)=\left(\begin{array}{ccc}
1+G_{0,0} & G_{0,1} & 0 \\
\vdots G_{1,0}^{\prime \prime} & 1+G_{1,1} & G_{1,2} \\
\vdots 0 & G_{2,1} & 1+G_{2,2}
\end{array}\right)\left(\begin{array}{l}
\lambda_{0}(f) \\
\lambda_{1}(f) \\
\lambda_{2}(f)
\end{array}\right)
\]
which we simplify to the form
\[
A_{+}=\Gamma \Lambda_{-}
\]
where the subscripts + and - indicate that there are only positive or negative powers of \(\mathrm{z}^{-1}\) in a particular vector. For our application the matrix \(\Gamma\) has real and constant, but probably unequal, diagonal terms since we are dealing with Nyquist channels. The off diagonal terms are conjugates of each other and therefore the matrix is Hermitian on the unit circle. The off diagonal terms \(G_{2,1}\) and \(G_{0.1}\) are related, but the relationship is not used.

Equations of thus type may be solved by spectral factorization[5] of the matrix \(\Gamma=\Gamma_{+} \Gamma\).
Then there is a solution if we can find a constant vector \(K\) such that
\[
\begin{aligned}
& \mathbf{A}_{+}=\Gamma_{+} \mathbf{K} \\
& \mathbf{K}=\Gamma_{-} \Lambda_{-}
\end{aligned}
\]
so that the dc coefficient of the second component of \(\mathrm{A}_{+}\)is -1 , and of the second and third components is zero. If we write the dc component of \(\Gamma_{+}\)as \(\Gamma_{+0}\), then
\[
\begin{aligned}
& \mathbf{K}=\Gamma_{+0}{ }^{-1}\left(\begin{array}{c}
0 \\
-1 \\
0
\end{array}\right) \\
& \therefore A_{+}=\Gamma_{+} \Gamma_{+0}^{-1}\left(\begin{array}{c}
0 \\
-1 \\
0
\end{array}\right)_{\cdots} \\
& \Lambda_{-}=\Gamma_{-}^{-1} \Gamma_{+0}{ }^{-1}\left(\begin{array}{c}
0 \\
-1 \\
0
\end{array}\right)=\left(\Gamma_{+0} \Gamma_{-}\right)^{-1}\left(\begin{array}{c}
0 \\
-1 \\
0
\end{array}\right)
\end{aligned}
\]

As in the Appendix, the mean squared error for this equalizer may be calculated. Here, however, when the calculation is performed, we recognize three terms in the integral. The first is
\[
\left|\lambda_{1}(f)\right|^{2}\left(1+G_{1,1}\right)+\lambda_{1}^{*}\left(G_{1,0} \lambda_{0}+G_{1,2} \lambda_{2}\right)=\lambda_{1}^{*}\left(-1-B_{+}\right)
\]

The second is
\[
\left|\lambda_{2}(f)\right|^{2}\left(1+G_{2,2}\right)+\lambda_{2}^{*} \lambda_{1} G_{2,1}=\lambda_{2}^{*}\left(-D_{+}\right)
\]
and the third is
\[
\left|\cdot \lambda_{0}(f)\right|^{2}\left(1+G_{0,0}\right)+\lambda_{0}^{*} \lambda_{1} G_{0,1}=\lambda_{2 .}^{*}\left(-C_{+}\right)
\]

Thus, the mean squared error is
\[
\therefore \int_{\substack{2 / 1 / 2 T}}^{1 / 2 T} \Lambda_{-}^{H}(f) A_{+}(f) d f T=
\]

But notice that this simply is the negative of the dc coefficient of \(\lambda_{1}\), which we label \(\lambda_{1-0}\), since \(\Lambda^{H}\) has only positive terms, the product has only positive terms, and the integral of all but the dc term will be zero. This may also be seen by considering
\[
\begin{aligned}
& \Lambda_{-}^{H} \mathbf{A}_{+}=\left(\begin{array}{lll}
0 & -1 & 0
\end{array}\right) \Gamma_{+0}^{-H} \Gamma_{-}^{-H} \Gamma_{+} \Gamma_{+0}^{-1}\left(\begin{array}{c}
0 \\
-1 \\
0
\end{array}\right) \\
& =\left(\begin{array}{lll}
0 & -1 & 0
\end{array}\right) \Gamma_{-0}^{-1} \Gamma_{+0}^{-1}\left(\begin{array}{c}
0 \\
-1 \\
0
\end{array}\right) \\
& =\left(\begin{array}{lll}
0 & 1 & 0
\end{array}\right)\left(\Gamma_{+0} \Gamma_{-0}\right)^{-1}\left(\begin{array}{l}
0 \\
1 \\
0
\end{array}\right) \\
& =\left(\begin{array}{lll}
0 & 1 & 0
\end{array}\right)\left(\Gamma_{+0} \Gamma_{+0}^{H_{0}^{\prime-1}}-\left(\begin{array}{l}
0 \\
1 \\
1 \\
0
\end{array}\right)\right.
\end{aligned}
\]

\section*{7. EXAMPLE}

The example of section 5, based on the communication system of [7], is continued. A separate \(\mathrm{C}++\) program, BAUER.EXE, described in an attached report[5], is fed the matrices of the matrix \(\Gamma\), and the spectral factorization \(\Gamma_{+}\)returned. In the MATLAB program CHECKBAU.M The matrix \(\Gamma_{+}\)is first verified, and the mean squared error for vector feedback calculated. The channel responses could also be calculated, if desired.

\section*{8. FINITE EQUALIZERŚ}

In a practical system[5] the equalizers are going to be finite, oversampled, and adaptive. Neglecting the "oversampling, the equalizer coefficients and feedback coefficients-assumed finite in number, form the scalar
\[
y_{n}=\sum_{k=0} f_{k} r_{n-k}-\sum_{k=1} b_{k} u_{n-k}-\sum_{k=1} c_{k} v_{n-k}-\sum_{k=1} d_{k} w_{n-k}=\mathbf{U}^{\mathrm{T}} \mathbf{X}_{n}
\]
and the mean squared error is the expectaiton of the square of \(\operatorname{ly}_{n}-u_{n}{ }^{2}\), where \(\left\{u_{n}\right\}\) are the transmitted sequence in the channel under study, \(\left\{v_{k}\right\}\) and \(\left\{w_{k}\right\}\) the transmitted sequences in the two adjacent channels, and \(f_{k}, c_{k}, d_{k}\) and \(e_{k}\) are to be determined. The received signal sequence is \(\left\{\mathrm{r}_{\mathrm{k}}\right\}\) which is assumed to be a linear combination of past present and future data sequences in all three transmitters and noise.
If we define \(A\) and \(V\) by
\[
\mathbf{A}=E\left\{\mathbf{X}_{\mathbf{n}} \mathbf{X}_{\mathrm{n}}{ }^{\mathbf{T}}\right\} \quad, \mathbf{V}=\mathrm{E}\left\{\mathbf{X}_{\mathrm{n}} u_{n}\right\}
\]
then the coefficients are given by \(\mathrm{U}_{\text {opt }}=\mathrm{A}^{-1} \mathrm{~V}\) and the minimum mean squared error is \(1-\mathrm{V}^{\mathrm{T}}\) Uopt, assuming that the transmitted data is binary 1 s .

\section*{9. CONCLUSIONS}

The ability of three equalizers to overcome the effects of adjacent channels being of the worst phase for a VSB multi-channel communication system has been determined. The performance measure is the unbiased mean squared error. The results are as follows, for a system operation in a signal-to-noise ratio of 5 , i.e. 7 dB.

TABLE 1 Mean squared error for VSB communication system with adjacent channels at equal level, 90 -degrees phase shifted, with various equalizers. There is no channel coding.
\begin{tabular}{|l|c|c|c|}
\hline \multicolumn{1}{|c|}{ System } & \begin{tabular}{c} 
Relative Level \\
of adjacent \\
channels \\
(dB)
\end{tabular} & \begin{tabular}{c} 
Relative phase of \\
adjacent \\
channels \\
(degrees)
\end{tabular} & \begin{tabular}{c} 
Mean Squared \\
Error \\
(Unbiased)
\end{tabular} \\
\hline Baseline & 0 & 0 & 0.2 \\
\hline Matched Filter & 0 & 90 & \(0.5689(2.45 \mathrm{~dB})\) \\
\hline \begin{tabular}{l} 
Minimum Mean Squared Error \\
Equalizer
\end{tabular} & 0 & 90 & \(0.3563(4.47 \mathrm{~dB})\) \\
\hline \begin{tabular}{l} 
Single Decision Feedback \\
(Main channel only)
\end{tabular} & 0 & 90 & \(0.3168(4.99 \mathrm{~dB})\) \\
\hline \begin{tabular}{l} 
Vector Decision Feedback \\
(Main+Adjacent Channels)
\end{tabular} & 0 & 90 & \(0.2200(6.57 \mathrm{~dB})\) \\
\hline
\end{tabular}

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\section*{APPENDIX 1}

\section*{CALCULATION OF MSE}

The purpose of this appendix is to record the calculations.

The MSE has three signal related terms and a noise term. The sum of the three signal related terms which we call MSES, is equal to
\[
\begin{aligned}
& \int_{-1 / 2 T}^{1 / 2 T}\left[\left|\lambda_{1}\right|^{2}+\left|\lambda_{2}\right|^{2}+\left|\lambda_{0}\right|^{2}\right] d f T \\
& =\int_{-1 / 2 T}^{1 / 2 T}\left|\lambda_{1}\right|^{2}\left[1+\frac{\left|G_{2,1}\right|^{2}}{\left(1+G_{2,2}\right)^{2}}+\frac{\left|G_{0,1}\right|^{2}}{\left(1+G_{0,0}\right)^{2}}\right] d f T
\end{aligned}
\]

The noise mean squared error is .
\[
M S E N=\int_{-\infty}^{\infty} N_{0}|E(f)|^{2} d f
\]

When the expression for \(E(f)\) are substituted, the cross terms that are zero neglected, and the infinite integral replaced with an integral over the finite range \(-1 / 2 \mathrm{~T}\) to \(1 / 2 \mathrm{~T}\), the integrand contains terms
\[
\left|\lambda_{1}\right|^{2} G_{1,1}+\left|\lambda_{0}\right|^{2} G_{0,0}+\left|\lambda_{2}\right|^{2} G_{2,2}+\lambda_{1} * \lambda_{2} G_{1,2}+\lambda_{1} * \lambda_{0} G_{1,0}+\lambda_{0} * \lambda_{1} G_{0,1}+\lambda_{2} * \lambda_{1} G_{2,1}
\]
and this may be written
\[
\left|\lambda_{1}\right|^{2}\left[G_{1,1}-\frac{\left|G_{2,1}\right|^{2}}{1+G_{2,2}}-\frac{\left|G_{2,1}\right|^{2}}{\left(1+G_{2,2}\right)^{2}}-\frac{\left|G_{0,1}\right|^{2}}{1+G_{0,0}}-\frac{\left|G_{0,1}\right|^{2}}{\left(1+G_{0,0}\right)^{2}}\right]
\]
and, when MSES and MSEN are added together the additions and cáncellations result in the following expression for the total mean squared error
\[
\begin{aligned}
& M S E=\int_{-1 / 2 T^{\prime} \cdot}^{1 / 2 T}\left|\lambda_{1}\right|^{2}\left[1+G_{1,1}-\frac{\left|G_{2,1}\right|^{2}}{1+G_{2,2}}-\frac{\left|G_{0,1}\right|^{2}}{1+G_{0,0}}\right] d f T \\
& =\int_{-1 / 2 T}^{1 / 2 T} \frac{\ddots}{\left[1+G_{1,1}:-\frac{d f T}{1+G_{2,2}}-\frac{\left|G_{0,1}\right|^{2}}{1+G_{0,0}}\right]}
\end{aligned}
\]

It is perhaps a bit clearer to keep the \(\lambda\) 's separate, and then the integrand is
\[
\begin{aligned}
& \left|\lambda_{1}\right|^{2} G_{1,1}+\left|\lambda_{0}\right|^{2} G_{0,0}+\left|\lambda_{2}\right|^{2} G_{2,2}+\lambda_{1} * \lambda_{2} G_{1,2}+\lambda_{1} * \lambda_{0} G_{1,0}+\lambda_{0} * \lambda_{1} G_{0,1}+\lambda_{2} * \lambda_{1} G_{2,1} \\
& +\left|\lambda_{1}\right|^{2}+\left|\lambda_{0}\right|^{2}+\left|\lambda_{2}\right|^{2} \\
& =\lambda_{1}^{*}\left(\left(1+G_{1,1}\right) \lambda_{1}+\lambda_{2} G_{1,2}+\lambda_{0} G_{1,0}\right] \\
& \left.+\lambda_{0}^{*}\left[\left(1+G_{0,0}\right) \lambda_{0}+\lambda_{1} G_{0,1}\right)\right] \\
& +\lambda_{2}^{*}\left[\left(1+G_{2,2}\right) \lambda_{2}+\lambda_{1} G_{2,1}\right] \\
& =-\lambda_{1}^{*}
\end{aligned}
\]

\section*{APPENDIX 2 PROPERTIES OF THE ELEMENTS \(G_{i j}\).}

The report makes us of the elements Gij. Here we obtain some properties.
From the definition
\[
\begin{aligned}
& G_{i, j}(f)=\frac{1}{2 T N_{0}} \sum H_{i}(f+m / T) H_{j}^{*}(f+m / T)+H_{i}^{*}(-f+m / T) H_{j}(-f+m / T) \\
& =G_{j, i}^{*}(f)=G_{i, j}^{*}(-f)
\end{aligned}
\]

In the communication system under consideration, the transfer functions are due to half (or root) Nyquist filters, with an additional gain A, phase \(\theta\) and delay \(\tau\). Thus, a root Nyquist filter with a channel
\[
A_{i} e^{j \theta_{i}} e^{-j \omega \tau_{i}}
\]

In the diagonal elements the phase and delay are eliminated, and the elements are just
\[
\because A^{2} / N_{0} .
\]

The off-diagonal elements are more complicated. The elements \(G_{0,1}\) and \(G_{1,0}\) are mirrors of each other. So are \(\mathrm{G}_{2,1}\) and \(\mathrm{G}_{1,2}\). In general, \(\mathrm{G}_{0,1}\) and \(\mathrm{G}_{2,1}\) have coefficients that are similar, but in which every second coefficient is of opposite sign.
Thus, the matrix appearing in the equations
\[
\left(\begin{array}{c}
-C_{+} \\
-1-, B_{+} \\
-D_{+}
\end{array}\right)=\left(\begin{array}{ccc}
1+G_{0,0} & G_{0,1} & 0 \\
G_{1,0} & 1+G_{1,1} & G_{1,2} \\
0 & G_{2,1} & 1+G_{2,2}
\end{array}\right)\left(\begin{array}{l}
\lambda_{0}(f) \\
\lambda_{1}(f) \\
\lambda_{2}(f)
\end{array}\right)
\]
may be written
\[
\left(\begin{array}{ccc}
1+A_{0}^{2} / N_{0} & A_{0} A_{1} G_{0,1} & 0 \\
A_{1} A_{0} G_{1,0} & 1+A_{1}^{2} / N_{0} & A_{1} A_{2} G_{1,2} \\
0 & A_{2} A_{1} G_{2,1} & 1+A_{2}^{2} / N_{0}
\end{array}\right)
\]

Thus, in the matrices used in the Bauer factorization, the matrices that are related to powers of \(z\) that are not zero have no diagonal, no 1,3 or 3,1 term, and, are otherwise arbitrary, consistent with the overall matrix
polynomial being positive definite. We haven't discovered any properties that follow from these observations.

Pulse response Channel 1

3.1


Sampled Denominator pulse response



Frequency response of first term in numerator

\[
3.5
\]

\[
3 . i
\]
third term in numerator

\[
3.7
\]

\(3.8\)

Receive MSE equalizer with/without interference




DFE reppisie

DFE equalizer with/without interference

5.2

\(5 \cdot 3\)
ar all
mat compact
Initialize the filters
hitializing filters'
ilt;
mp=transmit;
transmit=receive;
eceive=temp;
mp= [];
mulation parameters
heta=pi/2;
; \% a is the strength of the interference
0.2 ; \% signal-to-noise ratio
modify per parameters
nsmit (32,: ) =a*exp(j*theta)*transmit (32,:);
Insmit (2,:) \(=\) a* \(\exp (j * t h e t a) * t r a n s m i t(2,:) ;\)
eceive ( \(32,:\) ) =a*exp (-j*theta)*receive (32,:);
feive (2,:)=a*exp(-j*theta)*ieceive(2,:);
-conv(transmit(1,:), receive(1,:)); \% the main channel
r-conv(transmit ( \(32,:\) ), receive(1,:)); \% one interferer \(12=-\operatorname{conv}(t r a n s m i t(2,:), r e c e i v e(1,:)) ; \%\) the other interferer se=1ength(transmit(1,:))-16*floor(length(transmítr(1,:))|16); se=16-phase;
Lhe peak is at index 244 so phase \(=16-244 \bmod 15=12\)
i=real (h0) ; Jol=g01/n0;
for \(i=1: 1\) ength (g01)
f \(-((i+p h a s e)-16 * f l o o r((i+p h a s e) / 16)==0)\)
g01(i) \(=0\);
end
end
l=real (h2);
g21=g21/n0;
for \(i=1: 1\) ength (g21)
if \(\sim((i+p h a s e)-16 * f l o o r((i+p h a s e) / 16)==0)\)
\(g 21(i)=0\);
end
gnd
11
fi=1: length (g11)
if \(-((i+p h a s e)-16 *\) floor \(((i+p h a s e) / 16)==\) =
g11 \((i)=0\);
end
\({ }_{9}^{4}\)
since these are all the same size the code could be simplified a lot
sum=0;
for \(k=1\) : length (g21)
sum=sum+n0*n0*g21(k)*conj(g21(k))+n0*n0*g01(k)*conj(g01(k));
d
sum=n0+a*a*sum
factor \(1=(1+1 / \mathrm{n} 0)\);
ctor0 \(=(1+a * a / n 0)\);
\(\operatorname{CCO} 2=(1+a * a / n 0)\);
scompute the numerator of the equalizer
\(=\operatorname{conv}(n 0 * g 11\), receive (1, :)) ;
=conv (g01, receive ( \(32,:\) )) /factor0;
\(\mathrm{e}=\mathrm{2}=\operatorname{conv}(\mathrm{g} 21\), receive \((2,:)) /\) factor 2 ;
enum=e1-e2-e0;
um=enum/n0;
\%compute the denominator
Aen=factor1*( \(\operatorname{conv}(\operatorname{conj}(f l i p l r(g 11 * n 0)), g 11 * n 0))-\ldots\)
onv(conj(fliplr(g01)),g01)/factor0) - ...
```

    e1=eden (floor ((length (eden) +1 )/2));
    ```
    \(\pm 1=(\) factor \(1-m s e 1) *(\) factor0) ;
    =1=n0+a*a*n0*n0*mse1
    jeenum/eden is \(I I R\), not \(F I R\)
    Eft (enum, 4096) ;
Hfft (eden, 4096) ;
=u./v; \%, fft of the equalizer transfer function
    Eft (transmit (1,:),4096);
    =u. *V;
\(N_{2}^{2}=f f t(t r a n s m i t(32,:), 4096)\);
    \(0=f f t(t r a n s m i t(2,:), 4096)\);
    =u. *w2;
    =u. *W0;
    tback to the time domain
    =ifft (u, 4096) ;
    =ifft(w1, 4096);
    =ifft(w0,4096);
v2=ifft (w2,4096);
    infactor \(=\max ((w 1))\)
    =w1/gainfactor;
    u/gainfactor;
w \(0=w 0 /\) gainfactor;
\(\mathfrak{r}=\mathrm{w} 2 /\) gainfactor;
    ot (-200:200, center(real(w1), 401));
    ld on;
plot(-200:200, center(abs (w0), 401));
5ot (-200:200, center(abs(w2), 401)); .
    figure ready'
    fid
        ld off;
        gure
un=8.68*log (abs(fft (u, 4096)));
\(\mathrm{v}=8.68 * \log (\mathrm{abs}(\mathrm{fft}(\mathrm{receive}(1,:), 4096)))\);
    ot ( \(0:(\operatorname{length}(u 1)-1)) /\) length(iu1), ul,'g-', (0:(length(u)-1))/length(u),v,'x-');
    abel ('dB')
XLabel('relative frequency')
title('Receive MSE equalizer with/without interference');
    is ([0 \(11-50\) 50]) ;
    alculate mean squared errors
w=abs(fft (eder, 4096));
msemse=0;
    edfe \(=0\);
    ri=1:4096
    msemse=msemse+1/w(i);
    msedfe=msedfe+log(1/w(i));
    d;
    emse=msemse/4096
msedfe=msedfe/4096;
    medfe=exp (msedfe)
\(z z z=\left[' N 0='\right.\) num2str(n0) ' MSE \(='^{\prime}\) num2str(mseu(msemse)) ' \(a=\) ' num2str(a) ' theta = ' num2str(18
    onter location'
    ext(zzz);
    spectral factorization roots inside UC
    inding spectral factors'
r=roots (sample (eden, 16,0));
\(\mathrm{p}=[]\);
    ir \(i=1\) : length \((r)\)
    if abs(r(i))<1
        \(\mathrm{p}=[\mathrm{p} \mathrm{r}(\mathrm{i})]\);
    end
    d
    huffle the roots before multiplying again
for \(i=1\) : length ( \(p\) )
    \(j=1+\) length \((p) *\) rand;
    if \(j<=\) length ( \(p\) )
```

    ke it a row vector
    ```
prus=poly \((p)\);
infactor=eplus(1);
    us=real (eplus/gainfactor);
    h=zeros(1, floor ((length (eden) +1\() / 2)\) );
    \(i=1\) : length (eplus)
    \(1+16 *(i-1))=\) eplus(i);
and
    2lus=ep;
    numdfe=conv(enum, eplus);
    ( \(=f f t(u, 4096)\). *fft(eplus, 4096);
    contains the DFE equalizer frequency response
srepeat the calculations above
fif(transmit (1,:), 4096);
    =u. *v;
    dise response from main channel
dW \(\mathcal{W}=\) fft (transmit \((32,:), 4096)\);
结=ft(transmit (2,:), 4096);
    ? \(=\mathrm{u}\). * dw2;
zwu =u. * dwo;
\% back to the time domain
    (fft(u, 4096);
    [=ifft(dw1,4096);
dwo=ifft (dw0,4096);
dw2=ifft (dw2, 4096);
infactor=dw1(1)
    l=dwl/gainfactor;
\(d w 0=d w 0 /\) gainfactor;
dw2=dw2/gainfactor;
    u/gainfactor;
    gure
Dlot (-200:200; center(real(dw1), 401));
rold on;
bt ( \(-200: 200\), center (abs (dw0), 401));
(bt (-200:200, center (abs (dw2), 401));
grid
とN/gainfactor;
    figure ready'
hold off;
fingure
    \(=8.68 * \log (\operatorname{abs}(\operatorname{fft}(u, 4096))) ;\)
    \(8.68 * \log (\operatorname{abs}(f f t(r e c e i v e(1,:), 40,96)))\);
plot( ( \(0:(\operatorname{length}(u 1)-1)) /\) length(ul), ul,'g-', ( \(0:(\operatorname{length}(v)-1)) /\) length (v), v, 'r-');
- abel('dB')
خabel('relative frequency')
title('DFE equalizer with/without interference');
is ([0 \(\left.\left.1 \begin{array}{lll}0 & -50 & 50\end{array}\right]\right)\);

'enter location'
    ext (zzz);
turns \(n\) values of a pulse cencrea at origin co cencer ur screen rmally \(n\) is odd
\(=1\) loor \((\mathrm{n} / 2)\);
\(\mathrm{b}((1\) ength \((\mathrm{b})-\mathrm{m}+1):\) length \((\mathrm{b})) \mathrm{b}(1:(\mathrm{m}+1))]\);
sample: samples a function so the middle is included
Unction [output]=sample(a,k,offset)
dale=offset+floor ((length(a)+1)/2);
a=middle-k*floor(middle/k);
\((\bmod ==0)\)
d \(=\bmod +k\);
Cput=[];
Gor (i=mod:k:length(a))
utput=[output a(i)];
```

    nted 10:33 23 Jan 97
    Hecks the output from the bauer program
    Because MATLAB doesn't do 3d matrices, must fudge it
    d results.dta
    esults(1): \%order of the matrices
    -results(2); \% the number of matrices
    3 ;
    \(\mathrm{kkk}=0:(\mathrm{m}-1)\)
    \(=[' b '\) num2str (kkk) \(\quad '=z e r o s(3,3) ; '] ;\)
    eval(s);
    for \(k k=1: n\)
        for \(j=1: n\)
            \(s=[' b '\) num2str \((k k k) \quad 1(k k, j)=r e s u l t s(k) ; "]\);
            eval(s);
            \(k=k+1\);
    end
        nd
            nd
        \(k k=0:(m-1)\)
        = [];
        for \(j=k k:(m-1)\)
        if \((s==[])\)
        \(s=[' b '\) num2str (j) '*b' num2str(j-kk) ''''];
        else
        \(s=\left[s{ }^{\prime}+b^{\prime}\right.\) num2str (j) \(i * b^{\prime}\) num2str(j-kk) ''''];
        end
        nd
        eval(s)
    ```

\title{
STUDY INTO IMPROVED SPECTRUM EFFICIENCY FOR FDMA/TDMA TRANSMISSION IN MOBILE SATELLITE AND MOBILE ENVIRONMENTS:
}

\section*{3d. ZERO FORCING, MINIMUM MSE AND DECISION FEEDBACK MULTI-USER QAM RECEIVERS}

\author{
W F McGee \\ Contract: U6800-6-1604
}

13 March 1997

\begin{abstract}
This report is a progress report of work towards milestone 3 of this contract. In a previous report we studied VSB filter banks. Here were study the same filterbanks uised for quadrature amplitude modulation (QAM).
\end{abstract}

The reason for comparing QAM and VSB filterbanks's that, if the phasing between channels cannot be maintained in a VSB system, the interference between adjacent channels is very like that of a QAM signal.

The results indicate that QAM MSE receivers are different in philosophy from VSB MSE receivers, in that the bandwidth of the receive equalizers is not constrained to the bandwidth of the channel being received, but includes all the channels.

A decision feedback design based on whitening the output from a MSE equalizer has been analyzed. The performance is good, if timing synchronization can be maintained.

The theoretical foundations for the study have been made and are presented herein.
This work is of peripheral interest for the project since QAM signals are not as useful as VSB signals; their envelope fluctuations are larger. The report does document work that may be useful in some other context; perhaps OFDM.

An appendix shows how to design \(M / 2\) channel \(M\)-band filters if a good \(M\)-band design is in hand.

\section*{1. THE DIFFERENCE BETWEEN QAM AND VSB SIGNALING USING FILTERBANKS}

When QAM signals are sent through a fifierbank, each complex QAM signal \(\mathrm{Q}_{\mathrm{i}}\left(\mathrm{z}^{\mathrm{M}}\right)\) is applied to one channel of the filterbank at a rate \(1 / \mathrm{M}\), where M is the number of complex channels. Thus, the transmitted signal is the sum of M filtered QAM signals. In the receiver, the received signal is passed through a receive filterbank, and the \(M\) different outputs are sampled every \(M\) samples.

In contract, for VSB filterbanks, each of \(M\) real input sequences is applied at a rate \(2 / \mathrm{M}\) to each complex channel, and the real part of the output is sampled every \(2 / \mathrm{M}\) samples.

When VSB filterbanks are properly designed and phased, there is no intersymbol interference for each channel, and there is no crosstalk between adjacent channels, because the crosstalk is purely imaginary at the sample times. The root-Nyquist prototype filters may be designed so that there is little crosstalk between channels that are further apart than the adjacent channels.

For QAM signals, on the other hand, when the same root-Nyquist filters are used, there is no intersymbol interference in the channel, and little crosstalk between channels that are not adjacent. But there is adjacent channel interference. However, since it may not be possible to maintain the phasing between adjacent VSB
channels in a multi-user environment, there is interest in comparing QAM and VSB signaling when there is no phase coordination.

Much may be learned by considering the minimum bandwidth filters required for pulse communication. Assume that the filter passband is not split. Then, if the spectrum of the real VSB input signals at rate \(2 / \mathrm{M}\) real symbols per seconds examined, it is clear that to recover the input with a narrowband filter the bandwidth may be as small as \(1 / \mathrm{M}\), but the position of the passband must be from \(\mathrm{k} / \mathrm{M}\) to \((\mathrm{k}+1) / \mathrm{M}\), where k is an integer. On the other hand, for complex QAM symbols at rate \(1 / \mathrm{M}\), the filters of the filterbank may be anywhere, and the bandwidth must be \(1 / \mathrm{M}\). Thus, let us considered QAM filterbanks where the filters are the same as those used for VSB filterbanks, i.e., with the \(3-\mathrm{dB}\) crossover frequencies at \(\mathrm{k} / \mathrm{M}\) and \((\mathrm{k}+1) / \mathrm{M}\).

Consider some special sequences. In particular, examine what happens when a steady signal 1 is sent on both channel 0 and channel \(\mathrm{M}-1\). The VSB input is a 1 appearing every \(\mathrm{M} / 2\) samples, and has harmonics equally spaced about the unit circle at frequencies that are multiples of \(2 / \mathrm{M}\). The dc signal generated from the input to the \(\mathrm{M}-1\) st channel is discerned from that of the symbols of channel 0 , in the receiver, by insisting that there be a 90 -degrees phase shift between them at this frequency, and this allows the two symbol sequences to be discriminated. If the 90 -degrees phase shift is not maintained, then the discrimination is not maintained, and the two detected symbols may destructively interfere with each other.

For QAM, on the other hand, because the samples occur only every \(M\) samples instead of M/2 as for VSB, there are harmonics at dc and equally spaced about the unit circle at frequencies that are multiples of \(1 / \mathrm{M}\), each of magnitude \(1 / \mathrm{M}\). In particular, the channel in question will have two harmonics, which, added together, form the recovered symbol 1 again.

Now consider crosstalk from channel M-1. Because of the restricted bandwidth assumed for the filters, only the tone at 0 -frequency is passed by the transmit filter for channel \(\mathrm{M}-1\); the other tone at ( \(\mathrm{M}-1\) )/M is not detected by the receive filter for channel 0 . With no phase coordination between adjacent sidebands, the worst case would be a 45 -degree phase difference, when the interference in the I channel would be as much as \(\sqrt{2 / 2}\) when the \(Q\) channel interference would be zero.

Thus, the worst interference from one adjacent channel in QAM systems that are not phase aligned is 3 dB less than the corresponding VSB system.

\section*{2. MSE LINEAR RECEIVER FOR MAXIMALLY DECIMATED QAM SIGNALS}

In contrast to the VSTB filterbanks, all the transmitting channels must be included in'the definition of the mean squared error. Therefore, considéi \(M\) input data sequences \(\mathrm{Q}_{\mathrm{i}}\left(\mathrm{z}^{\mathrm{M}}\right)\) acting on M input filters or equalizers [we use the two terms interchangeably] \(H_{i}(z)\); the transmitted signal is
\[
\sum_{i=0}^{M-1} Q_{i}\left(z^{M}\right) H_{i}(z)
\]

Furthermore, assume that the filters \(\mathrm{H}_{\mathrm{i}}(\mathrm{z})\) are frequency shifted versions of a prototype filter \(\mathrm{P}(\mathrm{z})\) with the polyphase representation
\[
P(z)=\sum_{r=0}^{M-1} z^{-r} P_{r}\left(z^{M}\right)
\]
with
\[
H_{i}(z)=P\left(W^{i} z\right)
\]
and
\[
W=e^{-j 2 \pi / M}
\]
is an Mth root of unity. The filter \(\mathrm{H}_{\mathrm{i}}(\mathrm{z})\) is centered at the frequency \(\mathrm{i} / \mathrm{M} . \mathrm{P}(\mathrm{z})\) is a lowpass filter that meets the requirements for a root-Nyquist filter and with 100 -percent or less excess bandwidth. This means that \(\mathrm{H}_{\mathrm{i}}\) and \(\mathrm{H}_{\mathrm{i}+2}\) do not overlap; their product \(\mathrm{H}_{\mathrm{i}}(\mathrm{z}) \mathrm{H}_{\mathrm{i}+2}(\mathrm{z})\) is essentially zero, where zero is defined by the stopband loss..

Note that
\[
\begin{aligned}
& H_{j}(z)=\sum_{k=0}^{M-1} W^{-j k} z^{-k} P_{k}\left(z^{M}\right) \\
& z^{-k} P_{k}\left(z^{M}\right)=\frac{1}{M} \sum_{j=0}^{M-1} W^{j k} H_{j}(z)
\end{aligned}
\]

We can establish that
\[
H_{j}\left(1=e^{j 0}\right)=\sqrt{M} \delta_{j, 0}
\]
and therefore
\[
P_{k}\left(1=e^{j 0}\right)=\frac{1}{\sqrt{M}}
\]

On the other hand,
\[
P\left(e^{j 2 \pi / 2 M}\right)=\sqrt{\frac{M}{2}} e^{j \theta}
\]
so that
\[
\begin{aligned}
& P_{k}\left(e^{j 2 \pi M / 2 M}\right)=P_{\dot{k}}\left(e^{j \pi}\right)=e^{j 2 \pi k / 2 M} \frac{1}{M} \sqrt{\frac{M}{2}}\left[e^{j \theta}+e^{-j \theta^{\prime}-j 2 \pi k / M}\right] \\
& =e^{j 2 \pi k / 2 M} \frac{1}{M} \sqrt{\frac{M}{2}} e^{-j 2 \pi k / 2 M}\left[e^{j \theta+j 2 \pi k / 2 M}+e^{-j \theta-j 2 \pi k / 2 M}\right] \\
& =\sqrt{\frac{2}{M}} \cos ^{1}(\theta+2 \pi k / 2 M)
\end{aligned}
\]

For any frequency between 0 and \(1 / 2 \mathrm{M}\), in fact
\[
P_{k}\left(e^{j 2 \pi \pi M}\right)=\frac{e^{j 2 \pi k f}}{M}\left[H_{0}\left(e^{j 2 \pi f}\right)+e^{-j 2 \pi k / M} H_{1}\left(e^{j 2 \pi f}\right)\right]
\]
and therefore, for linear phase \(\mathrm{P}(\mathrm{z})\),
\[
\left|P_{k}\left(e^{j 2 \pi M}\right)\right|^{2}=\frac{1}{M^{2}}\left[\left|H_{0}\left(e^{j 2 \pi f}\right)\right|^{2}+\left|H_{1}\left(e^{j 2 \pi f}\right)\right|^{2}+2\left|H_{0}\left(e^{j 2 \pi f}\right) \| H_{1}\left(e^{j 2 \pi f}\right)\right| \cos (2 \theta+2 \pi k / M)\right]
\]
where \(2 \theta\) is the phase angle between HO and H 1 , which may be easily determined for a linear phase filter. For example, for a length 256 filter, with \(\mathrm{M}=32\), this phase angle is
\[
2 \theta=\pi / M .
\]
\(\left|P_{k}\right|^{2}\) thus has a more or less universal shape for various \(k\), since it is only the cosine factor which depends on k , and, in addition depends only on \(\mathrm{H}_{0}(\mathrm{f})\) since in this frequency band
\[
\left|H_{0}\left(e^{j 2 \pi f}\right)\right|^{2}+\left|H_{1}\left(e^{j 2 \pi f}\right)\right|^{2}=M
\]

The matched filters for the receiver are \(\mathrm{H}_{\mathrm{j}^{*}}(\mathrm{z})\) are the polynomials Hj with the coefficients conjugated and z replaced with \(1 / z\), thus,
\[
\begin{aligned}
& H_{j^{*}}(z)=\sum_{k=0}^{M-1} W^{j k} z^{k} P_{k}\left(1 / z^{M}\right) \\
& z^{k} P_{k}\left(1 / z^{M}\right)=\frac{1}{M} \sum_{j=0}^{M-1} W^{-j k} H_{j^{*}}(z)
\end{aligned}
\]

Under the polyphase assumption, the transmitted signal may be represented as
\[
\begin{aligned}
& \sum_{i=0}^{M-1} Q_{i}\left(z^{M}\right) H_{i}(z) \\
& =\sum_{i=0}^{M-1} Q_{i}\left(z^{M}\right) P\left(W^{i} z\right) \\
& =\sum_{i=0}^{M-1} \sum_{r=0}^{M-1} z^{-r} W^{-i r} P_{r}\left(z^{M}\right) Q_{i}\left(z^{M}\right) \\
& =\sum_{r=0}^{M-1} z^{-r} P_{r}\left(z^{M}\right) \sum_{i=0}^{M-1} W^{-i r} Q_{i}\left(z^{\ddot{M}}\right)
\end{aligned}
\]

Noise is added to this signal and this received signal \(R(z)\) is passed through an equalizer \(E(z)\) for channel 0 . The equalizer for the other channel willithy symmetry, be \(E\left(W^{i} z\right)\) but we only consider the design of the single equalizer. Suppose that there is a polyphase expansion of the equalizer \(E(z)\) given by
\[
E(z)=\sum_{r=0}^{M-1} z^{r} E_{r}\left(z^{M}\right)
\]

If the variance of the white noise is \(\mathrm{N}_{0}\), then the noise power of the complex signal at the output of the equalizer is just \(N_{0}\) times the sum of squared absolute values of the coefficients of \(E(z)\), the sum of the sum of squared coefficients of the polynomials \(\mathrm{E}_{\mathrm{i}}\left(\mathrm{z}^{\mathrm{M}}\right)\).


Figure 1Complex (e.g. \(\pm 1 \pm \mathrm{j}\) ) data is presented to the input DFT matrix. The M outputs are pasised through the filters \(\mathrm{Pk}(\mathrm{zM})\) and then added, with the appropriate delay to form the transmitted signal. At the receiver, the equalizer is represented in its polyphase representation. The outputs are sampled.

As far as the signals are concerned, the above may be replaced with the simpler equivalent circuit shown in the next figure.


Figure 2 Equivalent circuit to Figure 1Complex (e.g. \(\pm 1 \pm \mathrm{j}\) ) data is presented to the input DFT matrix. The M outputs are passed through the filters \(\operatorname{Pk}(\mathrm{zM})\) and then added, with the appropriate delay to form the transmitted signal. At the receiver, the equalizer is represented in its polyphase representation. The outputs are sampled.

The problem of minimizing the mean squared error in the first (the 0th) channel at the output is a standard problem[1] in noise theory. The output sequence is the sum of the samples of the noise passing through the equalizer \(E(z)\) and
\[
\sum_{r=0}^{M-1} Q_{r}\left(z^{M}\right) e^{j r k 2 \pi / M} P_{k}\left(z^{M}\right) E_{k}\left(z^{M}\right)
\]

If we use the notation SS to represent the sum of squared magnitude of coefficients of a z-transform, then the mean squared error is
\[
\begin{aligned}
& M S E=N_{0} \sum_{r=0}^{M-1} S S\left(E_{r}\right) \\
& +S S\left(1-\sum_{k} P_{k} E_{k}\right) \\
& +\sum_{r=1}^{M-1} S S\left(\sum_{k} P_{k} E_{k} e^{j 2 \pi k r / M}\right)
\end{aligned}
\]
and the problem posed is to determine \(E_{k}\) to minimize the mean squared error MSE. In this analysis we assume that the mean squared value of the transmitted symbols is unity. And the mean squared error is the sum of the square of fluctuations in the real and the imaginary received signal.

The first term is the noise passing through the equalizer, which, for our purposes, we assume white with spectral density \(\mathrm{N}_{0}\), the second term represents the intersymbol interference in the channel under study, and the other terms represent the crosstalk from the other channels.

For subsequent analysis we use the notation \(P^{*}(z)\) to represent the polynomial with \(z\) replace by \(1 / z\), and the coefficients conjugated.

The minimization is done by taking partial derivatives with respect to \(\mathrm{E}_{\mathrm{k}}{ }^{*}\) and results in the equation
\[
\begin{aligned}
& 0=N_{0} E_{k} \\
& +P_{k:}^{*}\left[\sum_{s=0}^{M-1} P_{s} E_{s}-1\right] \\
& +\sum_{r=1}^{M-1} P_{k}^{*} e^{-j 2 \pi k r / M}\left[\sum_{l=0}^{M-1} e^{j 2 \pi r s r / M} E_{s} P_{s}\right]
\end{aligned}
\]
which we rewrite as
\[
\begin{aligned}
& 0=N_{0} E_{k} \\
& -P_{k}^{*} \\
& +P_{k}^{*} \sum_{r=0}^{* M-1} e^{-j 2 \pi k r / M}\left[\sum_{l=0}^{M-1} e^{j 2 \pi s s r / M} E_{s} P_{s}\right]
\end{aligned}
\]

The sum on \(r\) is zero unless \(s=k\), and so these equations imply
\[
N_{0} E_{k}=P_{k}^{*}\left(1-M P_{k} E_{k}\right)
\]
that is,
\[
E_{k}=P_{k}^{*} /\left(N_{0}+M P_{k} P_{k}^{*}\right)
\]

Consequently the equalizer \(\mathrm{E}(\mathrm{z})\) is given by
\[
\begin{gathered}
E(z)=\sum_{k=0}^{M-1} z^{k} E_{k} \\
=\sum_{k=0}^{M-1} z^{r} P_{k}^{*} /\left(N_{0}+M P_{k} P_{k}^{*}\right) \\
=\frac{1}{M} \sum_{r=0}^{M-1} H_{r^{*}}(z) \sum_{k=0}^{M-1} \frac{W^{-r k}}{N_{0}+M P_{k} P_{k^{*}}} \\
=\sum_{r=0}^{M-1} H_{r^{*}}(z) V_{r}\left(z^{M}\right) \\
\cdots \\
\text { where } \\
\vdots
\end{gathered} \quad \begin{aligned}
& W^{-r k} \\
& V_{r}\left(z^{M}\right)=\frac{1}{M} \sum_{k=0}^{M-1} \frac{W_{0}+M P_{k}\left(z^{M}\right) P_{k^{*}}\left(1 / z^{M}\right)}{N_{0}+} .
\end{aligned}
\]

When the noise is large, the MSE equalizer is the matched filter to the transmit filter,
\[
\sum_{k=0}^{M-1} z^{k} P_{k}^{*} N_{0}
\]
and if \(\mathrm{P}(\mathrm{z})\) is a FIR filter, so is the receive equalizer a FIR filter, whereas when the noise is small the equalizer is
\[
\sum_{k=0}^{M-1} z^{k} / M P_{k}
\]
which is, for P FIR, IIR. This is the proof, of the theorem that is not possible to transmit QAM signal through a FIR transmitter and a FIR receiver, in the absence of noise, without crosstalk interference.

The last expression expresses the equalizer as, a sum of frequency weighed matched filters, each matched to the k transmitting channel.

These weighting functions \(V_{j}\left(z^{M}\right)\) are clearly periodic about the unit circle, and satisfy some relations based on the fact that we are dealing with 100 -percent linear phase filterbanks. In the appendix we show that, if the phase is linear, we have
\[
P_{k}\left(z^{M}\right) P_{k^{*}}\left(z^{M}\right)=P_{M-1-k}\left(z^{M}\right) P_{M-1-k^{*}}\left(z^{M}\right)
\]
and because of the Nyquist property
\[
P_{k}\left(z^{M}\right) P_{k^{*}}\left(z^{M}\right)+P_{k+M / 2}\left(z^{M}\right) P_{k+M / 2^{*}}\left(z^{M}\right)=2 / M
\]
and for linear phase and Nyquist
\[
P_{k}\left(z^{M}\right) P_{k^{*}}\left(z^{M}\right)+P_{M / 2-1-k}\left(z^{M}\right) P_{M / 2-1-k^{*}}\left(z^{M}\right)=2 / M
\]

For linear phase, therefore, we may represent \(\mathrm{V}_{\mathrm{j}}\left(\mathrm{z}^{\mathrm{M}}\right)\) as
\[
\begin{aligned}
& V_{j}\left(z^{M}\right)=\frac{1}{M} \sum_{k=0}^{M-1} \frac{W^{-j k}}{N_{0}+M P_{k}\left(z^{M}\right) P_{k^{*}}\left(1 / z^{M}\right)} \\
& =\frac{1}{M} \sum_{k=0}^{M / 2-1} \frac{W^{-j k}+W^{-j(M-1-k)}}{N_{0}+M P_{k}\left(z^{M}\right) P_{k^{*}}\left(1 / z^{M}\right)} \\
& =\frac{1}{M} W^{-j(M-1) / 2} \sum_{k=0}^{M / 2-1} \frac{W^{-j(k-(M-1) / 2)}+W^{-j((M-1) / 2-k)}}{N_{0}+M P_{k}\left(z^{M}\right) P_{k^{*}}\left(1 / z^{M}\right)} \\
& =\frac{2}{M} W^{-j(M-1) / 2} \sum_{k=0}^{M / 2-1} \frac{\cos (j(k+1 / 2) 2 \pi / M)}{N_{0}+M P_{k}\left(z^{M}\right) P_{k^{*}}\left(1 / z^{M}\right)}
\end{aligned}
\]

The coefficients satisfy
\[
\begin{aligned}
& V_{M-k}\left(z^{M}\right)=V_{k}^{*}\left(z^{M}\right) \\
& V_{M / 2}\left(z^{M}\right)=0
\end{aligned}
\]

Using the previous approximation for \(\mathrm{P}_{\mathrm{k}}\), this may be written in term sof \(\mathrm{H}_{0}\) and \(\mathrm{H}_{1}\) as
\[
V_{r}\left(e^{j 2 \pi f}\right)=\frac{2}{M} W^{-r(M-1) / 2} \sum_{k=0}^{M / 2-1} \frac{\cos (r(k+1 / 2) 2 \pi / M)}{\therefore} \frac{:}{N_{0}+1+\frac{2}{M}\left|H_{0}\left(e^{j 2 \pi f}\right) \| H_{1}\left(e^{j 2 \pi f}\right)\right| \cos ((k+1 / 2) 2 \pi / M)}
\]
which may be approximated by
\[
V_{r}\left(e^{j 2 \pi f}\right) \cong W^{-r(M-1) / 2} \int_{0}^{2 \pi} \frac{\cos (r \theta) d \theta / 2 \pi}{N_{0}+1+\frac{2}{M}\left|H_{0}\left(e^{j 2 \pi f}\right) \| H_{1}\left(e^{j 2 \pi f}\right)\right| \cos (\theta)}
\]

This may be evaluated using the integral
\[
\int_{0}^{2 \pi} \frac{\cos (r \theta) d \theta / 2 \pi}{1+a \cos (\theta)}=\frac{1}{\sqrt{1-a^{2}}}(-1)^{r}\left(\frac{a}{\sqrt{1-a^{2}}+1}\right)^{r}
\]
thus
\[
V_{r}\left(e^{j 2 \pi J M}\right) \cong \dot{W}^{-r(M-1) / 2} \frac{1}{N_{0}+1} \frac{(-1)^{r}}{\sqrt{1-a^{2}}}\left(\frac{a}{\sqrt{1-a^{2}}+1}\right)^{r}
\]
where
\[
a=\frac{1}{N_{0}+1} \frac{2}{M}\left|H_{0}\left(e^{j 2 \pi f}\right) \| H_{1}\left(e^{j 2 \pi f}\right)\right|
\]

Two frequencies are of special interest. When \(f\) is zero, \(H_{1}\) vanishes, and so \(a=0\), and all the \(V_{r}\) are zero except for \(\mathrm{r}=0\). When f is \(1 / 2 \mathrm{M}, \mathrm{H}_{1}\) and \(\mathrm{H}_{0}\) are equal to each other and their product is a maximum and equal to \(\mathrm{M} / 2\), so a is equal to \(1 /(\mathrm{N} 0+1)\). Thus
\[
\left|V_{r}\left(e^{j \pi}\right)\right| \cong \frac{1}{\sqrt{N_{0}^{2}+2 N_{0}}}\left(\frac{1}{1+N_{0}+\sqrt{2 N_{0}+N_{0}^{2}}}\right)^{r}
\]

The phase angle is
\[
W^{-r(M-1) / 2}(-1)^{r}=W^{r / 2}
\]

Let us now examine again the equation for the equalizer
\[
E(z)=\sum_{j=0}^{M-1} H_{j^{*}}(z) V_{j}\left(z^{M}\right)
\]

The factors Vj have been analyzed, and it has been determined that when j is not zero, they are zero at the frequencies \(1 / 2 \mathrm{M}\), and a maximum in absolute value at the interband frequencies between the kth and the ( \(k+1\) )st translated response \(\mathrm{H}_{k}(z)^{\circ}\) and \(\mathrm{H}_{\mathrm{k}+1}(\mathrm{z})\).

The mean squared error is given by
\[
\begin{aligned}
& M S E_{M S E}=\int_{-1 / 2 M}^{1 / 2 M} d f T\left(\frac{1}{M} \sum_{k=0}^{M-1} \frac{N_{0}}{N_{0}+M P_{k} P_{k}^{*}}\right) \\
& =\int_{-1 / 2 M}^{1 / 2 M} N_{0} V_{0}\left(e^{j 2 \pi J M}\right) d f M \\
& =\int_{-1 / 2}^{1 / 2} N_{0} V_{0}\left(e^{j 2 \pi f}\right) d f
\end{aligned}
\]

This is the so-called biased MSE[3,4], and is related to the unbiased MSEU by
\[
\frac{1}{M S E U}=\frac{1}{M S E}-1
\]

The time has probably come to draw some pictures.


Figure 3 Typical frequency responses. In a are plotted the responses of the transmit and receive filters, showing the normalization that results in a received pulse of height unity. The crossover frequency represents a loss of 3 dB , as indicated. Part b shows the response of the overall channels. The bandwidth is \(1 / T\), the height is \(T\) and the area is unity as required. The third figure indicates the response of the adjacent channel filters. These are narrow, but have a fixed height, \(\mathrm{T} / 2\).

Examine the differences between the corresponding relationship for the mean squared error for VSB signals. For the sake of argument, assume that, \(T=32\) samples for QAM:

Here is a derivation that does not explicitly use polyphase components.
The mean squared error is
\[
\begin{aligned}
& M S E=\int_{-\infty}^{-\infty} N(f)|E(f)|^{2} d f \\
& +\int_{-1 / 2 M}^{1 / 2 M}\left|\frac{1}{M} \sum_{m=0}^{M-1} H_{0}(f+m / M) E(f+m / M)-1\right|^{2} d f M \\
& +\sum_{r=1}^{M-1} \int_{-1 / 2 M}^{1 / 2 M}\left|\frac{1}{M} \sum_{m=0}^{M-1} H_{r}(f+m / M) E(f+m / M)\right|^{2} d f M
\end{aligned}
\]
and the problem posed is to determine \(\mathrm{E}(\mathrm{f})\) to minimize the mean squared error MSE. The first term is the noise passing through the equalizer, which, for our purposes, we assume white with spectral density \(\mathrm{N}_{0}\), the second term represents the intersymbol interference in the channel under study, and the last term represent the crosstalk from the adjacent channels.

The minimization is done by taking partial derivatives with respect to \(\mathrm{E}^{*}(\mathrm{f})\) and results in the equation
\[
\begin{aligned}
& 0=N_{0} E(f) \\
& +H_{0}^{*}(f)\left[\frac{1}{M} \sum_{m=0}^{M-1} H_{0}(f+m / M) E(f+m / M)-1\right] \\
& +\sum_{r=1}^{M-1} H_{r}^{*}(f)\left[\frac{1}{M} \sum_{m=0}^{M-1} H_{r}(f+m / M) E(f+m / M)\right]
\end{aligned}
\]

Each of the expression in square brackets is periodic in \(f\) with period \(1 / \mathrm{M}\), and we represent them by \(\lambda_{0}(\mathrm{f})\), \(\lambda_{1}(f), \lambda_{2}(f)\), and so on.. This equation may be written
\[
\begin{aligned}
& 0=N_{0} E(f)+\sum_{r=0}^{M-1} H_{r}^{*}(f) \lambda_{r}(f) \\
& \lambda_{0}(f)=\frac{1}{M} \sum_{m=0}^{M-1} H_{0}(f+m / M) E(f+m / M)-1 \\
& \lambda_{r}(f)=\frac{1}{M} \sum_{m=0}^{M-1} H_{r}(f+m / M) E(f+m / M) \text { for } 0<\mathrm{r} \leq \mathrm{M}-1
\end{aligned}
\]

The equalizer \(E(f)\) is obtained by substituting the first equation in the \(M\) following, resulting in \(M\) equations for the \(M\) unknowns \(\lambda_{i}(f)\); and then putting these equations back into the first to solve for \(E(f)\).

There is another important result that is obtained from the expression for \(E(f)\). Since the parameters \(\lambda_{i}(f)\) are : periodic in f, they may be realized with (perhaps infinitely long) FIR filters with delays M. Thus, when samples are taken at the equalizer output, this is equivalent to sampling the outputs of all the matched filters .and passing them to FIR filters with delays \(M\) [i.e. in \(z^{M}\) ].

These \(M\) equations are of the form
\[
\begin{aligned}
& \lambda_{0}(f)=-G_{0,0} \lambda_{0}(f)-G_{0,1}(f) \lambda_{1}(f)-G_{1, M-1}(f) \lambda_{M-1}(f)-1 \\
& \lambda_{1}(f)=-G_{1,0} \lambda_{0}(f)-G_{1,1}^{\prime}(f) \lambda_{1}(f)-G_{1,2}(f) \lambda_{2}(f) \\
& \Lambda \\
& \lambda_{M-1}(f)=-G_{M-1, M, 2} \lambda_{M-2}(f)-G_{M-1, M-1}(f) \lambda_{M-1}(f)-G_{M-1,0}(f) \lambda_{0}(f)
\end{aligned}
\]
i.e.,
\[
\left(\begin{array}{c}
-1 \\
0 \\
\Lambda \\
0
\end{array}\right)=\left(\begin{array}{cccc}
1+G_{0,0}(f) & G_{0,1}(f) & \Lambda & G_{0, M-1}(f) \\
G_{1,0}(f) & 1+G_{1,1}(f) & \Lambda & 0 \\
\Lambda & \Lambda & \Lambda & \Lambda \\
G_{M-1,0}(f) & 0 & \Lambda & 1+G_{M-1, M-1}(f)
\end{array}\right)\left(\begin{array}{c}
\lambda_{0}(f) \\
\lambda_{1}(f) \\
\Lambda \\
\lambda_{M-1}(f)
\end{array}\right)
\]
where
\[
\begin{aligned}
& G_{i, j}(f)=\frac{1}{M N_{0}} \sum_{m} H_{i}(f+m / M) H_{j} *(f+m / M) \\
& =G_{j, i} *(f)
\end{aligned}
\]

We rewrite these equations as
\[
\left(\begin{array}{c}
-1 \\
0 \\
\Lambda \\
0
\end{array}\right)=\Gamma \Lambda
\]

We have made use of the assumption that the productiof transfer functions \(\mathrm{H}_{\mathrm{i}} \mathrm{H}_{\mathrm{j}}\) of non-adjacent channels (i.e. li-jl>1is zero.

If we neglect the amplitude and phase of the channels, then the matrix \(\Gamma\) is circulant and Toeplitz.
Being circulant, it is diagonalized by the DFT matrix D , where \(\mathrm{DD}^{\mathrm{H}}=\mathrm{MI}\).
Thus,
\[
\begin{aligned}
& \Gamma=\frac{D G D^{H}}{M} \\
& \left(\begin{array}{c}
-1 \\
0 \\
0
\end{array}\right)=\frac{D G D^{H}}{M} \Lambda \\
& \vdots \\
& D^{H} \Lambda=G_{n}^{-1} D^{H}\left(\begin{array}{c}
-1 \\
0 \\
\ddots
\end{array}\right)=G^{-1}\left(\begin{array}{l}
-1 \\
-1 \\
\Lambda \\
-1
\end{array}\right)=-\left(\begin{array}{l}
1 / G_{0} \\
1 / G_{1} \\
1 / G_{M-1}
\end{array}\right) .
\end{aligned}
\]
where \(G\) is a diagonal matrix whose elements \(G_{i}\) are the \(D F T\) of the first column of the circulant matrix
\[
G_{r}=\sum_{r=0}^{M=1} \Gamma_{k, 0}^{\prime} e^{j r k 2 \pi / M}=1+G_{0,0}+e^{j r 2 \pi / M} G_{1,0}+e^{-j r 2 \pi / M} G_{M-1,0}
\]

The design equation for the equalizer satisfies
\[
\begin{aligned}
& E(f)=-\left(H_{0}^{*}(f) \quad H_{1}^{*}(f) \quad \Lambda \quad H_{M-1}^{*}(f)\right) \Lambda \\
& =\frac{1}{M}(D H)^{H}\left(\begin{array}{c}
1 / G_{0} \\
1 / G_{1} \\
\Lambda \\
1 / G_{M-1}
\end{array}\right) \\
& =\frac{1}{M}\left(H_{0}^{*}(f) \quad H_{1}^{*}(f) \quad \Lambda \quad H_{M-1}^{*}(f)\right) D\left(\begin{array}{c}
1 / G_{0} \\
1 / G_{1} \\
\Lambda \\
1 / G_{M-1}
\end{array}\right)
\end{aligned}
\]
and we can interpret the factors \(\mathrm{V}_{\mathrm{j}}\) as Fourier Transforms of the function \(1 / \mathrm{G}_{\mathrm{j}}\).

\section*{3. DECISION FEEDBACK EQUALIZATION}

The improvements that result with the use of data decisions from the channel and the two adjacent channels is a complicated problem, and has been solved by Kavehrad and Salz[2]in another context. But first, we consider a simpler approach, in which only that decoded data from the channel itself is used, and this approach is based on the use of a prediction filter to whiten the error sequence resulting from the MSE equalizer.

Since
\[
\begin{aligned}
& M S E_{M S E}=\int_{-1 / 2 M}^{1 / 2 M} d f T\left(\frac{1}{M} \sum_{k=0}^{M-1} \frac{N_{0}}{N_{0}+M P_{k} P_{k}^{*}}\right) \\
& =\int_{-1 / 2 M}^{1 / 2 M} N_{0} V_{0}\left(e^{j 2 \pi j M}\right) d f M \\
& =\int_{-1 / 2}^{1 / 2} N_{0} V_{0}\left(e^{j 2 \pi f}\right) d f
\end{aligned}
\]
we have the mean squared error with a spectrum
\[
\begin{aligned}
& M S E(f)=\frac{1}{M} \sum_{k=0}^{M-1} \frac{N_{0}}{N_{0}+M P_{k} P_{k}^{*}} \\
& =N_{0} V_{0}\left(e^{j 2 \pi \mu M}\right)
\end{aligned}
\]
and this is the power spectrum of the deviations from the transmitted sequence of the output from the linear minimum MSE equalizer.

A spectral factorization of the denominator is of the form
\[
1 /\left[N_{0} V_{0}(z)\right]=A_{0}\left(1+B_{+}\right)\left(1+B_{-}\right)=A_{0} \prod_{\left|z_{i}\right|<1}\left(1-z^{-1} z_{i}\right) \prod_{\mid z_{i}>1}\left(1-z / z_{i}\right)
\]
where \(1+B_{+}\)is causal and \(1+B_{\text {. is anti-causal. If the output of the MSE equalizer is followed with the }}\) equalizer \(\left(1+B_{+}\right)\)the resulting sequence will have an error sequence that is white and the mean squared error is \(1 / \mathrm{A}_{0}\). This is equivalent to the following DFE structure


Figure 4 Decision Feedback Receiver. The causal filter B+ whitens the error sequence from the nonDFE equalizer \(\mathrm{E}(\mathrm{f})\) making the sequence spectrum white. The decision feedback removes the correlated part of the signal that results.

The mean squared error for the decision feedback receiver is less than that of the MSE receiver[1] and is equal to
\[
M S E_{D F E}=\exp \left[M \int_{-1 / 2 M}^{1 / 2 M} \ln \left(N_{0} V_{0}\left(e^{j 2 \pi f / \dot{M}}\right)\right) d f\right]
\]

This expression may be derived without spectral factorization, but is a useful check on the factorization.

\section*{4. VECTOR FEEDBACK}

With vector feedback we assume that we have access to all the previous received digits of all the channels.


Figure 4 Vector Feedback. In Figure 3 only the decoded output from the main channel is used for reducing mean squared error. In this model, all the relevant adjacent channel decoded data are used.

With a mean-squared error design criterion, the goal is to minimize
\[
\begin{aligned}
& M S E=\int_{-\infty}^{-\infty} N(f)|E(f)|^{2} d f \\
& +\int_{-1 / 2 T}^{1 / 2 T}\left|\frac{1}{T} \sum_{m} H_{0}(f+m / T) E(f+m / T)-\left(1+C_{0+}(f)\right)\right|^{2} d f T \\
& +\sum_{r \neq 0}^{1 / 2 T} \int_{-1 / 2 T}\left|\frac{1}{T} \sum_{m} H_{r}(f+m / T) E(f+m / T)-C_{R+}(f)\right|^{2} d f T
\end{aligned}
\]
where \(\mathrm{C}_{\mathrm{r}+}\) are causal and have only positive exponents of \(\mathrm{z}^{-1}\)
\[
C_{r,+}=\sum_{k=1}^{\infty} c_{r, k} z^{-k}
\]

When the partial derivative with respect to the equalizer \(E\) is taken we find that
\[
0=N_{0} E(f)+\sum_{r=0}^{M-1} H_{r^{*}}(f) \lambda_{r}(f)
\]
where
\[
\begin{aligned}
\lambda_{0}(f) & =\frac{1}{T} \sum_{m}\left[H_{0}(f+m / T) E(f+m / T)\right]-\left(1+C_{0,+}(f)\right) \\
\cdots & \\
\lambda_{r}(f) & =\frac{1}{T} \sum_{m}\left[H_{r}(f+m / T) E(f+m / T)\right]-C_{r,+}(f) \text { for } r \neq 0
\end{aligned}
\]

Thus
where
\[
\left(\begin{array}{c}
-1-C_{0+} \\
-C_{1+} \\
\Lambda \\
-C_{M-1+}
\end{array}\right)=\left(\begin{array}{ccccc}
1+G_{0,0}(f) & G_{0,1}(f) & \Lambda & \cdot G_{0, M-1}(f) \\
G_{1,0}(f) & 1+G_{1,1}(f) & \Lambda & 0 \\
\Lambda & \vdots & \Lambda & \Lambda & \Lambda \\
G_{M-1,0}(f) & \ddots & \ddots & \vdots & 0 \\
\vdots & \Lambda & 1+G_{M-1, M-1}(f)
\end{array}\right)\left(\begin{array}{c}
\lambda_{0}(f) \\
\lambda_{1}(f) \\
\Lambda \\
\lambda_{M-1}(f)
\end{array}\right)
\]
\[
\begin{aligned}
& G_{i, j}(f)=\frac{1}{M N_{0}} \sum_{m} H_{i}(f+m / M) H_{j} *(f+m / M) \\
& =G_{j, i} *(f .)
\end{aligned}
\]
\[
A_{+}=\Gamma \Lambda_{-}
\]
where the subscripts + and - indicate that there are only positive or negative powers of \(z^{-1}\) in a particular vector. For our application the matrix \(\Gamma\) has real and constant, but probably unequal, diagonal terms since we are dealing with Nyquist channels. The off diagonal terms are conjugates of each other and therefore the matrix is Hermitian on the unit circle.

Equations of thus type may be solved by spectral factorization[5] of the matrix \(\Gamma=\Gamma_{+} \Gamma\).

Then there is a solution if we can find a constant vector \(K\) such that
\[
\begin{aligned}
& \mathbf{A}_{+}=\Gamma_{+} \mathbf{K} \\
& \mathbf{K}=\Gamma_{-} \Lambda_{-}
\end{aligned}
\]
so that the de coefficient of the second component of \(A_{+}\)is -1 , and of the second and third components is zero. If we write the de component of \(\Gamma_{+}\)as \(\Gamma_{+0}\), then
\[
\begin{aligned}
& \mathbf{K}=\Gamma_{+0}{ }^{-1}\left(\begin{array}{c}
-1 \\
0 \\
\Lambda \\
0
\end{array}\right) \ldots \\
& A_{+}=\Gamma_{+} \Gamma_{+0}\left(\begin{array}{c}
-1 \\
0 \\
\Lambda \\
0
\end{array}\right)
\end{aligned}
\]
\[
\Lambda_{-}=\Gamma_{-}^{-1} \Gamma_{+0}^{-1}\left(\begin{array}{c}
-1 \\
0 \\
\Lambda \\
0
\end{array}\right)=\left(\Gamma_{+0} \Gamma_{-}\right)^{-1}\left(\begin{array}{c}
-1 \\
0 \\
\Lambda \\
0
\end{array}\right)
\]

As in the Appendix, the mean squared error for this equalizer may be calculated. Here, however, when the calculation is performed, we recognize three terms in the integral. The first is
\[
\left|\lambda_{1}(f)\right|^{2}\left(1+G_{1,1}\right)+\lambda_{1}^{*}\left(G_{1,0} \lambda_{0}+G_{1,2} \lambda_{2}\right)=\lambda_{1}^{*}\left(-1-B_{+}\right)
\]

The second is
\[
\left|\lambda_{2}(f)\right|^{2}\left(1+G_{2.2}\right)+\lambda_{2}^{*} \lambda_{1} G_{2,1}=\lambda_{2}^{*}\left(-D_{+}\right) \vdots
\]
and the third is
\[
\left|\lambda_{0}(f)\right|^{2}\left(1+G_{0,0}\right)+\lambda_{0}^{*} \lambda_{1} G_{0,1}=\lambda_{2}^{*}\left(-C_{+}\right)
\]

Thus, the mean squared error is
\[
-\int_{-1 / 2 T}^{1 / 2 T} \Lambda_{-}^{H}(f) A_{+}(f) d f T
\]

But notice that this simply is the negative of the dc coefficient of \(\lambda_{1}\), which we label \(\lambda_{0-0}\), since \(\Lambda^{H}\) has only positive terms, the product has only positive terms, and the integral of all but the de term will be zero. This may also be seen by considering
\[
\begin{aligned}
& \Lambda_{-}^{H} \mathbf{A}_{+}=\left(\begin{array}{llll}
-1 & 0 & 0 & 0
\end{array}\right) \Gamma_{+0}^{-H} \Gamma_{-}^{-H} \Gamma_{+} \Gamma_{+0}^{-1}\left(\begin{array}{c}
-1 \\
0 \\
0 \\
0
\end{array}\right) \\
& =\left(\begin{array}{llll}
-1 & 0 & 0 & 0
\end{array}\right) \Gamma_{-0}^{-1} \Gamma_{+0}^{-1}\left(\begin{array}{c}
-1 \\
0 \\
0 \\
0
\end{array}\right) \\
& \therefore \\
& =\left(\begin{array}{llll}
1 & 0 & 0 & 0
\end{array}\right)\left(\Gamma_{+0} \Gamma_{-0}\right)^{-1} \cdots\left(\begin{array}{l}
1 \\
0 \\
0 \\
0
\end{array}\right), \\
& =\left(\begin{array}{llll}
1 & 0 & 0 & 0
\end{array}\right)\left(\Gamma_{+0} \Gamma_{+0}^{H}\right)^{-1}\left(\begin{array}{l}
1 \\
0 \\
0 \\
0
\end{array}\right) \\
& \therefore \\
& \therefore
\end{aligned}
\]

We have calculated the performance and response of a system using a 32 -band prototype linear phase filter derived from the MPEG filter coefficients.

First, note that if a pulse has zeros at multiples of time M , at times kM , its transform satisfies the Nyquist criterion
\[
\sum_{k=0}^{M-1} H\left(W^{k} z\right)=0
\]
\[
\stackrel{\cdots}{i}
\]
\(\therefore\) "多
which states that the sum of shifted transforms is zero.
What the MSE QAM equalizer \(\mathrm{E}(\mathrm{z})\) appears to be doing is the following
1. Maintain zeros crossings in \(\mathrm{P}(\mathrm{z}) \mathrm{E}(\mathrm{z})\).
2. Minimize noise.
3. Force the crosstalk transfer functions to satisfy the above relation. This is done by making the crosstalk transfer functions asymmetrical in frequency about the frequencies ( \(\mathrm{k}+1 / 2\) )/M. This makes the envelope of the crosstalk small at sampling times.

The problem with this strategy is that if the crosstalk from the adjacent channel is so minimized, the symmetry will allow crosstalk from the second channel, and so. on. Thus the equalizer becomes very broadband.

TABLE 1 Mean squared error for QAM communication system with adjacent channels at equal level with various equalizers. There is no channel coding. The phase shift between channels is zero. The noise is 7 dB below the signal.
\begin{tabular}{|l|c|c|c|}
\hline \multicolumn{1}{|c|}{ System } & \begin{tabular}{c} 
Relative Level \\
of adjacent \\
channels \\
(dB)
\end{tabular} & \begin{tabular}{c} 
Phase shift \\
(degrees)
\end{tabular} & \begin{tabular}{c} 
Mean Squared \\
Error \\
(Unbiased)
\end{tabular} \\
\hline Baseline & \(-\infty\) & 0 & \(0.2(7 \mathrm{~dB})\) \\
\hline \begin{tabular}{l} 
Minimum Mean Squared Error \\
Equalizer
\end{tabular} & 0 & 0 & \(0.2434(6.27 \mathrm{~dB})\) \\
\hline \begin{tabular}{l} 
Single Decision Feedback \\
(Main channel only)
\end{tabular} & 0 & 0 & \(0.2001(7 \mathrm{~dB})\) \\
\hline
\end{tabular}

The calculation of the factors \(\mathrm{Vk}(\mathrm{z})\) is shown in the Figure, and it agrees well with the theory. The equalizer response, in the next figure, is dramatically different from the VSB equalizers of the previous reports. The mean squared error is so good with the equalizers that there appears to be little point to using vector feedback.


Figure 5 The absolute value of the factors \(V_{k}(f)\). In 'real' frequency these are periodic with period \(1 / \mathrm{M}\). Thus, all the factors except the first have a zero at frequencies \(\mathrm{k} / \mathrm{M}\), and they associate a weighting near the mid-band frequencies \((k+1 / 2) / M\). The \(V_{k}\) 's alternate in sign.

Equalizer frequency response


Figure 5 The frequency response of the QAM equalizer for the channel centered at \(\mathbf{0}\) frequency. This is what the noise sees. In order to understand the effect of this equalizer on the signals, it must be convolved with the transmitter transfer functions for each channel in turn.


QAM multichannel communication is fundamentally different from the VSB designs previously considered. The equalizers are broadband, not narrowband, and, especially with decision feedback, appear to offer good performance. But this is accomplished by making the crosstalk envelope ring in time, and so timing is important for QAM systems; we have argued that it is not critical for adjacent channel interference in VSB systems.

We have not studied the effects of phasing between the channels.
In this report we have a (relatively) simple transmitter and a (relatively) more complex receiver; many times this may be reversed and a more complex transmitter be coupled with a simpler receiver. This is of particular interest when the relative cost of transmitter are significantly different; for example, in an earthsatellite link.

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\section*{APPENDIX 2 PROPERTIES OF THE ELEMENTS \(\mathrm{G}_{\mathrm{ij}}\).}

The report makes us of the elements Gij. Here we obtain some properties.
-From the definition
\[
\begin{aligned}
& G_{i, j}(f)=\frac{1}{T N_{0}} \sum_{i} H_{i}(f+m / T) H_{j} *(f+m / T) \\
& =G_{j, i} *(f)_{i} \\
& =G_{i+r, j+r}(f)
\end{aligned}
\]

Since \(\mathrm{Gi}, \mathrm{j}=0\) if \(\mathrm{i}-\mathrm{j} \mid>1\) there is just one canonical response given by \(\mathrm{C}(\mathrm{f})=\mathrm{G}_{0,1}(\mathrm{f})\) when \(\mathrm{j}-\mathrm{i}=1\), and another \(A(f)\) for \(\mathrm{j}-\mathrm{i}=0\). Thus the matrix
\[
\left(\begin{array}{cccc}
1+G_{0,0}(f) & G_{0,1}(f) & \Lambda & G_{0, M-1}(f) \\
G_{1,0}(f) & 1+G_{1,1}(f) & \Lambda & 0 \\
\Lambda & \Lambda & \Lambda & \Lambda \\
G_{M-1,0}(f) & 0 & \Lambda & 1+G_{M-1, M-1}(f)
\end{array}\right)=\left(\begin{array}{cccc}
1+A(f) & C(f) & \Lambda & C^{*}(f) \\
C^{*}(f) & 1+A(f) & \Lambda & 0 \\
\Lambda & \Lambda & \Lambda & \Lambda \\
0 & \Lambda & C^{*}(f) & 1+A(f)
\end{array}\right)
\]

These would be the correct analysis if the channel transfer functions were unity.

In the communication system under consideration, the transfer functions are due to half (or root) Nyquist filters, with an additional gain A, phase \(\theta\) and delay \(\tau\). Thus, a root Nyquist filter with a channel
\[
A_{i} e^{j \theta_{i}} e^{-j \omega \tau_{i}}
\]

These should be included in any more detailed analysis of system behavior.

\section*{APPENDIX 3}

More Properties of Polyphase Components of 100 -percent half-Nyquist Prototype Polynomials
In this appendix we recall some more properties of polyphase components. It amounts to showing that, given an \(M\)-band prototype filter with restricted stopband loss, we may generate \(M / 2\) band filter by taking the even or odd numbered coefficients.

First, note that \(\mathrm{P}(\mathrm{z})\) is a lowpass polynomial. In the frequency domain, since the excess bandwidth is 100 percent, the stopband starts at the frequency \(1 / \mathrm{M}\), and extends to \(-1 / \mathrm{M}\). The \(3-\mathrm{dB}\) frequencies are \(1 / 2 \mathrm{M}\) and \(-1 / 2 \mathrm{M}\).

The Nyquist property means that \(\mathrm{P}(\mathrm{z}) \mathrm{P}^{*}(\mathrm{z})\) has zeros spaced M apart except for the coefficient of \(\mathrm{z}^{0}\) which is unity. In terms of the polyphase components this means that
\[
\sum_{k=0}^{M-1} P_{k}\left(z^{M}\right) P_{k^{*}}\left(z^{M}\right)=1
\]

Consider the polynomial \(\mathrm{Q}\left(\mathrm{z}^{2}\right)\) formed by taking the even coefficients of \(\mathrm{P}(\mathrm{z})\). Thus
\[
Q\left(z^{2}\right)=\frac{P(z)+P(-z)}{\sqrt{2}}=\sqrt{2} \frac{P(z)+P(-z)}{2}
\]

We argue that if \(M\) is 4 or more, then \(Q(z)\) is an \(M / 2\)-band 100 -percent half-Nyquist filter.
This follows because
\[
\begin{aligned}
& Q\left(z^{2}\right) Q\left(1 / z^{2}\right)=\frac{\dot{[(P(z)+P(-z)][P(1 / z)+\dot{P}(-1 / z)]}}{2} \\
& =\frac{P(z) P(1 / z)+P(-z) P(-1 / z)+P(z) P(-1 / z)+P(-z) P(1 / z)}{2} \\
& =\frac{P(z) P(1 / z)+P(-z) P(-1 / z)}{2}
\end{aligned}
\]

The third line follows because \(\mathrm{P}(\mathrm{z}) \mathrm{P}(-1 / \mathrm{z})\) has the same magnitude on the unit circle as \(\mathrm{P}(\mathrm{z}) \mathrm{P}(-\mathrm{z})\) and this will be zero since the stopband of \(P(z)\) overlaps the passband of \(P(-z)\). This holds for \(M\) that is 4 or more. For \(M=4\), the stopband of \(P(z)\) is at \(1 / 4\), and the stopband of \(P(-z)\) is at \(1 / 4\). This does not hold for \(M=2[M\) must be even.].

The last line is just the even numbered coefficients of \(\mathrm{P}(\mathrm{z}) \mathrm{P}(1 / \mathrm{z})\) which has zeros at time samples \(\mathrm{M} / 2\), and thus \(Q(z)\) is as required.

Note that we may take the even or odd coefficients.
Starting from, for example, a 32 -band filter prototype we generate at 16 -channel, and 8 -channel, a 4 channel and a 2 -channel filter by taking respectively even second, every fourth, every eighth and every 16th coefficient of \(P(z)\). If we had taken every thirty-second coefficient we would have the polyphase components. But the polynomial formed with every 16 th coefficient is just
\[
R(z)=\sqrt{\frac{M}{2}} P_{0}\left(z^{2}\right)+z^{-1} P_{M / 2}\left(z^{2}\right)
\]
and this is a half band filter, thus
\[
\frac{R(z) R(1 / z)+R(-z) R(-1 / z)}{2}=1
\]
that is to say
\[
P_{0}\left(z^{M}\right) P_{0^{*}}\left(z^{M}\right)+P_{M / 2}\left(z^{M}\right) P_{M 12^{*}}\left(z^{M}\right)=2 / M
\]

By making the other decimation choices, one can show that
\[
P_{k}\left(z^{M}\right) P_{k^{*}}\left(z^{M}\right)+P_{k+M / 2}\left(z^{M}\right) P_{k+M / 2^{*}}\left(z^{M}\right)=2 / M
\]

If, in addition, \(P(z)\) is linear phase of order \(D\), we have
\[
P(1 / z)=z^{D} P(z)
\]
thus if \(D\) is one less than a multiple of \(M\)
\[
P_{k}\left(z^{M}\right) P_{k^{*}}\left(z^{M}\right)=P_{M-1-k}\left(z^{M}\right) P_{M-1-k^{*}}\left(z^{M}\right)
\]

\title{
STUDY INTO IMPROVED SPECTRUM EFFICIENCY FOR FDMA/TDMA TRANSMISSION IN MOBILE SATELLITE AND MOBILE ENVIRONMENTS:
}

\title{
3e. MINIMUM MSE AND DECISION FEEDBACK MULTI-USER RECEIVERS USING SPECTRUM CONTROL
}

\author{
W F McGee \\ Contract: U6800-6-1604
}

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\begin{abstract}
This report is a progress report of work towards milestone 3 of this contract. It records the calculation of optimal receivers for coded VSB signals. The coding in question is channel coding using, for example, partial response codes.

Such coding is useful in the system under study, because the interference from an adjacent channel, with the worst data sequences, may completely close the data eye; in fact, just one adjacent channel may close the data eye. We have previously found that the performance improvement will be more as the transition bandwidth between adjacent channels is less, that is, if the excess bandwidth is reduced.

The purpose of this report is to record the details of the theory that have been performed, which allow the design of the equalizers for such a receiver. But the detailed design work has not been done.

The case of the minimum mean-squared error equalizer is different from our previous studies[6], but we indicate an equivalence between this problem and the previously studied problem to use the earlier results with a few changes of notation. The reason, basically, is that a decision feedback or vector feedback receiver can undo the spectrum control coding, turning what would be a three-level signal back to a twolevel signal.

It should be mentioned that these equalizers are often front ends for Viterbi receivers since the error -sequence is white. Further, it should be noted that the same theory.applied to filter bank systems using staggered QAM modulation.
\end{abstract}

\section*{1. CODING FOR SPECTRUM CONTROL}

The power spectrum of random data fends to be flat. However, if correlation is introduced into the data sequence, the power spectrum is not flat; most often, the correlation is deliberately introduced to create spectral nulls[3]. This spectrum control is essential for certain technologies, for example magnetic recording, and in baseband transmission when dc transmission is not feasible due to transformers, such as balanced telephone lines.

For filterbank applications in which the adjacent channels may not be phased properly, it is an option to consider spectrum coding that introduces a spectral null in the transition band of the channel filters, since this will create a null in the crosstalk transfer function at mid frequency.

For this work we use a definition of message spectrum \(M(f)\) of a stationary data sequence \(\left\{B_{k}\right\}\) given by
\[
\begin{gathered}
M(f)=\sum m_{k} e^{-j 2 \pi f T} \\
m_{k}=E\left(B_{k} B_{0}^{*}\right)=E\left(B_{k+l} B_{l}^{*}\right)=m_{-k}^{*}
\end{gathered}
\]

For a random sequence of uncorrelated binary digits \(\{+1,-1\}\), the message spectrum is unity.
Alternatively, the message spectrum is
\[
M(z)=\sum_{k} m_{k} z^{-k}
\]
where \(z=\mathrm{e}^{\mathrm{i} \omega}, \omega=2 \pi \mathrm{f}\).
The message spectrum of a data sequence that has passed through a linear filter \(\mathrm{H}(\mathrm{z})\) is
\[
M(f)|H(f)|^{2}
\]
[We abuse notation and set \(\mathrm{H}(\mathrm{f})=\mathrm{H}\left(\mathrm{e}^{\mathrm{j} 2 \pi \mathrm{f}}\right)\).]
For M-channel VSB filterbank applications, an attractive spectrum control is achieved by passing a random sequence with the filter
\[
\left(1-z^{-M}\right) / 2
\]
producing the message spectrum with random input data of
\[
M(z)=-0.25 z^{M}+0.5-0.25 z^{-M}
\]
which produces zeros at the frequencies
\[
\left\{0, \frac{2 \pi}{M}, \frac{4 \pi}{M}, \ldots, \frac{2 \pi(M-1)}{M} \equiv-\frac{2 \pi}{M}\right\}
\]
.and results in duobinary line coding.

\section*{2. MSE LINEAR RECEIVER FOR MAXIMALLY DECIMATED CODED VSB SIGNALS多}

Three coded data streams are presented to filters with responses \(\mathrm{H}_{0}(\omega), \mathrm{H}_{1}(\omega), \mathrm{H}_{2}(\omega)\), added together. Noise is added to the combination, and an equalizer \(\mathrm{E}(\omega)\) to minimize the mean square difference between the real part of its output and the input sequence to the filter \(\mathrm{Hl}(\omega)\).


Figure 2.1 Correlated data is presented to the input equalizers. The equalizers include channel gain, phase and delay, on a per channel basis. After an equalizer \(E(\omega)\) we try to determine the digits sent through the equalizer H1. The filters without the channel gain, phase, and delay, are assumed to be those of adjacent channels of a VSB multi-channel communication system with no intersymbol or interchannel interference, and with restricted bandwidth. If there were no channel delay or phase shift, the performance would be optimum with an equalizer matched to the transmitted pulse shaping filter H1.

A coded system like duobinary has an inherent \(3-\mathrm{dB}\) performance loss. There is a \(6-\mathrm{dB}\) penalty because the distance of the signal to the decision level is reduced by half, and a 3-dB increase in performance because, if the coding is done in the transmitter, the signal power is reduced by \(3-\mathrm{dB}\), and if the coding is done in the receiver, the noise power is reduced by 3 dB . The \(3-\mathrm{dB}\) performance loss may be received with a maximal likelihood sequence detector (Viterbi) .

The design of the minimum mean squared error equalizer is a classic problem[1] in noise theory. The mean squared error is
\[
\begin{aligned}
& M S E=\int_{-\infty}^{-\infty} N(f)|\cdot E(f)|^{2} d f \\
& +\int_{-1 / 2 T}^{1 / 2 T}\left|\frac{1}{2 T} \sum_{m} H_{1}(f+m / T) E(f+m / T)+H_{1} *(-f+m / T) E^{*}(-f+m / T)-1\right|^{2} M(f) d f T \\
& +\int_{-1 / 2 T}^{1 / 2 T} \frac{1}{2 T} \sum_{m} H_{0}(f+m / T) E(f+m / T)+\left.H_{0} *(-f+m / T) E^{*}(-f+m / T)\right|^{2} M(f) d f T \\
& +\int_{-1 / 2 T}^{1 / 2 T}\left|\frac{1}{2 T} \sum_{m} H_{2}(f+m / T) E(f+m / T)+H_{2} *(-f+m / T) E^{*}(-f+m / T)\right|^{2} M(f) d f T
\end{aligned}
\]
and the problem is to determine \(E(f)\) to minimize the mean squared error MSE.
The sole complication in the theory results from the second term, and the quantity. 1 in the intersymbol interference term. The mean squared error is that which results from a multilevel system, and is with respect to the average pulse under the coding constraint.

The first term is the noise passing through the equalizer, which, for our purposes, we assume white with spectral density \(\mathrm{N}_{0}\), the second term represents the intersymbol interference ir the channel under study, and the other two terms represent the crosstalk from the adjacent channels.

Before we start, it would appedr reasonable that the receive filter be matched to the transmitted signal filter \(\mathrm{H}_{1}(\mathrm{f})\), and in particular, it will not pass frequencies beyond the bandedge of its filters.

The minimization is done by taking partial derivatives with respect to \(\mathrm{E}^{*}(\mathrm{f})\) and results in the equation
\[
\begin{aligned}
& 0=N_{0} E(f) \\
& +M(f) H_{1} *(f)\left[\frac{1}{2 T} \sum_{m} H_{1}(f+m / T) E(f+m / T)+H_{1} *(-f+m / T) E *(-f+m / T)-1\right] \\
& +M(f) H_{0} *(f)\left[\frac{1}{2 T} \sum_{m} H_{0}(f+m / T) E(f+m / T)+H_{0}^{*}(-f+m / T) E^{*}(-f+m / T)\right] \\
& +M(f) H_{2} *(f)\left[\frac{1}{2 T} \sum_{m} H_{2}(f+m / T) E(f+m / T)+H_{2} *(-f+m / T) E^{*}(-f+m / T)\right]
\end{aligned}
\]

Each of the expression in square brackets is periodic in \(f\) with period \(1 / T\), and we represent them by \(\lambda_{1}(f)\), \(\lambda_{0}(f)\) and \(\lambda_{2}(f)\). As well as being periodic, they also satitisfy \(\left.\lambda_{i}{ }^{*}(-f)=\lambda_{i} f\right)\). Finally, the message spectrum \(M(f)\) is also periodic with period \(1 / T\). This equation may be written as four equations, •
\[
\begin{aligned}
& 0=N_{0} E(f)+M(f) H_{1} *(f) \lambda_{1}(f)+M(f) H_{0} *(f) \lambda_{0}(f)+M(f) H_{2} *(f) \lambda_{2}(f) \\
& \lambda_{1}(f)=\left[\frac{1}{2 T} \sum_{m} H_{1}(f+m / T) E(f+m / T)+H_{1} *(-f+\dot{m} / T) E *(-f+m / T)-1\right] \\
& \lambda_{0}(f)=\left[\frac{1}{2 T} \sum_{m} H_{0}(f+m / T) E(f+m / T)+H_{0} *(-f+m / T) E^{*}(-f+m / T)\right] \\
& \lambda_{2}(f)=\left[\frac{1}{2 T} \sum_{m} \dot{H}_{2}(f+m / T) E(f+m / T)+H_{2} *(-f+m / T) E^{*}(-f+m / T)\right]
\end{aligned}
\]
: The equalizer \(\mathrm{E}(\mathrm{f})\) is obtained by substituting the first equation in the three following, resulting in three equations for the three unknowns \(\lambda_{i}(f)\), and then putting these equations back into the first to solve for \(E(f)\).

There is another important result that is obtained from the expression for \(E(f)\). Since the parameters \(\lambda_{i}(f)\) and \(M(f)\) are periodic in \(f\) with period \(1 / F\), they may be realized with (perhaps infinitely long) FIR filters. Thus, when samples are taken at the equalizer output, this is equivalent to sampling the outputs of the three matched filters and passing them to FIR filters with delays times T.
‘uspor

These last three equations are of the form':
\[
\begin{aligned}
& \lambda_{1}(f)=-G_{1,1} \lambda_{1}(f)-G_{1,0}(f) \lambda_{0}(f)-G_{1,2}(f) \lambda_{2}(f)-1 \\
& \lambda_{0}(f)=-G_{0.1} \lambda_{1}(f)-G_{0,0}(f) \lambda_{0}(f)-G_{0,2}(f) \lambda_{2}(f) \\
& \lambda_{2}(f)=-G_{2,1} \lambda_{1}(f)-G_{2,0}(f) \lambda_{0}(f)-G_{2,2}(f) \lambda_{2}(f)
\end{aligned}
\]
i.e.,
\[
\left(\begin{array}{c}
-1 \\
0 \\
0
\end{array}\right)=\left(\begin{array}{ccc}
1+G_{1,1} & G_{1,0} & G_{1,2} \\
G_{0,1} & 1+G_{0,0} & G_{0,2} \\
G_{2,1} & G_{2,0} & 1+G_{2,2}
\end{array}\right)\left(\begin{array}{l}
\lambda_{1}(f) \\
\lambda_{0}(f) \\
\lambda_{2}(f)
\end{array}\right)
\]
where
\[
\begin{aligned}
& G_{i, j}(f)=\frac{M(f)}{2 T N_{0}} \sum H_{i}(f+m / T) H_{j}^{*}(f+m / T)+H_{i}^{*}(-f+m / T) H_{j}(-f+m / T) \\
& =G_{j, i}^{*}(f)=G_{i, j} *(-f)=G_{i, j}(f+1 / T)
\end{aligned}
\]

But, because of the properties of the filters, the elements whose indices are separated by 2 or more are zero. Thus, \(\mathrm{G}_{0,2}=\mathrm{G}_{2.0}=0\). [If the problem had been written in the frequency-ordered way, the matrix would be a bordered diagonal matrix, i.e., tridiagonal.] Also, because of the filters, there will only be one term in the sum for most of the elements, except perhaps for the main diagonal term. In any case, the equations are
\[
\left(\begin{array}{c}
-1 \\
0 \\
0
\end{array}\right)=\left(\begin{array}{ccc}
1+G_{1,1} & G_{1,0} & G_{1,2} \\
G_{0,1} & 1+G_{0,0} & 0 \\
G_{2,1} & 0 & 0
\end{array}\right)\left(\begin{array}{l}
\lambda_{1}(f) \\
\lambda_{0}(f) \\
\lambda_{2}(f)
\end{array}\right) .
\]

These equations may be solved and result in
\[
\left(\begin{array}{l}
\lambda_{1} \\
\lambda_{0} \\
\lambda_{2}
\end{array}\right)=\frac{1}{\Delta}\left(\begin{array}{ccc}
\left(1+G_{00}\right)\left(1+G_{22}\right) & -G_{10}\left(1+G_{22}\right) & -G_{12}\left(1+G_{00}\right) \\
-G_{01}\left(1+G_{22}\right) & \left(1+G_{11}\right)\left(1+G_{22}\right) & G_{0,1} G_{1,2}(=0) \\
-G_{21}\left(1+G_{00}\right) & G_{2,1} G_{1,0}(=0) & \left(1+G_{11}\right)\left(1+G_{00}\right)
\end{array}\right)\left(\begin{array}{c}
-1 \\
0 \\
0
\end{array}\right)
\]
and the determinant
\[
\Delta=\left(1+G_{11}\right)\left(1+G_{00}\right)\left(1+G_{22}\right)-G_{01} G_{10}\left(1+G_{22}\right)-G_{21} G_{12}\left(1+G_{00}\right)
\]
is real and positive.

Also, if the transmitting filters are normalized root-Nyquist filters, the diagonal terms are equal to \(1+\mathrm{M}(\mathrm{f}) \mathrm{N}_{0}\). The normalization makes the sum of squares of the coefficients of H unity.

In any case, we can solve exactly, with the result
\[
\begin{array}{r}
\lambda_{0}=-\frac{\lambda_{1} G_{0,1}}{1+G_{0,0}} \\
\lambda_{2}=-\frac{\lambda_{1} G_{2,1}}{1+G_{2,2}}
\end{array}
\]

Solving for \(\lambda_{1}(f)\)
\[
\lambda_{1}(f)=\frac{-1}{1+G_{1,1}-\frac{G_{0,1} G_{1,0}}{1+G_{0,0}}-\frac{G_{1,2} G_{2,1}}{1+G_{2,2}}}
\]
and substituting back to find \(E(f)\) we have
\[
E(f)=M(f) \frac{H_{1}^{*}(f)-H_{0} *(f) \frac{G_{0,1}}{1+G_{0,0}}-H_{2} *(f) \frac{G_{2,1}}{1+G_{2,2}}}{N_{0}\left[1+G_{1,1}-\frac{\left|G_{0,1}(f)\right|^{2}}{1+G_{0,0}}-\frac{\left|G_{2,1}(f)\right|^{2}}{1+G_{2,2}}\right]}
\]

An explicit calculation of the mean squared error is tedious, and the details are in an appendix. The result is
\[
M S E=\int_{-1 / 2 T}^{1 / 2 T} \frac{M(f) d f T}{\left[1+G_{1,1}-\frac{\left|G_{0,1}(f)\right|^{2}}{1+G_{0,0}}-\frac{\left|G_{2,1}(f)\right|^{2}}{1+G_{2,2}}\right]}
\]
and this also puts the power spectrum of the error in view; it is not flat. This is the so-called biased MSE[3,4], and is related to the unbiased MSEU by
\[
\frac{1}{M S E U}=\frac{1}{M S E}-1
\]

\section*{4. DECISION FEEDBACK EQUALIZATION}

The improvements that result with the use of data decisions is a complicated problem, and has been solved by Kavehrad and Salz[2]in a related context. But first, we consider a simpler approach, in which only that decoded data from the channel itself is used, and this approach is based on the use of a prediction filter to whiten the error sequence resulting from the MSE equalizer.

As we derived, the error sequence from the linear minimum MSE equalizer has the spectrum
\[
\operatorname{MSE}(f)=\frac{M(f) T}{\left[1+G_{1,1}-\frac{\left|G_{0,1}(f)\right|^{2}}{1+G_{0,0}}-\frac{\left|G_{2,1}(f)\right|^{2}}{1+G_{2,2}}\right]}
\]

A spectral factorization of the denominator is of the form
\[
\frac{\left[1+G_{1,1}-\frac{\left|G_{0,1}(f)\right|^{2}}{1+G_{0,0}}-\frac{\left|G_{2,1}(f)\right|^{2}}{1+G_{2,2}}\right]}{M(f)}=A_{0}\left(1+B_{+}\right)\left(1+B_{-}\right)=A_{0} \prod_{z_{i} \mid<1}\left(1-z^{-1} z_{i}\right) \prod_{\mid z_{i}>1}\left(1-z / z_{i}\right)
\]

Thus, \(1+B_{+}\)is causal and \(1+B\). is anti-causal. If the output of the MSE equalizer is followed with the equalizer \(\left(1+B_{+}\right)\)the resulting sequence will have an error sequence that is white and the mean squared error is \(1 / \mathrm{A}_{0}\). This is equivalent to the following DFE structure


Figure 3. Decision Feedback Receiver. The causal filter B+ whitens the error sequence from the nonDFE equalizer \(E(f)\) making the sequence spectrum white. The decision feedback removes the correlated part of the signal that results.

This is the receiver that minimizes
\[
\begin{aligned}
& M S E=\int_{-\infty}^{-\infty} N(f)|E(f)|^{2} d f \\
& +\int_{-1 / 2 T}^{1 / 2 T} \mathrm{I} \frac{1}{2 T} \sum_{m} H_{1}(f+m / T) E(f+m / T)+H_{1}^{*}(-f+m / T) E^{*}(-f+m / T)-\left.\left(1+B_{+}(f)\right)\right|^{2} M(f) d f T \\
& +\int_{-1 / 2 T}^{1 / 2 T} \frac{1}{2 T} \sum_{m} H_{0}(f+m / T) E(f+m * / T)+\left.H_{0} *(-f+m / T) E^{*}(-f+m / T)\right|^{2} M(f) d f T \\
& +\int_{-1 / 2 T}^{1 / 2 T} \mathrm{l} \frac{1}{2 T} \sum_{m} H_{2}(f+m / T) E(f+m / T)+\left.H_{2}^{*}(-f+m / T) E^{*}(-f+m / T)\right|^{2} M(f) d f T
\end{aligned}
\]

If factorize \(\mathrm{M}(\mathrm{z})\) as
\[
M(z)=M_{0} M_{+}\left(z^{-1}\right) M_{-}(z)
\]
where \(M_{+}\)and \(M\) have the coefficient of \({ }_{r}^{\prime}\) z0 equal to unity.
Then a simplification in the argument may be made.
"ix. . .
Note that for the example quoted at the beginning, this factorization would be
\[
\begin{aligned}
& M(z)=\frac{1}{4}\left(1-z^{-M}\right)\left(1-z^{M}\right) \\
& M_{0}=1 / 4
\end{aligned}
\]
\[
\begin{aligned}
& \frac{M S E}{M_{0}}=\frac{\int_{-\infty}^{-\infty} N(f)|E(f)|^{2} d f}{M_{0}} \\
& +\int_{-1 / 2 T}^{1 / 2 T}\left|\frac{1}{2 T} \sum_{m} H_{1}(f+m / T) E(f+m / T) M_{+}+H_{1}^{*}(-f+m / T) E^{*}(-f+m / T) M_{-}-M_{+}\left(z^{-1}\right)\left(1+B_{+}(f)\right)\right|^{2} d f T \\
& +\int_{-1 / 2 T}^{1 / 2 T} \frac{1}{2 T} \sum_{m} H_{0}(f+m / T) E(f+m / T) M_{+}+H_{0}^{*}(-f+m / T) E^{*}(-f+m / T) M_{-} 1^{2} d f T \\
& +\int_{-1 / 2 T}^{1 / 2 T} \frac{1}{2 T} \sum_{m} H_{2}(f+m / T) E(f+m / T) M_{+}+H_{2} *(-f+m / T) E^{*}(-f+m / T) M_{-} 1^{2} d f T
\end{aligned}
\]

Then the solution may be expressed in terms of the solution to the corresponding problem in the uncoded case if the following substitutions are made.
\begin{tabular}{|l|l|}
\hline In Report [6] replace & with \\
\hline\(M S E\) & \(M S E / M_{0}\) \\
\hline\(H_{i}(f)\) & \(H_{i}(f) M_{+}(f)\) \\
\hline\(N(f)\) & \(N(f) / M_{0}\) \\
\hline \(1+B_{+}\) & \(\left(1+B_{+}\right) M_{+}(z)\) \\
\hline
\end{tabular}

In particular, the expression for \(G\) is unchanged, and the mean squared error for the decision feedback receiver is less than that of the MSE receiver[1] and is equal to
\[
\begin{aligned}
& M S E_{D F E}=M_{0} \exp \left(-\int_{-1 / 2 T}^{1 / 2 T} \ln \left(1+G_{1,1}-\frac{\left|G_{0,1}(f)\right|^{2}}{1+G_{0,0}}-\frac{\left|G_{2,1}(f)\right|^{2}}{1+G_{2,2}}\right) d f T\right) \\
& =\exp \left(\int_{-1 / 2 T}^{1 / 2 T} \ln \left(\frac{.}{1+G_{1,1}-\frac{\left|G_{0,1}(f)\right|^{2}}{1+G_{0,0}}-\frac{\left|G_{2,1}(f)\right|^{2}}{1+G_{2,2}}}\right) d f T\right) \quad \therefore
\end{aligned}
\]

\section*{5. VECTOR FEEDBACK}

With vector feedback we assume that we have access to all the previous received digits of all the channels.


Figure 4 Vector Feedback. In Figure 3 only the decoded output from the main channel is used for reducing mean squared error. In this model, all the relevant adjacent channel decoded data are used.

With a mean-squared error design criterion, the goal is to minimize
\[
\begin{aligned}
& M S E=\int_{-\infty}^{\infty} N(f)|E(f)|^{2} d f \\
& +\int_{-1 / 2 T}^{1 / 2 T} M(f)\left|\frac{1}{2 T} \sum_{m} H_{1}(f+m / T) E(f+m / T)+H_{1} *(-f+m / T) E^{*}(-f+m / T)-\left(1+B_{+}(f)\right)\right|^{2} d f T \\
& +\int_{-1 / 2 T}^{1 / 2 T} M(f)\left|\frac{1}{2 T} \sum_{m} H_{0}(f+m / T) E(f+m / T)+H_{0} *(-f+m / T) E^{*}(-f+m / T)-C_{+}(f)\right|^{2} d f T \\
& +\int_{-1 / 2 T}^{1 / 2 T} M(f)\left|\frac{1}{2 T} \sum_{m} H_{2}(f+m / T) E(f+m / T)+H_{2} *(-f+m / T) E^{*}(-f+m / T)-D_{+}(f)\right|^{2} d f T
\end{aligned}
\]
where \(\mathrm{B}+, \mathrm{C}+\) and \(\mathrm{D}+\) are causal and have only positive exponents of \(\mathrm{z}^{-1}\) and thus the unknowns are the (real) numbers \(b_{k}, c_{k}\) and \(d_{k}\), where
\[
\begin{aligned}
& B_{+}=\sum_{k=1}^{\infty} b_{k} z^{-k} \\
& C_{+}=\sum_{k=1}^{\infty} c_{k} z^{-k} \\
& D_{+}=\sum_{k=1}^{\infty} d_{k} z^{-k}
\end{aligned}
\]

It will be noticed that, since
\[
\begin{aligned}
& \frac{M S E}{M_{0}}=\int_{-\infty}^{-\infty} \frac{N(f)}{M_{0}}|E(f)|^{2} d f \\
& +\int_{-1 / 2 T}^{1 / 2 T}\left|\frac{1}{2 T} \sum_{m} H_{1}(f+m / T) E(f+m / T) M_{+}+H_{1}^{*}(-f+m / T) E *(-f+m / T) M_{-}-\left(1+B_{+}(f) M_{+}\right)\right|^{2} d f T \\
& +\int_{-1 / 2 T}^{1 / 2 T} 1 \frac{1}{2 T} \sum_{m} H_{0}(f+m / T) E(f+m / T) M_{+}+H_{0} *(-f+m / T) E^{*}(-f+m / T) M_{-}-\left.C_{+}(f) M_{+}\right|^{2} d f T \\
& +\int_{-1 / 2 T}^{1 / 2 T} 1 \frac{1}{2 T} \sum_{m} H_{2}(f+m / T) E(f+m / T) M_{+}+H_{2} *(-f+m / T) E^{*}(-f+m / T) M_{-}-D_{+}(f) M_{+} I^{2} d f T
\end{aligned}
\]
we may use the results of report [6] to calculate relevant performance characteristics, as follows.
\begin{tabular}{|l|l|}
\hline In Report [6] replace & with \\
\hline\(M S E\) & \(M S E / M_{0}\) \\
\hline\(H_{i}(f)\) & \(H_{i}(f) M_{+}(f)\) \\
\hline\(N(f)\) & \(N(f) / M_{0}\) \\
\hline
\end{tabular}
\begin{tabular}{|l|l|}
\hline \(1+B_{+}\) & \(\left(1+B_{+}{ }^{\prime}\right) M_{+}(z)\) \\
\hline\(C_{+}\) & \(C_{+} M_{+}(z)\) \\
\hline\(D_{+}\) & \(D_{+} M_{+}(z)\) \\
\hline
\end{tabular}
and the calculations of the previous report made. We do not repeat the details.

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\section*{CALCULATION OF MSE}

The purpose of this appendix is to record the calculations.
The MSE has three signal related terms and a noise term. The sum of the three signal related terms which we call MSES, is equal to
\[
\begin{aligned}
& \int_{\text {MSES }}= \\
= & {[\mid / 2 T} \\
= & \left.\int_{-1 / 2 T}^{1 / 2 T}\left|\lambda_{1}\right|^{2}+\left|\lambda_{2}\right|^{2}+\left|\lambda_{0}\right|^{2}\right] M(f) d f T \\
\left(1+G_{2,2}\right)^{2} & \left.\frac{\left|G_{2,1}\right|^{2}}{\left(1+G_{0,0}\right)^{2}}\right] M(f) d f T
\end{aligned}
\]

The noise mean squared error is
\[
M S E N=\int_{-\infty}^{\infty} N_{0}|E(f)|^{2} d f .
\]

When the expression for \(E(f)\) are substituted, the cross terms that are zero neglected, and the infinite integral replaced with an integral over the finite range \(-1 / 2 \mathrm{~T}\) to \(1 / 2 \mathrm{~T}\), the integrand contains terms
\[
\left|\lambda_{1}\right|^{2} G_{1,1}+\left|\lambda_{0}\right|^{2} G_{0,0}+\left|\lambda_{2}\right|^{2} G_{2,2}+\lambda_{1} * \lambda_{2} G_{1,2}+\lambda_{1} * \lambda_{0} G_{1,0}+\lambda_{0} * \lambda_{1} G_{0,1}+\lambda_{2} * \lambda_{1} G_{2,1}
\]
and this may be written
\[
\left|\lambda_{1}\right|^{2} \cdot\left[G_{1,1}-\frac{\left|G_{2,,}\right|^{2}}{1+G_{2,2}}-\frac{\left|G_{2,1}\right|^{2}}{\left(1+G_{2,2}^{\prime}\right)^{2}}-\frac{\left|G_{0,1}\right|^{2}}{1+G_{0,0}}-\frac{\left|G_{0,1}\right|^{2}}{\left(1+G_{0,0}\right)^{2}}\right]
\]
and, when MSES and MSEN are added together the additions and cancellations result in the following expression for the total mean squared error
\[
\begin{aligned}
& M S E=\int_{-1 / 2 T}^{1 / 2 T}\left|\lambda_{1}\right|^{2} M(f)\left[1+G_{1,1}-\frac{\mid G_{2,1} 1^{2}}{1+G_{2,2}}-\frac{\left|G_{0,1}\right|^{2}}{1+G_{0,0}}\right] d f T \\
& =\int_{-i / 2 T}^{1 / 2 T} \frac{M(f) d f T}{\left[1+G_{1,1}-\frac{\left|G_{2,1}\right|^{2}}{1+G_{2,2}}-\frac{\left|G_{0,1}\right|^{2}}{1+G_{0,0}}\right]}
\end{aligned}
\]

It is perhaps a bit clearer to keep the \(\lambda\) 's separate, and then the integrand is, besides \(M(f)\),
\[
\begin{aligned}
& \left|\lambda_{1}\right|^{2} G_{1,1}+\left|\lambda_{0}\right|^{2} G_{0,0}+\left|\lambda_{2}\right|^{2} G_{2,2}+\lambda_{\mathrm{r}}^{*} * \lambda_{2} G_{1,2}+\lambda_{1} * \ddot{\lambda}_{0} G_{1,0}+\lambda_{0} * \lambda_{1} G_{0,1}+\lambda_{2} * \lambda_{1} G_{2,1} \\
& +\left|\lambda_{1}\right|^{2}+\left|\lambda_{0}\right|^{2}+\left|\lambda_{2}\right|^{2} \\
& =\lambda_{1}^{*}\left(\left(1+G_{1,1}\right) \lambda_{1}+\lambda_{2} G_{1,2}+\lambda_{0} G_{1,0}\right]_{0} \\
& \left.+\lambda_{0}^{*}\left[\left(1+G_{0,0}\right) \lambda_{0}+\lambda_{1} G_{0,1}\right)\right] \\
& +\lambda_{2}^{*}\left[\left(1+G_{2,2}\right) \lambda_{2}+\lambda_{1} G_{2,1}\right] \\
& =-\lambda_{1}^{*}
\end{aligned}
\]

\section*{APPENDIX 2 PROPERTIES OF THE ELEMENTS \(\mathrm{G}_{\mathrm{ij}}\).}

The report makes us of the elements Gij. Here we obtain some properties.
From the definition
\[
\begin{aligned}
& G_{i, j}(f)=\frac{M(f)}{2 T N_{0}} \sum H_{i}(f+m / T) H_{j}^{*}(f+m / T)+H_{i}^{*}(-f+m / T) H_{j}(-f+m / T) \\
& =G_{j, i} *(f)=G_{i, j}^{*}(-f)
\end{aligned}
\]

In the communication system under consideration, the transfer functions are due to half (or root) Nyquist filters, with an additional gain A, phase Өand delay \(\tau\). Thus, a root Nyquist filter with a channel
\[
A_{i} e^{j \theta_{i}} e^{-j \omega r_{i}}
\]

In the diagonal elements the phase and delay are eliminated, and the elements are just
\[
A^{2} / N_{0}
\]

The off-diagonal elements are more complicated. The elements \(\mathrm{G}_{0,1}\) and \(\dot{\mathrm{G}}_{1,0}\) are mirrors of each other. So are \(\mathrm{G}_{2,1}\) and \(\mathrm{G}_{1,2}\). In general, \(\mathrm{G}_{0,1}\) and \(\mathrm{G}_{2,1}\) have coefficients that are similar, but in which every second coefficient is of opposite sign.

Thus, the matrix appearing in the equations
\[
\left(\begin{array}{c}
-C_{+} \\
-1-B_{+} \\
-D_{+}
\end{array}\right)=\left(\begin{array}{ccc}
1+G_{0,0} & G_{0,1} & 0 \\
G_{1,0} & 1+G_{1,1} & G_{1,2} \\
0 & G_{2,1} & 1+G_{2,2}
\end{array}\right)\left(\begin{array}{c}
\lambda_{0}(f) \\
\lambda_{1}(f) \\
\lambda_{2}(f)
\end{array}\right)
\]
may be written
\[
\left(\begin{array}{ccc}
1+A_{0}^{2} / N_{0} & A_{0} A_{1} G_{0,1} & 0 \\
A_{1} A_{0} G_{1,0} & 1+A_{1}^{2} / N_{0} & A_{1} A_{2} G_{1,2} \\
0 & A_{2} A_{1} G_{2,1} & 1+A_{2}^{2} / N_{0}
\end{array}\right)
\]

Thus, in the matrices used in the Bauer factorization, the matrices that are related to powers of \(z\) that are not zero have no diagonal, no 1,3 or 3,1 term, and, are otherwise arbitrary, consistent with the overall matrix polynomial being positive definite. We haven't discovered any properties that follow from these observations.

Finally note that
\[
\therefore \%
\]
\[
G_{i, i} G_{j, j} \geqslant\left|G_{i, j}\right|^{2},
\]

Minkowski's inequality.

\section*{MODIFIED DUOBINARY ERROR RATE CALCULATION}

\section*{W F McGee}

19 March 1997

The purpose of this report is to give the rationale of calculating duobinary error rates.
The calculation of error rate for modified duobinary coding is more difficult than for binary. We imagine that we are given the received pulse response including the filter
\[
M(z)=0.5-0.5 z^{-8}
\]
where the exponent 8 reflects the use of a system in which the data is removed by taking every fourth sample, i.e., decimation by 4 . The waveform is itself sampled every fourth sample, and results in a sequence \(\mathrm{A}=\{\ldots\}\). For a no-noise, no intersymbol interference, no crosstalk receiver, this sequence will have the structure \(\{\ldots, 0,0,0,0,0,1 / 2,0,-1 / 2,0,0,0,0, \ldots\}=\left\{a_{k}\right\}\). In the theory section we consider the \(1 / 2\) response to occur at time \(\mathrm{t}=0\).

The modified duobinary receiver may be though of as an interleaved set of bipolar signals.
The received signal, neglecting noise and crosstalk, (i.e., considering only intersymbol, interference) is
\[
s_{k}=\sum a_{k-k^{\prime}} b_{k^{\prime}}=\sum b_{k-k^{\prime}} a_{k}
\]

With random input data \(\left\{b_{k}\right\}\) the received data sequence \(s_{k}\) will, with random binary inputs, have three basic values; \(\{+1,0,-1\}\). A three level receiver will determine, at a sample time, which of the three is to be \(\therefore\) decoded. Examine the sample \(s_{0}\) at time 0 .

If , in the ideal case, \(s_{0}=+1\), this means that the transmitted binary pulses must have been \(b_{0}=1, b_{-2}=-1\), and the received signal will, in fact be, \(\mathrm{a}_{0}-\mathrm{a}_{2}\).

If , in the ideal case, \(s_{0}=-1\), the transmitted pulses must have been \(b_{0}=-1, b_{-2}=1\), and the received signal will be \(-\left(a_{0}-a_{2}\right)\).
\[
\therefore
\]

If , in the ideal case, \(s_{0}=0\), there are two choices, \(b_{0}=1, b_{-2}=1\) or \(b_{0}=-1, b_{-2}=-1\), and the received signal will be \(a_{0}+a_{2}\) or \(-\left(a_{0}+a_{2}\right)\).

The threshold would, presumably, be set halfway between these signal levels. Thus, the positive threshold \(d\) will be set so that \(\operatorname{la} 0-\mathrm{a} 2|-\mathrm{d}=\mathrm{d}-|\mathrm{la} 0+\mathrm{a} 2|\), i.e., \(\mathrm{d}=(1 / 2)(|a 0-\mathrm{a} 2|+|a 0+\mathrm{a} 2|)=\max (|a 00|,|a 2|)\).

The average error probability will be
\[
\begin{aligned}
& (1 / 4) \operatorname{Pr}\left(\left|a_{0}-a_{2}\right|+n<d\right)+(1 / 2) \operatorname{Pr}\left(a_{0}+a_{2}+n>d\right)+ \\
& (1 / 2) \operatorname{Pr}\left(a_{0}+a_{2}+n<-d\right)+(1 / 4) \operatorname{Pr}\left(-\left|a_{0}-a_{2}\right|+n>-d\right) \\
& =(1 / 4) \operatorname{Pr}\left(\left|a_{0}-a_{2}\right|-d+n<0\right)+(1 / 2) \operatorname{Pr}\left(d-\left(a_{0}+a_{2}\right)-n<0\right)+ \\
& (1 / 2) \operatorname{Pr}\left(d+a_{0}+a_{2}+n<0\right)+(1 / 4) \operatorname{Pr}\left(\left|a_{0}-a_{2}\right|-d-n<0\right) \\
& =(1 / 2) \operatorname{Pr}\left((1 / 2)\left(\left|a_{0}-a_{2}\right|+\left|a_{0}+a_{2}\right|\right)+n<0\right)+(1 / 2) \operatorname{Pr}\left(d-\left(a_{0}+a_{2}\right)-n<0\right)+ \\
& (1 / 2) \operatorname{Pr}\left(d+a_{0}+a_{2}+n<0\right) \\
& \leq(3 / 2) \operatorname{Pr}\left(\min \left(\left|a_{0}\right|,\left|a_{2}\right|\right)+n<0\right)
\end{aligned}
\]

For example, if \(\mathrm{a} 0=.52, \mathrm{a} 2=-0.4 \dot{8}, \mathrm{~d}=0.52\), and the three probabilities are
\((1 / 2) \operatorname{Pr}(0.48+n<0)+(1 / 2) \operatorname{Pr}(0.48+n<0)+(1 / 2) \operatorname{Pr}(0.56+n<0)<(3 / 2) \operatorname{Pr}(0.48+n<0)\)

For our calculations we often use the approximation \((3 / 2) \operatorname{Pr}\left(\left(\left|a_{0}\right|+\left|a_{2}\right|\right) / 2+n<0\right)\) for the error probability. This is reasonable in Gaussian noise.

\section*{REVIEW OF INTERSYMBOL INTERFERENCE MITIGATION}

\author{
W F McGee
}

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Contract U6800-6-3505

There is a great similarity between the problem of reducing the effects of intersymbol interference in data communications and multi-user interference reduction. This arises because the interference generated by other symbols in a sequence depends on the pulse response and the data symbols; for multiple users the intereference depends on the transmitted data and the response of different matched filters. In this appendix we recall the theory of intersymbol interference.

The theory has as its basis the detection of a single symbol in noise. For example, a pulse is transmitted with one of two polarities, and the optimum receiver in Gaussian noise is a pulse-matched (Wiener) filter that is matched to the transmitted pulse shape, followed by a sign detector. There a five commonly studied receivers.
1. Linear Zero-Forcing Equalizer. (ZF)
2. Minimum Mean Squared Error Receiver. (MMSE)
3. Decision Feedback Receiver. (DF)
4. Maximum Likelihodd SEquence Estimator (MLSE)
5. Maximum A Priori Receiver. (MAP)

Matched filter.
For a pulse with shape \(h(t)\), transform \(H(f)\) in noise with spectrum \(N(f)\), the matched filter is
\(E(f)=H^{*}(f) / N(\dddot{f})\)
and the signal to noise ratio at the decision time is
\[
\rho=\int \frac{|H(f)|^{2}}{N(f)} d f
\]
1. Linear Zero-Forcing Equalizer. (ZF)

A receive filter is designed to maximize the signal to noise ratio at the decision time, and to eliminate interference at other sampling times. The resultant pulse shape has zeros in the pulse response at other sample times. This receiver eliminates infersymbol interference but increases the thermal noise compared to the matched filter. For transmitted pulse shape \(h(t)\), transform \(H(f)\), noise power spectrum \(N(f)\), the \(Z F\) equalizer is
\[
E(f)=\frac{H^{*}(f) / N(f)}{\sum_{m} \frac{|H(f+m / T)|^{2}}{N(f+m / T)}}
\]

\section*{2. Minimum Mean Squared Error Receiver. (MMSE)}

The MSE expressed in the frequency domain is
\[
M S E=\int N(f)|E(f)|^{2} d f+\int_{-1 / 2 T}^{1 / 2 T}\left|\frac{1}{T} \sum_{m} H(f+m / T) E(f+m / T)-1\right|^{2} d f
\]
and the optimum equalizer is
\[
E(f)=\frac{H^{*}(f) / N(f)}{\frac{1}{T}+\sum_{m} \frac{|H(f+m / T)|^{2}}{N(f+m / T)}}
\]

This receiver has gain bias, and it is necessary to convert the mean squared error with bias to the mean squared error without bias, a larger number, and more indicative of system operation.

\section*{3. Decision Feedback Receiver. (DF)}

The decision Feedbach Receiver assumes that the receiver has available the previously transmitted symbols, and minimizes the mean squared error assuming that these are known. Hence the intersymbol interference comes only from symbols yet to be received, that is, from the precursors of the pulse response.
\[
M S E=\int N(f)|E(f)|^{2} d f+\int_{-1 / 2 T}^{1 / 2 T}\left|\frac{1}{T} \sum_{m} H(f+m / T) E(f+m / T)-\sum_{k=0} B_{k} e^{-j 2 \pi T T}\right|^{2} d f
\]

\section*{4. Maximum Likelihood Sequence Estimator (MLSE)}

This receiver considers a whole sequence to be detected, and minimizes the mean squared error over all possible transmitted sequences. The receiver has a pre-equalizer which minimizes noise and the precursor samples, a sampler, and a Viterbi receiver for the sequence.

This receiver determines that the sequence was transmitted which is most likely to have resulted in the given received sequence. Because of the pre-equalizer that is used, the conditional probability does not depend on future transmitted symbols. Thus, the potential transmitted sequences may be pruned by only considering those which are most likely, at each sample time, to have resulted in the received sequence up to that time. Viterbi algorithm is used because the calculation of the likelihood of the output sequence given a particular input sequence.

\section*{5. Minimum Bit-Error Probabilty Receiver.}

The MLSE determines which input sequence is most likely, but MLSE does not determine the input bit sequences which result in the lowest error probability on a symbol basis. The minimum bit-error probability receiver looks at all the received signals, and determines, for a particular transmitted symbol, what was most likelyu to have caused the received signal. The effect of the other transmitted symbols is averaged. There are iterative algorithms to perform thé catculation.

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1. ELee and D Messerschmitt, Digital Communications, Kluwer, 1994.

\title{
Maximum Likelihood Intersymbol Interference Receivers W F McGee \\ Contract U6800-6-1604 \\ 19/03/97 1:27 PM
}

The purpose of this note is to summarize the arguments in Lee and Messerschmitt[1].
Assume that the received signal is
\[
y(t)=\sum_{k=1}^{K} \dot{a}_{k} h(t-k T)+e(t)
\]
where \(h(t)\) is the transmitted pulse, \(a_{k}\) the transmitted data, and \(e(t)\) white Gaussian noise.
The optimum receiver calculates that input sequence \(\left\{a_{k}\right\}\) of length \(K\) that maximizes
\[
2 \operatorname{Re}\left\{\sum_{k=1}^{K} u_{k} a_{k}\right\}-\sum_{k=1}^{K} \sum_{m=1}^{K} a_{k} a_{m} r(m-k)
\]
where
\[
u_{k}=\int_{-\infty}^{\infty} y(t) h^{*}(t-k T) d t
\]
and \(r(k)\) is the autocorrelation of the pulse \(h(t)\).
For each possible sequence, the computation requires in the order of \(\mathrm{K}^{2}\) operations.
The further analysis is an attempt to turn the quadratic form into a sum of squares.
-If we represent the correlation matrix as a Toeplitz matrix
\[
\cdots \mathrm{R} \text { with } R_{i, j}=r(i-j)
\]
then the sum may be recognized as equivalentito
\[
-\left\{u^{H} u-2 \operatorname{Re}\left(u^{H} a\right)+a^{H} R a\right\}
\]
and with an LU (Cholesky) decomposition of \(\mathrm{R}=\mathrm{U}^{\mathrm{H}} \mathrm{U}\)
the expression to be minimized is
\[
\begin{aligned}
& -\left\{u^{H} u-2 \operatorname{Re}\left(\left(U^{-H} u\right)^{H} U a\right)+(U a)^{H} U a\right\} \\
& =-\left(U^{-H} u-U \dot{a}\right)^{H}\left(U^{-H} u-U a\right) \\
& =-(w-U a)^{H}(w-U a)
\end{aligned}
\]
where
\(w=U^{-H} u\)

Suppose that the correlation coefficients \(r(n)\) become zero as \(n\) increases; thus \(r(n)=0, n \geq M\). Then the matrix \(R\) used in the calculation of \(\mathrm{a}^{\mathrm{H}} \mathrm{Ra}\) becomes banded, and so does the Cholesky factor U . In fact, as the data sequence becomes longer and longer, the Cholesky factor \(U\) has the last column that does not change; it simply shifts down.If we examine the Cholesky factor for \(M=3\) and for a sequence of length 8 , it appears as follows
\[
U=\left(\begin{array}{llllllll}
x & x & x & 0 & 0 & 0 & 0 & 0 \\
0 & x & x & x & 0 & 0 & 0 & 0 \\
0 & 0 & x & x & x & 0 & 0 & 0 \\
0 & 0 & 0 & x & x & x & 0 & 0 \\
0 & 0 & 0 & 0 & x & x & x & 0 \\
0 & 0 & 0 & 0 & 0 & x & x & x \\
0 & 0 & 0 & 0 & 0 & 0 & x & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x
\end{array}\right)
\]
where the \(x\) 's indicate the non-zero entries, and, as mentioned, the last column approaches unchanging values. Define a non-square matrix \(\mathrm{R}+\) as a Toeplitz matrix with the columns the asymptotic value.

Then it can be shown that
\[
R_{+}=\left(\begin{array}{cccccccc}
x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y & x & 0 & 0 & 0 & 0 & 0 & 0 \\
z & y & x & 0 & 0 & 0 & 0 & 0 \\
0 & z & y & x & 0 & 0 & 0 & 0 \\
0 & 0 & z & y & x & 0 & 0 & 0 \\
0 & 0 & 0 & z & y & x & 0 & 0 \\
0 & 0 & 0 & 0 & z & y & x & 0 \\
0 & 0 & 0 & 0 & 0 & z & y & x \\
0 & 0 & 0 & 0 & 0 & 0 & z & y \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & z
\end{array}\right)
\]
\[
\therefore \text { 矿 } \cdots
\]
\[
R=R_{+}^{H} R_{+}
\]

The elements of the last column are the coefficients of a spectral factor of \(S(z)\). Also note that
\[
\left(\begin{array}{llll}
1 & z^{-1} & \mathrm{~K} & z^{-(R-1)}
\end{array}\right) R_{+} R_{+}^{H}\left(\begin{array}{c}
1 \\
z \\
\Lambda \\
z^{R-1}
\end{array}\right)
\]
is proportional to \(S(z)\).
The quadratic form then is
\[
\begin{aligned}
& a^{H} R_{+}^{H} R_{+} a-2 \operatorname{Re}\left(u^{H}\left(R_{+}^{H} R\right)^{-1} R_{+}^{H} R_{+} a\right)+\text { terms not depending on a } \\
& =\left(R_{+} a-R_{-} u\right)^{H}\left(R_{+} a-R_{-} u\right)
\end{aligned}
\]
where we have defined the non-square upper diagonal Toeplitz matrix R-by
\[
R_{-} R_{+}=I
\]

The advantage of this approach is that it involves a convoulution of the input sequence and the data sequence, whereas the use of the Cholesky factorization involves the formation of inner products in which the coefficients are all different.


Figure 1 Implementation of the maximum likelihood sequence estimator using the augmented Cholsky matrices \(\mathrm{R}_{+}(\mathrm{z})\) and \(\dot{\mathbf{R}} .(\mathrm{z})\).

The communications approach is in termis of spectral factorization, as follows.
Define transforms as follows
\[
\begin{aligned}
A(z) & =\sum_{k=1}^{K} a_{k} z^{-k} \\
U(z) & =\sum_{k=1}^{K} u_{k} z^{-k} \\
S(z) & =\sum_{k=-\infty}^{\infty} \rho_{k} z^{-k}
\end{aligned}
\]

Then the likelihood is proportional to the dc coefficient of
\[
2 \operatorname{Re}\left\{A_{*}(z) U(z)\right\}-A_{*}(z) A(z) S(z)
\]

A simplification occurs if the spectral factorization of the folded pulse spectrum is used, thus
\[
S(z)=G(z) G_{*}(z)
\]

The likelihood is then proportional to the dc coefficient of
\[
-\left[A(z) G(z)-U(z) / G_{*}(z)\right]\left[A_{*}(z) G_{*}(z)-U_{*}(z) / G(z)\right]
\]

The second factor is just the first written backwards and with the coefficients conjugated.
Now, \(A(z) G(z)\) will have coefficients of \(z^{-1}, z^{-2}, \ldots\) but \(W(z)=U(z) / G^{*}(z)\) will have all powers of \(z\). But as we vary the elements of the data sequence \(A(z)\), only the positive coefficients are of interest, and therefore only the causal part of \(W(z)\) is of interest. Thus, the quantity to be evaluated to determine the most likely data sequence is
\[
\sum_{m=1}^{\infty}\left|\sum_{k=1}^{K} a_{k} \dot{g}_{m-k}^{\prime}-w_{m}\right|^{2}
\]

The block diagram is


Figure 2. The transmitted is a sequencé of pulse amplitude modulated signals, whetre the pulse is \(h(t)\). White noise \(e(t)\) is added. The receiver determines that set of transmitted symbols that makes the indicated sum of squares a minimum.

\section*{Vector channels}

In this case there are \(M\) interfering channels, and the receiver signal may be written in the form
\[
y(t)=\sum_{k=1}^{K} \mathbf{a}_{k}^{T} \mathbf{h}(t-k T)+e(t)
\]
where \(\mathbf{h}\) and \(\mathbf{a}\) are now M -vectors.
The log-likelihood is proportional to
\[
\int\left|y(t)-\sum_{k=1}^{K} \mathbf{a}_{k}^{T} \mathbf{h}(t-k \dot{T})\right|^{2} d t
\]
and the receiver is to determine the set of transmitted data that minimizes this real number. Neglecting the terms that don't depend on the transmitted data, this is equivalent to minimizing
\[
\begin{aligned}
& -2 \operatorname{Re}\left\{\sum_{k=1}^{K} \mathbf{a}_{k}^{H} \int y(t) \mathbf{h}^{*}(t-k t) d t\right\}+\sum_{k=1}^{K} \sum_{k^{\prime}=1}^{K} \mathbf{a}_{k}^{H} \int \mathbf{h}^{*}(t-k t) \mathbf{h}^{T}\left(t-k^{\prime} T\right) d t \mathbf{a}_{k^{\prime}} \\
& -2 \operatorname{Re}\left\{\sum_{k=1}^{K} \mathbf{a}_{k}^{H} \mathbf{u}_{k}\right\}+\sum_{k=1}^{K} \sum_{k^{\prime}=1}^{K} \mathbf{a}_{k}^{H} \mathbf{R}\left(k-k^{\prime}\right) \mathbf{a}_{k^{\prime}}
\end{aligned}
\]
where
\[
\mathbf{u}_{k}=\int y(t) \mathbf{h}^{*}(t-k T) d t
\]
is an \(M\)-vector with the outputs of the \(M\) matched filters \(h_{i}{ }^{*}(-t)\) and the matrix \(R\) is given by
\[
\begin{gathered}
-2 \operatorname{Re}\left\{\sum_{k=1}^{K} \mathbf{a}_{k}^{H} \int y(t) \mathbf{h}^{*}(t-k T) d t\right\}+\sum_{k=1}^{K} \sum_{k^{\prime}=1}^{K} \mathbf{a}_{k}^{H} \int \mathbf{h}^{*}(t-k T) \mathbf{h}^{\dot{T}}\left(t-k^{\prime} T\right) d t \mathbf{a}_{k^{\prime}} \\
-2 \operatorname{Re}\left\{\sum_{k=1}^{K} \mathbf{a}_{k}^{H} \mathbf{u}_{k}\right\}+\sum_{k=1}^{K} \sum_{k^{\prime}=1}^{K} \mathbf{a}_{k}^{H} \mathbf{R}\left(k-k^{\prime}\right) \mathbf{a}_{k^{\prime}} \\
\vdots \\
\cdots \quad \mathbf{R}\left(k-k^{\prime}\right)=\int \mathbf{h}^{*}(t-k t) \mathbf{h}^{T}\left(t-k^{\prime} T\right) d t
\end{gathered}
\]

To make further analysis easier, we now extend the set of K M -vectors to a KM -vector. This may be done in many ways, but the most logical would be to either group those entries of the same channel, or those occuring at the same time; we opt for the first choice.
.. |lThus, a is a vector consisting of the K transmitted symbols for the first channel, the K transmitted sym,bols for the second channel, etc. The vector \(u\) consists of \(K\) outputs from the first matched filter, \(K\) outputs from the second matched filter; 'ęt. The matrix \(\mathbf{R}\) is of the form
\[
\mathbf{R}=\left(\begin{array}{ccccc}
\Lambda_{11} & \Lambda_{12} & 0 & \Lambda & \Lambda_{1, M} \\
\Lambda_{21} & \Lambda_{22} & \Lambda_{23} & \Lambda & 0 \\
0 & \Lambda_{32} & \Lambda_{33} & \Lambda & 0 \\
\Lambda & \Lambda & \Lambda & \Lambda & \Lambda \\
\Lambda_{M, 1} & 0 & 0 & \Lambda & \Lambda_{M, M}
\end{array}\right)
\]

Each element of the matrix is an K -by- K Toeplitz matrix consisting of the shifts of the pulse correlation matrices
\[
\Lambda_{i, j}\left(k-k^{\prime}\right)=\int h_{i}^{*}(t-k T) h_{j}\left(t-k^{\prime} T\right) d t=\Lambda_{j, i}^{*}\left(k^{\prime}-k\right)
\]
thus
\[
\Lambda_{j, i}=\Lambda_{i, j}^{H}
\]

If we assume that each receiver is matched to the transmitted waveform for that channel including gain \(A_{i}\), phase \(\varphi_{\mathrm{i}}\) and delay \(\tau_{\mathrm{i}}\), and if we ignore the different delays [essentially quantize the delays to a symbol time], and if, in addition, we assume that the channel filters are otherwise frequency shifted root-Nyquist filters then the channel responses for QAM communication are given by
\[
\Lambda_{i, j}\left(k-k^{\prime}\right)=A_{i} A_{j} e^{j\left(\varphi_{j}-\varphi_{i}\right)} R(i-j)
\]
where \(R(i-j)\) is a \(K\)-by- \(K\) matrix given by the sampled pulse response of the adjacent receiver in channel \(j\) to a sequence in channel \(i\).

In the case of VSB filterbanks the matrices are more complicated.
The matrix \(R(i)\) vanishes if \(i>1\), and is a unit matrix if \(i=0\). So the only complicated portion is \(R(1)=C\) and \(R(-1)=R^{H}(1)\). The matrix \(C\) itself is Toeplitz and if the transmitting and receiving filters are linear phase, \(C\) is also Hermitian.

\section*{REFERENCES}
1. E. Lee and D Messerschmitt, "Digital Communication-Second Edition", Kluw'er, Boston, 1994.

\section*{APPENDIX 1 REVIEW OF MULTI-USER CDMA}

The analysis of multi-user detection systems for direct sequence CDMA starts with an elementary system. It consists of a \(M\) transmitters of binary \([\{+1,-1\}]\) waveforms using real sequences \(s_{k}(t)\) of unit power, real additive white Gaussian noise. These signals are transmitted through flat channels with gains \(c_{k}=\sqrt{w_{k}}\), added together with the noise. The waveforms \(\mathrm{s}_{\mathrm{k}}(\mathrm{t})\) are assumed to be over one symbol only. Hence intersymbol interferences is not relevant, and it is sufficient to consider the transmission of only M bits, one for each transmitter, which is considered a vector x with components \(\mathrm{x}_{\mathrm{k}}\).

A sufficient statistic for the determination of the transmitted bits is obtained by passing the received signal through M matched filters whose outputs, when sampled, form the vector with components \(y_{k}\). The components \(y_{k}\) are a linear combination of the transmitted signal bits \(x_{k}\), and a set of correlated noise samples with correlation kmatrix \(\mathbf{R}\).

If we define a diagonal matrix \(\mathbf{C}\) with elements \(c_{k}\), the channel gain for the \(k\) th transmitter, the output from the matched filters has the log-likelihood function proportional to
\[
(\mathbf{y}-\mathbf{R C x})^{T} \mathbf{R}^{-1}(\mathbf{y}-\mathbf{R C x})
\]

This implies that the best estimate of the transmit sequence based on the received signals \(y\) is that possible transmit vector x which maximizes
\[
2 y^{\mathrm{T}} \mathrm{Cx}-\mathrm{x}^{\mathrm{T}} \mathrm{CRCx}
\]

This is called the maximum likelihood receiver, and is generally considered to be too complicated to implement.

The so-called conventional receiver simply looks at the sign of \(y\).
Yet another receiver is the decorrelating receiver which operates with the variables
\[
\because \mathbf{d}=\mathbf{R}^{-1} \mathbf{y}
\]
which is the sum of \(\mathbf{C x}\) and a Gaussian noise with covariance matrix \(\mathbf{R}^{-1}\); the covariance of \(\mathbf{y}\), on the other hand, is \(\mathbf{R}\). The log-likelihood of \(d\) is
\[
{ }^{\prime}(\mathbf{d}-\mathbf{C x})^{\mathbf{T}} \mathbf{R}(\mathbf{d}-\mathbf{C x})
\]
so the maximum likelihood receiver is based on maximizing
\[
2 d^{T} R C x-x^{T} C R C x
\]
which is the same as the previous function. But the decorrelating receiver operates by taking the sign of \(\mathbf{d}\). Thus
\[
\overline{\mathbf{x}}=\operatorname{sign}(\mathbf{d}) .
\]

If we compare the decorrelating receiver to the conventional receiver, the interference from the other users has been eliminated, but the noise power is increased.

A minimum mean-squared error MMSE detector actually estimates \(\mathbf{x}\) by taking a linear combination of the vector \(y\) to minimize
\[
(x-\mathbf{L y})^{T}(x-\mathbf{L} y)
\]

B

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W F McGee \\ 10 March 1997 \\ Contract U6800-6-1604
}

This note summarizes properties of correlation matrices and sequences.
There are two main approaches in the literature: one uses matrices, the other uses sequences. We include vectors. I am not aware of anything new in this summary.

\section*{1. MATRICES}

In this work we need the reversal operator * on matrices
\[
A_{*, i, j}=A_{i}^{*} *
\]
defined for scalars (conjugation, vectors, matrices)
\[
(\mathrm{AB})_{*}=\mathrm{A}_{*} \mathrm{~B}_{*}
\]

A persymmetric matrix satisfies \(A=A *\)

\section*{Correlation matrix \(\mathbf{R}\)}

This is a Toeplitz, positive definite, Hermitian, persymmetric matrix with the following properties.
There are two main models of random processes, the non-white noise model, and the sum of sinusoids model.

Non-white noise model
\(\mathrm{R}_{\mathrm{i} . \mathrm{j}}=\mathrm{r}(\mathrm{i}-\mathrm{j})=E\left[Z(\mathrm{i}) \mathrm{Z}^{*}(\mathrm{j})\right]\) circularly complex Gaussian noise
\[
r(-i)=r^{*}(i)
\]
\[
R_{M+1}=\left(\begin{array}{cccc}
r(0) & r(-1) & \Lambda & \dot{r}(-M) \\
r(1) & r(0) & \Lambda & r(-(M-1)) \\
\Lambda & \Lambda & \Lambda & \Lambda \\
r(M) & r(M-1) & \Lambda & r(0)
\end{array}\right)=\left(\begin{array}{cc}
r(0) & r_{M}^{H} \\
r_{M} & R_{M}^{-}
\end{array}\right)=\left(\begin{array}{cc}
R_{M} & r_{M^{*}} \\
r_{M^{*}}^{H} & r(0)
\end{array}\right)
\]
\(S(f)\) is the power spectrum \(S(f)\) real; non-negative,
\[
R_{n}=\int S(f) e^{j 2 \pi f n} d f
\]

Independent sine waves in white noise model
\[
R_{n}=\sum_{k}\left|a_{k}\right|^{2} e^{j 2 \pi n_{k}}+\sigma^{2} \delta_{n}
\]
\[
S(f)=\sum\left|a_{k}\right|^{2} \delta\left(f-f_{k}\right)+\sigma^{2}
\]

\section*{Linear Prediction}

The Generalized Yule Walker equations are
\[
R_{M+1} a_{M}=\left(\begin{array}{c}
P_{M} \\
0 \\
\Lambda \\
0
\end{array}\right)
\]
leading coefficient of \(\mathrm{a}_{\mathrm{M}}\) is 1 .
\[
a_{M}=\binom{\frac{1}{2}}{-w}
\]
the solution of
\[
R_{M+1} a_{M}=\left(\begin{array}{cc}
r(0) & r_{M}^{H} \\
r_{M} & R_{M}
\end{array}\right)\binom{1}{-f}=\binom{r(0)-r_{M}^{H} f}{r_{M}-R_{M} f}=\binom{P_{M}}{0}
\]

The backward prediction problem is the vector \(\mathrm{a}_{\mathrm{M}^{*}}\) which satisfies
\[
R_{M+1} a_{M^{*}}=\left(\begin{array}{c}
0 \\
\Lambda \\
0 \\
P_{M}
\end{array}\right)
\]

\section*{Order update solutions}

\section*{Levinson-Durbin}
\[
a_{M}=\binom{a_{M-1}}{0}+\beta_{M}\binom{0}{a_{M-1}}
\]
with
\[
R_{M+1} a_{M}=\binom{P_{M}}{0}=\left(\begin{array}{cc}
R_{M} & r_{M^{*}} \\
r_{M^{*}}^{H} & r(0)
\end{array}\right)\binom{a_{M-1}}{0}+\beta_{M}\left(\begin{array}{cc}
r(0) & r_{M}^{H} \\
r_{M} & R_{M}
\end{array}\right)\binom{0}{a_{M-1^{*}}}=\left(\begin{array}{c}
P_{M-1}+\beta_{M} r_{M}^{H} a_{M-1^{*}} \\
0 \\
\Lambda \\
r_{M^{*}}^{H} a_{M-1}+\beta_{M} P_{M-1}
\end{array}\right)
\]
thus
\[
\begin{aligned}
& \beta=-r_{M^{*}}^{H} a_{M-1} / P_{M-1} \\
& P_{M}=P_{M-1}\left(1-\left.1 \beta_{M}\right|^{2}\right)
\end{aligned}
\]

\section*{Solution of Levinson equations}

Levinson equations are of the form
\[
R_{M+1} x_{M+1}=y_{M+1}
\]

We assume that we have a solution to the linear prediction problem and the solution to the equations
\[
R_{M} x_{M}=y_{M}
\]
where \(\mathrm{y}_{\mathrm{M}}\) is the first M elements of \(\mathrm{y}_{\mathrm{M}+1}\). Then, since
\[
R_{M+1} x_{M+1}=y_{M+1}=\binom{y_{M}}{y(M+1)}=\left(\begin{array}{cc}
R_{M} & r_{M^{*}} \\
r_{M^{*}}^{H} & r(0)
\end{array}\right)\binom{\ddot{x}_{M}+\delta x_{M}}{x(M+1)}=\binom{y_{M}+R_{M} \delta x_{M}+r_{M^{*}} x(M+1)}{r_{M^{*}}^{H} x_{M}+r_{M^{*}}^{H} \delta x_{M}+r(0) x(M+1)}
\]
this implies that
\[
\begin{aligned}
& \delta x_{M}=-x(M+1) f_{*} \\
& x(M+1)=-\frac{r_{M^{*}}^{H} x_{M}}{P_{M}}
\end{aligned}
\]
solving the problem.

\section*{Trench algorithm for inverse \(\mathbf{R}_{\mathrm{M}}{ }^{-1}\).}

The algorithm assumes that we have the solution of the linear prediction problem.
Since
\[
\begin{aligned}
: & R_{M} R_{M}^{-1}=I \\
\hdashline & \text { and } \\
& R_{M^{*}}=R_{M} \\
& \text { thus } \\
\quad & \left(R_{M}^{-1}\right)_{*}=R_{M}^{-1}
\end{aligned}
\]
i.e., \(\mathrm{R}_{\mathrm{M}}{ }^{-1}\) is persymmetric.

Let
\[
R_{M+1}^{-1}=\left(\begin{array}{cc}
B & v \\
v^{H} & \dot{\gamma}
\end{array}\right)
\]
then
\[
I_{M+1}=R_{M+1}\left(\begin{array}{cc}
B & v \\
v^{H} & \gamma
\end{array}\right)=\left(\begin{array}{cc}
R_{M} & r_{M^{*}} \\
r_{M^{*}}^{H} & r(0)
\end{array}\right)\left(\begin{array}{cc}
B & v \\
v^{H} & \gamma
\end{array}\right)=\left(\begin{array}{cc}
R_{M} B+r_{M^{*}} v^{H} & R_{M} v+\gamma_{M^{*}} \\
r_{M^{*}}^{H} B+r(0) v^{H} & r_{M^{*}}^{H} v+\gamma r(0)
\end{array}\right)=\left(\begin{array}{cc}
I_{M} & 0 \\
0 & 1
\end{array}\right)
\]

Consequently,
\[
\begin{aligned}
& v=\frac{-f_{M^{*}}}{P_{M^{*}}} \\
& \gamma=1 / P_{M}
\end{aligned}
\]

This in turn requires that
and
\[
B=R_{M}^{-1}+\frac{f_{M^{*}} f_{M^{*}}^{H}}{P_{M}}
\]
\[
B_{*}=R_{M}^{-1}+\frac{f_{M} f_{M}^{H}}{P_{M}}
\]
which means that we can eliminate the inverse correlation matrix and determine that
\[
B=B_{*}+\frac{f_{M^{*}} f_{M^{*}}^{H}}{P_{M}}-\frac{f_{M} f_{M}^{H}}{P_{M}}
\]

By using the persymmetry in \(\mathrm{R}_{\mathrm{M}+1}{ }^{-1}\) and these relations for the reversed matrix, the whole of the inverse matrix may be calculated: Consider a 6-by-6 matrix. We know the 6th row and column by the formulas for v. Using persymmetry of \(\mathrm{R}_{\mathrm{M}+1}{ }^{-1}\), we know the first row and column. This allows the calculation of the 5th row and column, using the relation for B . This then gives the second row and column. Using the relationship for \(B\) we get the fourth row and column, for \(\mathrm{R}_{\mathrm{M}+1}{ }^{-1}\) the third row, and we are done. Consequently the order of calculation is columns \(6,1,5,2,4,3\).

As Golub [3] points out, only the upper quarter neëds to be calculated for a persymmetric Hermitian matrix. If we have access to all the predictor polynomials then the inverse matrix is given by
\[
\therefore\left(\begin{array}{cc}
R_{M}^{-1} & 0 \\
0 & 0
\end{array}\right)+\frac{a_{M^{*}} a_{M^{*}}^{H}}{P_{M}^{-1}}
\]
which is an explicit writing out of the Cholesky decomposition of the inverse of the correlation matrix
\[
R^{-1}=U D^{-1} U^{H}=L^{H} D^{-1} L
\]

\section*{Spectral factorization}

As noted above, the elements of the correlation matrix are the Fourier transform of the spectrum. Letting
\[
\begin{aligned}
& S_{M+1}(z)=\sum_{k=-M}^{M} r(k) z^{-k}=S_{+}\left(z^{-1}\right) S_{-}(z) \\
& S_{+}\left(z^{-1}\right)=\sum_{k=0}^{M} s_{+}(k) z^{-k} \\
& S_{-}(z)=\sum_{k=0}^{M} s_{+}^{*} z^{k}
\end{aligned}
\]

Bauer shows that the spectral decomposition may be performed by finding the Cholesky decomposition of aa matrix which becomes larger and larger as follows
\[
\left(\begin{array}{cccccc}
r(0) & r(-1) & r(-2) & 0 & 0 & 0 \\
r(1) & r(0) & r(-1) & r(-2) & 0 & 0 \\
r(2) & r(1) & r(0) & r(-1) & r(-2) & 0 \\
0 & r(2) & r(1) & r(0) & r(-1) & r(-2) \\
0 & 0 & r(2) & r(1) & r(0) & r(-1) \\
0 & 0 & 0 & r(2) & r(1) & r(0)
\end{array}\right) \therefore\left(\begin{array}{cccccc}
x & 0 & 0 & 0 & 0 & 0 \\
x & x & 0 & 0 & 0 & 0 \\
x & x & x & 0 & 0 & 0 \\
0 & x & x & x & 0 & 0 \\
0 & 0 & x & x & x & 0 \\
0 & 0 & 0 & x & x & x
\end{array}\right)\left(\begin{array}{cccccc}
x & x & x & 0 & 0 & 0 \\
0 & x & x & x & 0 & 0 \\
0 & 0 & x & x & x & 0 \\
0 & 0 & 0 & x & x & x \\
0 & 0 & 0 & 0 & x & x \\
0 & 0 & 0 & 0 & 0 & x
\end{array}\right)
\]
where x indicates an unknown; we are only interested in the last row of the first matrix.
When the spectral factorization has been accomplished then the correlation matrix may be written as the product of two Toeplitz matrices as follows
\[
R_{M+1}=\left(\begin{array}{ccccc}
s_{+}(0) & s_{+}(1) & s_{+}(2) & 0 & 0  \tag{Bin}\\
0 & s_{+}(0) & s_{+}(1) & s_{+}(2) & 0 . \\
0 & 0 & s_{+}(0) & s_{+}(1) & s_{+}(2)
\end{array}\right)\left(\begin{array}{ccc}
s_{+}^{*}(0) & 0 & 0 \\
s_{+}^{*}(1) & s_{+}^{*}(0) & 0 \\
s_{+}^{*}(2) & s_{+}^{*}(1) & s_{+}^{*}(0) \\
0 & s_{+}^{*}(2) & s_{+}^{*}(1) \\
0 & \vdots 0 & s_{+}^{*}(2)
\end{array}\right)
\]

\section*{Eigenvalues}

Let the eigenvectors of the correlation matrix be \(\mathrm{q}_{\mathrm{i}}\). Then
\[
R q_{i}=\lambda_{i} q_{i}
\]

The eigenvalues must be real and positive, since R is positive definite [and Hermitian]. If the eigenvectors are unit-normalized, then the square matrix of eigenvectors \(Q\) given by
\[
Q=\left(\begin{array}{llll}
q_{1} & q_{2} & \Lambda & q_{M}
\end{array}\right)
\]
is unitary and
\[
R=Q \Lambda Q^{H}=\sum_{i} \lambda_{i} q_{i} q_{i}^{H}
\]
and
\[
R^{-1}=Q \Lambda^{-1} Q^{H}=\sum \frac{q_{i} q_{i}^{H}}{\lambda_{i}}
\]

Since
\[
R q_{i^{*}}=\lambda_{i} q_{i^{*}}
\]
an eigenvectors must be a multiple of itself backwards; typically they are the same or the negative of each other.

Since
\[
\begin{aligned}
& R_{M+1} a_{M}=P_{M}\left(\begin{array}{c}
1 \\
0 \\
\Lambda \\
0
\end{array}\right) \\
& a_{M}=P_{M} \sum \frac{q_{i} q_{i}^{H}}{\lambda_{i}}\left(\begin{array}{c}
1 \\
0 \\
\Lambda \\
0
\end{array}\right)=P_{M} \sum \frac{q_{i} q_{i, 1}^{*}}{\lambda_{i}}
\end{aligned}
\]
and \(\mathrm{a}_{\mathrm{M}^{*}}\) is the same sum but with a weighting equal to the last element in the eigenvector.
The eigenfilter associated with an eigenvector is
\[
Q_{i}(z)=\left(\begin{array}{llll}
1 & z^{-1} & \Lambda & \left.z^{-(M-1)}\right)
\end{array}\right) q_{i}
\]
and the conjugate filter is
\[
Q_{i^{*}}(z)=q_{i}^{H}\left(\left(1 \quad z \quad \Lambda \quad z^{M+1}\right)=z^{M-1}\left(\begin{array}{llll}
1 & z^{-1} & \Lambda & \left.z^{-(M-1)}\right)
\end{array}\right) q_{i^{*}}\right.
\]

The eigenfilters with the smallest and largest eigenvalues have their zeros on the unit circle.
The zeros of the eigenfilter with the smaffest eigenvalue are associated with the frequencies of sinusoids in a decomposition of a random process into sine waves and white noise. The eigenfilter with the largest eigenvalue maximizes the signal to noise ratio for a signal with correlation matrrix R and white noise; its zeros are where the signal is not.

Since
\[
r(n)=\int S(f) e^{j 2 \pi n n} d f
\]
it follows that
\[
\lambda_{k}=q_{k}^{H} R q_{k}=\sum_{r, s} q_{k, r}^{*} r(r-s) q_{k, s}=\int \sum_{r, s} q_{k, r}^{*} q_{k, s} e^{j 2 \pi f(r-s)} S(f) d f=\int\left|Q_{k}\left(e^{j 2 \pi f}\right)\right|^{2} S(f) d f
\]
and since
\[
1=q_{k}^{H} q_{k}=\int\left|Q_{k}\left(e^{j 2 \pi f}\right)\right|^{2} d f
\]
it follows that
\[
S_{\min } \leq \lambda_{k} \leq S_{\max }
\]
where \(\mathrm{S}_{\min }\) and \(\mathrm{S}_{\text {max }}\) are the minimum and maximum of the power spectrum.
The argument that the zeros of the eigenfilter of the smallest eigenvalue are on the unit circle is the following[4]. Let \(U\) be an eigenvector, and let the zeros of the eigenfilter \(U(z)\) be determined. Suppose that there is a zero off the unit circle. If they exist, they must occur in mirror pairs. Then create a new polynomial \(U^{\prime}(z)\) with the same zeros as \(U\), but with the zeros inside the unit circle reflected to those outside, i.e., replace \(z\) with \(1 / \mathrm{z}^{*}\). Then form the polynomial with these zeros, and a corresponding vector \(U^{\prime}(z)\). The magnitude of this vector is normalized to unity. Then \(U^{H} R U '\) is the same as \(U^{H} R U\). But \(U\) is the minimum eigenvector, and, if its multiplicity is one, this means that \(U\) and \(U\) ' must be proportional, since the eigenvectors are unique to within a scale factor. Thus, the zeros must all be on the unit circle. \(\%\);
This argument also applies to the maximum eigenvalue, but not to any other. The reason is that although \(U^{\prime}{ }^{\prime} R U\) ' and UHRU have the same value, one is an eigenvalue but the other is not necessarily an eigenvalue.

\section*{2. USE OF CORRELATION Z-TRANSFORMS INSTEAD OF MATRICES.}

The theory may be usefully interpreted in terms of z-transforms. These are useful because the Toeplitz nature of the correlation matrices in inherent in this formulation, and we may deal with familiar transfer functions instead of vectors.

Let X be a stationary sequence of zero-mean circularly complex random variables. \(\{\mathrm{x}(\mathrm{i})\}\). The z -transform of \(X\) is
\[
X(z)=\sum_{n=-\infty}^{\infty} x(n) z^{-n}
\]

The autocorrelation function \(\mathrm{R}_{x x}(\mathrm{z})\) is defined as
\[
\therefore R_{z K X}(z)=E\left[X(z) x^{*}(0)\right]
\]

Since
\[
R_{X X}(z)=\sum_{n=-\infty}^{\infty} r_{X X}(n) z^{-n}
\]
we have
\[
r_{X X}(n)=r_{X X}^{*}(-n)
\]
and define the power of the sequence as \(\mathrm{r}_{\mathrm{xx}}(0)\) which is real and positive and larger in magnitude than all the other correlation coefficients \(\mathrm{r}_{\mathrm{xx}}(\mathrm{n})\).

There is an alternative understanding of an autocorrelation function. Suppose that \(X\) is a sum of sinusoids of random phase, amplitude \(A i\), and frequency fi, and white noise. Then \(r_{x x}(n)\) is given by
\[
r_{x x}(n)=\sum_{k=1}^{M}\left|A_{k}\right|^{2} e^{2 \pi j j_{k}}+N_{0} \delta_{n}
\]
and
\[
R_{x X}(z)=N_{0}+\sum_{k=1}^{M} \sum_{n=-\infty}^{\infty}\left|A_{k}\right|^{2} z^{-n} e^{2 \pi j j_{k} n}
\]
and
\[
\begin{array}{r}
R_{X x}\left(e^{2 \pi i f}\right)=N_{0}+\sum_{k=1}^{M}\left|A_{k}\right|^{2} \delta\left(f-f_{k}\right) \\
\cdots
\end{array}
\]

In general, for two sequences \(X\) and \(Y\) we define the crosscorrelation as
\[
R_{X Y}(z)=E\left(X(z) y^{*}(0)\right)
\]

If \(Y\) and \(Z\) are filtered versions of the same sequence \(X\) with transfer function \(G(z)\) and \(H(z)\) then
\[
\begin{equation*}
R_{Y Z}(z)=G(z) H_{*}(z) R_{X X}(z) \tag{2}
\end{equation*}
\]
where \(\mathrm{H}^{*}(\mathrm{z})\) is \(\mathrm{H}(\mathrm{z})\) with its coefficients conjugated and z replaced with \(1 / \mathrm{z}\). Thus
\[
G(z) H_{*}(z)=\sum g(n) z^{-n} \sum_{m} h^{*}(\dot{m}) z^{m}=\sum_{n, m} z^{-(n-m)} g(n) h^{*}(m)=\sum_{r r . n} z^{-r} g(m+r) h *(m)
\]

Then
\[
\begin{aligned}
& E\left[Y(z) z^{*}(0)\right]:=
\end{aligned}
\]
\[
\begin{aligned}
& =\sum_{n=-\infty}^{\infty} z^{-n} \sum_{m=-\infty}^{\infty} g(n-m) x(m) \sum_{r=-\infty}^{\infty} h *(-r) x^{*}(r) \\
& =\sum_{n=-\infty}^{l^{\infty}} z^{-n \cdot} \sum_{m=-\infty}^{\infty} g(n-m) \sum_{r=-\infty}^{\infty} h^{*}(-r) r_{x x}(m-r) \\
& =\sum_{m, r, n} z^{-n} r_{X X}(\dot{m}+r) g(n-r) h *(m) \\
& =\sum_{m, s, n} z^{-n} r_{X X}(n-s) g(s+m) h^{*}(m) \\
& =G(z) H_{*}(z) R_{X X}(z)
\end{aligned}
\]

In particular, if \(\mathrm{Y}=\mathrm{GX}\)
\[
R_{Y Y}(z)=G(z) G_{*}(z) R_{X X}(z)
\]

Since the power of \(F X-\lambda X\) is positive, we must have
\[
\left|P_{F X X}\right|^{2} \leq P_{X X} P_{F X F X}
\]

The power spectrum is defined as
\[
S_{X}(f)=R_{X X}\left(e^{2 \pi i f}\right)
\]
and is real and positive on the unit circle.
Also note that \(\mathrm{H} *\) is the conjugate of H on the unit circle; i.e., \(\mathrm{H}^{*}\) is the analytic continuation of the conjugate of \(H\) off the unit circle, i.e.,
\[
H_{*}\left(e^{2 \pi j f}\right)=\left[H\left(e^{2 \pi i f}\right)\right]^{*}
\]

\section*{Predictor polynomials.}

The forward predictor polynomial \(F_{M}\left(z^{-1}\right)\) of order \(M\) is defined as that polynomial of order \(M\) in \(z^{-1}\) with the coefficient of \(z^{0}\) unity which minimizes the power of \(F_{M} X\), called the prediction error power.

As an example, the first order predictor may be written \(1+a_{1} z^{-1}\) and the prediction error power is the \(z^{0}\) coefficient of
\[
\left(1+a_{1} z^{-1}\right)\left(1+a_{1}^{*} z\right) R_{X}(z)
\]
i.e.
\[
\left(1+\left|a_{1}\right|^{2}\right) r(0)+r(1) a_{1}^{*}+r(-1) a_{1}
\]
.-which is minimized when
\[
: a_{1}=-\frac{r(1)}{r(0)}
\]
and the prediction error power is
\[
P_{1}=r(0)-\frac{|r(1)|^{2}}{r(0)}=P_{0}\left(1-\left|\frac{r(1)}{r(0)}\right|^{2}\right)
\]

The crosscorrelation between \(X\) and the prediction error sequence \(F_{1} X\) is \(F_{1}\left(z^{-1}\right) R_{x x}(z)\) and the coefficient of \(\mathrm{z}^{-1}\) vanishes since it is
\[
r(1)-\frac{r(1)}{r(0)} r(0)=0
\]
and in general the coefficients of \(\mathrm{z}^{-1}\) to \(\mathrm{z}^{-\mathrm{M}}\) vanish for the crosscorrelation of the prediction error sequence of order M and the sequence being predicted.

This general result follows because the power to be minimized is the coefficient of \(\mathrm{z}^{0}\) of the autocorrelation of \(F_{M}\left(z^{-1}\right) X(z)\), i.e. is the coefficient of \(z^{0}\) of \(F_{M^{*}}\left(z^{-1}\right) F_{M}\left(z^{-1}\right) R(z)\). By taking partial derivatives with respect
to conjugate of the filter coefficients, this means that the \(z^{0}\) coefficient of \(F_{M}\left(z^{-1}\right) R(z) z^{k}\) is zero, for \(\mathrm{k}=1,2, \ldots, \mathrm{M}\). But this then means that the coefficient of \(z^{-k}\) of \(\mathrm{F}_{\mathrm{M}}\left(\mathrm{z}^{-1}\right) \mathrm{R}(\mathrm{z})\) is zero, as stipulated.

The backward predictor polynomial is defined as \(\mathrm{B}_{\mathrm{M}}(\mathrm{z})\) and is a polynomial in z with coefficient of \(\mathrm{z}^{0}\) equal to unity which minimizes the power of \(\mathrm{B}_{\mathrm{M}}(\mathrm{z}) \mathrm{X}\), called the backward prediction error power. Since \(\mathrm{R} *(\mathrm{z})=\mathrm{R}(\mathrm{z})\), the backward predictor is the conjugate of the forward predictor, i.e., \(\mathrm{B}_{\mathrm{M}}(\mathrm{z})=\mathrm{F}_{\mathrm{M}^{*}}\left(\mathrm{z}^{-1}\right)\), and the coefficients of \(z\) to \(z_{M}\) of \(B_{M}(z) R(z)\) vanish.
\[
r(n)
\]


Figure 1 Correlation coefficients


Figure 2 The forward predictor \(F_{M}\left(z^{-1}\right)\) annihilates coefficients of \(F_{M}\left(z^{-1}\right) R(z)\)


Figure 3 The backward predictor \(B_{M}(z)\) annihilates some coefficients of \(B_{M}(z) R(z)\)

The forward predictors may be derived using the Levinson-Durbin algorithm which has a simple explanation using these figures. The annihilate the \(\mathrm{M}+1\) st coefficient, simply take \(\mathrm{FM}(\mathrm{z}-1) \mathrm{R}(\mathrm{z})\) which has
\(M\) zeros, and the \(M+1\) st coefficient is \(x\), say. Then subtract \(\left(x / P_{M}\right) z^{-(M+1)} B(z) R(z)\) which has zeros at \(1,2, \ldots, M\), and where \(P M\) is the coefficient of \(z^{0}\) of \(B_{M}(z) R(z)\) and \(\left.F_{M}\left(z^{-1}\right) R(z)\right]\). Thus
\[
\begin{aligned}
& F_{M+1}\left(z^{-1}\right)=F_{M}\left(z^{-1}\right)-\Gamma_{M} z^{-(M+1)} B_{M}(z) \\
& P_{M+1}=P_{M}\left(1-\left|\Gamma_{M}\right|^{2}\right) \\
& \Gamma_{M}=\frac{\text { coefficient of } z^{-(M+1)} \text { of } F_{M}\left(z^{-1}\right) R(z)}{P_{M}}
\end{aligned}
\]

Other statistical properties of forward and backward predictors may be obtained by noting that polynomials of lower order operating on \(\mathrm{F}_{\mathrm{M}}\left(\mathrm{z}^{-1}\right) \mathrm{R}(\mathrm{z})\), in particular the predictor polynomials, still leave some zeros, and these may be interpreted using Equation 2as crosscorrelation properties of prediction error sequences.

If only the reflection coefficients must be calculated from the correlation coefficients, the Leroux-Gueguen technique is preferred and works directly with the variable of \(\mathrm{F}_{\mathrm{M}} \mathrm{R}_{\mathrm{xx}}\); defined as \(\mathrm{H}_{\mathrm{M}}\left(\mathrm{z}^{-1}\right)\). We have
\[
\begin{aligned}
& H_{M+1}(z)=H_{M}(z)-\Gamma_{M} z^{-(M+1)} B_{M}(z) R(z) \\
& =H_{M}(z)-\Gamma_{M} z^{-(M+1)}\left[F_{M}\left(z^{-1}\right) R(z)\right]_{*} \\
& =H_{M}(z)-\Gamma_{M} z^{-(M+1)} H_{M^{*}}(z)
\end{aligned}
\]

Thus
the coefficient of \(\mathrm{z}^{-\mathrm{j}}, \mathrm{h}_{\mathrm{j} . \mathrm{N}}\) is. governed by
\[
\begin{gathered}
h_{j, M+1}=h_{j, M}-\Gamma_{M} h_{M-j, M}^{*} \\
\text { that is } \\
h_{j, M 1}=h_{j, M-1}-\Gamma_{M-1} h_{M-1-j, M-1}^{*}
\end{gathered}
\]
.In particular
\[
\begin{aligned}
& h_{M, M+1} \equiv 0 \xlongequal{\rightrightarrows} h_{M, M}-\Gamma_{M} h_{0, M}^{*} \\
& h_{0, M+1}=h_{0, M}-\Gamma_{M} h_{M, M}^{*}=h_{0, M}^{*}\left(1-\left|\Gamma_{M}\right|^{2}\right)=P_{M+1} \\
& h_{j, M+1}=0 \text { for } 1 \leq j \leq M
\end{aligned}
\]

If up to the N th reflection coefficient is required, then the indices on h vary from -( \(\mathrm{N}-1\) ) to N , and initially \(h_{0}=R\). These coefficients are bounded by the square root of \(P_{0} P_{M}\), and are thus less than \(P_{0}=r(0)\).

The analytic properties of predictor polynomials follow by considering \(\mathrm{z}^{\mathrm{M}+1} \mathrm{~F}_{\mathrm{M}+1}\left(\mathrm{z}^{-1}\right)\) which is a polynomial in z . An application of Rouche's theorem shows that \(\mathrm{z}^{\mathrm{M}+1} \mathrm{~F}_{\mathrm{M}+1}\left(\mathrm{z}^{-1}\right)\) has as many zeros inside the unit circle as \(z^{M+1} F_{M}\left(z^{-1}\right)=z z^{-M} F_{M}\left(z^{-1}\right)\). Since \(F_{0}\left(z^{-1}\right)\) has no zeros inside, \(F_{1}\) has one, etc. and \(F_{M}\) has \(M\) zeros inside the unit circle, as long as the reflection coefficients are less than unity in magnitude. That requirement follows because prediction error powers cannot increase as \(M\) increases. If we consider \(M\) approaching infinity, then all the coefficients of \(z^{-1}\) vanish in the product \(F_{M}\left(z^{-1}\right) R(z)\), and we label this
\[
\begin{aligned}
& \frac{1}{S_{+}\left(z^{-1}\right)}=\lim _{M \rightarrow \infty} F_{M}\left(z^{-1}\right) / \sqrt{P_{M}} \\
& \frac{1}{S_{-}(z)}=\lim _{M \rightarrow \infty} B_{M}(z) / \sqrt{P_{M}}=\frac{1}{S_{+^{*}}\left(z^{-1}\right)}
\end{aligned}
\]
and if we then determine the backward predictor polynomial that annihilates all the coefficients of power of \(z\), we are left with on the coefficient of \(z^{0}\).
\[
\frac{1}{S_{+}\left(z^{-1}\right) S_{-}(z)} R(z)=1
\]
which shows that \(\mathrm{R}(\mathrm{z})\) has been decomposed into the product of a forward and a backward prediction function, but probably not a polynomial. There is a lafge literature on determining these spectral factors.

In statistical theory.we say that two sequences are orthogonal if the coefficient of \(z^{0}\) in the crosscorrelation is zero. From equation 2this means that two sequences \(\mathrm{H}(\mathrm{z})\) and \(\mathrm{G}(\mathrm{z})\) are uncorrelated if the coefficient of \(\mathrm{z}_{0}\) of \(\mathrm{H}_{\mathrm{z}}(\mathrm{z}) \mathrm{G}_{*}(\mathrm{z}) \mathrm{R}(\mathrm{z})\) is zero. Define the sequence
\[
b_{M}=z^{-M} B_{M}(z) X
\]

Then \(b_{M}\) and \(b_{i k}\) are orthogonal, \(k=0, \ldots, M\) - 1 since the coefficient of \(z^{0}\) of
\[
z^{-M} B_{M}(z) z^{k} F_{k}\left(z^{-1}\right) R(z)
\]
is zero unless \(k\) is \(M\) when the crosscorrelation is \(P_{M}\).
Robinson and Treitel deal with two other properties of these sequences. The first is to recognize that the coefficient of \(z^{-N}\) of \(\mathrm{FN}(\mathrm{z}-1) 2 \mathrm{R}(\mathrm{z})\) is the prediction power times the negative reflection coefficient.
 \({ }^{{ }^{N}} \mathrm{~B}_{\mathrm{N}}(\mathrm{z}) \mathrm{X}\), the forward and backward predictors.

They also consider the problem: of all correlation functions, with the first N correlation coefficients fixed, which set produces a prediction error that is maximum. Their conclusion is that it'is the set of coefficients with vanishing reflection coefficients form \(k>N\), resulting in a prediction error of \(\mathrm{P}_{\mathrm{N}}\). Since maximum unpredictability corresponds to maximum entropy, they call \(\mathrm{FN}\left(\mathrm{z}^{-1}\right)\) the maximum entropy filter, and
\[
S_{M E M}(\omega)=\frac{P_{M}}{\left|F_{M}\left(e^{-j \omega}\right)\right|^{2}}
\]
is the maximum entropy estimate of the spectrum of the process of order \(M\). As \(M\) approaches \(\infty\) the power spectrum and the maximum entropy estimate are the same.

\section*{Eigenfilters}

The minimum eigenfilter associated with the random sequence \(X\) is that causal filter \(E\left(z^{-1}\right)\) of unit energy which minimizes the power of \(E X\). Such a filter thus minimizes the \(z^{0}\) coefficient of \(E\left(z^{-1}\right) E=(z) R(z)\) with the \(z^{0}\) coefficient of \(E\left(z^{-1}\right) E *(z)\) set to unity. By taking partial derivatives of the Lagrange multiplier formulation of this minimization problem, we determine that \(E\) is proportional to \(R E\) for the coefficients of \(z^{0}\) to \(z^{-K}\), where K is the order of \(\mathrm{E}\left(\mathrm{Z}^{-1}\right)\).

There are K such polynomials satisfying \(\mathrm{ER}=\lambda \mathrm{E}\) over the coefficients of E (and undefined outside this range); they are the K eigenfilters. The K eigenvalues \(\lambda_{\mathrm{K}}\) are real and non-negative, and are in the range of the power spectrum \(\mathrm{S}(\mathrm{f})\).

Since \(R(z)=R_{*}(z)\), if \(E\left(z^{-1}\right)\) is an eigenfilter then so is \(z^{-K} E^{*}(z)\) with the same eigenvalue as \(E\), and if the eigenvalues are distinct, then \(E\left(z^{-1}\right)\) and \(z^{-K} E^{*}(z)\) must be proportional to each other, and the constant of proportionality must have unit magnitude. With a suitable choice of constant multiplier, the eigenfilters are conjugate symmetric, that is, linear phase, with
\[
E_{k^{*}}(z)=z^{K} E_{k}\left(z^{-1}\right)
\]

The choice of the suitable constant multiplier is needed because antisymmetric filters, with the proper choice of multiplier, are symmetric. For example \(1-z^{-1}\) has conjugate \(1-z=-z\left(1-z^{-1}\right)\), but \(j\left(1-z^{-1}\right)=z^{-1}(-j+j z)\) is conjugate symmetric.

The zeros on the unit circle of the minimum polynomal will tend to be at the maxima of \(S(f)\) and the zeros of the maximum eigenfilter will tend to be at the minima of the spectrum \(S(f)\).

If the minimum eigenvalue is zero then \(R E\) is annihilated for \(z^{0}\) to \(z^{-K}\). Thus the minimum eigenfilter is also the forward predictor of order \(\mathrm{M}_{\text {; }}\) and the prediction error is zero.

As the minimum eigenfilter becomes longer and longer, it will tend to approximate an impulse at the frequency of the minimum of the spectrum \(\mathrm{S}(\mathrm{f})\).

If we plot the convolution of the eigenfilter with the correlation sequence, the middle portion over the extent of the eigenfilter tends to be large for the maximum eigenfilter and small for the minimum eigenfilter.

Thus, the maximum eigenfilter is rather related to a matched filter, and the minimum eigenfilter to a prediction filter.

Test
\[
R=\left(\begin{array}{ccc}
1 & 0.5 & 0.2 \\
0.5 & 1 & 0.5 \\
0.2 & 0.5 & 1
\end{array}\right)
\]
\[
\because
\]


Yule Walker
\[
\begin{aligned}
a_{0} & =(1), P_{0}=1 \\
a_{1} & =\binom{1}{-0.5}, P_{1}=0.75 \\
a_{2} & =\left(\begin{array}{c}
1 \\
-0.5333 \\
0.0667
\end{array}\right), P_{2}=0.75 * 224 / 225
\end{aligned}
\]

To find spectral factorization set \(a=\left[\begin{array}{ll}1 & 0.5 \\ 0.2\end{array}\right]\) and successively compute
\(\mathrm{a}=[\mathrm{a} 0]\)
chol(toeplitz(a))
using MATLAB
and take the last column.
\[
\begin{aligned}
S_{+} & =0.8577+0.4853 z^{-1}+0.2332 z^{-2} \\
R^{-1} & =\left(\begin{array}{ccc}
1.3393 & -0.7143 & 0.0893 \\
-0.7173 & 1.7143 & -0.7173 \\
0.0893 & -0.7273 & 1.3393
\end{array}\right)
\end{aligned}
\]

\section*{3. SEQUENCES OF VECTORS.}

Since all that is needed is a definition of correlation, we may use vectors of matrices, and define correlation in terms of \(\mathrm{VU}^{\mathrm{H}}\), where H represents complex conjugation.

Thus let \(\mathbf{X}\) be a stationary sequence of zero-mean circularly complex random vectors \(\{\mathbf{x}(\mathrm{i})\}\). The z transform of \(\mathbf{X}\) is
\[
\mathbf{X}(z)=\sum_{n=-\infty}^{\infty} \mathbf{x}(n) z^{-n}
\]

The autocorrelation function \(\mathrm{R}_{\mathrm{xx}}(\mathrm{z})\) is defined as
\[
\mathbf{R}_{X X}(z)=E\left[\mathbf{X}(z) \mathbf{x}^{H}(0)\right]
\]

Since
we have
\[
\ldots \mathbf{R}_{X X}(z)=\sum_{n=-\infty}^{\infty} \mathbf{r}_{X X}(n) z^{-n}
\]
\[
\begin{aligned}
& \quad \cdot \mathbf{r}_{X X}(n)=\mathbf{r}_{X X}^{H}(-n)
\end{aligned}
\]

Define the power of the sequence as the trace (sum of diagonal elements) of \(\mathbf{r}_{\mathrm{XX}}(0)\) which is real and positive and equal to \(E\left(x^{H}(0) x(0)\right)\), and
\[
\mid \operatorname{Tr}\left(\left.\mathbf{r}_{X X}(n)\right|^{2} \leq \operatorname{Tr}\left(\left.\mathbf{r}_{X X}(0)\right|^{2} .\right.\right.
\]

In general, for two sequences \(\mathbf{X}\) and \(\mathbf{Y}\), define the crosscorrelation as
\[
\mathbf{R}_{X Y}(z)=E\left(\mathbf{X}(z) \mathbf{y}^{H}(0)\right)
\]

If \(Y\) and \(Z\) are filtered versions of the same sequence \(X\) with transfer matrices \(G(z)\) and \(H(z)\) then
\[
\begin{equation*}
\mathbf{R}_{Y Z}(z)=\mathbf{G}(z) \mathbf{R}_{X X}(z) \mathbf{H}_{*}(z) \tag{2}
\end{equation*}
\]
where \(H^{*}(z)\) is \(H(z)\) with its coefficients Hermitian conjugated, \(z\) replaced with \(1 / z\).
Then
\[
\begin{aligned}
& E\left[\mathbf{Y}(z) \mathbf{z}^{H}(0)\right]= \\
& E\left[\sum_{n=-\infty}^{\infty} \mathbf{y}(n) \mathbf{z}^{H}(0) z^{-n}\right] \\
& =\sum_{n=-\infty}^{\infty} z^{-n} \sum_{m=-\infty}^{\infty} \mathbf{g}(n-m) \mathbf{x}(m) \sum_{r=-\infty}^{\infty} \mathbf{x}^{H}(r) \mathbf{h}^{H}(-r) \\
& =\sum_{n=-\infty}^{\infty} z^{-n} \sum_{m=-\infty}^{\infty} \mathbf{g}(n-\ddot{m}) \sum_{r=-\infty}^{\infty} \mathbf{r}_{X X}(m-r) \mathbf{h}^{\dot{H}}(-r) \\
& =\sum_{m, r, n} z^{-n} \mathbf{g}(n-r) \mathbf{r}_{X X}(m+r) \mathbf{h}^{H}(m) \\
& =\sum_{m, s, n} z^{-n} \mathbf{g}(s+m) \mathbf{r}_{X X}(n-s) \mathbf{h}^{H}(m) \\
& =\mathbf{G}(z) \mathbf{R}_{X X}(z) \mathbf{H}_{*}(z)
\end{aligned}
\]

In particular, if \(Y=G X\)
\[
\mathbf{R}_{Y Y}(z)=\mathbf{G}(z) \mathbf{R}_{X X}(z) \mathbf{G}_{*}(z)
\]

With scalars, the power in a signal is equal to \(r(0)\). With vectors, the power of all the signals is equal to the trace of the correlation matrix at time 0 ; thus, the trace will appear. This follows because
\[
E\left[\mathbf{X}(0)^{H} \mathbf{X}(0)\right]=E\left[\sum_{i} x_{i}(0) x_{i}^{*}(0)\right]=\sum_{i} \mathbf{R}_{i i}(0)=\operatorname{Tr}[\mathbf{R}(0)]
\]

This also follows form the relationship "trat if"AB is square, \(\operatorname{Tr}(\mathbf{A B})=\operatorname{tr}(\mathbf{B A})\).
If \(A\) is a square matrix, then the expectation of \(x^{H} A x\) is
\[
\begin{gathered}
E\left(\mathbf{x}^{H} \mathbf{A x}\right)=\operatorname{Tr}[\mathbf{A R}(0)] \\
\text { and } \\
E\left(\mathbf{X}^{H} \mathbf{A X}\right)=\operatorname{Tr}[\mathbf{A R}]
\end{gathered}
\]

Since the power of \(F X-\lambda X\) is positive, we must have
\[
\begin{aligned}
& \operatorname{Tr}\left[F_{*} F R\right]-\lambda^{*} \operatorname{Tr}\left[F_{*} R\right]-\lambda \operatorname{Tr}[F R]+\lambda \lambda^{*} \operatorname{Tr}[R(0)] \\
& |\operatorname{Tr}(F R)|^{2} \leq \operatorname{Tr}\left[F_{*} F R\right] \operatorname{Tr}(R)
\end{aligned}
\]
where all traces are over the elements that are the coefficients of \(z^{0}\).

The power spectrum is defined as
\[
\mathbf{S}_{X}(f)=\mathbf{R}_{X X}\left(e^{2 \pi j f}\right)
\]
and the diagonal terms are real and positive on the unit circle.

\section*{Predictor polynomials.}

The forward predictor polynomial \(\mathrm{F}_{\mathrm{M}}\left(\mathrm{z}^{-1}\right)\) of order M is defined as that polynomial of order M in \(\mathrm{z}^{-1}\) with the coefficient of \(z^{0}\) unity which minimizes the power of \(\mathbf{F}_{\mathrm{M}} \mathbf{X}\), called the prediction error power.

This general result follows because the power to be minimized is the coefficient of \(z^{0}\) of the autocorrelation of \(F_{M}\left(z^{-1}\right) \mathbf{X}(z)\), ie. is the coefficient of \(z^{0}\) of \(F_{M}\left(z^{-1}\right) \mathbf{R}(z) \cdot F_{M^{*}}\left(z^{-1}\right)\) By taking partial derivatives with respect to conjugate of the filter coefficients, this means that the \(z^{0}\) coefficient of \(F_{M}\left(z^{-1}\right) \mathbf{R}(z) z^{k}\) is zero, for \(k=1,2, \ldots, M\). But this then means that the coefficient of \(z^{-k}\) of \(F_{M}\left(z^{-1}\right) \mathbf{R}(z)\) is zero, as stipulated.

For example, with the coefficient of \(z^{-1}\) vanishing, we have, for the first order predictor
\[
\begin{aligned}
& \mathbf{R}(1)-\mathbf{F}(1) \mathbf{R}(0)=0 \\
& \mathbf{F}(1)=\mathbf{R}(1) \mathbf{R}(0)^{-1} \\
& \mathbf{F}=\mathbf{I}-\mathbf{R}(1) \mathbf{R}(0)^{-1} z^{-1} \\
& \mathbf{B}=\mathbf{I}-\mathbf{R}(0)^{-1} \mathbf{R}(-1) z
\end{aligned}
\]
\(\therefore\) The backward predictor polynomial is defined as \(\mathrm{B}_{\mathrm{M}}(\mathrm{z})\) and is a matrix polynomial in z with the coefficient of \(z^{0}\) equal to the unit matrix which minimizes the power of \(\mathbf{X} B_{M}(z)\), called the backward prediction error power. Since \(\mathbf{R}^{*}(z)=\mathbf{R}(z)\), the backward predictor is the conjugate -of the forward predictor, i.e., \(\mathbf{B}_{\mathrm{M}}(\mathrm{z})=\mathrm{F}_{\mathrm{M}^{*}}\left(\mathrm{z}^{-1}\right)\), and the coefficients of z to \(\mathrm{z}^{\mathrm{M}^{\mathbf{M}}}\) of \(\ddot{\mathbf{R}}(\mathrm{z}) \mathbf{B}_{\mathrm{M}}(\mathrm{z})\) vanish.

After annihilation up to degree \(M\), the matrix coefficient of \(z^{0}\) is denoted \(P_{M}\). The trace of \(P_{M}\) is the forward (or backward) prediction error.


Figure 4 Correlation coefficients


Figure 5 The forward predictor \(F_{M}\left(z^{-1}\right)\) annihilates coefficients of \(F_{M}\left(z^{-1}\right) R(z)\)


Figure 6 The backward predictor \(B_{M}(z)\) annihilates some coefficients of \(R(z) B_{M}(z)\)
\(\therefore\) The forward predictors may be derived using the Levinson-Durbin algorithm which has a simple explanation using these figures. To annihilate the \(\mathrm{M}+1\) st coefficient, simply take \(\mathrm{F}_{\mathrm{M}}\left(\mathrm{z}^{-1}\right) \mathbf{R}(\mathrm{z})\) which has M zeros, and the \(M+1\) st coefficient is \(x\), say. Then subtract \(x P_{M}^{-1} z^{-(M+1)} \mathbf{R}(z) B(z)\) which has zeros at \(1,2, \ldots, M\), and where \(P_{M}\) is the coefficient of \(z^{0}\) of \(R(z) B_{M}^{\prime}(z)\left[\right.\) and \(\left.F_{M}\left(z^{-1}\right) R(z)\right]\). Thus
\[
\begin{aligned}
& \mathbf{F}_{M+1}\left(z^{-1}\right)=\mathbf{F}_{M}\left(z^{-1}\right)-\mathbf{G}_{M^{\prime}} z^{-(M+1)} \mathbf{F}_{M^{*}}(z) \\
& \mathbf{P}_{M+1}=\left(I-\operatorname{CNM}_{M}^{H} \mathbf{F}_{M}\right) \mathbf{P}_{M} \\
& \Gamma_{M}=\left[\text { coefficient of } z^{-(M+1)} \text { of } \mathbf{F}_{M}\left(z^{-1}\right) \mathbf{R}(z)\right] \mathbf{P}_{M}^{-1}
\end{aligned}
\]

Other statistical properties of forward and backward predictors may be obtained by noting that polynomials of lower order operating on \(\mathbf{F}_{\mathrm{M}}(\mathrm{z}-1) \mathbf{R}(\mathrm{z})\), in particular the predictor polynomials, still leave some zeros, and these may be interpreted using Equation 2as crosscorrelation properties of prediction error sequences.

If only the reflection coefficients must be calculated from the correlation coefficients, the Leroux-Gueguen technique is preferred and works directly with the variable of \(\mathbf{F}_{\mathrm{M}} \mathbf{R}_{\mathrm{Xx}}\), defined as \(\mathrm{H}_{\mathrm{M}}\left(\mathrm{z}^{-1}\right)\). We have
\[
\begin{aligned}
& \mathbf{H}_{M+1}(z)=\mathbf{H}_{M}(z)-\Gamma_{M} z^{-(M+1)}\left[\mathbf{F}_{M}\left(z^{-1}\right) \mathbf{R}(z)\right]_{*} \\
& =\mathbf{H}_{M}(z)-\mathbf{G}_{M} z^{-(M+1)} \mathbf{H}_{M^{*}}(z)
\end{aligned}
\]

Thus
the coefficient of \(z^{-j}, h_{j, N}\) is governed by
\[
\begin{gathered}
\mathbf{h}_{j, M+1}=\mathbf{h}_{j, M}-\Gamma_{M} \mathbf{h}_{M-j, M}^{*} \\
\text { that is } \\
\mathbf{h}_{j, M 1}=\mathbf{h}_{j, M-1}-\Gamma_{M-1} \mathbf{h}_{M-1-j, M-1}^{*}
\end{gathered}
\]

In particular
\[
\begin{aligned}
& \mathbf{h}_{M, M+1} \equiv \mathbf{0}=\mathbf{h}_{M, M}-\Gamma_{M} \mathbf{h}_{0, M}^{H} \\
& \mathbf{h}_{0, M+1}=\mathbf{h}_{0, M}-\Gamma_{M} \mathbf{h}_{M, M}^{H}=\left(\mathbf{I}-\Gamma_{M}^{H} \Gamma_{M}\right) \mathbf{h}_{0, M}=P_{M+1} \\
& \mathbf{h}_{j, M+1}=\mathbf{0} \text { for } 1 \leq j \leq M
\end{aligned}
\]

If up to the Nth reflection coefficient is required, then the indices on h vary from.-( \(\mathrm{N}-1\) ) to N , and initially \(\mathrm{H}_{0}=\mathbf{R}\).

As before, define
\[
\begin{aligned}
& \mathbf{S}_{+}\left(z^{-1}\right)^{-1}=\lim _{M \rightarrow \infty} \mathbf{F}_{M}\left(z^{-1}\right){\sqrt{\mathbf{P}_{M i}}}^{-1} \\
& \mathbf{S}_{-}(z)^{-1}=\lim _{M \rightarrow \infty} \mathbf{B}_{M}(z){\sqrt{P_{M}}}^{-1}
\end{aligned}
\]
and if we then determine the backward predictor polynomial that annihilates all the coefficients of power of z , we are left with on the coefficient of \(\mathrm{z}^{0}\).
\[
\begin{aligned}
& \mathbf{S}_{+}\left(z^{-1}\right)^{-1} \mathbf{R}(z) \mathbf{S}_{-}(z)^{-1}=I \\
& \mathbf{R}(z)=\mathbf{S}_{+}\left(z^{-1}\right) \mathbf{S}_{-}(z)
\end{aligned}
\]
which shows that \(R(z)\) has been decomposed into the product of a forward and abackward prediction function, but probably not a polynomial, There is a large literature on determining these spectral factors. We have impolemented a program to do tḥ̣is[5].

In statistical theory we say that two sequences are orthogonal if the coefficient of \(z^{0}\) in the crosscorrelation is zero. From equation 2this means that two sequences \(\mathbf{H}(z)\) and \(\mathbf{G}(z)\) are uncorrelated if the coefficient of \(z^{0}\) of \(\mathbf{H}(\mathrm{z}) \mathbf{R}(\mathrm{z}) \mathbf{G}\). z\()\) is zero. Define the sequence
\[
\mathbf{b}_{M}=z^{-M} \mathbf{B}_{M}(z) \mathbf{X}
\]

Then \(b_{M}\) and \(b_{k}\) are orthogonal, \(k=0, \ldots, M\) - 1 since the coefficient of \(z^{0}\) of
\[
z^{-M} z^{k} \mathbf{F}_{k}\left(z^{-1}\right) \mathbf{R}(z) \mathbf{B}_{M}(z)
\]
is zero unless \(k\) is \(M\) when the crosscorrelation is \(\mathbf{P}_{\mathrm{M}}\).
Robinson and Treitel deal with two other properties of these sequences. The first is to recognize that the coefficient of \(z^{-N}\) of \(\mathbf{F}_{N}\left(\mathrm{z}^{-1}\right)^{2} \mathbf{R}(\mathrm{z})\) is the prediction power times the negative reflection coefficient. Therefore
they associate the reflection coefficient with the correlation coefficient of \(\mathbf{F}_{N}\left(z^{-1}\right) \mathbf{X}\) and \(z^{-N} \mathbf{B}_{N}(z) \mathbf{X}\), the forward and backward predictors.

As to eigenvectors, one approach is to consider the vector
\[
\begin{aligned}
& \mathbf{H}(z)=\sum_{k=0}^{M-1} \mathbf{H}_{k} z^{-k} \\
& \mathbf{H}_{*}(z)=\sum_{k=0}^{M-1} \mathbf{H}_{k}^{H} z^{k}
\end{aligned}
\]
and minimize the dc coefficient of the scalar
\[
\mathbf{H}_{*}(z) \mathbf{R}(z) \mathbf{H}(z)
\]
subject to the dc coefficient of
\[
\mathbf{H}_{*}(z) \mathbf{H}(z)
\]
being fixed.

These lead to equations of the form
\[
\left(\begin{array}{cccc}
\mathbf{r}(0) & \mathbf{r}(-1) & \Lambda & \mathbf{r}(-(N-1)) \\
\dot{\mathbf{r}}(1) & \mathbf{r}(0) & \Lambda & \mathbf{r}(-(N-2)) \\
\Lambda & \Lambda & \Lambda & \Lambda \\
\mathbf{r}(N-1) & & \Lambda & \mathbf{r}(0)
\end{array}\right)\left(\begin{array}{c}
\mathbf{h}(0) \\
\mathbf{h}(1) \\
\Lambda \\
\mathbf{h}(n)
\end{array}\right)=\lambda\left(\begin{array}{c}
\mathbf{h}(0) \\
\mathbf{h}(1) \\
\Lambda \\
\mathbf{h}(n)
\end{array}\right)
\]

Another approach would be to consider H as a matrix polynomial and minimize the trace of the dccoefficient.
\[
\because
\]

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\section*{APPENDIX 2 MARKOV DETECTION}

Here is a short review of the theory [1,2] of Markov detection. It covers Viterbi and least-error-bitprobability detection.

Assume that there is an underlying Markov process with \(M\) states \(u_{k}\), and associate with each transition a known output \(x_{k}\). Because of the Markov property \(x_{k}\) depends only on the states \(u_{k-1}\) and \(u_{k}\), and is stationary. The received signal is \(z_{k}=x_{k}+n_{k}\), where \(n_{k}\) is uncorrelated Gaussian noise. The problem is to determine the sequence \(\mathbf{u}=\left\{\mathrm{u}_{0}, \mathrm{u}_{1}, \mathbf{u}_{2}, \ldots, \mathrm{u}_{\mathrm{K}}\right\}\) of Markov states from the received sequence \(\mathrm{z}=\left\{\mathrm{z}_{0}, \mathrm{z}_{1}, \ldots, z_{\mathrm{K}}\right\}\). Let \(\mathbf{u}_{0}{ }^{\text {k }}\) represent the subsequence states from time 0 to time k .

The joint probability \(\mathrm{P}(\mathrm{u}, \mathrm{z})\) satisfies
\(P(\mathbf{u}, \mathbf{z})=P(\mathbf{u}) P(\mathbf{z} \mid \mathbf{u})=P(\mathbf{u}) \prod_{k=0}^{K-1} P\left(z_{k} \mid u_{k}, u_{k-1}\right) \doteq \prod_{k=0}^{K-1} P\left(u_{k+1} \mid u_{k}\right) \prod_{k=0}^{K-1} P\left(z_{k} \mid u_{k}, u_{k-1}\right)\)
A given transition is assigned a length \(\lambda\left(\mathrm{u}_{\mathrm{k}}, \mathrm{u}_{\mathrm{k}-1}\right)\) given by
\[
\lambda\left(u_{k}, u_{k-1}\right) \equiv-\ln \left(P\left(u_{k}, u_{k-1}\right)\right)-\ln \left(P\left(z_{k} \mid u_{k}, u_{k-1}\right)\right)
\]
so that
\[
-\ln (P(\mathbf{u}, \mathbf{z}))=\sum \lambda\left(u_{k}, u_{k-1}\right)
\]

The quantity \(-\ln (\mathrm{P}(\mathbf{u}, \mathbf{z}))\) is the likelihood function, i.e., the length of the path. The expression above states that the length of the path is the sum of the length of the individual links in the path.

The prototype problem is as follows. The beginning and end states are known. A sequence is received. The intermediate states are to be estimated. The probability of error is also to be determined.

The Viterbi algorithm determines the most likely sequence. It does this by proceeding, in time, to find those sequences which were most likely to have led to all the states, and their likelihood; these sequences are called the survivor sequences. The procedure is then repeated for the next time, making use of the previously determined most likely sequences terminating in a particular state. Thús, at any time k , it is only necessary to remember the M survivor sogúceices.

There are other algorithms for these problems. Let \(y_{j}^{k}\) represent the output sequence from time \(j\) to time \(k\), inclusive, and let \(\xi_{k}\) represent the pair of states \(\left(u_{k}, u_{k+1}\right)\), i.e., the transitions. Then another approach calculates the state and transition probabilities \(\mathrm{P}\left(\mathrm{u}_{\mathrm{k}} \mid \mathrm{z}\right)\) and \(\mathrm{P}\left(\xi_{k} \mid \mathrm{z}\right) \mathrm{m}\), allowing a variety of performance measures to be optimized. This calculation is done as follows.

Firstly, it is more convenient to use the probabilities without conditioning. Three intermediate probabilities are used. These are
\[
\begin{aligned}
& \dot{\alpha}_{k}\left(u_{k}\right)=P\left(u_{k}, \mathbf{z}_{0}^{k-1}\right) \\
& \beta_{k}\left(u_{k}\right)=P\left(\mathbf{z}_{k}^{K} \mid u_{k}\right) \\
& \gamma_{k}\left(u_{k-1}, u_{k}\right)=P\left(u_{k}, z_{k} \mid u_{k-1}\right)
\end{aligned}
\]
\[
\begin{aligned}
& P\left(u_{k}, \mathbf{z}\right)=P\left(u_{k}, \mathbf{z}_{0}^{k-1}\right) P\left(\mathbf{z}_{k}^{K} \mid u_{k}, \mathbf{z}_{0}^{k-1}\right) \\
& =P\left(u_{k}, \mathbf{z}_{0}^{k-1}\right) P\left(\mathbf{z}_{k}^{K} \mid u_{k}\right) \\
& =\alpha_{k}\left(u_{k}\right) \beta_{k}\left(u_{k}\right)
\end{aligned}
\]
since the outputs \(z_{k}{ }^{K}\), given \(u_{k}\), are independent of the earlier outputs.
Also
\[
\begin{aligned}
& P\left(\zeta_{k}, \mathbf{z}\right)=P\left(u_{k}, u_{k+1}, \mathbf{z}\right) \\
& =P\left(u_{k}, \mathbf{z}_{0}^{k-1}\right) P\left(u_{k+1}, z_{k} \mid u_{k}, \mathbf{z}_{0}^{k}\right) P\left(\mathbf{z}_{k+1}^{K} \mid u_{k+1}, u_{k}, \mathbf{z}_{0}^{k}\right) \\
& =P\left(u_{k}, \mathbf{z}_{0}^{k-1}\right) P\left(u_{k+1}, z_{k} \mid u_{k}\right) P\left(\mathbf{z}_{k+1}^{K} \mid u_{k+1}\right) \\
& =\alpha_{k}\left(u_{k}\right) \gamma_{k}\left(u_{k}, u_{k+1}\right) \beta_{k}\left(u_{k+1}\right)
\end{aligned}
\]

The three unknown are computed recursively by
\[
\alpha_{k}\left(u_{k}\right)=\sum_{u_{k-1}} \alpha_{k-1}\left(u_{k-1}\right) \gamma_{k}\left(u_{k-1}, u_{k}\right)
\]
using forward recursion, with the values for \(\mathrm{k}=0\) assumed known,
\[
\beta_{k}\left(u_{k}\right)=\sum_{u_{k+1}} \beta_{k+1}\left(u_{k+1}\right) \gamma_{k+1}\left(u_{k}, u_{k+1}\right)
\]
using backward recursion; and.
\[
\gamma_{t}\left(u_{k-1}, u_{k}\right)=\sum_{x_{k}} \operatorname{Pr}\left(u_{k} \mid u_{k-1}\right) \operatorname{Pr}\left(x_{k} \mid u_{k-1}, u_{k}\right) \operatorname{Pr}\left(z_{k} \mid x_{k}\right)
\]
or
\[
\gamma_{t}\left(u_{k-1}, u_{k}\right)=P\left(u_{k}, z_{k} \mid u_{k-1}\right)=P\left(z_{k} \mid \cdot u_{k}^{\cdot}, u_{k-1}^{\circ}\right) P\left(u_{k} \mid u_{k-1}\right) .
\]

More details also appear in [3].

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\title{
POLYPHASE FILTERS FOR COMMUNICATIONS EW SYSTEMS \\ W F McGee \\ 10/03/97 2:05 PM \\ Contract U6800-6-3505
}

\begin{abstract}
This note is a short investigation of the use of filterbanks for spectrum analysis for an application described by Inkol[1]. After reviewing filterbanks and windowed FFTs, we determine that they differ only in that filterbanks tend to use Nyquist, or root-Nyquist prototype filters, which eliminates the picket-fence effect, at then expense of filter length for a given stopband suppression. With root-Nyquist filters, a quadratic output appears natural; full Nyquist filters would tend towards envelope outputs.

For realization, the use of polyphase filterbanks is a desirable feature, since they are very efficient. It may be beneficial to do the polyphase processing after the FFT, and this is discussed. But their use is independent of the use of windows or Nyquist filters.

Wavelets are analyzed for this application, but they are not attractive, since the frequency channels are linearly spaced.

Finally, we include an estimate of the total delay that would be suffered by typical.linear phase, equiripple stopband prototype filters.
\end{abstract}

\section*{0. PERFORMANCE OF SPECTRUM ANALYZERS.}

In our minds we have a picture of the time-frequency plane, with a positive function of time and frequency, unfolding in time. But there is no universal definition for this intuitive concept, and therefore the application drives the tools.

This is commonly displayed using a waterfall plot.

\section*{burst of sine wave}

\(=\)

Issues include
-response time -spectral leakage
-output sampling rate
-picket fence effect

\section*{1. ANALYSIS/SYNTHESIS FILTERBANKS AND SPECTRUM ANALYZERS}

A spectrum analyzer measure signal power spectrum in time. A common spectrum analyzer passes the signal to a filterbank, and takes the instantaneous squared magnitude (or perhaps the envelope) of the output as an estimate of the power spectrum.

Analysis/synthesis or synthesis/analysis filterbanks are used in signal processing and communications. The analysis filterbank expresses an input signal as a linear sum of signals, each of which has a restricted frequency response, where the frequency response is characterized by the filterbank filter responses. A synthesis filterbank, on the other, combines signals together to form an output signal.

In an analysis/synthesis filterbank (first analyze, then synthesize) it is useful if the back-to-back cascade results in an output signal that is the same as the input. Similarly, it is useful if a synthesis/analysis filterbank cascade results in a set of signal that is unchanged.

If we focus attention on the filters, a block diagram may be made.


Figure 2 Each block is a filter. The four filters in the analysis filterbank generate four channel signals from the composite input signal; the synthesis filterbank generates a composite signal from the four channel signals. Often the frequency response of the two corresponding filters in the analysis and synthesis filterbank are matched to each other, with the same magnitude frequency response, and opposite phase frequency response.


Analysis
Figure 3 One implementation of a spectrum analyzer. The squared magnitude of the output signals are used.

If we consider the requirements on the filters, it is useful that the sum of the filter responses is a constant, and also that the sum of squares magnitude responses is constant in frequency.


Figure 4 Another implementation of a spectrum analyzer. The magnitude of the output of the cascade of corresponding analysis and synthesis filters used. In this realization the sum of the four outputs is equal to the input signal in a good analysis/synthesis filterbank.


Figure 5 The filter responses may be combined. \(\because\)


Most filterbanks are based on a lowpass prototype filter, a Nyquist filter \(\mathrm{N}(\mathrm{z})\), or a pair of prototype rootNyquist (sometimes called half-Nyquist) filters \(\mathrm{H}(\mathrm{z})\) and \(\mathrm{G}(\mathrm{z})\) such that \(\mathrm{H}(\mathrm{z}) \mathrm{G}(\mathrm{z})\) is a Nyquist filter and H and G have the same magnitude response on then unit circle. If there are \(M\) channels, the Nyquist filters is called an M-band (or Mth band) filter has all the coefficients at times kM from the center coefficient equal to zero.

This requirement in the pulse response may be interpreted in the frequency domain, and the sum of frequency shifted (by \(\mathrm{k} / \mathrm{M}\) ) versions of the frequency response is a constant, usually set to unity. Thus
\[
\sum_{\mathrm{k}} N\left(e^{j \omega-2 \pi k / M}\right)=\mathrm{constant}
\]


Figure 6 A set of translated Nyquist filters sums to a constant, is positive, and crossover at the \(\mathbf{6 - d B}\) frequency.

For low-pass filters that cutoff before the frequency \(1 / \mathrm{M}\), this means that the loss at the frequency \(1 / 2 \mathrm{M}\) is 6 dB.

This in turn implies that the loss of a half-Nyquist prototype filter at this frequency is 3 dB with respect to the loss at 0 frequency.

If we use a root-Nyquist based filterbank, it makes sense, it seems to me, to take the squared magnitude of the filter outputs, since, with fixed-magnitude but variable frequency sinuisoidal input, the sum of all the outputs is a constant as the frequency is swept about the unit circle.

On the other hand, is we base the design on a full-Nyquist prototype filter it would seem reasonable to me to take the (unsquared) envelope of the filter outputs to represent the spectrum equalizer output, and the sum of all the envelopes is a constant with a variable frequency sinusoid sweeping across the band of the spectrum analyzer.

\section*{3. FFT-BASED FILTERBANKS}
'Because of the great efficiency of the FFT, there is tremendous interest in using this signal processing approach to frequency selection, and to generalize it in many ways. We are going to concentrate on the frequency response of these filterbanks, and for that purpose it is useful to draw the typical FFT processor in the following block diagram.


Figure 7 A representation of the usual application of the unwindowed DFT filterbank putting the delays explicitly in view.

The transfer function from the input to the first channel output is
\[
P(z)=\sum_{s=0}^{N-1} z^{-s}=\frac{1-z^{-M}}{1-z^{-1}}
\]
and to the other filter outputs, for \(\mathrm{k}=1 . \mathrm{M}\), are
\[
P\left(W^{k-1} z\right)
\]
where
\[
W=e^{-j 2 \pi / M}
\]
and the various frequency responses are simply those of the lowpass \(\mathrm{P}(\mathrm{z})\) shifted around the unit circle.
The first generalization is the concept of windowing, where weights are included in the input arms, as follows


Figure 8 The first generalization is to include weights \(w_{0}, w_{1}\), etc. on the inputs and/or outputs.

This gives a frequency response
\[
P(z)=\sum_{s=0}^{M-1} w_{s} z^{-s}
\]

The synthesis filterbank responses are shifted versions of the prototype polynomial \(P^{*}(z)\) given by
\[
P_{*}(z)=\sum_{s=0}^{M-1} z^{-s} w_{M-1-s}
\]
that is, \(P(z)\) with its coefficients written backwards.
The extension of this idea is to use weights that are polynomials in \(z^{-M}\). This is the basis of Bellanger's polyphase expansion; given any \(\mathrm{H}(\mathrm{z})\), form the polyphase expansion
\[
H(z)=\sum_{s=0}^{M-1} z^{-s} H_{k}\left(z^{M}\right)
\]
and set the weights equal to \(\mathrm{H}_{\mathrm{k}}\left(\mathrm{z}_{\mathrm{M}}\right)\); this allows any frequency response.
If the only technique available is a windowed FFT without the delays required for a polyphase FFT , then one approach is to use an FFT whose size is the same as the length of the prototype filters, and weight it with the prototype filter coefficients, and throw away many of the output values.

\section*{4. WINDOWING}

The use of windows to improve frequency response is common in spectrum analyzers. This is usually presented by noting that the use of the ordinary.FFT with all the weights equal to unity is like a convolution with a time function that is a square pulse, and that this is reason for the poor frequency response. If we take the filterbank approach then the polyphase representation of the synthesis filterbank is not useful, because the summing operation is built in. The idea then is to combine the analysis and synthesis filters in one branch as one filter, and simply use this in the polyphase structure.

Consequently, the windowed DFT has the following block diagram.


Figure 9 Windowed FFT spectrum analyzer. The squared outputs are used for spectrum analysis.

The prototype filter is
\[
H(z)=\sum_{k=0}^{L-1} w_{k} z^{-k}
\]
and this is what is drawn in the window literature[1]. But the translated responses are ordinarily not drawn.
The windowed DFTs are used to minimize spectral leakage. This is commonly viewed by examining the response of prototype filter. But in the filterbank design area, we would describe the sidelobe suppression by the maximum interference that may be exerted in the passband of a filter due to channels separated by two or more channels away. We do not ordinarily worry about the effect of the next channel, because it has to share a signal anyway. We have attached some responses of common windows, but we have kept the stopband frequency fixed at \(1 / \mathrm{M}\). In the example figures, the number \(\mathrm{M}=32\), so there are 32 channels.

By design, then, the placement of the stopband is fixed. The remaining factors to include are the 'picket fence' effect, and the spectral leakage.

The details of the examination are in Appendix 1
It appears that there are standard windows that achieve any required spectral leakage requirement, but they do not have control of the 'picket fence' effect. The class of half-Nyquist filters achieves both the required spectral leakage and eliminate the picket fence effect. Notice that the length \(\mathrm{L}=124\) Dolph-Chebyshev filter is too long for the standard windowed FFT and so a polyphase realization would be required.

The polyphase FFT is a technique to realize filterbanks efficiently which allows the use of 'window' functions that are longer than the dimensions of the FFT , and which allow less spectral leakage.

\section*{5 AN ALTERNATIVE INTERPRETATION OF WINDOWING.}

If we examine the previous figure, it can be seen that the portion after the delay lines may be represented as a diagonal matrix operating on N inputs, followed with an N -dimension DFT .

The signal processing, in matrix terms, is
\[
F D
\]
where F is the FFT matrix and D is a diagonal matrix.
This is equal to
\[
\therefore: \quad C F
\]
where C is a circulant matrix, i.e., each row is simply a shifted version of the previous row. According to this view, we may consider a windowed FFT as one in which the input is applied directly to the FFT without windowing, and processing is applied on the outputs. The processing is identical for all the outputs, and consists of taking weighted sums of outputs; the weights are the same for each output. For example, with the "Hamming window, this is equivalent to taking the output as the same output of the DFT multiplied by 0.54 and 0.23 times the sum of the two adjacent outputs.



Figure 10 A Windowed DFT may also be considered as a DFT applied first, followed with a frequency-domain weighed sum, for each output. This may be useful if not all the outputs are required, for example, or for hardware reasons.

\section*{6. WAVELETS.}

The elements of wavelet theory are summarized as follows. The wavelet theory is based on a prototype halfband filter \(\mathrm{H}(\mathrm{z})\) satisfying the 2-Nyquist condition; every second coefficient of \(\mathrm{H}(\mathrm{z}) \mathrm{H}^{*}(\mathrm{z})\) except the center one vanishes. And decimation is by 2 . Here is the block diagram of an analysis/synthesis system.

\[
=
\]

Figure 11 Half-band analysis synthesis. The lower * indicates the filter with its coefficients written in reverse order. The filter orders are odd.

For an input sequence \(\mathrm{X}(\mathrm{z})\), the output is, with even samples on the top and on the bottom
\[
\begin{aligned}
& X(z)\left[H(z) H_{*}(z)-H(-z) H_{*}(-z)\right]+X(-z)\left[H(-z) H_{*}(z)-H_{*}(z) H(-z)\right] \\
& =X(z)\left[H(z) H_{*}(z)-H(-z) H_{*}(-z)\right]
\end{aligned}
\]
and the condition for perfect reconstruction is that
\[
\left[H(z) H_{*}(z)-H(-z) H_{*}(-z)\right]
\]
which has only odd coefficients is to be a monomial.

The concept of half-band filtering may be continued by constructing an analysis tree or pyramid as follows


Figure 12 Tree of half-band filters and decimators. The symbol rate is reduced by 2 .

For analysis purposes it is actually easier to draw the sampling at the tips of the branches. As well the filters are reordered so that the frequency bands of adjacent outputs are touching.


Figure 13 This process of pyramidal analysis may be continued indefinitely.

In the theory of wavelet the same prototype filter is used at every level, and results in a rather elegant theory. But for most signal processing applications this elegance is wasteful, and it is possible to relax the filter design requirements for each level. The reason that this is desirable is the following. If we write the transfer function from the input to output for one output channel it will consist of a product of filter transfer functions \(\mathrm{Hi}\left(\mathrm{z}^{\mathrm{K}}\right)\) or \(\mathrm{G}_{\mathrm{i}}\left(\mathrm{z}^{\mathrm{K}}\right)\) where \(\mathrm{G}_{\mathrm{i}}(\mathrm{z})=\mathrm{Hin}^{*}(-\mathrm{z})\). Suppose that the transfer function is
\[
H_{1}(z) G_{2}\left(z^{2}\right) H_{3}\left(z^{4}\right)
\]

The transfer function for the next channel will have only one of the constituent transfer functions changed. This means that the transition band between the two channels is determined by the transition band of only one of the filters. Because in most applications the transition bands should be similar, and filters that are higher in the pyramid have inherently narrower transition bands due to the presence of powers of \(z\) ion the transfer function, this means that the prototypes for the leaves of the tree (the lower layers of the pyramid) may have wider transition bands.

For example, the four filters for the tree in the figure are \(\mathrm{HH}, \mathrm{HG}, \mathrm{GG}, \mathrm{GH}\). Thus if we use the notation \{\}* to mean a sequence backwards, the next sequence of filter transfer function is obtained from the previous sequence A by forming \(\left\{\mathrm{HA}, \mathrm{GA}^{\prime}\right\}\).

We mention in passing that the theory of wavelets makes use of iterated transfer functions, in particular an iterated lowpass transfer function. This requires that the frequency response have its maximum at 0
frequency, which implies that \(G(z)\) has a zero at -1 . The number of zeros at -1 is the regularity of the wavelet. The Daubeschies wavelets are formed from a half-band half-Nyquist filter that has all its zeros in the stop band at -1 ; the other zeros are used to obtain the Nyquist property.

If we examine the frequency response of the pyramidal filterbank, then the response is that of real bandpass filters. Since in many communications examples the resulting two sidelovbes are not useful, the input signal would have to restricted to positive (or negative) frequency using a Hilbert transformer; in factor, a halfband, or quarter-band Nyquist filter may be used. This would allow envelopes to be determined.

If it is not useful to obtain the intermediate stages in the pyramidal analysis, there appears to be no advantage of wavelet transforms over polyphase filtering.

\section*{7. THE INFLUENCE OF FILTER PARAMETERS ON DELAY.}

Real time applications imply the need for fast response, and short delays. By delay we mean the time taken for a burst of sine wave to cause the output of detector to rise to an acceptable value. In general, the more filter coefficients that there are, the more the delay. The length of narrowband filters with good stopband performance is proportional to the attenuation in the stopband and the width of the transition band relative to the total bandwidth. The transition band is the frequency between overlapping passbands and the beginning of the high loss frequency band, and is commonly expressed as a percent of the distance from the band center to the passband edge; Typical values range up to 100 percent. An approximate results for linear phase filters is that the length L is approximately
\[
L=\frac{A_{s}}{15} \frac{M}{\alpha}
\]

For example the MPEG linear phase filters with a stopband loss of 120 dB , a cutoff of 100 percent and 64 channels has \(L=8 \mathrm{M}=512\). The DFT filterbanks with a stopband loss of about 15 dB have a length M . The ., delay of such a filter is about \(\mathrm{L} / 2\).

For even less delay the use of minimum.phase filterbanks is desirable. They are not as long as a linear phase filterbank for the same stopband loss, and have the least delay.

For an M-band Nyquist filter, the pulse response is small except for a sequence of 2 M samples. Therefore the risetime will be about 2 M samples. There will be an associated delay equal to about half the length of the Nyquist filter.

2M
Pulse response of a Nyquist filter

\section*{Step response. The rise time is about 2 M samples.}

Figure 14 Pulse responses

\section*{8. CONCLUSION: DESIRABLE PROPERTIES OF FILTERBANK SPECTRUM ANALYZERS}

These spectrum analyzers are for identifying narrowband signals.
1. A given input sinusoid should, in the steady state, produce a minimal number of outputs. The outputs should be real positive quantities indicating the presence of a signal in a particular frequency band. The minimum number of possible outputs that are excited for full coverage is two. The existence of other outputs due to an input sinusoid is 'spectral leakage' and should be minimized. A constant level sinusoid should produce a constant output.
2. The spectrum analyzer should be such that it permits narrow band signals to be recovered. In particular, the input should be recoverable.
3. If the sum of the analyzer outputs is added together, an input sinusoid swept across the band should result in a constant sum.
4. The delay from input to output should be small. This implies the use of minimum phase prototype filters.
5. The risetime should be small and predictable.

\[
\therefore x+x
\]

禺
6. The output should be linear or quadratic in the input signal.
7. If two sinusoids are separated in frequency and have random phase, the outputs should be independent of each other. If the sinusoids are in the same bin then the output should be the sum of the outputs for each sinusoid in isolation.

Based on these requirements, we recommend the use of M -band minimum-phase root-Nyquist filters, and a quadratic detector at the output of each filter.

For computing efficiency, the filters should be realized as FIR filters in a polyphase structure.

\section*{ACKNOWLEDGEMENTS}

This work has benefited from discussion with Robert Inkol, John Lodge and Mike Sablatash. John Lodge stressed the need for good stopband rejection, and fast response.

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In this examination, we keep the width of position of the stopband fixed. All the illustrations are of a 32channel system.


Figure 1 Frequency Response of the 'BOXCAR' width; equal weights, length 32.
The problem with this frequency response-is that the stopband loss is only 14 dB . The crossover loss is about 3 dB .


Figure 2 Frequency response of the Hamming window, length 64, for a 32-channel spectrum analyzer. The spectral leakage is about 43 dB , and the crossover loss is about 6 dB . This would be a
design for a full-Nyquist filterbank
Chebyshev length 32 cutoff \(2^{*}\) pi/32


Figure 3 An equiripple stopband results with the Dolph-Chebyshev design. The length is 32 for a 32channel system; the stopband performance is slightly better than the 'boxcar' weight.

Chebyshev length 96


Figure 4 Frequency response of a Chebyshev window of length 96. The stopband loss has increased to 75 dB . But note that the crossover loss is not controlled; it is about 10 dB , so there will be a significant picket fence effect.


Figure 5 Frequency response of a 32-channel spectrum analyzer with a Blackman window. The worst case sidelobe suppression is about 59 dB , and the crossover loss is about 10 dB , resulting in a picket fence effect. Although it is not clear in this figure, the stopband loss continues to fall off as we depart from the main channel, which falloff is often a feature desired by system designers.


Figure 6 Frequency response of an 8-channel spectrum analyzer using a root-Nyquist, equiripple stopband. The filter length is 37 . The loss at the crossover frequency is \(\mathbf{3 d B}\). A 32 -channel filter with the same loss would be about 4 times as long, about 150 coefficients.
;-"

Pulse and stip response for f32_8


Figure 7 Step and pulse response for the 8-channel filter of Figure 6 Frequency response of an 8channel spectrum analyzer using a root-Nyquist, equiripple stopband. The filter length is 37 . The loss at the crossover frequency is 3 dB . A 32 -channel filter with the same loss would be about 4 times as long, about 150 coefficients. The respponse is minimum phase, and so responds as soon as possible.


\title{
AN VESTIGIALLY ANALYTIC WAVELET
}

Issue 2
W F McGee

\section*{Contract U6800-6-1604}

18 March 1997

\begin{abstract}
The purpose of the note is to propose an vestigially analytic wavelet.
Wavelets are real functions, and thus bandpass wavelets have two sidebands, one for positive and one for negative frequency. This is unattractive. We propose the following definition of a vestigially analytic wavelet.
\end{abstract}

Let \(H(z)\) be a half-band Nyquist filter. Let \(G(z)\) be itsoiresponding high-pass counterpart; generally, \(G(z)=H(-1 / z)\).

Define
\[
\Phi(\omega)=H\left((-j z)^{1 / 2}\right) H\left((-j z)^{1 / 4}\right) H\left((-j z)^{1 / 8}\right) \ldots
\]
where
\[
z=e^{j \omega}
\]
and define
\[
W(\omega)=G(-j z) \Phi(\omega)
\]
"Consider the following analysis tree


Figure 1 Analytic Wavelet Tree The transfer function from the input to the output nodes is a scaled copy of the transform of \(W(\omega)\).

Then the transfer function from the input to the output nodes are the Fourier transforms of
\begin{tabular}{ll} 
node 1 & \(\mathrm{W}(\omega)\) \\
node 2 & \(\mathrm{W}(2 \omega)\) \\
node 3 & \(\mathrm{W}(4 \omega)\)
\end{tabular}
etc.
and has little negative frequency contenet; i.e., it is vestigially analytic.

\section*{Remaining work}

Does \(\Phi\) exist. Chancesa re better than the proposal of issue 1 .
This is just the theory of wavelets shifted by \(\pi / 2\).
Acknowledgements
Coffee discussions with Paul Guinand and Mike Sablatash led to this proposal.

\section*{LKC}

TK7872 .F5 M32 1997
Study into improved spectrum efficiency for FDMA/TDMA transmission in mobile satellite and mobile environments
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